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NASA TN D-4002
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A NUMERICAL THEORY OF SATELLITES IN BRENDEL'S COORDINATES

by Peter Musen

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1967

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ABSTRACT

A form of the numerical lunar theory is suggested for planetary satellites moving in orbital planes with higher inclinations relative to the orbit plane of the Sun. The number of terms given explicitly in final solutions is too small to warrant the noncritical application of the written portions of the series to the cases of higher inclinations. In addition, the convergence of these series is slow near resonance angles. Convergence problems can be remedied in part by arranging the theory in such a way that the squares of the mean motions of the perigee and node are determined instead of the first powers of these motions. Brendel's coordinates are used, and the equations of motion are reduced to the problem of two disturbed linear oscillators. Following Hansen the numerical values of all the basic parameters of the lunar theory are substituted from the outset and the process of iteration is used to solve the problem. The mean orbital plane and the constants of integration are chosen in such a way that the expansions of the coordinates contain only purely periodic terms. The polar angle v of the projection of the satellite on the mean orbital plane is the independent variable of the theory. With this choice of the independent variable, the differential equations of the problem become the equations for linear oscillators.

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INTRODUCTION

In this paper, we suggest a form of the numerical lunar theory that can be used for planetary satellites in orbits with high inclinations to the orbit plane of the Sun relative to the planet. The classical analytical lunar theories by Delaunay (1860), Hill (1878), and Brown (1895) assume the smallness of the inclination I, and $\sin I/2$ is considered to be one of the small literal parameters with respect to which the coordinates or the elements are being expanded.

Despite the elegance of these literal developments, the number of terms given explicitly is too limited to warrant an uncritical application of the written portions of the series to the cases of high inclinations. The theory of Delaunay—and, one might suspect, the theory of Hill and Brown—have resonances near $I \sim 39°$, $141°$; this affects their applicability for higher inclinations, because of the slow convergence of the expansions near these critical inclinations. This slow convergence can be partly remedied by determining the square of the mean motion of the pericenter instead of the motion of the perigee itself, because the square of the mean motion of the pericenter can be written in a more closed form than can the motion of the pericenter itself. Furthermore, the problem of convergence can be removed to a considerable extent if numerical values of the parameters are substituted from the outset—especially the parameter m, the ratio of mean motions, or its equivalent. This fact is well known from the Hill-Brown lunar theory, in which the numerical value of m is used. We suggest here the numerical substitution of all small parameters and a purely numerical theory. In this respect we decided to follow in Hansen's footsteps (1862).

An important advantage of the purely numerical approach is that the terms are computed in large groups at each iteration step. The number of cycles in the iterative process depends upon the values of the parameters and upon the final accuracy needed for computing the ephemeris of the satellite. In every lunar theory the small divisors and the long-period effects constitute a difficult computational problem. The only safeguard against distortion of the critical terms in a numerical theory is computation with a formal accuracy higher than that needed in the coefficients. In other words, one must program with "double precision."

We suggest also the use of Brendel's coordinates s, ζ, and ψ, where s is the perturbed value of the projection of the unit vector of the satellite on the Laplacian vector, ζ is the elevation of the satellite relative to its mean orbital plane, and ψ is proportional to the areal velocity of the projection of the satellite on its mean orbital plane. The mean orbital plane and the constants of integration are so chosen that the expansions of Brendel's coordinates contain only purely periodic terms.

Then the expansions of the satellite coordinates in the inertial system of coordinates will also consist of purely periodic terms.

Some points of affinity between Hansen's theory and the theory in Brendel's coordinates should be pointed out. Hansen's lunar theory makes use of the osculating orbit plane as a reference plane. The equations of motion of the satellite in the reference frame rigidly connected with the osculating orbit plane have the same form as in the inertial frame. For this reason a frame of reference rigidly connected with the osculating orbit plane was termed *ideal* by Hansen. Its introduction facilitates treatment of the perturbations of the satellite in the osculating orbit plane but requires determination of the position of the osculating orbit plane relative to the inertial system as an intermediary step before the expansion of the coordinates is done. In our theory we choose a reference frame with the ξ- and η-axes in the mean orbital plane and with the z-axis normal to this plane in such a way that the equations of motion of the satellite relative to the frame ($\xi\eta z$) will differ in form by the terms of higher orders from the corresponding equations of motion in an ideal system. As compared with Hansen's theory, such a choice of the reference frame, together with the use of Brendel's coordinates, permits us to obtain the rectangular coordinates in the inertial system in a more direct way, bypassing the determination of the position of the osculating orbit plane.

We choose the polar angle v of the projection of the satellite in the ($\xi\eta$) system to be the independent variable of our theory. With this choice the differential equations for s and ζ take the form of the differential equations of disturbed linear oscillators. These equations resemble the corresponding equations of Brendel's planetary theory (1925) but contain also some additional terms which reflect the motion of the "nearly ideal" system ($\xi\eta z$) relative to the inertial system.

The form of the differential equations of the lunar theory given here also favors the application of the Krylov-Bogolubov method (1961), if a purely analytical form of the solution is preferred. The expansion of Brendel's coordinates is obtainable in the form of trigonometric series in four arguments w_1, w_2, w_3, and w_4, each linear in v. The angle w_1 is the mean argument of the latitude, w_2 is the mean true anomaly, w_3 is the mean (relative to v) longitude of the Sun, and w_4 is the mean longitude of the ascending node.

The form of the differential equations for s and ζ, resembling the differential equations for oscillators, suggests the determination of squares of the frequencies of w_1 and w_2 rather than the frequencies themselves. The positive side of this is that the squares of frequencies have a more closed form relative to the mean inclination than do the frequencies themselves. We also might add that Brendel's planetary theory, as well as the lunar theory in Brendel's coordinates, resembles the Laplacian lunar theory (with the requirement for smallness of the inclination removed). It seems that in our time we are witnessing a revival of Laplacian ideas in more or less hidden forms.

There is a tendency now to represent the coordinates or the osculating elements in terms of arguments other than time. Thus, in the case of an artificial satellite, the perturbations of the elements are obtained in terms of the true anomaly and also in terms of the arguments which are linear with respect to time (Kozai, 1959, and Brouwer, 1959). In Musen's (1961) theory, the true orbital longitude is taken as the independent variable. There is a similar tendency to represent the perturbations of the elements of the satellite in terms of the true, or of the eccentric, anomaly of the satellite and in terms of the true anomaly of the disturbing body. All these trends belong to the circle of Laplacian ideas.

BASIC DIFFERENTIAL EQUATIONS

Let us assume that the trajectory of the Sun relative to the planet is a fixed ellipse. This also means that the satellite mass is considered negligible. We put the X- and Y-axes into the plane of the solar ellipse; the Z-axis will be normal to this plane. The system (XYZ) is the inertial system to which the motion of the satellite is referred.

The satellite's mean orbit plane (xy) in the present theory has a constant inclination I relative to the XY-plane and rotates around the Z-axis at such a speed that it approximates the osculating orbit plane with an accuracy up to the periodic perturbations in the longitude of the ascending node and in the inclination. Furthermore, all frequencies of the arguments of the theory, as well as the constants of integration, must be so defined that no purely secular or mixed terms appear in the expansions of the satellite's coordinates relative to the system (XYZ). Let

$$\vec{\gamma} = \gamma \vec{k}$$

be the angular speed of rotation of the xy-plane, $\vec{P}$ be the unit vector along the line of the ascending node of xy on the XY-plane, $\vec{R}$ be the unit vector normal to xy in the direction of the angular momentum, and

$$\vec{Q} = \vec{R} \times \vec{P} .$$

We place the axes x and y along $\vec{P}$ and $\vec{Q}$, respectively; the z-axis is along $\vec{R}$. We have

$$\vec{\gamma} = \lambda \vec{Q} + \nu \vec{R} , \tag{1}$$

where

$$\lambda = \gamma \sin I , \qquad \nu = \gamma \cos I . \tag{2}$$

The differential equation of motion of the satellite relative to the system (xyz) has the form

$$\frac{d^2 \vec{r}}{dt^2} + 2\vec{\gamma} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\gamma}}{dt} \times \vec{r} + \vec{\gamma} \times (\vec{\gamma} \times \vec{r}) = -\frac{\vec{r}}{r^3} + \mathrm{grad}_{\vec{r}}\, \Omega . \tag{3}$$

We decompose the position vector $\vec{r}$ into the component $\vec{\rho}$, which is the projection of $\vec{r}$ on the xy-plane, and the component $z\vec{R}$ normal to that plane; we have

$$\vec{r} = \vec{\rho} + z\vec{R} . \tag{4}$$

Substituting Equations 1 and 4 into Equation 3, gives

$$\frac{d^2\vec{\rho}}{dt^2} + \vec{R} \times \left(2\nu \frac{d\vec{\rho}}{dt} + \frac{d\nu}{dt}\vec{\rho}\right) + \left(2\lambda \frac{dz}{dt} + \frac{d\lambda}{dt} z\right)\vec{P} = -\frac{\vec{\rho}}{r^3} + \text{grad}_{\vec{\rho}} F , \tag{5}$$

$$\frac{d^2 z}{dt^2} - \vec{P} \cdot \left(2\lambda \frac{d\vec{\rho}}{dt} + \frac{d\lambda}{dt}\vec{\rho}\right) = -\frac{z}{r^3} + \frac{\partial F}{\partial z} , \tag{6}$$

where

$$F = \frac{1}{2}\left[\gamma^2 x^2 + (\nu y - \lambda z)^2\right] + \Omega . \tag{7}$$

The scalar differential equations of motion as obtained from Equations 5 and 6 are

$$\frac{d^2 x}{dt^2} - \left(2\nu \frac{dy}{dt} + \frac{d\nu}{dt} y\right) + \left(2\lambda \frac{dz}{dt} + \frac{d\lambda}{dt} z\right) = -\frac{x}{r^3} + \frac{\partial F}{\partial x} , \tag{8}$$

$$\frac{d^2 y}{dt^2} + \left(2\nu \frac{dx}{dt} + \frac{d\nu}{dt} x\right) = -\frac{y}{r^3} + \frac{\partial F}{\partial y} , \tag{9}$$

$$\frac{d^2 z}{dt^2} - \left(2\lambda \frac{dx}{dt} + \frac{d\lambda}{dt} x\right) = -\frac{z}{r^3} + \frac{\partial F}{\partial z} . \tag{10}$$

Equations 8 through 10 are almost identical in form to those suggested by Brouwer (1958) for the development of the Hill-Brown theory of motion of artificial satellites. In Brouwer's work λ and ν are constants, and thus the terms involving $d\lambda/dt$ and $d\nu/dt$ are absent.

Equations 5 and 6 can be transformed to a form closely resembling the equations of motion in an ideal system and more convenient for the determination of Brendel's coordinates. We refer the motion of the satellite to the system $(\xi\eta z)$ rotating relative to (xyz) with angular velocity $-\nu\vec{R}$. In order to transform the equations of motion from the system (xyz) to the "nearly ideal" system $(\xi\eta z)$ we must replace, in Equations 5 and 6,

$$\frac{d\vec{\rho}}{dt} \quad \text{by} \quad \frac{d\vec{\rho}}{dt} - \nu\vec{R} \times \vec{\rho}$$

and

$$\frac{d^2\vec{\rho}}{dt^2} \quad \text{by} \quad \frac{d^2\vec{\rho}}{dt^2} - 2\nu\vec{R} \times \frac{d\vec{\rho}}{dt} - \frac{d\nu}{dt}\vec{R} \times \vec{\rho} - \nu^2\vec{\rho}$$

in accordance with the basic kinematic rules for the relative velocity and relative acceleration. The derivatives

$$\frac{d\vec{\rho}}{dt} \quad \text{and} \quad \frac{d^2\vec{\rho}}{dt^2}$$

now mean the velocity and acceleration relative to the system (ξ, η, z). We obtain

$$\frac{d^2\vec{\rho}}{dt^2} = -\frac{\vec{\rho}}{r^3} + \mathrm{grad}_{\vec{\rho}}\,\Phi - \left(2\lambda \frac{dz}{dt} + \frac{d\lambda}{dt} z\right)\vec{P} , \tag{11}$$

$$\frac{d^2 z}{dt^2} = -\frac{z}{r^3} + \frac{\partial\Phi}{\partial z} + \left(2\lambda \frac{dx}{dt} + \frac{d\lambda}{dt} x\right) , \tag{12}$$

where we put

$$\Phi = \frac{1}{2}\lambda^2\left(x^2 + z^2\right) - \lambda\nu yz + \Omega . \tag{13}$$

The form of Equations 11 and 12 now resembles, as closely as possible under the circumstances, the form of the equations of motion in the ideal system of coordinates.

In the formation of the differential equations for Brendel's coordinates, it is convenient to choose the angle v between $\vec{\rho}$ and the ξ-axis as the independent variable of our theory. This seems to be a most natural choice, because v represents the closest analog of the true orbital longitude. In accordance with this choice, we set the angular speed of rotation of the (xy)-plane to be uniform with respect to v. In other words, we set

$$\nu = (1 - g)\frac{dv}{dt} ,$$

where g is a constant. Then we have

$$\lambda = (1 - g)\frac{dv}{dt}\tan I ,$$

$$\gamma = (1 - g)\sec I \frac{dv}{dt} .$$

The angular distance w of $\vec{\rho}$ from $\vec{P}$ is obtained from the equation

$$w = v - \int \nu\, dt = gv - \theta_0 ,$$

where θ_0 is a constant. This angle is one of the basic arguments in the expansions of Brendel's coordinates into trigonometric series.

From Equation 5 we obtain, by projecting the left and the right sides on the directions of $\vec{\rho}$ and of $\vec{R} \times \vec{\rho}$:

$$\frac{d^2 \rho}{dt^2} - \rho \left(\frac{dv}{dt}\right)^2 = -\frac{\rho}{r^3} + \frac{\partial \Phi}{\partial \rho} - \left(2\lambda \frac{dz}{dt} + z \frac{d\lambda}{dt}\right) \cos w \quad , \tag{14}$$

$$\frac{d}{dt}\left(\rho^2 \frac{dv}{dt}\right) = \frac{\partial \Phi}{\partial w} + \left(2\lambda \frac{dz}{dt} + z \frac{d\lambda}{dt}\right) \rho \sin w \quad ; \tag{15}$$

and, in addition, from Equation 6:

$$\frac{d^2 z}{dt^2} = -\frac{z}{r^3} + \frac{\partial \Phi}{\partial z} + \left(2\lambda \frac{d\rho \cos w}{dt} + \frac{d\lambda}{dt} \rho \cos w\right) \quad . \tag{16}$$

At this point we introduce, instead of ρ, v, z, Brendel coordinates ψ, s, ζ by means of the equations

$$\rho^2 \frac{dv}{dt} = \sqrt{p} \; , \tag{17}$$

$$p = p_0 (1 - \psi) \; , \qquad p_0 = \text{constant} \; , \tag{18}$$

$$\rho = \frac{p_0 (1 - \psi)}{1 + s} = \frac{p}{1 + s} \; , \tag{19}$$

$$z = \rho \zeta \; . \tag{20}$$

It must be emphasized that p is not the osculating semi-latus rectum in the sense of the theory of variation of astronomical constants, because the system $(\xi \eta z)$ is not ideal. However, we can put $p = p_0 (1 - \psi)$, where p_0 is a constant and ψ is the perturbations whose expansion contains only purely periodic terms. In Equation 19 s is analogous, but not identical, to the osculating value of $e \cos f$.

Pursuing this development further, we take ν either in the form

$$\nu = (1 - g) \frac{\sqrt{p}}{\rho^2} \tag{21}$$

or in the form

$$\nu = p_0^{-3/2} (1 - g) (1 - \psi)^{-3/2} (1 + s)^2 \; , \tag{22}$$

and we make use of the relation

$$\lambda = \nu \tan I . \tag{23}$$

From Equations 15, 17, 20, 21, and 23 we deduce

$$\frac{d\sqrt{p}}{dv} = \frac{\rho^2}{\sqrt{p}}\frac{\partial\Phi}{\partial w} + (1-g)\sqrt{p}\left(2\zeta\,\frac{d\log\rho}{dv} + 2\,\frac{d\zeta}{dv} + \zeta\,\frac{d\log\nu}{dv}\right)\sin w \tan I .$$

Making use of Equations 18, 19 and 22 we obtain from the last equation,

$$\frac{d\psi}{dv} = -\frac{2\rho^2}{p_0}\frac{\partial\Phi}{\partial w} + (1-g)\left[\zeta\,\frac{d\psi}{dv} - 4(1-\psi)\,\frac{d\zeta}{dv}\right]\sin w \tan I . \tag{24}$$

From Equations 19 and 17 we have

$$\frac{d\rho}{dt} = -\frac{1}{\sqrt{p}}\left(\frac{1+s}{1-\psi}\,\frac{d\psi}{dv} + \frac{ds}{dv}\right) . \tag{25}$$

Differentiating Equation 25 and taking Equation 17 into consideration again, we obtain

$$\rho^2\,\frac{d^2\rho}{dt^2} = -\frac{d^2 s}{dv^2} - \frac{1+s}{1-\psi}\,\frac{d^2\psi}{dv^2} - \frac{3}{2}\,\frac{1}{1-\psi}\,\frac{ds}{dv}\,\frac{d\psi}{dv} - \frac{3}{2}\,\frac{1+s}{(1-\psi)^2}\left(\frac{d\psi}{dv}\right)^2 . \tag{26}$$

The two last equations appear in Brendel's work. However, the meaning of v in Brendel's equations is different from the meaning assigned to it here.

From Equations 14 and 17 we have

$$\rho^2\,\frac{d^2\rho}{dt^2} = \frac{p}{\rho} - \left(\frac{\rho}{r}\right)^3 + \rho^2\,\frac{\partial\Phi}{\partial\rho} - \nu\left(2\,\frac{dz}{dv} + z\,\frac{d\log\nu}{dv}\right)\sqrt{p}\,\cos w \tan I , \tag{27}$$

and taking Equations 19 and 21 into account we can rewrite this equation in the form

$$\rho^2\,\frac{d^2\rho}{dt^2} = \frac{p}{\rho} - \left(\frac{\rho}{r}\right)^3 + \rho^2\,\frac{\partial\Phi}{\partial\rho} - (1-g)(1+s)\left(2\,\frac{d\zeta}{dv} + 2\zeta\,\frac{d\log\rho}{dv} + \zeta\,\frac{d\log\nu}{dv}\right)\cos w \tan I .$$

Substituting Equations 19, 22 and

$$r = \rho\sqrt{1+\zeta^2} \tag{28}$$

into the last equation, we obtain

$$\rho^2 \frac{d^2 \rho}{dt^2} = s + 1 - \left(1 + \zeta^2\right)^{-3/2} + \rho^2 \frac{\partial \Phi}{\partial \rho} - (1-g)(1+s)\left(2 \frac{d\zeta}{dv} - \frac{1}{2} \frac{\zeta}{1-\psi} \frac{d\psi}{dv}\right) \cos w \tan I \quad . \tag{29}$$

Comparing Equations 26 and 29 we deduce

$$\frac{d^2 s}{dv^2} + s = A + B - \rho^2 \frac{\partial \Phi}{\partial \rho} \quad , \tag{30}$$

where

$$A = -\frac{1+s}{1-\psi} \frac{d^2 \psi}{dv^2} - \frac{3}{2} \frac{1}{1-\psi} \frac{ds}{dv} \frac{d\psi}{dv} \quad , \tag{31}$$

$$B = \left(1 + \zeta^2\right)^{-3/2} - 1 - \frac{3}{2} \frac{1+s}{(1-\psi)^2} \cdot \left(\frac{d\psi}{dv}\right)^2$$

$$+ (1-g)(1+s)\left(2 \frac{d\zeta}{dv} - \frac{1}{2} \frac{\zeta}{1-\psi} \frac{d\psi}{dv}\right) \cos w \tan I \quad . \tag{32}$$

Next, from Equations 16, 17, and 21,

$$\frac{d^2 \zeta}{dt^2} + \frac{2}{\rho} \frac{d\rho}{dt} \frac{d\zeta}{dt} + \frac{\zeta}{\rho} \frac{d^2 \rho}{dt^2} = -\frac{\zeta}{r^3} + \frac{1}{\rho} \frac{\partial \Phi}{\partial z}$$

$$+ \frac{p}{\rho^4} (1-g)\left(2 \cos w \frac{d \log \rho}{dv} - 2g \sin w + \cos w \frac{d \log \nu}{dv}\right) \tan I.$$

Substituting Equations 19 and 22 into the last equation gives

$$\frac{d^2 \zeta}{dt^2} + \frac{2}{\rho} \frac{d\rho}{dt} \frac{d\zeta}{dt} + \frac{\zeta}{\rho} \frac{d^2 \rho}{dt^2} = -\frac{\zeta}{r^3} + \frac{1}{\rho} \frac{\partial \Phi}{\partial z}$$

$$- (1-g) \frac{p}{\rho^4} \left(\frac{1}{2} \frac{1}{1-\psi} \frac{d\psi}{dv} \cos w + 2g \sin w\right) \tan I \quad . \tag{33}$$

Differentiating

$$\frac{d\zeta}{dt} = \frac{\sqrt{p}}{\rho^2} \frac{d\zeta}{dv} \quad , \tag{34}$$

gives, after some easy transformations,

$$\frac{d^2\zeta}{dt^2} = \frac{p}{\rho^4}\left(\frac{d^2\zeta}{dv^2} + \frac{1}{2}\frac{d\zeta}{dv}\frac{d\log p}{dv} - 2\frac{d\zeta}{dv}\frac{d\log\rho}{dv}\right) . \tag{35}$$

Substituting Equations 18 and 19 into this equation gives

$$\frac{d^2\zeta}{dt^2} = \frac{p}{\rho^4}\left(\frac{d^2\zeta}{dv^2} + \frac{3}{2}\frac{1}{1-\psi}\frac{d\psi}{dv}\frac{d\zeta}{dv} + \frac{2}{1+s}\frac{ds}{dv}\frac{d\zeta}{dv}\right) . \tag{36}$$

Multiplying Equations 25 and 34 together gives

$$\frac{1}{\rho}\frac{d\rho}{dt}\frac{d\zeta}{dt} = -\frac{p}{\rho^4}\left(\frac{1}{1-\psi}\frac{d\psi}{dv}\frac{d\zeta}{dv} + \frac{1}{1+s}\frac{ds}{dv}\frac{d\zeta}{dv}\right) . \tag{37}$$

From Equation 27, taking Equation 21 into account,

$$\frac{1}{\rho}\frac{d^2\rho}{dt^2} = -\frac{1}{r^3} + \frac{p}{\rho^4}\left[\left(1 + \frac{\rho^3}{p}\frac{\partial\Phi}{\partial\rho}\right) - (1-g)\left(2\frac{dz}{dv} + z\frac{d\log\nu}{dv}\right)\frac{\cos w\tan I}{\rho}\right]$$

or

$$\frac{1}{\rho}\frac{d^2\rho}{dt^2} = -\frac{1}{r^3} + \frac{p}{\rho^4}\left[\left(1 + \frac{\rho^3}{p}\frac{\partial\Phi}{\partial\rho}\right) - (1-g)\left(2\frac{d\zeta}{dv} + 2\zeta\frac{d\log\rho}{dv} + \zeta\frac{d\log\nu}{dv}\right)\cos w\tan I\right] ;$$

and finally, eliminating ζ and ν in favor of s and ψ as before,

$$\frac{1}{\rho}\frac{d^2\rho}{dt^2} = -\frac{1}{r^3} + \frac{p}{\rho^4}\left[\left(1 + \frac{\rho^3}{p}\frac{\partial\Phi}{\partial\rho}\right) - (1-g)\left(2\frac{d\zeta}{dv} - \frac{1}{2}\frac{\zeta}{1-\psi}\frac{d\psi}{dv}\right)\cos w\tan I\right] . \tag{38}$$

Substituting Equations 35, 37, and 38 into Equation 33 gives

$$\frac{d^2\zeta}{dv^2} + \zeta + 2g(1-g)\sin w = P + \frac{\rho^3}{p}\left(\frac{\partial\Phi}{\partial z} - \zeta\frac{\partial\Phi}{\partial\rho}\right) , \tag{39}$$

where

$$P = \frac{1}{2}\frac{1}{1-\psi}\frac{d\psi}{dv}\frac{d\zeta}{dv} + (1-g)\left(\frac{d\zeta^2}{dv} - \frac{1}{2}\frac{1+\zeta^2}{1-\psi}\frac{d\psi}{dv}\right)\cos w\tan I . \tag{40}$$

If in the development of Φ we eliminate z in favor of ζ by means of Equation 20, then

$$\frac{\partial\Phi}{\partial z} \quad \text{and} \quad \frac{\partial\Phi}{\partial\rho}$$

in Equations 30 and 39 will be replaced by

$$\frac{1}{\rho}\frac{\partial\Phi}{\partial\zeta} \quad \text{and} \quad \frac{\partial\Phi}{\partial\rho} - \frac{\zeta}{\rho}\frac{\partial\Phi}{\partial\zeta} ,$$

respectively, and the complete set of differential equations of the problem is

$$\frac{d^2 s}{dv^2} + s = A + B - \rho\left(\rho\frac{\partial\Phi}{\partial\rho} - \zeta\frac{\partial\Phi}{\partial\zeta}\right) , \tag{30'}$$

$$\frac{d^2\zeta}{dv^2} + \zeta + 2g(1-g)\sin w = P + \frac{\rho^2}{p}\left[\left(1+\zeta^2\right)\frac{\partial\Phi}{\partial\zeta} - \zeta\left(\rho\frac{\partial\Phi}{\partial\rho}\right)\right] , \tag{39'}$$

$$\frac{d\psi}{dv} = -\frac{2\rho^2}{p_0}\frac{\partial\Phi}{\partial w} + (1-g)\left[\zeta\frac{d\psi}{dv} - 4(1-\psi)\frac{d\zeta}{dv}\right]\sin w \tan I , \tag{24'}$$

$$\frac{dt}{dv} = p_0^{3/2}(1-\psi)^{3/2}(1+s)^{-2} , \tag{41}$$

where A, B, and P are defined by Equations 31, 32, and 40, respectively.

Combining the equations

$$\rho'^2\frac{dv'}{dt} = \sqrt{(1+m')\,p_0'} ,$$

$$\rho' = \frac{p_0'}{1+s'} , \qquad s' = e'\cos v'$$

with Equations 17 through 19 gives

$$\frac{dv'}{dv} = m_0(1-\psi)^{3/2}\left(\frac{1+s'}{1+s}\right)^2 , \tag{42}$$

where

$$m_0 = \sqrt{1+m'}\left(\frac{p_0}{p_0'}\right)^{3/2} .$$

We now split the right side of Equation 42 into a constant part m, which is of the order of m_0, and a purely periodic part Q. We thus have

$$v' = w_3 + \delta v' \quad , \tag{42'}$$

where

$$w_3 = mv - v_0' \quad , \qquad v_0' = \text{Const} \quad ; \tag{43}$$

and

$$\delta v' = \int Q \, dv \quad . \tag{44}$$

The integral in Equation 44 is taken in a formal way, without adding any constant of integration. The angle w_4 is one of the basic arguments of the theory, analogous to the argument $mv + \text{const}$ of the Laplacian (1802) theory. However, the meanings of v in the theory presented here and in the Laplacian theory are different. The remaining arguments are

$$w_1 = cv - \pi_0 \ ,$$

which is introduced in the process of integration of Equation 30', and

$$w_4 = gv - \theta_0 \ ,$$

the mean longitude of the ascending node of the satellite. We shall see in one of the next sections how these four arguments appear naturally in the expansion of the disturbing function.

THE GENERALIZED CAUCHY NUMBERS

The Cauchy (1885) numbers $N_{-p,j,q}$ can be defined as the contour integrals

$$N_{-p,j,q} = \frac{1}{2\pi i} \oint_{|z|=1} z^{-p-1} \left(z + \frac{1}{z}\right)^j \left(z - \frac{1}{z}\right)^q dz \tag{45}$$

taken along the unit circle $|z| = 1$. Cauchy has shown that their introduction permits one to obtain in a simple way some basic developments associated with the elliptic motion.

In the development of the higher harmonics in the solar disturbing function into trigonometric series for a satellite in a highly inclined osculating orbital plane, we found it convenient to introduce the set of rational numbers

$$N^{\nu}_{-p,j,q} = \frac{1}{2\pi i} \oint_{|z|=1} z^{-p-1} C^{\nu}_{j} \left(\frac{z + \frac{1}{z}}{2}\right) \left(z - \frac{1}{z}\right)^{q} dz , \tag{46}$$

whose properties resemble closely the properties of the Cauchy numbers. The symbol C^{ν}_{j} designate a Gegenbauer polynomial; p, j, q are integers; j and q are nonnegative; and p can be positive, negative, or zero. The generalized Cauchy numbers $N^{\nu}_{-p,j,q}$ represent a useful instrument in obtaining the disturbing function of the lunar theory by means of electronic computers.

Writing the classical expansions of Gegenbauer polynomials (Erdely et al., 1953) in the form

$$C^{\nu}_{j}\left(\frac{z+\frac{1}{z}}{2}\right) = \frac{1}{2}\sum_{m=0}^{j} \frac{(\nu, m)(\nu, j-m)}{m!\,(j-m)!}\left(z^{j-2m} + \frac{1}{z^{j-2m}}\right) ,$$

$$C^{\nu}_{j}\left(\frac{z+\frac{1}{z}}{2}\right) = \sum_{m=0}^{[j/2]} \frac{(-1)^{m}(\nu, j-m)}{m!\,(j-m)!}\frac{1}{2^{j-2m}}\left(z + \frac{1}{z}\right)^{j-2m} ,$$

$$C^{\nu}_{2j}\left(\frac{z+\frac{1}{z}}{2}\right) = (-1)^{j}\frac{(\nu, j)}{j!}\sum_{m=0}^{j} \frac{(-j, m)(j+\nu, m)}{\left(\frac{1}{2}, m\right)(1, m)}\frac{1}{2^{2m}}\left(z + \frac{1}{z}\right)^{2m} ,$$

$$C^{\nu}_{2j}\left(\frac{z+\frac{1}{z}}{2}\right) = \frac{(2\nu, 2j)}{(2j)!}\sum_{m=0}^{j} \frac{(-j, m)(j+\nu, m)}{\left(\nu+\frac{1}{2}, m\right)(1, m)}\frac{(-1)^{m}}{2^{2m}}\left(z - \frac{1}{z}\right)^{2m} ,$$

$$C^{\nu}_{2j+1}\left(\frac{z+\frac{1}{z}}{2}\right) = (-1)^{j}\frac{(\nu, j+1)}{j!}\sum_{m=0}^{j} \frac{(-j, m)(j+\nu+1, m)}{\left(\frac{3}{2}, m\right)(1, m)}\frac{1}{2^{2m}}\left(z + \frac{1}{z}\right)^{2m+1} ,$$

$$C^{\nu}_{2j+1}\left(\frac{z+\frac{1}{z}}{2}\right) = \frac{(2\nu, 2j+1)}{(2j+1)!}\sum_{m=0}^{j} \frac{(-j, m)(j+\nu+1, m)}{\left(\nu+\frac{1}{2}, m\right)(1, m)}\frac{(-1)^{m}}{2^{2m+1}}\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)^{2m} ,$$

and substituting them in Equation 46, we obtain a set of formulas giving the numbers $N^{\nu}_{-p,j,q}$ in terms of the standard Cauchy numbers:

$$N^{\nu}_{-p,j,q} = \frac{1}{2}\sum_{m=0}^{j} \frac{(\nu, m)(\nu, j-m)}{m!\,(j-m)!}\left(N_{-p-j+2m,0,q} + N_{-p+j-2m,0,q}\right) , \tag{47}$$

$$N^{\nu}_{-p,\,j,\,q} = \sum_{m=0}^{[j/2]} \frac{(-1)^m (\nu,\, j-m)}{m!\,(j-m)!} \frac{1}{2^{j-2m}} N_{-p,\, j-2m,\, q} \quad , \tag{48}$$

$$N^{\nu}_{-p,2j,q} = (-1)^j \frac{(\nu,\, j)}{j!} \sum_{m=0}^{j} \frac{(-j,\, m)(j+\nu,\, m)}{\left(\frac{1}{2},\, m\right)(1,\, m)} \frac{1}{2^{2m}} N_{-p,\, 2m,\, q} \quad , \tag{49}$$

$$N^{\nu}_{-p,2j,q} = \frac{(2\nu,\, 2j)}{(2j)!} \sum_{m=0}^{j} \frac{(-j,\, m)(j+\nu,\, m)}{\left(\nu+\frac{1}{2},\, m\right)(1,\, m)} \frac{(-1)^m}{2^{2m}} N_{-p,\, 0,\, q+2m} \quad , \tag{50}$$

$$N^{\nu}_{-p,2j+1,q} = (-1)^j \frac{(\nu,\, j+1)}{j!} \sum_{m=0}^{j} \frac{(-j,\, m)(j+\nu+1,\, m)}{\left(\frac{3}{2},\, m\right)(1,\, m)} \frac{1}{2^{2m}} N_{-p,\, 2m+1,\, q} \quad , \tag{51}$$

$$N^{\nu}_{-p,2j+1,q} = \frac{(2\nu,\, 2j+1)}{(2j+1)!} \sum_{m=0}^{j} \frac{(-j,\, m)(j+\nu+1,\, m)}{\left(\nu+\frac{1}{2},\, m\right)(1,\, m)} \frac{(-1)^m}{2^{2m+1}} N_{-p,\, 1,\, 2m+q} \quad . \tag{52}$$

From these formulas it is evident that, as is the case for Cauchy numbers,

$$N^{\nu}_{-p,\,j,\,q} = 0 \tag{53}$$

if $j+q-p$ is negative or odd and

$$N^{\nu}_{p,\,j,\,q} = (-1)^q N^{\nu}_{-p,\,j,\,q} \quad . \tag{54}$$

In particular,

$$N^{\nu}_{-p,\,0,\,q} = N_{-p,\,0,\,q} \quad , \tag{55}$$

$$N^{\nu}_{-p,\,1,\,q} = \nu N_{-p,\,1,\,q} \quad . \tag{56}$$

A set of recurrence formulas can be established between the generalized Cauchy numbers. Making use of the recurrence formulas

$$(j+2)\, C^{\nu}_{j+2}(x) = 2(\nu+j+1)\, x\, C^{\nu}_{j+1}(x) - (2\nu+j)\, C^{\nu}_{j}(x) \quad ,$$

$$j\, C^{\nu}_{j}(x) = 2\nu \left[x\, C^{\nu+1}_{j-1}(x) - C^{\nu+1}_{j-2}(x)\right] \quad ,$$

$$(j + 2\nu)\, C_j^{\nu}(x) = 2\nu \left[C_j^{\nu+1}(x) - x\, C_{j-1}^{\nu+1}(x) \right] ,$$

$$j\, C_j^{\nu}(x) = (j - 1 + 2\nu)\, x\, C_{j-1}^{\nu}(x) - 2\nu \left(1 - x^2\right) C_{j-2}^{\nu-1}(x) ,$$

and setting

$$x = \frac{z + \frac{1}{z}}{2} ,$$

we deduce the recurrence relations

$$N_{-p,\, j+2,\, q}^{\nu} = \frac{\nu + j + 1}{j + 2} \left(N_{-p+1,\, j+1,\, q}^{\nu} + N_{-p-1,\, j+1,\, q}^{\nu} \right) - \frac{2\nu + j}{j + 2} N_{-p,\, j,\, q}^{\nu} , \tag{57}$$

$$N_{-p,\, j,\, q}^{\nu} = \frac{\nu}{j} \left(N_{-p+1,\, j-1,\, q}^{\nu+1} + N_{-p-1,\, j-1,\, q}^{\nu+1} - 2N_{-p,\, j-2,\, q}^{\nu+1} \right) , \tag{58}$$

$$N_{-p,\, j,\, q}^{\nu} = \frac{\nu}{j + 2\nu} \left(2N_{-p,\, j,\, q}^{\nu+1} - N_{-p+1,\, j-1,\, q}^{\nu+1} - N_{-p-1,\, j-1,\, q}^{\nu+1} \right) , \tag{59}$$

$$N_{-p,\, j,\, q}^{\nu} = \frac{j - 1 + 2\nu}{2j} \left(N_{-p+1,\, j-1,\, q}^{\nu} + N_{-p-1,\, j-1,\, q}^{\nu} \right) + \frac{\nu}{2j} N_{-p,j-2,q+2}^{\nu-1} , \tag{60}$$

which, combined with Equations 53 and 54, can be used to compute the table of the generalized Cauchy numbers.

A TRANSFORMATION OF THE ADDITION THEOREM FOR GEGENBAUER POLYNOMIALS

We shall transform the addition theorem for Gegenbauer polynomials to a form suitable for obtaining the expansion of the disturbing function of satellites in orbits with high inclinations.

The transformed addition theorem shows that the generalized Cauchy numbers represent a useful instrument for obtaining the expansion of the disturbing function, especially of the terms of higher order. We have

$$C_n^{\nu}(\cos H) = \sum_{m=0}^{n} 2^{2m} A_{n,m}^{\nu} \sin^m\theta\, C_{n-m}^{\nu+m}(\cos\theta) \sin^m\psi\, C_{n-m}^{\nu+m}(\cos\psi)\, C_m^{\nu-1/2}(\cos\phi) \quad , \tag{61}$$

where

$$\cos H = \cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi \tag{62}$$

and

$$A_{n,m}^{\nu} = (n-m)!\,\frac{\Gamma(2\nu-1)\cdot\Gamma^2(\nu+m)}{\Gamma^2(\nu)\cdot\Gamma(2\nu+n+m)}\,(2\nu+2m-1)\ . \tag{63}$$

We set

$$x = \exp i\theta \quad , \qquad y = \exp i\psi\ . \tag{64}$$

Then we have the expansion of the form

$$C_n^{\nu}(\cos H) = \sum_{j,k=-n}^{j,k=+n} [n,\ j,\ k]_{\nu}\, x^j\, y^k \quad , \tag{65}$$

where $[n,\ j,\ k]_{\nu}$ are polynomials in $\cos\phi$. We have from Equation 65:

$$[n,\ j,\ k]_{\nu} = \frac{1}{(2\pi i)^2}\oint\oint x^{-j-1}\, y^{-k-1}\, C_n^{\nu}(\cos H)\, dx\, dy\ . \tag{66}$$

Substituting Equation 64 into Equation 61,

$$C_n^{\nu}(\cos H) = \sum_{m=1}^{n} (-1)^m A_{n,m}^{\nu}\left(x - \frac{1}{x}\right)^m C_{n-m}^{\nu+m}\left(\frac{x+\frac{1}{x}}{2}\right)\left(y - \frac{1}{y}\right)^m C_{n-m}^{\nu+m}\left(\frac{y+\frac{1}{y}}{2}\right) C_m^{\nu-1/2}(\cos\phi)\ . \tag{67}$$

Substituting Equation 67 into Equation 66 and taking Equation 46 into account,

$$[n, j, k]_\nu = \sum_{m=1}^{n} (-1)^m A^\nu_{n,m} N^{\nu+m}_{-j, n-m, m} N^{\nu+m}_{-k, n-m, m} C_m^{\nu-1/2} (\cos\phi) \quad . \tag{68}$$

From Equation 53 we conclude that j and k must be integers of the form

$$j = 2p - n , \qquad k = 2q - n , \qquad p, q = 0, 1, \cdots , n .$$

Consequently

$$C_n^\nu (\cos H) = \sum_{p, q=0}^{n} [n, 2p - n, 2q - n]_\nu \, x^{2p-n} y^{2q-n} \quad ,$$

or

$$C_n^\nu (\cos H) = \sum_{p, q=0}^{n} [n, 2p - n, 2q - n]_\nu \cos\left[(2p - n)\theta + (2q - n)\psi\right] \quad . \tag{69}$$

This last relation together with Equation 68 supplies the form of the addition theorem for Gegenbauer polynomials to be used in the expansions in the lunar theory. The polynomials $[n, j, k]_\nu$ are expanded in terms of Gegenbauer polynomials $C_m^{\nu-1/2}(\cos\phi)$. There is no actual necessity to rearrange the right side of Equation 68 to represent an expansion in powers of $\cos\phi$. It is enough to provide a subroutine for the numerical evaluation of the polynomials $C_m^{\nu-1/2}(\cos\phi)$ for the given value of $\cos\phi$.

The case

$$\nu = \frac{1}{2} , \qquad C_n^{1/2} (\cos H) = P_n (\cos H)$$

requires a special consideration. In this case we have, from Equation 65,

$$P_n (\cos H) = \sum_{j, k=-n}^{j, k=+n} [n, j, k]_{1/2} \, x^j y^k \quad . \tag{70}$$

And by substituting

$$P_n^m(\cos\theta) = (-1)^m 2^m \left(\frac{1}{2}, m\right) \sin^m\theta\, C_{n-m}^{m+1/2}(\cos\theta)$$

$$= i^m \left(\frac{1}{2}, m\right) \left(x - \frac{1}{x}\right)^m C_{n-m}^{m+1/2}\left(\frac{x + \frac{1}{x}}{2}\right)$$

and

$$P_n^m(\cos\psi) = (-1)^m 2^m \left(\frac{1}{2}, m\right) \sin^m\psi\, C_{n-m}^{m+1/2}(\cos\psi)$$

$$= i^m \left(\frac{1}{2}, m\right) \left(y - \frac{1}{y}\right)^m C_{n-m}^{m+1/2}\left(\frac{y + \frac{1}{y}}{2}\right)$$

into the addition formula

$$P_n(\cos H) = P_n(\cos\theta) P_n(\cos\psi) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\psi) \cos m\phi \quad ,$$

we obtain

$$P_n(\cos H) = \sum_{m=0}^{n} A_{n,m}^{1/2} \left(x - \frac{1}{x}\right)^m C_{n-m}^{m+1/2}\left(\frac{x + \frac{1}{x}}{2}\right) \left(y - \frac{1}{y}\right)^m C_{n-m}^{m+1/2}\left(\frac{y + \frac{1}{y}}{2}\right) \cos m\phi \quad , \qquad (71)$$

where

$$A_{n,m}^{1/2} = 2(-1)^m \left(\frac{1}{2}, m\right)^2 \frac{(n-m)!}{(n+m)!} \quad , \qquad m \neq 0 \; ,$$

$$A_{n,0}^{1/2} = 1 \quad .$$

Combining the result of Equation 71 with Equation 66, with $\nu = 1/2$,

$$C_n^{1/2}(\cos H) = P_n(\cos H) \quad ,$$

we obtain

$$[n, j, k]_{1/2} = \sum_{m=0}^{n} A_{n,m}^{1/2} N_{-j,\, n-m,\, m}^{m+1/2} N_{-k,\, n-m,\, m}^{m+1/2} \cos m\phi \tag{72}$$

or, analogous to Equation 68,

$$[n, j, k]_{1/2} = \sum_{m=0}^{n} A_{n,m}^{1/2} N_{-j,\, n-m,\, m}^{m+1/2} N_{-k,\, n-m,\, m}^{m+1/2} T_m(\cos\phi) \quad , \tag{72'}$$

where T_m designates a Chebichef polynomial of the first kind. As before in the case of Equation 68, we conclude that

$$j = 2p - n , \qquad k = 2q - n , \qquad p, q = 0, 1, \cdots , n ;$$

and consequently

$$P_n(\cos H) = \sum_{p,q=0}^{n} [n,\, 2p-n,\, 2q-n]_{1/2} \cos\left[(2p-n)\theta + (2q-n)\psi\right] \quad . \tag{73}$$

Making use of the form of the addition theorem given in this section, we obtain the expansion

$$\left(1 + \alpha^2 - 2\alpha\cos H\right)^{-\nu} = \sum_{n=0}^{\infty} \alpha^n C_n^{\nu}(\cos H) = \sum_{n=0}^{\infty} \alpha^n \sum_{p,q=0}^{n} [n,\, 2p-n,\, 2q-n]_{\nu} \cos\left[(2p-n)\theta + (2q-n)\psi\right] . \tag{74}$$

This formula can be used to obtain the literal expansion of the disturbing function and its derivatives for the case of satellites in orbits highly inclined toward the orbital plane of the Sun. The polynomials $[n, j, k]_{\nu}$ can be expressed in terms of Tisserand polynomials $Q_{i,j}^{(n)}$, which are expanded by Tisserand (1960) in powers of $\sin^2 I/2$ and $\cos^2 I/2$. In the present work, the polynomials $[n, j, k]_{\nu}$ are expanded in powers of cos I, since this expansion is more convenient for the analytical treatment of the perturbations also in Delaunay's canonical variables.

If we decide to develop a purely numerical theory then we can also proceed in a different way: we can substitute the values e_0, e_0', p_0, p_0' , and I from the outset and use the formulas

$$\cos H = \cos^2\frac{I}{2}\cos\left(w_1 - v' + w_4\right) + \sin^2\frac{I}{2}\sin\left(w_1 + v' - w_4\right) \quad ,$$

$$C_j^{\nu}(\cos H) = \sum_{m=0}^{[j/2]} \frac{(-1)^m (\nu, j-m)}{m!\,(j-m)!} (2\cos H)^{j-2m}$$

or the recurrence formula

$$C_{j+2}^{\nu}(\cos H) = \frac{2\nu + 2j + 2}{j + 2} \cos H\, C_{j+1}^{\nu}(\cos H) - \frac{2\nu + j}{j + 2} C_j^{\nu}(\cos H) \quad .$$

DEVELOPMENT OF THE DISTURBING FUNCTION

In the theory presented in this work, the disturbing function has the form

$$\Phi = \frac{1}{2}\nu^2 \left(x^2 + z^2\right) \tan^2 I - \nu^2 yz \tan I + \Omega \quad , \tag{75}$$

where, with properly chosen units,

$$\Omega = m' \left(\frac{1}{\Delta} - \frac{\vec{r} \cdot \vec{r}\,'}{r'^3}\right) \quad , \tag{76}$$

$$\Delta = |\vec{r} - \vec{r}\,'| \quad .$$

In the system (x, y, z) the coordinates of the satellite are

$$x = \rho \cos w_1 \, , \qquad y = \rho \sin w_1 \, , \qquad z = \rho\zeta \, , \tag{77}$$

and the coordinates of the planet in the same system are

$$x' = \rho' \cos\left(v' - w_4\right) , \qquad y' = \rho' \sin\left(v' - w_4\right) \cos I \, , \qquad z' = -\rho' \sin\left(v' - w_4\right) \sin I \, .$$

We obtain

$$\vec{r} \cdot \vec{r}\,' = \rho\rho' \left[\cos H - \zeta \sin\left(v' - w_4\right) \sin I\right] \quad , \tag{78}$$

where

$$\cos H = \cos w_1 \cos\left(v' - w_4\right) + \sin w_1 \sin\left(v' - w_4\right) \cos I \quad . \tag{79}$$

We also have

$$\Delta^2 = \left(\rho^2 + \rho'^2 - 2\rho\rho' \cos H\right) + \left[2\rho\rho' \zeta \sin\left(v' - w_4\right) \sin I + \rho^2 \zeta^2\right] \quad ; \tag{80}$$

and setting

$$\Delta_0^2 = 1 + \frac{\rho^2}{\rho'^2} - \frac{2\rho}{\rho'} \cos H \quad , \tag{81}$$

$$\delta = 2 \frac{\rho}{\rho'} \zeta \sin\left(v' - w_4\right) \sin I + \frac{\rho^2}{\rho'^2} \zeta^2 \quad , \tag{82}$$

we obtain from Equation 80

$$\frac{1}{\Delta} = \frac{1}{\rho'} \left(\frac{1}{\Delta_0} - \frac{1}{2} \frac{\delta}{\Delta_0^3} + \frac{3}{8} \frac{\delta^2}{\Delta_0^5} - \frac{5}{16} \frac{\delta^3}{\Delta_0^7} + \cdots \right) \quad . \tag{83}$$

Substituting

$$\frac{1}{\Delta_0^{2k+1}} = 1 + \frac{\rho}{\rho'} C_1^{k+1/2} (\cos H) + \frac{\rho^2}{\rho'^2} C_2^{k+1/2} (\cos H) + \cdots$$

into Equation 83 gives

$$\begin{aligned} \frac{1}{\Delta} = & \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{1/2} + \frac{\rho^2}{\rho'^3} C_2^{1/2} + \frac{\rho^3}{\rho'^4} C_3^{1/2} + \frac{\rho^4}{\rho'^5} C_4^{1/2} \right) \\ & - \frac{1}{2} \delta \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{3/2} + \frac{\rho^2}{\rho'^3} C_2^{3/2} + \frac{\rho^3}{\rho'^4} C_3^{3/2} + \cdots \right) \\ & + \frac{3}{8} \delta^2 \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{5/2} + \frac{\rho^2}{\rho'^3} C_2^{5/2} + \cdots \right) \\ & - \frac{5}{16} \delta^3 \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{7/2} + \cdots \right) + \cdots \quad . \end{aligned} \tag{84}$$

Taking Equation 78 in the form

$$\vec{r} \cdot \vec{r}\,' = \rho \rho' C_1^{1/2} - \frac{1}{2} \rho'^2 \delta + \frac{1}{2} \rho^2 \zeta^2 \tag{85}$$

and substituting Equations 84 and 85 into Equations 76 and 75 gives

$$\Phi = \Phi_0 + \Phi_1 \quad , \tag{86}$$

where

$$\Phi_0 = m' \left\{ \left(\frac{\rho^2}{\rho'^3} C_2^{1/2} + \frac{\rho^3}{\rho'^4} C_3^{1/2} + \frac{\rho^4}{\rho'^5} C_4^{1/2} + \cdots \right) \right.$$

$$- \frac{1}{2} \left(\frac{\rho}{\rho'^2} C_1^{3/2} + \frac{\rho^2}{\rho'^3} C_2^{3/2} + \frac{\rho^3}{\rho'^4} C_3^{3/2} + \cdots \right) \delta$$

$$+ \frac{3}{8} \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{5/2} + \frac{\rho^2}{\rho'^3} C_2^{5/2} + \cdots \right) \delta^2$$

$$\left. - \frac{5}{16} \left(\frac{1}{\rho'} + \frac{\rho}{\rho'^2} C_1^{7/2} + \cdots \right) \delta^3 + \cdots \right\} , \tag{87}$$

and

$$\Phi_1 = \frac{1}{4} \nu^2 \rho^2 \left(1 + \cos 2w_1\right) \tan^2 I$$

$$- \nu^2 \rho^2 \zeta \sin w_1 \tan I$$

$$+ \frac{1}{2} \rho^2 \zeta^2 \left(\nu^2 \tan^2 I - \frac{m'}{\rho'^3} \right) . \tag{88}$$

The quality Φ_0 constitutes the main part of the disturbing function, and it is of the second order relative to the ratio of mean motions of the Sun and of the satellite; Φ_1 is of the second order relative to Φ_0.

Let us introduce the set of the following auxiliary quantities and series which appear in the expansion of the disturbing function and in the expansion of its derivatives:

$$\mu^2 = m' \left(\frac{p_0}{p_0'} \right)^3 , \qquad \alpha = \frac{p_0}{p_0'} \tag{89}$$

$$\Gamma_{0j} = \left(\frac{\rho}{p_0}\right)^j \left(\frac{p_0'}{\rho'}\right)^{j+1} \alpha^{j-2} C_j^{1/2} (\cos H) , \qquad \Gamma_{1j} = \left(\frac{\rho}{p_0}\right)^j \left(\frac{p_0'}{\rho'}\right)^{j+1} \alpha^{j-1} C_j^{3/2} (\cos H) , \tag{90}$$

$$\Gamma_{ij} = \left(\frac{\rho}{p_0}\right)^j \left(\frac{p_0'}{\rho'}\right)^{j+1} \alpha^j C_j^{i+1/2} (\cos H) , \qquad i > 1 , \tag{91}$$

$$\Gamma_0 = \Gamma_{02} + \Gamma_{03} + \Gamma_{04} + \cdots , \tag{92}$$

$$\Gamma_1 = \Gamma_{11} + \Gamma_{12} + \Gamma_{13} + \cdots , \tag{93}$$

$$\Gamma_i = \Gamma_{i0} + \Gamma_{i1} + \Gamma_{i2} + \cdots , \qquad i > 1 , \tag{94}$$

$$\Theta_0 = 2\Gamma_{02} + 3\Gamma_{03} + 4\Gamma_{04} + \cdots , \tag{95}$$

$$\Theta_1 = \Gamma_{11} + 2\Gamma_{12} + 3\Gamma_{13} + \cdots , \tag{96}$$

$$\Theta_i = \Gamma_{i1} + 2\Gamma_{i2} + 3\Gamma_{i3} + \cdots , \qquad i \geqq 1 , \tag{97}$$

$$\epsilon = \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \zeta \left[2 \sin\left(v' - w_4\right) \sin I + \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \alpha\zeta \right] , \tag{98}$$

$$\Lambda = \mu^2 \left(-\frac{1}{2} \Gamma_1 + \frac{3}{4} \epsilon \Gamma_2 - \frac{15}{16} \epsilon^2 \Gamma_3 + \cdots \right) , \tag{99}$$

$$\Psi = \mu^2 \left(\Theta_0 - \frac{1}{2} \epsilon \Theta_1 + \frac{3}{8} \epsilon^2 \Theta_2 - \frac{5}{16} \epsilon^3 \Theta_3 + \cdots \right) . \tag{100}$$

These two last series are the basic ones in the theory presented here. The quantity μ is analogous to the ratio of mean mean motions of the theory of Delaunay, and α is the parallactic factor of our theory. We obtain, after some easy transformations,

$$p_0 \Phi_0 = \mu^2 \left(\Gamma_0 - \frac{1}{2} \epsilon \Gamma_1 + \frac{3}{8} \epsilon^2 \Gamma_2 - \frac{5}{16} \epsilon^3 \Gamma_3 + \cdots \right) . \tag{101}$$

From the last equation and from Equations 91 through 98, we deduce

$$\rho^2 \frac{\partial \Phi_0}{\partial \rho} = \frac{\rho}{p_0} \left[\left(\epsilon + \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \alpha \zeta^2 \right) \Lambda + \Psi \right] , \tag{102}$$

from which, taking Equations 19 and 21 into account, we obtain

$$\rho^2 \frac{\partial \Phi_1}{\partial \rho} = (1-g)^2 (1+s) \left[\frac{1}{2} \left(1 + \cos 2w_1\right) \tan^2 I - 2\zeta \sin w_1 \tan I + \zeta^2 \tan^2 I \right]$$

$$- \mu^2 \zeta^2 \left(\frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \right)^3 .$$

Taking the two last equations and Equations 19 and 21 into account, we obtain

$$\rho^2 \frac{\partial \Phi}{\partial \rho} = \frac{\rho}{p_0} \left[\left(\epsilon + \frac{\rho}{p_0} \cdot \frac{p_0{}'}{\rho'} \alpha \zeta^2 \right) \Lambda + \Psi \right]$$

$$+ (1-g)^2 (1+s) \left[\frac{1}{2} \left(1 + \cos 2w_1 \right) \tan^2 I - 2\zeta \sin w_1 \tan I + \zeta^2 \tan^2 I \right]$$

$$- \mu^2 \zeta^2 \left(\frac{\rho}{p_0} \cdot \frac{p_0{}'}{\rho'} \right)^3 \quad . \qquad \textbf{(103)}$$

Now making use of

$$\frac{d\, C_n^{\nu}(x)}{dx} = 2\nu\, C_{n-1}^{\nu+1}(x)$$

we obtain, from Equations 90 through 94,

$$\frac{\partial \Gamma_i}{\partial \cos H} = (2i+1) \frac{\rho}{p_0} \cdot \frac{p_0{}'}{\rho'} \alpha \Gamma_{i+1} \qquad i = 0, 1, 2, \cdots \quad . \qquad \textbf{(104)}$$

Differentiating Equation 101 relative to w and taking Equations 79, 99, and 104 into account, we have

$$\frac{\partial (p_0 \Phi_0)}{\partial w} = 2 \left[\sin^2 \frac{I}{2} \sin\left(w_1 + v' - w_4\right) + \cos^2 \frac{I}{2} \sin\left(w_1 - v' + w_4\right) \right] \frac{\rho}{p_0} \frac{p_0{}'}{\rho'} \alpha \Lambda \; ; \qquad \textbf{(105)}$$

and from Equations 17, 21, and 88

$$\frac{\partial (p_0 \Phi_1)}{\partial w} = - \frac{1}{2} \frac{(1-g)^2}{1-\psi} (1+s)^2 \left(\sin 2w_1 \tan^2 I + 2\zeta \cos w_1 \tan I \right) \quad . \qquad \textbf{(106)}$$

From these two last equations we deduce

$$2 \frac{\rho^2}{p_0} \frac{\partial \Phi}{\partial w} = \left(\frac{\rho}{p_0} \right)^2 \left\{ 4\Lambda \left[\sin^2 \frac{I}{2} \sin\left(w_1 + v' - w_4\right) + \cos^2 \frac{I}{2} \sin\left(w_1 - v' + w_4\right) \right] \frac{\rho}{p_0} \cdot \frac{p_0{}'}{\rho^1} \alpha \right.$$

$$\left. - \frac{(1-g)^2}{1-\psi} (1+s)^2 \left(\sin 2w_1 \tan^2 I + 2\zeta \cos w_1 \tan I \right) \right\} \cdot \qquad \textbf{(107)}$$

Substituting

$$\frac{\partial (p_0 \Phi_0)}{\partial \epsilon} = \Lambda \quad ,$$

$$\frac{\partial \epsilon}{\partial \zeta} = \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \left[2 \sin (v' - w_4) \sin I + 2 \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \zeta \right]$$

into

$$\frac{\partial (p_0 \Phi_0)}{\partial \zeta} = \frac{\partial (p_0 \Phi_0)}{\partial \epsilon} \frac{\partial \epsilon}{\partial \zeta} \quad ,$$

we have

$$\frac{\partial (p_0 \Phi_0)}{\partial \zeta} = \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \left[2 \sin(v' - w_4) \sin I + 2 \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \zeta \right] \Lambda \quad . \tag{108}$$

From Equation 88 we have

$$\frac{\partial (p_0 \Phi_1)}{\partial \zeta} = p_0 \left[-\nu^2 \rho^2 \sin w_1 \tan I + \rho^2 \zeta \left(\nu^2 \tan^2 I - \frac{m'}{\rho'^3} \right) \right] ,$$

and, taking Equation 22 into consideration,

$$\frac{\partial (p_0 \Phi_1)}{\partial \zeta} = \frac{(1-g)^2}{1-\psi} (1+s)^2 \left(-\sin w_1 \tan I + \zeta \tan^2 I \right) - \mu^2 \left(\frac{\rho}{p_0} \right)^2 \left(\frac{p_0'}{\rho'} \right)^3 .$$

Combining the last equation with Equation 108 we obtain

$$\frac{\partial (p_0 \Phi)}{\partial \zeta} = 2 \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \left[\sin(v' - w_4) \sin I + \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \zeta \right] \Lambda$$
$$+ \frac{(1-g)^2}{1-\psi} (1+s)^2 \left(-\sin w_1 \tan I + \zeta \tan^2 I \right) - \mu^2 \left(\frac{\rho}{p_0} \right)^2 \left(\frac{p_0'}{\rho'} \right)^3 ,$$

and finally

$$\rho \frac{\partial \Phi}{\partial \zeta} = 2 \left(\frac{\rho}{p_0} \right)^2 \frac{p_0'}{\rho'} \Lambda \left[\sin(v' - w_4) \sin I + \frac{\rho}{p_0} \cdot \frac{p_0'}{\rho'} \zeta \right]$$
$$+ (1-g)^2 (1+s) \left(-\sin w_1 \tan I + \zeta \tan^2 I \right) - \mu^2 \left(\frac{\rho}{p_0} \right)^3 \left(\frac{p_0'}{\rho'} \right)^3 .$$

The limit of the lower index of Gegenbauer polynomials in Equations 90 and 91 depends upon the values of α and ζ. If there is a necessity for developments for higher indices, we can use Equation 69 for the trigonometrical expansion of Gegenbauer polynomials in terms of the generalized Cauchy numbers.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS

As the final output we must obtain the expansions of Brendel's coordinates s, ψ, and ζ in Fourier series in four basic arguments: w_1, w_2, w_3, w_4. The arguments w_1, w_2, w_4 have a direct geometric meaning in the "nearly ideal" system of coordinates, which we have discussed previously. However, w_3 requires additional discussion. The argument w_3, which enters into the expansion of s, ψ, and ζ, is intimately connected with the argument v'. The angle v' defines the position of the Sun in its ellipse, and it is this "elliptic" v' which enters into the expansions given in Equations 87 through 103 of the disturbing function Φ and of its derivatives. Thus, these developments contain five arguments: w_1, w_2, w_3, w_4, and v'. We can set

$$v' = w_3 + \delta v' \quad , \tag{42'}$$

where again $\delta v'$ is a series in four arguments, and we can eliminate v' in favor of w_3 by expanding Φ_0 and its derivatives into a Taylor series relative to $\delta v'$. However, in this expansion we must distinguish between w_3 as an argument in the series representing Brendel's coordinates on the one hand, the "elliptic" values of v' and w_3 in Equation 42' on the other. We have to eliminate the "elliptic" v' and, because we have two types of w_3, we must do so by making use of two operators. One is the replacement operator R which means the formal replacement of the "elliptic" v' by w_3:

$$RF(w_1, w_2, w_3, w_3; v') = F(w_1, w_2, w_3, w_4; w_3) \quad .$$

The other is the Taylor operator

$$T = \exp\left(\delta v' \frac{\partial}{\partial v'}\right)$$

which transforms the function $F(w_1, w_2, w_3, w_4, v')$ into the function $F(w_1, w_2, w_3, w_4, v' + \delta v')$:

$$TF(w_1, w_2, w_3, w_4, v') = F(w_1, w_2, w_3, w_4, v' + \delta v') \quad .$$

Making use of the "product"

$$K = RT$$

of both operators, we have from the last equation:

$$KF(w_1, w_2, w_3, w_4, v') = RF(w_1, w_2, w_3, w_4, v' + \delta v')$$

$$= F(w_1, w_2, w_3, w_4, w_3 + \delta v') = F(w_1, w_2, w_3, w_4, v')$$

$$= R \sum_{k=0}^{+\infty} \frac{\delta v'^k}{k!} \frac{\partial^k F}{\partial v'^k} = \sum_{k=0}^{+\infty} \frac{\delta v'^k}{k!} R \left(\frac{\partial^k F}{\partial v'^k}\right) .$$

The operator R is analogous to the "bar-operator" of Hansen's lunar theory. Thus

$$K = \sum_{k=0}^{\infty} \frac{\delta v'^k}{k!} R \frac{\partial^k}{\partial v'^k} , \tag{109}$$

and Equations 24, 30', and 39" are rewritten in the form

$$\frac{d^2 s}{dv^2} = A + B - \rho K \left(\rho \frac{\partial \Phi}{\partial \rho} - \zeta \frac{\partial \Phi}{\partial \zeta}\right) , \tag{30''}$$

$$\frac{d^2 \zeta}{dv^2} + \zeta + 2g(1-g)\sin w = P + \frac{\rho^2}{p} K \left[\left(1+\zeta^2\right)\frac{\partial \Phi}{\partial \zeta} - \zeta \left(\rho \frac{\partial \Phi}{\partial \rho}\right)\right] , \tag{39''}$$

$$\frac{d\psi}{dv} = - \frac{2\rho^2}{p_0} K \frac{\partial \Phi}{\partial v} + (1-g)\left[\zeta \frac{d\psi}{dv} - 4(1-\psi)\frac{d\zeta}{dv}\right] \sin w \tan I , \tag{24''}$$

where the expansion (109) of K is to be used. The expansion of $\delta v'$ in terms of the basic arguments and the value of m are determined from Equations 42 and 44.

In Equations 30", 39", and 24", all arguments are now linear in v, and this facilitates the problem of integration. As the method of solving these equations, we suggest the process of iteration, partly because it makes the program homogenous and partly to avoid the question of convergence relative to the parameters e_0, e_0', α, m, and I. We can substitute their values from the outset, before we start the process of iteration. At each iteration step, the frequency g of the argument w_1 is so determined that no term of the form $A \sin w_1$ is present in ζ. In a similar manner we determine the frequency c of w_2, by requiring that the only term of the form $A \cos w_2$ in s be $e_0 \cos w_2$. The frequency of w_4 is simply $(1-g)/\cos I$.

EXPANSION OF THE RECTANGULAR COORDINATES

The theory of satellites in Brendel's coordinates also favors expansion of the rectangular coordinates in the inertial system (XYZ) into Fourier series. We have

$$\vec{r} = \rho \left[\vec{P} \cos w_1 + \vec{Q} \sin w_1 + \zeta \vec{R} \right] ,$$

$$\rho = \frac{p_0 (1 - \psi)}{1 + s} ;$$

but

$$[\vec{P}, \vec{Q}, \vec{R}] = \begin{bmatrix} \cos w_4 & -\cos I \sin w_4 & \sin I \sin w_4 \\ \sin w_4 & \cos I \cos w_4 & -\sin I \cos w_4 \\ 0 & \sin I & \cos I \end{bmatrix} ,$$

and we have finally

$$x = \rho \left[\cos^2 \frac{I}{2} \cos (w_1 + w_4) + \sin^2 \frac{I}{2} \cos (w_1 - w_4) + \zeta \sin I \sin w_4 \right] , \quad (110)$$

$$y = \rho \left[\cos^2 \frac{I}{2} \sin (w_1 + w_4) - \sin^2 \frac{I}{2} \sin (w_1 - w_4) - \zeta \sin I \cos w_4 \right] , \quad (111)$$

$$z = \rho (\sin w_1 \sin I + \zeta \cos I) . \quad (112)$$

Substituting here the expansion of ρ, we have the expansions of the form

$$x = \sum C_{i_1 i_2 i_3 i_4} \cos (i_1 v_1 + i_2 v_2 + i_3 v_3 + i_4 v_4) , \quad (113)$$

$$y, z = \sum S_{i_1 i_2 i_3 i_4} \sin (i_1 v_1 + i_2 v_2 + i_3 v_3 + i_4 v_4) , \quad (114)$$

where we set, as suggested by Equations 110 and 111,

$$w_1 = v_1 + v_4 , \qquad w_4 = v_1 - v_4 , \qquad w_2 = v_2 , \qquad w_3 = v_3 .$$

In these last expansions, three arguments v_1, v_2, v_4, have nearly equal frequencies.

COLLECTION OF FORMULAS

$$S = \cos^2 \frac{I}{2} \cos\left(w_1 - v' + w_4\right) + \sin^2 \frac{I}{2} \cos\left(w_1 + v' - w_4\right)$$

$$S^2,\ S^2,\ \cdots$$

$$F = \frac{\rho}{p_0}$$

$$G = \frac{p_0'}{\rho'}$$

$$Q = FG$$

$$GQ = Q_1$$

$$GQ^2 = Q_2$$

$$GQ^3 = Q_3$$

. . . .

$$L = 2 \sin\left(v' - w_4\right) \sin I + \alpha \zeta Q$$

$$\epsilon \;\square\; \zeta L Q$$

$$M = \epsilon + \alpha \zeta^2 Q$$

$$T = (1 - g)^2 (1 + s) \left(-\sin w_1 \tan I + \zeta \tan^2 I\right)$$

$$U = \sin\left(v' - w_4\right) \sin I + \zeta Q$$

$$W = -(1 - g)^2 (1 - \psi) \left(\sin 2w_1 \tan^2 I + 2\zeta \sin w_1 \tan I\right) \mu^2 \zeta Q^3$$

$$A = -\frac{1 + s}{1 - \psi} \frac{d^2 \psi}{dv^2} - \frac{3}{2} \frac{1}{1 - \psi} \frac{ds}{dv} \frac{d\psi}{dv}$$

$$B = -\frac{3}{2}\zeta^2 + \frac{15}{8}\zeta^4 - \frac{3}{2}\frac{1+s}{(1-\psi)^2}\left(\frac{d\psi}{dv}\right)^2$$

$$+ (1-g)(1+s)\left(2\frac{d\zeta}{dv} - \frac{1}{2}\frac{\zeta}{1-\psi}\frac{d\psi}{dv}\right)\cos w \tan I$$

$$P = +\frac{1}{2}\frac{1}{1-\psi}\frac{d\psi}{dv}\frac{d\zeta}{dv} + (1-g)\left(\frac{d\zeta^2}{dv} - \frac{1}{2}\frac{1+\zeta^2}{1-\psi}\frac{d\psi}{dv}\right)\cos w \tan I$$

$$C = (1-g)\left[\zeta\frac{d\psi}{dv} - 4(1-\psi)\frac{d\zeta}{dv}\right]\sin w \tan I$$

$$C_2^{1/2} = \frac{3}{2}S^2 - \frac{1}{2}$$

$$C_3^{1/2} = \frac{5}{2}S^2 - \frac{3}{2}S$$

$$C_4^{1/2} = \frac{35}{8}S^4 - \frac{15}{4}S^2 + \frac{3}{8}$$

. . . .

$$C_1^{3/2} = 3S$$

$$C_2^{3/2} = \frac{15}{2}S^2 - \frac{3}{2}$$

$$C_3^{3/2} = \frac{35}{2}S^3 - \frac{15}{2}S$$

. . . .

$$C_1^{5/2} = 5S$$

$$C_2^{5/2} = \frac{35}{2}S^2 - \frac{5}{2}$$

. . . .

$$C_1^{7/2} = 7S$$

$$\Gamma_{02} = Q_2 C_2^{1/2}$$

$$\Gamma_{03} = \alpha Q_3 C_3^{1/2}$$

$$\Gamma_{04} = \alpha^2 Q_4 C_4^{1/2}$$

. . . .

$$\Gamma_0 = \Gamma_{02} + \Gamma_{03} + \Gamma_{04} + \cdots$$

$$\Theta_0 = 2\Gamma_{02} + 3\Gamma_{03} + 4\Gamma_{04} + \cdots$$

$$\Gamma_{11} = Q_1 C_1^{3/2}$$

$$\Gamma_{12} = \alpha Q_2 C_2^{3/2}$$

$$\Gamma_{13} = \alpha^2 Q_3 C_3^{3/2}$$

. . . .

$$\Gamma_1 = \Gamma_{11} + \Gamma_{12} + \Gamma_{13} + \cdots$$

$$\Theta_1 = \Gamma_{11} + 2\Gamma_{12} + 3\Gamma_{13} + \cdots$$

$$\Gamma_{20} = G$$

$$\Gamma_{21} = \alpha Q_1 C_1^{5/2}$$

$$\Gamma_{22} = \alpha^2 Q_2 C_2^{5/2}$$

$$\Gamma_{23} = \alpha^3 Q_3 C_3^{5/2}$$

. . . .

$$\Gamma_2 = \Gamma_{20} + \Gamma_{21} + \Gamma_{22} + \Gamma_{23} + \cdots$$

$$\Theta_2 = \Gamma_{21} + 2\Gamma_{22} + 3\Gamma_{23} + \cdots$$

$$\Gamma_{30} = G$$

$$\Gamma_{31} = \alpha Q_1 C_1^{7/2}$$

$$\Gamma_{32} = \alpha^2 Q_2 C_2^{7/2}$$

$$\Gamma_{33} = \alpha^3 Q_3 C_3^{7/2}$$

$$\Gamma_3 = \Gamma_{30} + \Gamma_{31} + \Gamma_{32} + \cdots$$

$$\Theta_3 = \Gamma_{31} + 2\Gamma_{32} + 3\Gamma_{33} + \cdots$$

$$\Lambda = \mu^2 \left(- \frac{1}{2} \Gamma_1 + \frac{3}{4} \epsilon \Gamma_2 - \frac{15}{16} \epsilon^2 \Gamma_3 + \cdots\right)$$

$$\Psi = \mu^2 \left(\Theta_0 - \frac{1}{2} \epsilon \Theta_1 + \frac{3}{8} \epsilon^2 \Theta_2 - \frac{5}{16} \epsilon^3 \Theta_3 + \cdots\right)$$

$$\rho^2 \frac{\partial \Phi}{\partial \rho} = F(M\Lambda + \Psi)$$

$$2 \frac{\rho^2}{p_0} \frac{\partial \Phi}{\partial w} = 4F^2 S\alpha Q\Lambda + W$$

$$\rho \frac{\partial \Phi}{\partial \zeta} = 2FQ\Lambda U + T$$

$$K = R\left(1 + \delta v' \frac{\partial}{\partial v'} + \frac{1}{2} \delta v'^2 \frac{\partial^2}{\partial v'^2} + \cdots\right)$$

$$\frac{d^2 \zeta}{dv^2} + s = A + B - K\left[\rho^2 \frac{\partial \Phi}{\partial \rho} - \zeta \left(\rho \frac{\partial \Phi}{\partial \zeta}\right)\right]$$

$$\frac{d^2 \zeta}{dv^2} + \zeta\, 2g(1-g) \sin w = P$$

$$+ K\left[\frac{1+\zeta^2}{1+s}\left(\rho \frac{\partial \Phi}{\partial \zeta}\right) - \frac{\zeta}{1+s}\left(\rho^2 \frac{\partial \Phi}{\partial \rho}\right)\right]$$

$$\frac{d\psi}{dv} = -\frac{2\rho^2}{p_0} K \frac{\partial \Phi}{\partial v} + C$$

$$m = \text{const. part in} \left\{\mu \sqrt{1 + \frac{1}{m'}}\,(1-\psi)^{3/2} \left(\frac{1+s'}{1+s}\right)^2\right\}$$

$$\delta v' = \int \text{period. part in} \left[\mu \sqrt{1 + \frac{1}{m'}} (1 - \psi)^{-3/2} \left(\frac{1 + s'}{1 + s} \right)^2 \right]$$

$$\frac{dt}{dv} \; p_0^{3/2} (1 - \psi)^{3/2} (1 + s)^{-2}$$

CONCLUSION

We have suggested a numerical theory of satellites which gives the expansion of the rectangular coordinates in terms of four arguments, using the expansion of Brendel's coordinates as an intermediary step. This theory, like Hansen's, can be used for highly inclined orbits up to the critical inclination, and it makes use of a "nearly ideal" system of coordinates. Unlike Hansen, we do not make use of the rotating ellipse as a reference orbit, and we have discarded the use of the w-function. Thus, the theory given here leads to the numerical expansion of the rectangular coordinates in a more direct way. Recently, the author has suggested a modification of Hansen's lunar theory which was programmed and compared with Hansen's theory to the extent and to the accuracy necessary for computing ephemerides of the satellites of outer planets (1×10^{-5} in r, $1° \times 10^{-4}$ in angles). Our next goal is the programming of the theory presented here and comparison of the expansion with those given by other theories.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, January 10, 1967
188-43-01-01-51

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Appendix A

Symbol List

m'	the mass of the Sun, the mass of the planet, and the gravitational constant are set to be one.
XYZ	the inertial system of coordinates, with the X- and Y-axes in orbital plane of the Sun.
I	the mean inclination of the orbital plane of the satellite toward the orbital plane of the Sun.
$\vec{P}$	the unit vector directed from the planet toward the ascending node of the mean orbital plane of the satellite.
$\vec{R}$	the unit vector normal to the mean orbital plane of the satellite.
$\vec{Q} = \vec{R} \times \vec{P}$	
$\vec{r}$	the position vector of the satellite with respect to the planet.
$\vec{r} = \lvert\vec{r}\rvert$	
$\vec{\rho}$	the projection of $\vec{r}$ on the mean orbital plane of the satellite.
$\vec{r}\,'$	the position vector of the Sun with respect to the planet.
$r' = \rho' = \lvert\vec{r}\,'\rvert$	
p_0'	the semi-latus rectum of the solar orbit.
v'	the true longitude of the Sun.
x	the projection of $\vec{r}$ on $\vec{P}$.
y	the projection of $\vec{r}$ on $\vec{Q}$.
z	the projection of $\vec{r}$ on $\vec{R}$.
$\vec{\gamma}$	the angular speed of rotation of the mean orbital plane of the satellite around the Z-axis.
λ	the projection of $\vec{\gamma}$ on $\vec{Q}$.
ν	the projection of $\vec{\gamma}$ on $\vec{R}$.

$\xi\eta$	the "nearly ideal" system of coordinates located in the mean orbital plane of the satellite and rotating with the angular velocity $-\nu\vec{R}$ relative to the frame $(\vec{P}, \vec{Q})$.
v	the polar angle of $\vec{\rho}$ in the system $(\xi\eta)$.
$w = w_1 = gv + c_1$	the mean argument of the latitude of the satellite.
$w_2 = cv + c_2$	the mean true anomaly of the satellite.
$w_3 = mv + c_3$	the mean longitude (relative to v) of the Sun.
$w_4 = (1 - g)\, v \sec I + c_4$	the longitude of the mean ascending node.
p	the areal velocity of $\vec{\rho}$ in the system $(\xi\eta)$.
p_0	the mean areal velocity of $\vec{\rho}$ in the system $(\xi\eta)$.
$-p_0 \psi$	the perturbations in the mean areal velocity of $\vec{\rho}$ relative to $(\xi\eta)$ $p = p_0 (1 - \psi)$.
s	the Brendel's coordinate defined by the equation: $\rho = \frac{p}{1 + s}$
$\zeta = \frac{z}{\rho}$	
$\Omega = m' \left(\frac{1}{\lvert \vec{r}\,' - \vec{r} \rvert} - \frac{\vec{r} \cdot \vec{r}\,'}{r'^3} \right)$	the disturbing function associated with the solar perturbations in the motion of the satellite.
$\Phi = \frac{1}{2} \lambda^2 \left(x^2 + z^2\right) - \lambda\nu yz + \Omega$	the modified disturbing function.
$N_{-p, j, q}$	the Cauchy number.
$N^{\nu}_{-p, j, q}$	the generalized Cauchy number.