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BY
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Project Status

The present report is one of a series of finite element modeling studies which is to be used as a basis for a handbook of modeling techniques.

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ABSTRACT

The dynamic mode shapes and frequencies of a circular ring are obtained using finite element techniques. The results obtained from models making use of half symmetry and of quarter symmetry are compared with those for the complete ring and with the exact solution.

The half-ring model provides the same solution as the complete ring. The quarter-ring model, however, must be subjected to two separate sets of boundary conditions in order to obtain the complete set of mode shapes.

This same quarter-ring model is then subjected to a third set of boundary conditions to obtain the same solution as an infinitely long "corrugated" wire.

CONTENTS

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1. INTRODUCTION

In the finite element modeling of symmetric structures, a saving in computer time can often be obtained by determining the response of a segment of the structure constrained by appropriate boundary conditions.

A class of such structures is characterized by radial symmetry (e.g., cylindrical shells, spherical shells, circular disks and circular rings.) The present investigation demonstrates a method of approach in applying symmetry-oriented modeling techniques to structures possessing radial symmetry.

Computations were performed on the IBM 7094 computer using the Martin Company SB038 program for the analysis of finite element models by the force method.[1, 2]

2. ANALYTICAL SOLUTION

The in-plane vibrations of a circular ring are discussed by Timoshemko in Vibration Problems in Engineering, p. 425 ff.[3]

The frequency of pure radial vibration is:

$$p = \sqrt{\frac{Eg}{\gamma r^2}}$$

All displacements are in the radial direction, and hence the motion is described by a circle of periodically varying radius.

The frequencies for longitudinal (circumferential) vibrations are given by

$$p_i = \sqrt{\frac{Eg}{\gamma r^2}} \sqrt{1 + i^2}$$

where i denotes the number of wave lengths to the circumference. All displacements in this case are in the circumferential direction. The modal frequencies for flexural vibrations are:

$$p_i = \sqrt{\frac{Eg}{\gamma} \left(\frac{I}{Ar^4}\right) \frac{i^2 (1 - i^2)^2}{(1 + i^2)}}$$

where the mode shapes are described by:

$u_i = a_i \cos i\phi$ = radial displacement

$v_i = a_i/i \sin i\phi$ = circumferential displacement in which a_i is a constant coefficient.

I = area moment of inertia of cross section

g = acceleration of gravity

E = Young's modulus

r = radius of the ring

γ = specific weight of the ring

A = cross sectional area

3. PARAMETER VALUES

The following values of the parameters were chosen for present investigations:

r = 10.0 in.

E = 1.0×10 lb/in.

$g = 386.06$ in./sec^2

$A = 0.19634954$ in.^{2}

$\gamma = 25.9382231$ lb/in.^{3}

$I = 0.0654498467$ in.^{4}

The values for the last three parameters were chosen so as to provide convenient values for the grid point masses and element compliances in the model.

4. SYMMETRY METHODS

The first finite element model to be investigated is a representation of the complete ring, and from this model the entire set of radial, flexural and circumferential mode shapes and associated frequencies are obtained.

It will be demonstrated in this report that the same information may be obtained by modeling only a portion of the structure, constrained by appropriate boundary conditions, which results in an appreciable savings in computer time.

The boundary conditions on the segment are chosen so as to be compatible with the deflections experienced by the complete structure in each mode shape desired. Hence, any modes whose deflections are not compatible with a given set of boundary conditions cannot be obtained from that set. It is obvious from the statements above that symmetry methods are applicable only in cases where the nature of the mode shapes has been predetermined.

Consider the two representative mode shapes of the ring shown in Figure 1. These are the flexural modes with i equal to 3 and 4 respectively. The radial displacements, u_i, are indicated by a solid line, and the circumferential displacements v_i, by a broken line.

Assuming the half-symmetry model to lie in the region $0 \leq \phi \leq \pi$, it is necessary to prescribe the boundary conditions at $\phi = 0$ and at $\phi = \pi$. Refering to the modes shown in Figure 1 it will be noticed that the slope $(du/d\phi)$ and the circumferential displacement (v) are zero at both points. It can be shown that this condition is also true for the circumferential modes. Hence, the half-symmetry model constrained such that these boundary conditions are satisfied may be expected to produce the entire set of mode shapes.

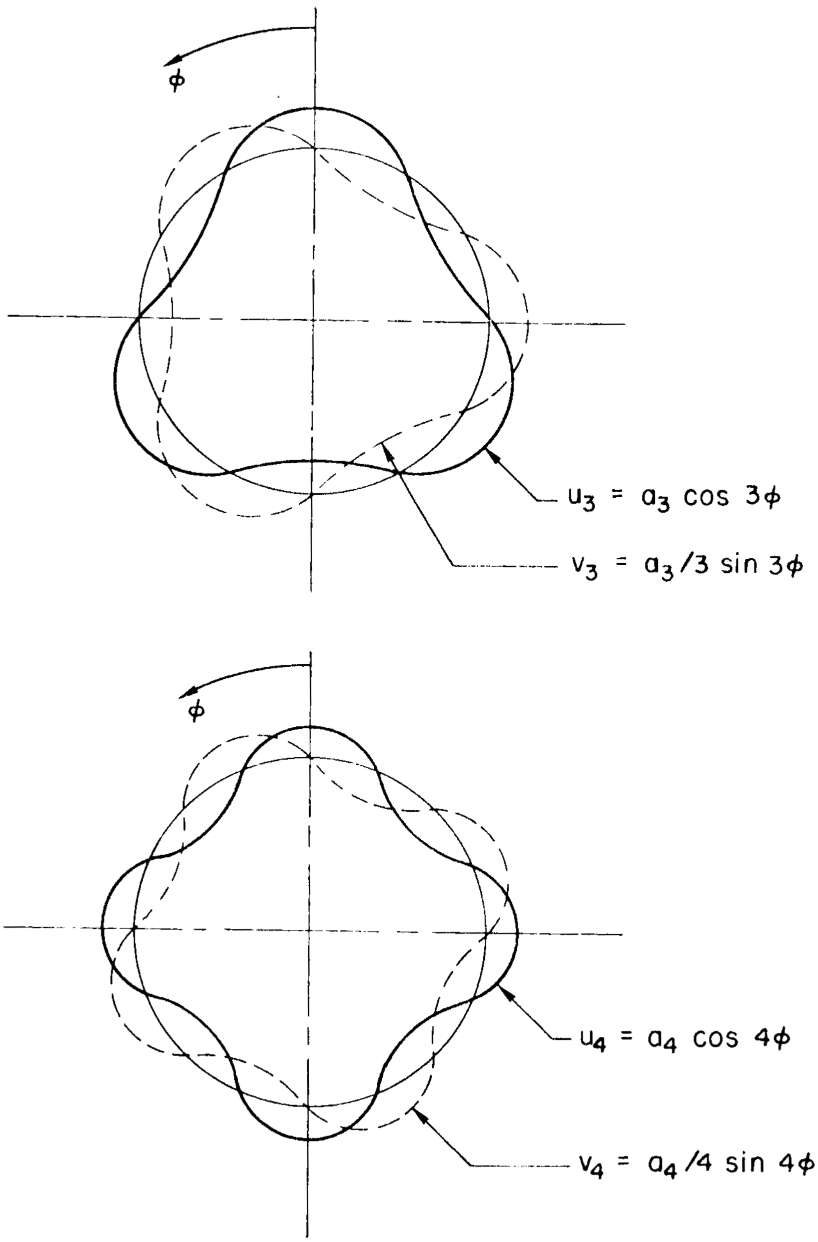


Figure 1—Third and Fourth Flexural Mode Shapes

In the case of a quarter-symmetry model in the region $0 \le \phi \le \pi/2$, boundary conditions must be determined at $\phi = 0$ and at $\phi = \pi/2$. As seen in Figure 1 the third flexural mode satisfies the condition that the slope ($du/d\phi$) and the circumferential displacement (v) are zero at $\phi = 0$, and the radial displacement (u) is zero at $\phi = \pi/2$. This set of boundary conditions will produce all the odd-numbered flexural and circumferential modes.

The fourth flexural mode (as well as all the remaining modes) may be obtained by constraining the slope and the circumferential displacement to be zero at both ends.

Great care should be taken that all components of displacement are investigated in the determination of the boundary conditions. For instance, from the fact that the radial displacement at $\phi = \pi/2$ is identically zero for the odd numbered modes, it might mistakenly be infered that a node exists at that point (i.e., that both u and v are zero.)

It will be shown that the same quarter-symmetry model discussed above, when constrained in just such a manner so that $du/d\phi$ and v are zero at $\phi = 0$, and both u and v are zero at $\phi = \pi/2$, will no longer represent a circular ring. The mode shapes in this case will be those of a structure formed by an anti-symmetric extension of the segment about $\phi = \pi/2$.

5. FINITE ELEMENT MODELS COMPLETE RING

The model for the complete ring consists of 32 grid points equally spaced about a circle of radius 10.0 inches, connected by 32 primary tension elements and 64 two-digit bending elements, 61 of which are primary (Figure 2).

The structure is supported by 3 reaction elements which are later removed for free-body analysis.

The values for the element compliances and grid point masses are shown in Table I.

Half Symmetry

The half-symmetry model consists of 17 grid points evenly spaced about a semi-circular arc of radius 10.0 inches, connected by 16 primary tension elements and 32 primary 2-digit bending elements (Figure 3).

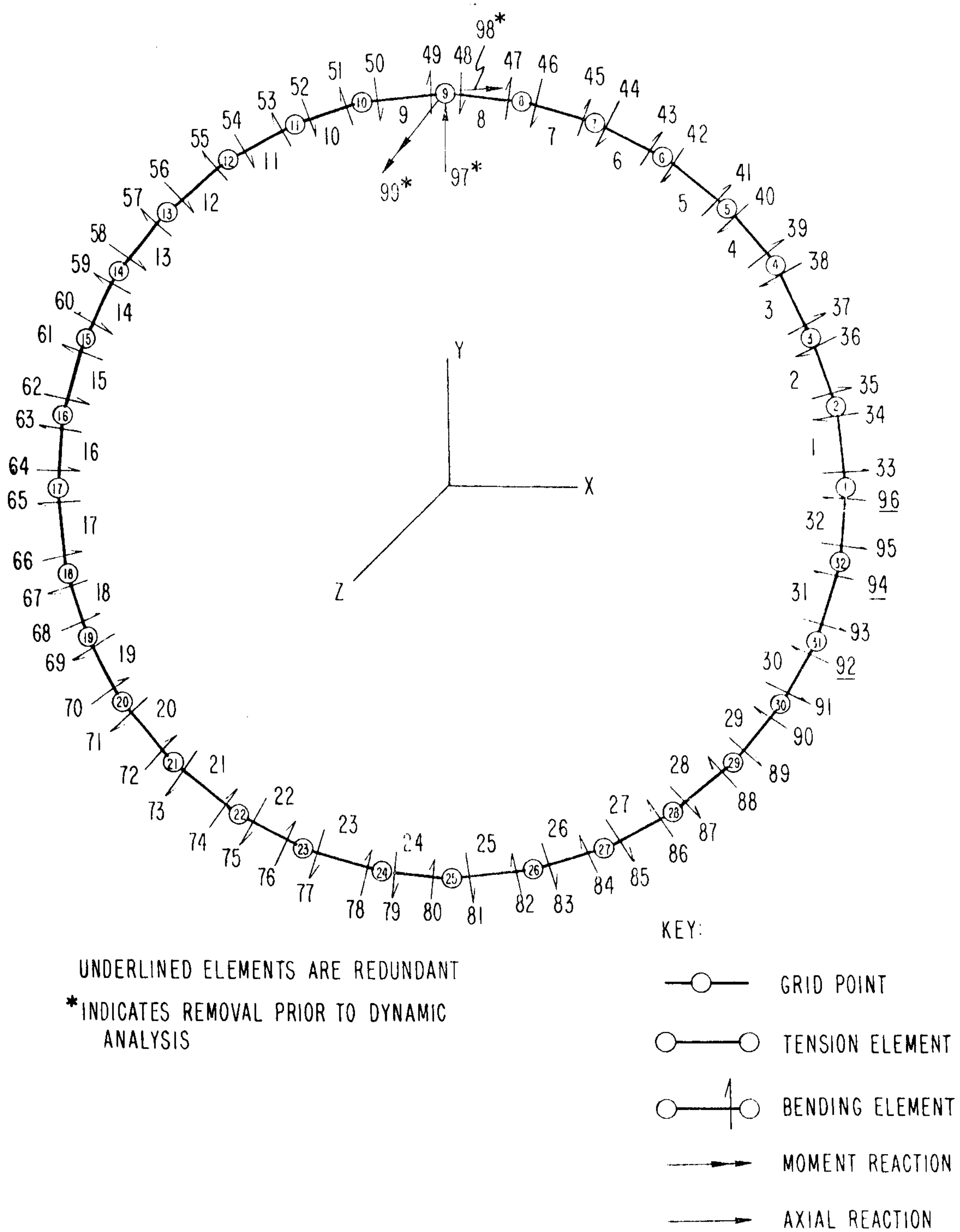


Figure 2–Model for Complete Ring

Table I
Compliance and Mass Data for Complete Ring

DIAGONAL COMPLIANCE

Element Numbers	Formula	Value
1-32	$C = L/AE$	1.0×10^{-6}
33-96	$C = L/3EI$	1.0×10^{-6}

COUPLED COMPLIANCE

Element Numbers	Formula	Value
33 to 34, 35 to 36, . . . 95 to 96	$C = L/6EI$	0.5×10^{-6}

MASS

Grid Points	Mass
1-32	10.0/g

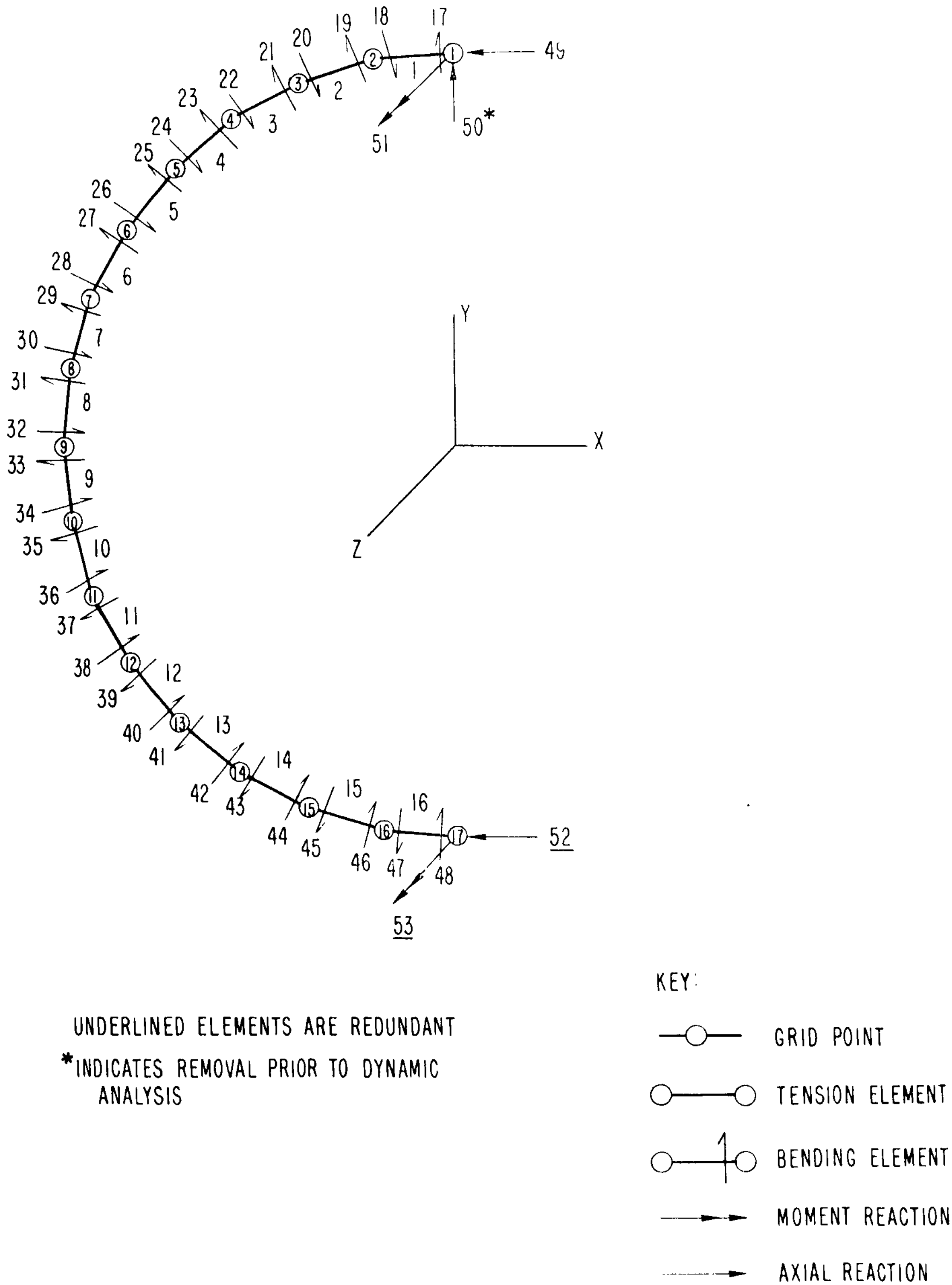


Figure 3–Model for Half Symmetry

Axial reactions 49 and 52 prevent circumferential motion at the ends of the ring, and moment reactions 51 and 53 prevent rotational motion at these points. Axial reaction 50 is necessary only for stability and is removed prior to determination of the mode shapes.

The compliance and mass values for this case are shown in Table II. The reactions in all of the models have zero compliances.

Table II
Compliance and Mass Data for Half Symmetry
and for S-Shaped Curve

DIAGONAL COMPLIANCE

Element Numbers	Formula	Value
1-16	$C = L/AE$	1.0×10^{-6}
17-48	$C = L/3\ EI$	1.0×10^{-6}

COUPLED COMPLIANCE

Element Numbers	Formula	Value
17 to 18, 19 to 20, . . . 47 to 48	$C = L/6EI$	0.5×10^{-6}

MASS

Grid Points	Mass
1, 17	5.0/g
2-16	10.0/g

Quarter Symmetry (Even Modes)

The circular ring is represented in this case by a quarter segment.

In order to obtain the radial mode and the even numbered flexural and circumferential modes, the ends of the segment are constrained such that rotation and circumferential translation are zero.

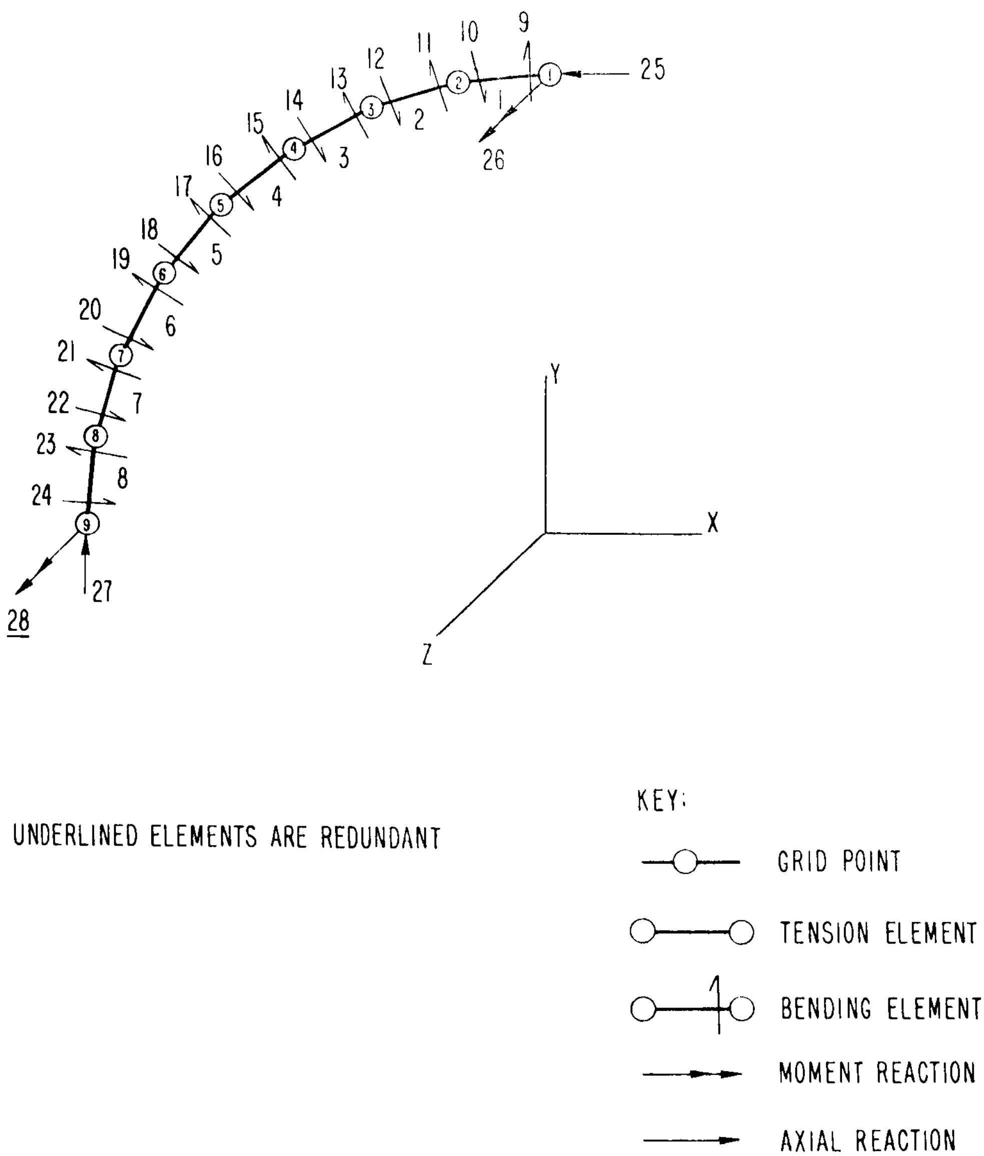


Figure 4–Model for Quarter Symmetry Even Modes

The model (Figure 4) consists of 9 grid points and 28 elements including 8 tension, 16 bending, and 4 reaction elements (3 of which are primary).

The compliance and mass values for this case are shown in Table III.

Table III
Compliance and Mass Data for Quarter Symmetry

DIAGONAL COMPLIANCE

Element Numbers	Formula	Value
1-8	$C = L/AE$	1.0×10^{-6}
9-24	$C = L/3EI$	1.0×10^{-6}

COUPLED COMPLIANCE

Element Numbers	Formula	Value
9 to 10, 11 to 12, . . . 23 to 24	$C = L/6EI$	0.5×10^{-6}

MASS

Grid Points	Mass
1, 9	5.0/g
2-8	10.0/g

Quarter Symmetry (Odd Modes)

The same quarter segment used to obtain the even modes is now subjected to a different set of boundary conditions to obtain the odd numbered modes (Figure 5).

One end of the segment is constrained as before, while at the other end an axial reaction (element 28) prevents motion in the radial direction. Reaction element 27 is necessary only for static stability and is removed prior to determination of the mode shapes. Mass and compliance values are the same as in Table III.

Quarter Symmetry with Boundary Conditions to Simulate S-Shaped Curve

The model for this structure is identical to that shown in Figure 5, except that reaction element 27 is not removed prior to determination of the mode shapes. In other words, all translational motion is arrested at grid point 9.

The mode shapes obtained using this set of boundary conditions are those of a structure formed by an antisymmetric extension of the quarter segment, i.e., of the s-shaped curve described in the following section.

Mass and compliance values are the same as those in Table III.

S-Shaped Curve

This model is the same as the half-symmetry model except that the last eight grid points are shifted so as to form an S-shaped curve (Figure 6). Mass and compliance values are the same as those in Table II.

The purpose of this exercise is to provide a comparison for the mode shapes obtained in the preceding quarter symmetry mode. The extension of that model into an antisymmetric geometry results in the curve considered here.

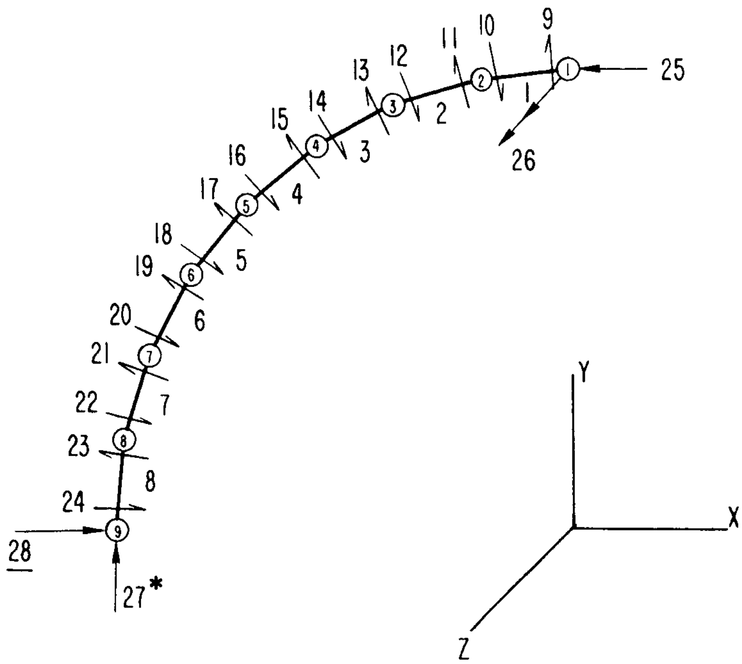
It should be noted that mode shapes obtained in this analysis also describe the motion of the figure obtained by extending the curve infinitely in either direction in a symmetrical manner. Such a figure may be described as a "corrugated" wire.

6. COMPARISON OF RESULTS

The values of the modal frequencies obtained from both the exact and the finite element solutions are compared in Table IV.

The half-symmetry and quarter-symmetry models are seen to yield the same results as the full circle model.

The differences between the finite element results and the exact results for the circular ring may be accounted for by the coarseness of the partition in the models.



UNDERLINED ELEMENTS ARE REDUNDANT
*INDICATES REMOVAL PRIOR TO DYNAMIC ANALYSIS

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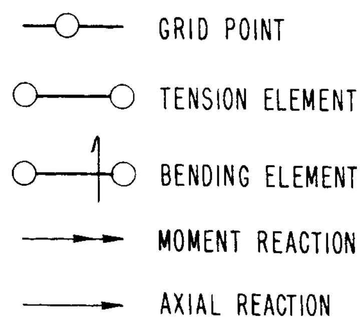

Figure 5–Model for Quarter Symmetry Odd Modes

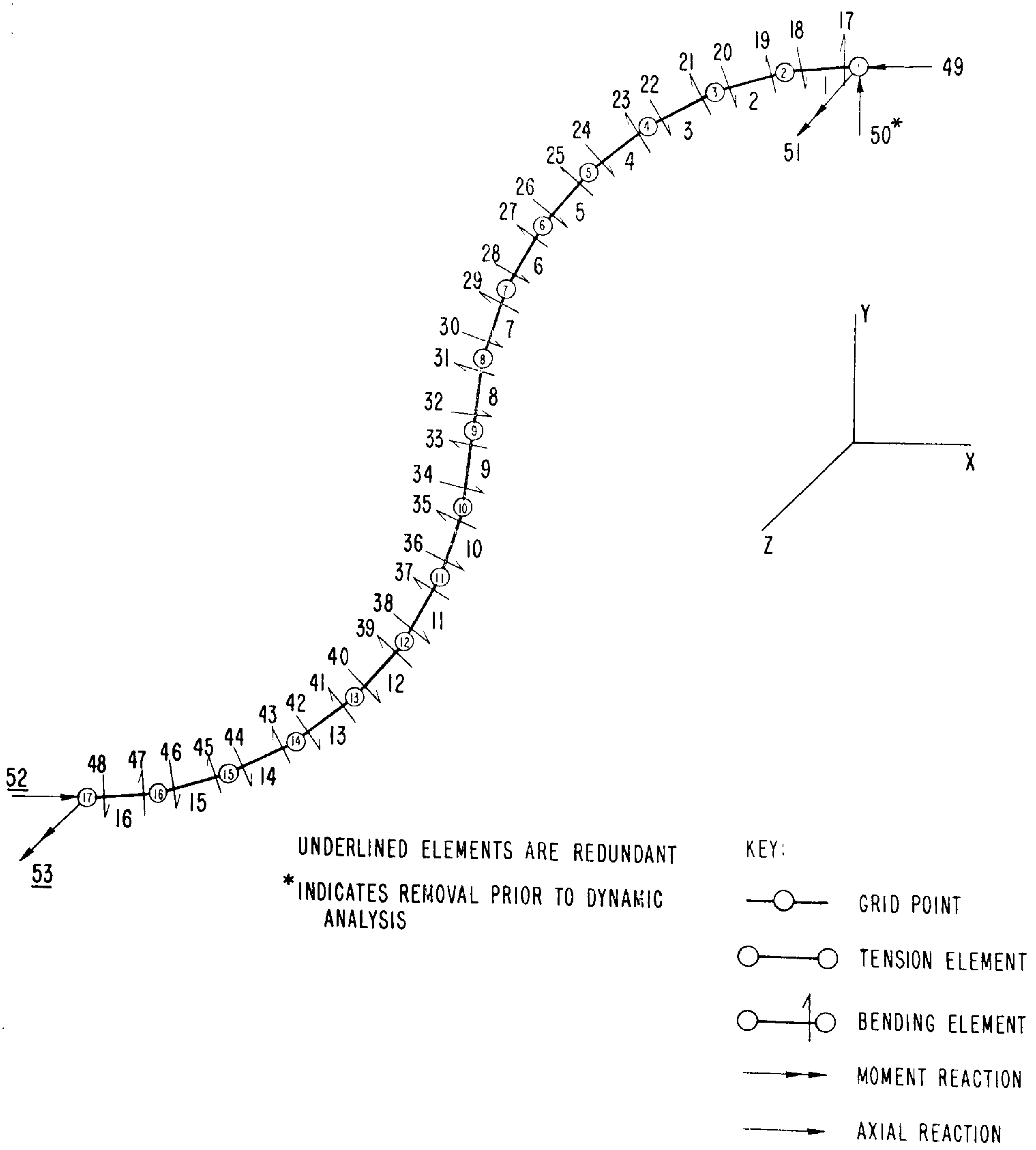

Figure 6–Model for S-Shaped Curve

Table IV
Modal Frequencies (Rad/Sec)

Circular Ring							Corrugated Wire	
Mode Type	i	Theory	Full Circle	Half Symmetry	Quarter (Even Modes)	Quarter (Odd Modes)	Quarter Symmetry	S-Shaped Curve
Flex	2	189.00	189.15	189.16	189.16	534.69		223.67
Flex'	3	534.57	534.67	534.69			478.71	478.71
Flex	4	1025.0	1024.6	1024.7	1024.7			1028.9
Radial		1220.0	1218.0	1218.0	1218.0		1143.0	1143.0
Flex	5	1657.6	1655.9	1655.9		1655.9	1661.2	1661.2
Circum	1	1725.3	1717.1	1717.1		1717.1		1725.2
Flex	6	2431.7	2426.1	2426.3	2426.3			2430.5
Circum	2	2728.0	2703.8	2703.8	2703.8		2664.0	2664.0
Flex	7	3347.0	3332.1	3332.3		3332.3	3349.4	3349.4
Circum	3	3858.0	3794.9	3794.9		3794.9		3793.3
Flex	8	4403.2	4367.3	4367.6	4367.6			4379.7
Circum	4	5030.2	4893.5	4893.5	4893.5		4870.9	4870.9
Flex	9	5600.4	5520.1	5520.4		5520.4	5534.0	5534.0
Circum	5	6220.8	5964.7	5964.7		5964.7		5963.9
Flex	10	6938.6	6768.8	6769.1	6769.1			6775.3
Circum	6	7420.9	6989.0	6989.1	6989.2		6982.4	6982.4
Flex	11	8417.6	7952.9	7952.9		7953.0		7961.5
Circum	7	8626.7	8076.2	8076.5		8076.5	8072.5	8072.6
Circum	8	9835.9	8844.4	8844.6	8844.6		8853.3	8853.3
Flex	12	10037	9381.6	9382.4	9382.4			9344.8

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