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ABSTRACT

A numerical approach to solving Kepler's universal transcendental equation is presented. A universal first approximation to the solution is obtained by solving a cubic equation which resembles the parabolic conic equation. A quadratic Newton-Raphson iteration technique is discussed. Results indicate that this technique gives a solution to 8-digit accuracy for most practical cases by only a few repeated iterations.

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CONTENTS

A NUMERICAL SOLUTION OF KEPLER'S PROBLEM IN UNIVERSAL VARIABLES

I. INTRODUCTION

In the classical approach to the problem of finding the position on a Keplerian orbit at a given time one is led to the necessity of solving one of the three following transcendental equations, corresponding to eccentricities smaller, equal and larger than one respectively,

$$\tau = \begin{cases} q^{3/2}(1-e)^{-3/2}(E - e\sin E) \\ q\beta + \beta^3/6 \\ q^{3/2}(e-1)^{-3/2}(e\sinh H - H) \end{cases} \tag{1.1}$$

where

$$\tau = \sqrt{\mu}\,(t - T)$$

$$\beta = \sqrt{2q}\,\tan v/2 \tag{1.2}$$

E and H are the eccentric anomalies of the elliptic, and hyperbolic orbits respectively; in the parabolic case v is the true anomaly. The symbols q and e represent as usual the periapsis distance and the eccentricity respectively. Since Gauss' time, a wealth of methods have been developed to deal with the difficulties which arise especially when the eccentricity of the orbit is large or close to 1. All these methods are essentially based on the use of formulas and tables especially devised for particular cases. A good account of these methods may be found in Watson (1964), Herget (1948), or Dubyago (1961). They give satisfactory answers when one has to deal for a long time with the same orbit or with a certain limited type of orbits. However in, some actual applications, as, for example, in the calculations of interplanetary trajectories where the orbit may suffer qualitative and quantitative drastic changes, a more uniform approach is necessary to solve Kepler's problem.

Herrick (1965) published a comprehensive paper on "universal variables" reviewing many different proposals and including a bibliography which goes back to his own early papers of 1945 and 1948 and those of Stumpff and others. By the use of the universal variables, the equations (1.1) are substituted by a unique transcendental equation which is valid in all cases and allows a more uniform treatment of the problem. In general the solution of these fundamental equations is obtained through a Newton-Raphson iteration process starting with an approximation of the solution. If this approximation is not too good the entire process may require a large number of iterations or even worse it may not converge.

In the present paper we reduce the problem to a standard form and then the solution is obtained in two steps. In the first step an initial approximation to the unknown is obtained by solving a cubic equation which is close to the fundamental equation in universal variables. Then by solving a second polynomial equation we obtain a correction of the first approximation which gives in most of the practical cases 8 or more correct significant figures in the result. By this method we have treated uniformly all orbits with eccentricities ranging from near zero to 1.5 and with sizes covering the motion of all bodies in the whole solar system. It is worth noticing that in its most general form Kepler's problem relates any point P_0 of the orbit to any other point P. We have chosen the point P_0 to be the periapsis to reduce the problem to a standard form and make all our numerical results comparable. On the other hand in many problems the points of interest on an orbit are centered around periapsis, thus making this point the best practical choice.

II. REDUCTION OF THE PROBLEM TO A STANDARD FORM

We notice first that due to the symmetry properties of conic sections, we may assume that τ is always positive. By the same reason in the elliptic case we may set as an upper limit for τ the value

$$\tau_c = a^{3/2}\pi \tag{2.1}$$

which corresponds to half a period of revolution around the primary. For the parabolic and hyperbolic cases we set also an upper limit τ_c which corresponds to a fixed heliocentric distance equal to 40 a.u. in order to reach Pluto's orbit.

From the form of the equations (1.1) we observe that for a given τ the corresponding solutions, E, β, or H depend on the values of the parameters q and e. The influence of q can be easily eliminated by the transformation

$$\tau_1 = \frac{\tau}{q^{3/2}} \tag{2.2}$$

and we obtain

$$\tau_1 = \begin{cases} (1-e)^{-3/2}(E - e\sin E) \\ B + B^3/6, \quad B = \sqrt{2}\tan v/2 \\ (e-1)^{-3/2}(e\sinh H - H) \end{cases} \tag{2.3}$$

These equations are formally identical to (1.1) and they become identical to (1.1) when $q = 1$. In what follows we shall always consider $q = 1$; in practical cases where $q \neq 1$ it is only necessary to apply the transformation (2.2).

The universal form of these equations may be obtained by expanding sin E and sinh H in powers of E and H respectively. Then we may define a variable B by

$$B = |1-e|^{-1/2}\,\epsilon, \tag{2.4}$$

where ϵ in this instance is equivalent to E or H of equations (2.3) and another variable

$$\zeta = -B^2(1-e). \tag{2.5}$$

Then introducing the special functions,

$$Z_i = eB^i \sum_{n=0}^{\infty} \frac{\zeta^n}{(2n+i)!} \tag{2.6}$$

we have the universal equation

$$\tau_1 = B + Z_3(B) \tag{2.7}$$

which represents the three equations (2.3).

It is interesting to note that by writing

$$F_i = \sum_{n=0}^{\infty} \frac{\zeta^n}{(2n+i)!}$$

we have

$$Z_i = eB^i F_i$$

The F_i's are the functions introduced by Stumpf (1959) in his treatment of the two-body problem in universal variables; see also, Blanchard and Wolf (1967). The functions Z_i have some properties similar to those of the functions F_i and we shall use them in what follows. For continuity of thought, these properties are included in Appendix A.

III. THE UNIVERSAL FIRST APPROXIMATION

The reduction of Kepler's problem to a standard form results in solving the equation

$$\tau = B + Z_3(B) \tag{3.1}$$

where the subindex of τ has been omitted.

Figure 1 is a graph of the above function for selected values of the eccentricity. It is apparent from this graph that a possible universal first guess for solving the problem would be to chose the parabolic solution.

In this case $e = 1$ and (3.1) reduces to the cubic equation

$$\tau = B + \frac{B^3}{6} \tag{3.2}$$

It is evident from the graph that the solution of this equation as a first approximation to the other cases is good or acceptable only for small values of τ. Logically the situation improves when e is closer to 1.

We have found that a much better approximation can be obtained. We may write the fundamental equation (3.1) in the form

$$\phi(B) = B + \frac{eB^3}{3!} - \tau + \left[Z_3 - \frac{eB^3}{3!} \right] = 0$$

and due to property III of Appendix A, we may write

$$\phi(B) = B + \frac{eB^3}{3!} - \tau + (e-1) Z_5(B) = 0 . \tag{3.3}$$

For eccentricities close to 1 or for values of $|B| < 1$ the last term may be neglected in a first approximation. This suggests as an approximation to the fundamental equation

$$B + \frac{eB^3}{3!} - \tau = 0 \quad ; \tag{3.4}$$

we shall refer to this as the "modified parabolic equation" because it differs from the parabolic equation (3.2) by the factor e introduced in the second term. We may note immediately that if B_0 is the solution of (3.4) then

$$\phi(B_0) = (e-1) Z_5(B_0) \tag{3.5}$$

a property that we shall use later. The modified parabolic equation (3.4) has the desirable property that for $e = 0$ and $e = 1$ it represents exactly the circular and parabolic cases respectively and its solution, B_0, gives in general an approximation to the true solution B of the fundamental equation (3.1). It is important to note that (3.4) has the reduced form of a cubic with a discriminant

$$D = \frac{24 + 27\tau^2 e}{3e^3} \quad ;$$

as can be seen D is always positive. Thus, the cubic equation has always a single real root and two complex conjugate roots that we ignore. The resolvent formula for the real root reduces in this case to

$$B_0 = \sqrt[3]{\frac{3\tau + A}{e}} + \sqrt[3]{\frac{3\tau - A}{e}}$$

where

$$A = \sqrt{\frac{8 + 9\tau^2 e}{e}} \tag{3.6}$$

Graphs 2a, 2b, and 2c represent a comparison of the approximate solution with the true solution. The ordinate is the difference between the true solution and the approximate solution given by (3.6) while the abscissa is a normalized time, that is,

$$\tilde{\tau} = \frac{\tau}{\tau_c} \tag{3.7}$$

where τ_c is defined in Section 2. For the elliptic case τ_c is given by (2.1) or due to our assumption that $q = 1$ results in

$$\tau_c = (1 - e)^{-3/2}\pi \ .$$

Superimposed on the graphs are various constant true anomaly curves to indicate the relative geometry between different conics. For clarity, the elliptic and hyperbolic cases are presented separately. In Figures 2b and 2c we show the hyperbolic cases in two different scales, the latter is enlarged to show more details. Note that near periapsis the approximation is best, as might be expected from the nature of the first approximation. For true anomalies of less than approximately 40° the approximation gives roughly 4 digits of accuracy, or better; approaching apoapsis, that is for values of close to 180°, the error becomes, of course, larger. For the hyperbolic cases the trend is similar although for distances close to Pluto the accuracy is still good to one digit.

IV. IMPROVEMENTS TO THE UNIVERSAL FIRST APPROXIMATION

Due to the particular features of this problem it is possible to obtain a correction ΔB, to improve the first approximation B_0 by solving a polynomial equation where the degree may be chosen according to the desired accuracy in the result. In fact, consider (3.1) written in the form

$$\phi(B) = B + Z_3(B) - \tau = 0 . \tag{4.1}$$

Consider now the Taylor Series expansion of the equation about the first guess B_0 obtained by solving the modified parabolic equation (3.3), that is,

$$\phi(B + \Delta B) = \phi\left(B_0\right) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{d^k\phi}{dB^k}\right)_0 \Delta B^k = 0 \tag{4.2}$$

where the subindex zero means that the corresponding function is to be calculated for $B = B_0$. In view of the form of (4.1) and the Property II of Appendix A we have for the first three derivatives the following;

$$\left.\begin{aligned} \phi' &= 1 + Z_2 \\ \phi'' &= Z_1 \\ \phi''' &= Z_0 \end{aligned}\right\} . \tag{4.3}$$

Theoretically it is possible to invert this series of ΔB; however, by considering only a few terms and solving the resultant polynomial equation for ΔB and adding this correction to B_0 we obtain a new value of B accurate to enough significant digits for all practical application. In other words we may establish the iteration formula

$$B_{\nu+1} = B_\nu + \Delta B_\nu , \qquad \nu = 0, 1, 2, \cdots \tag{4.4}$$

where B_0 is the solution of the modified cubic equation (3.3) and ΔB_ν, is the solution of a polynomial equation the degree of which depends on the number of terms we consider in the series of (4.2). It is important to notice here that a similar but not entirely equivalent procedure has been proposed by Stumpff, based on an ingenious rearrangement of the series (4.2) into a so-called closed form (See Appendix B).

If we take just the first term of the series (4.2) we have the linear equation

$$\phi_1(\Delta B_0) = \phi(B_0) + \phi'(B_0)\Delta B_0 = 0 \tag{4.5}$$

which solution,

$$\Delta B_0 = -\frac{\phi(B_0)}{\phi'(B_0)}$$

is the familiar Newton-Raphson formula.

By taking two terms of the series we obtain the quadratic equation

$$\phi_2(\Delta B_0) = \frac{1}{2}(Z_1)_0 \Delta B_0^2 + (1 + Z_2)_0 \Delta B_0 + \phi(B_0) = 0. \tag{4.6}$$

The solutions of this equation obtained by the usual resolvent formula may be written in the following forms;

$$\Delta B_0 = \frac{-2\phi(B_0)}{(1+Z_2)_0 \pm \sqrt{(1+Z_2)_0^2 - 2Z_{1_0}\phi(B_0)}}. \tag{4.7}$$

The solution corresponding to the minus sign is a spurious one as it results from the simple consideration that if $\phi(B_0)$ is small or zero then ΔB_0 must be also small or zero and that can occur if we adopt the solution with the positive sign. On the other hand on account of (3.5) we obtain;

$$\Delta B_0 = \frac{-2(e-1)Z_5(B_0)}{(1+Z_2)_0 + \sqrt{(1+Z_2)_0^2 - 2Z_{1_0}\phi(B_0)}} \tag{4.8}$$

Thus, in view of property I of Appendix A we have the result that for the hyperbolic case it is negative; for the parabolic case (e = 1) it reduces of course to zero. These results can also be observed from Figure 1.

If the correction furnished (4.7) is not sufficiently accurate we may iterate this formula together with (4.4). We have found that in most of the cases only one more iteration was necessary to achieve an accuracy of 8 significant digits. The numerical results after a correction of the first approximation by formula (4.7) are given in Figures 3a, 3b and 3c. The arrangement of these figures is similar to that of Figures 2a, 2b, 2c. In the elliptic cases one can see that just one application for formula (4.7) yields at least 6 correct digits in the result for true anomalies up to 120°. We remark that some apparent discontinuities might appear in the graph of Figure 3a. The reason is that in certain places formula (4.7) overcorrects beyond the true solution thus resulting in negative differences, which however are still small. For the sake of uniformity of the graphic representation of the results we inserted dotted lines where these circumstances occur. For the hyperbolic cases we have also satisfactory results but for extreme eccentricities the results indicate that additional corrections are necessary.

By taking one more term in the series (4.2) we obtain the cubic equation

$$\phi_3\left(\Delta B_0\right) = \left(\frac{Z_0}{6}\right)_0 \Delta B_0^3 + \left(\frac{Z_1}{2}\right)_0 \Delta B_0^2 + \left(1 + Z_2\right)_0 \Delta B_0 + \phi\left(B_0\right) = 0 \,. \quad (4.9)$$

The direct solution of this equation turns out to be impractical especially due to the tests which are necessary to apply to eliminate the spurious roots. Instead we may use as a first approximation the value obtained from the quadratic equation (given by (4.7)) and improve it by applying a Newton-Raphson scheme to the cubic equation (4.9). Thus we obtain an improved root by a few repeated applications of the following formula,

$$\Delta B_1 = \Delta B_0 - \frac{\phi_3\left(\Delta B_0\right)}{\phi_3'\left(\Delta B_0\right)}$$

The numerical results obtained by this procedure are shown on Figures 4a, 4b and 4c. The ordinate on these graphs are the difference between the true solutions and the approximate solution obtained by solving the two cubics given by equations (3.4) and (4.9). As on the graphs presented earlier, there are curves of constant true anomaly values superimposed on the graph to indicate the relative geometry. For the elliptic cases, an accuracy of 8 digits is obtained for anomalies up to approximately 110° for all eccentricities. A similar result is obtained for the hyperbolic orbits. At the extreme positions in the orbit, that is, near apoapsis, there is 2 or 3 digit accuracy in the approximation. Further iterations would be needed to achieve the desired 8-digit accuracy. One application of the quadratic correction given by Equation (4.7) will completely solve the problem, except for the extreme positions of the hyperbolic conics greater than 1.2 where an additional iteration is necessary. To reach our goal of obtaining a solution with at least 8 accurate digits both procedures, the double iteration on the quadratic equation or the first approximation from the quadratic and the second from the cubic, have approximately the same efficiency.

V. CONCLUSIONS

It is possible to normalize the general Kepler problem and express the result in universal variables such that a uniform procedure can be established to solve this equation. A modified parabolic equation is used as a universal first approximation for all conics of practical interest, with sizes covering the solar system. The modification of the parabolic conic equation is due to an inclusion of a linear dependence of the eccentricity of the orbit. This approximation is well suited for near periapsis situations. A substantial improvement to this approximation can be made by solving a quadratic or cubic equation. In the case of the cubic correction this increases the range of applicability of the solution for better than 8-digit accuracy to true anomalies of about 110°. Two repeated applications of the quadratic correction completely solves the problem for 8-digit accuracy for all considered conics. Finally we can summarize the entire procedure to solve Kepler's problem as follows:

1) Data:
   - q = periapsis distance
   - e = eccentricity (in the range 0 to 1.5)
   - T = time of periapsis passage
   - t = current time

2) Calculate

$$\tau = \frac{k(t - T)}{q^{3/2}} \tag{1}$$

where k = Gauss's constant.

3) Solve the modified cubic equation

$$B_0^3 + \frac{6}{e} B_0 - \frac{6\tau}{e} = 0 \tag{2}$$

by the resolvent formula

$$B_0 = \sqrt[3]{\frac{3\tau + A}{e}} + \sqrt[3]{\frac{3\tau - A}{e}} \quad \text{where } A = \sqrt{\frac{8 + 9\tau^2 e}{e}} \;. \tag{3}$$

4) If $e = 1$ (parabolic case)

true anomaly $$v = 2\tan^{-1}\left(\frac{B_0}{\sqrt{2}}\right) \;. \tag{4}$$

5) If $e \neq 1$ (elliptic or hyperbolic cases) Find a correction ΔB_ν by the iteration formulas

$$B_{\nu+1} = B_\nu + \Delta B_\nu \,, \qquad \nu = 0, 1, 2, \cdots$$

$$\Delta B_\nu = \frac{-2\phi(B_\nu)}{[1 + Z_2(B_\nu)] + \sqrt{[1 + Z_2(B_\nu)]^2 - 2Z_1(B_\nu)\,\phi(B_\nu)}} \tag{5}$$

where

$$\phi(B_\nu) = B_\nu + Z_3(B_\nu) - \tau \;. \tag{6}$$

For the definition, properties and methods of calculating the functions $Z_i(B_\nu)$ see Appendix A. The iteration proceeds until ΔB_ν is smaller than a certain fixed tolerance (say 10^{-8}).

6) Alternative to Step 5

After obtaining ΔB_0 calculate the following approximations by the iteration defined by

$$\Delta B_{\nu+1} = \Delta B_\nu - \frac{\phi_3(\Delta B_\nu)}{\phi_3'(\Delta B_\nu)} \,, \qquad \nu = 0, 1, 2, \cdots$$

where

$$\phi_3(\Delta B_\nu) = \left(\frac{Z_3}{6}\right)_\nu \Delta B_\nu^{\,3} + \left(\frac{Z_1}{2}\right)_\nu \Delta B_\nu^{\,2} + \left(1 + Z_2\right)_\nu \Delta B_\nu + \phi(B_\nu)$$

$$\phi_3'(\Delta B_\nu) = \left(\frac{Z_3}{2}\right)_\nu \Delta B_\nu^{\,2} + \left(Z_1\right)_\nu \Delta B_\nu + \left(1 + Z_2\right)_\nu$$

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We are indebted to Mr. James Moore for making numerous programming changes and runs thereby obtaining the data and preparing the figures.

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EXPLANATION OF FIGURES

Figure 1: Graph of Kepler's Equation in Universal Variables

Several curves corresponding to some selected values of the eccentricity showing their relationship to the parabolic case. The upper limits for these curves correspond, in the elliptic case, to half a period and, in the hyperbolic and parabolic cases, to a fixed heliocentric distance equal to 40 a.u.

Figures 2a, 2b, 2c: Differences Between the First Approximation (modified parabolic equation) and the True Solutions

In this figure the ordinates are the differences expressed on a logarithmic scale so that the number of correct significant figures of the approximate solution can be obtained readily; the abscissas correspond to the normalized time defined in Section 2. Figure 2a corresponds to the elliptic cases. Constant true anomaly curves indicate the relative geometry of the problem. Near periapsis the approximation is best; however for true anomalies of less than 40°, say, the approximation is still good to 4 digits. The approximation is equally good for near circular and near parabolic cases.

Figures 2b and 2c correspond to the hyperbolic cases. The trend is similar to that of the elliptic case but with increasing values of τ the loss of accuracy is more rapid depending on the values of the eccentricity, and requires a further correction.

Figures 3a, 3b, 3c: Differences between Approximate and True Solutions After Quadratic Correction Given by Formula (4.7)

The graphic arrangement of this figure is similar to that of the previous ones. In the elliptic cases we have 6 significant digits for true anomalies up to 120°. For the hyperbolic cases we have also satisfactory results but for extreme eccentricities the results indicate that additional corrections are necessary.

Figures 4a, 4b, 4c: Differences between Approximate and True Solutions After Cubic Correction Given by Formula (4.9)

The graphic arrangement is similar as above. For the elliptic cases an accuracy of 8 digits is obtained for anomalies up to 110° for all eccentricities. A similar result is obtained for the hyperbolic orbits, except for extreme positions where still an accuracy of 2 or 3 digits is obtained.

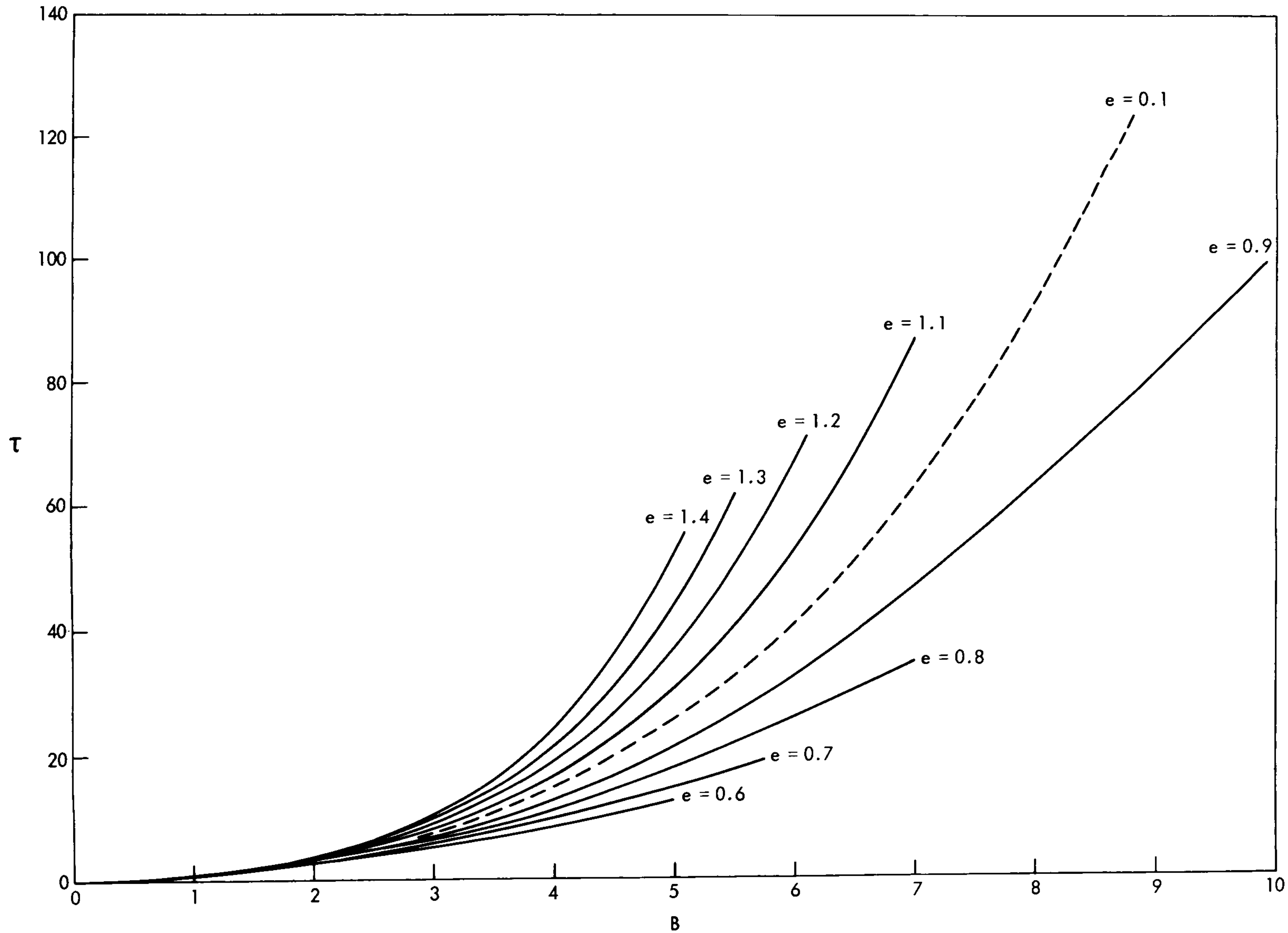

Figure 1. Graph of Kepler's Equation in Universal Variables

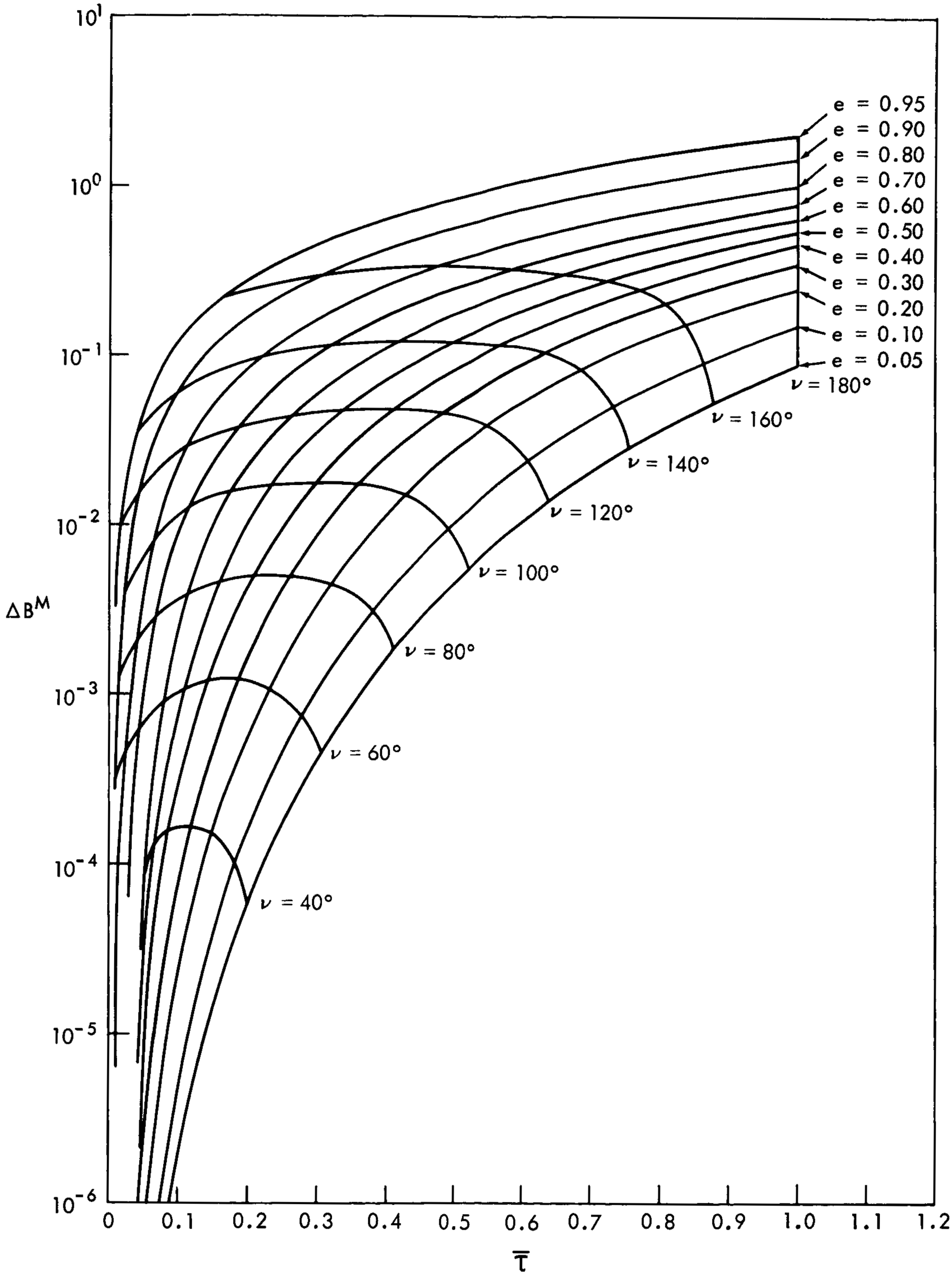

Figure 2a. Differences between the first approximation, i.e., modified parabolic equation, and the true solutions. (Elliptic Cases)

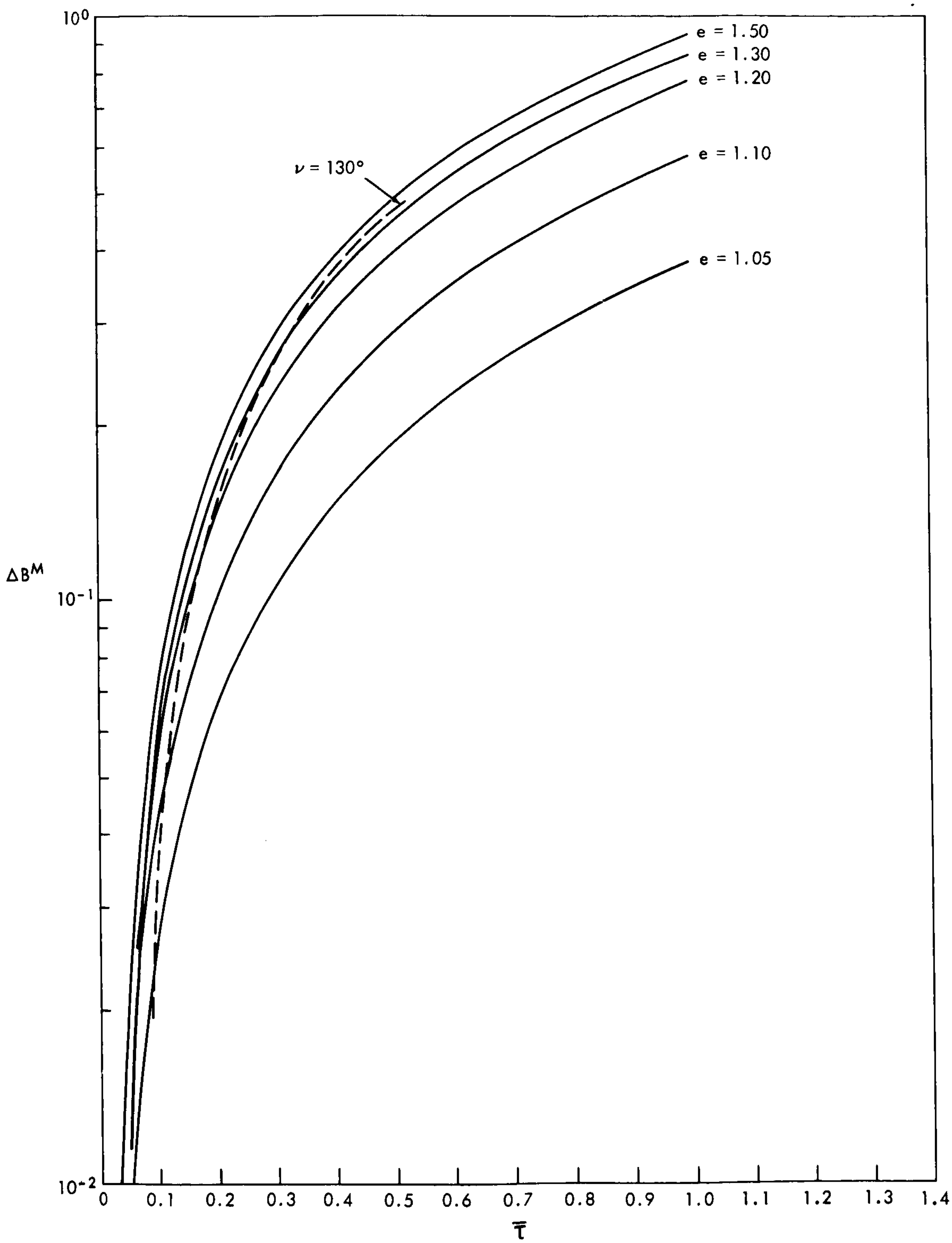

Figure 2b. Differences between the first approximation, i.e., modified parabolic equation, and the true solutions. (Hyperbolic Cases)

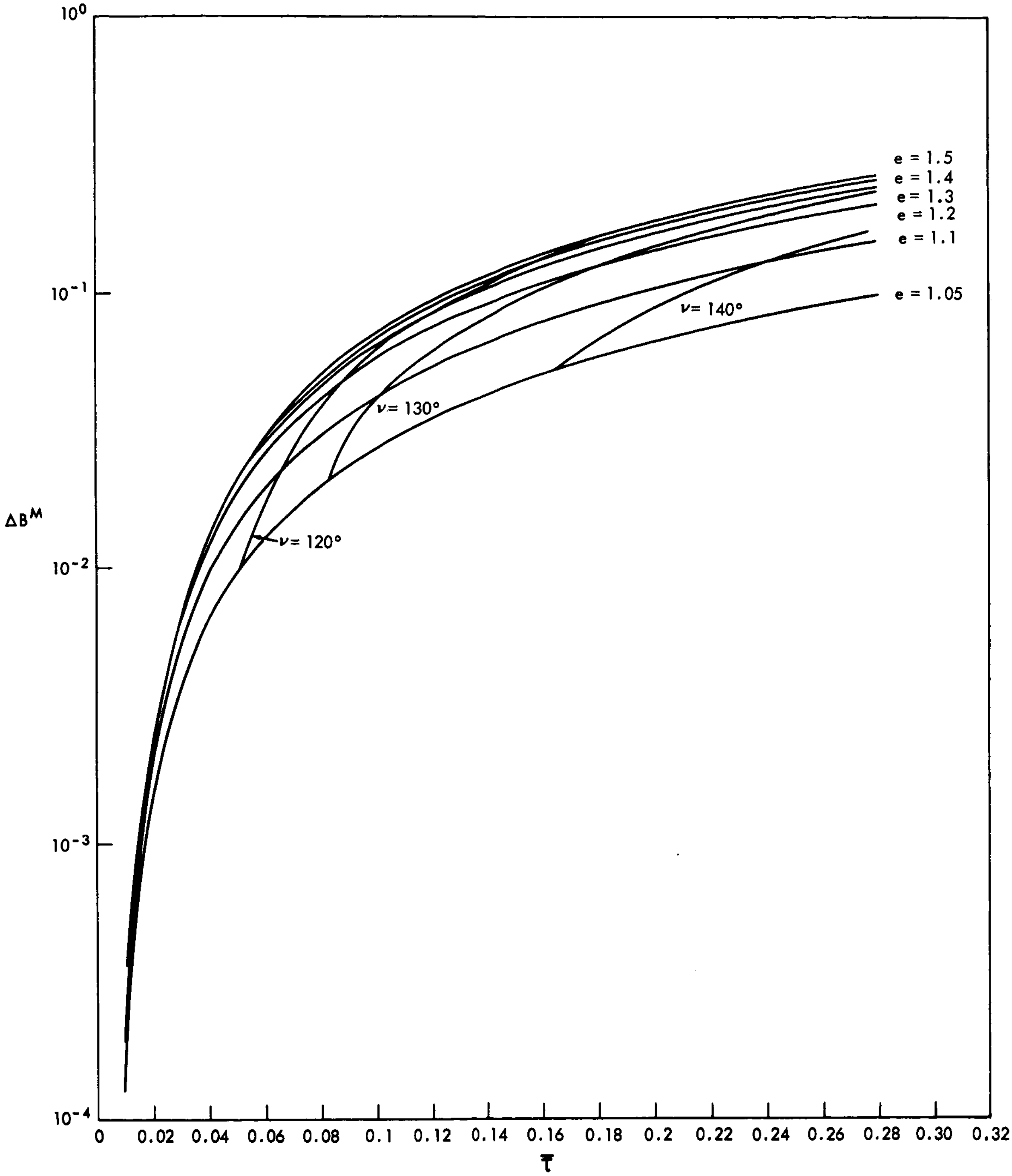

Figure 2c. Differences between the first approximation, i.e., modified parabolic equation, and the true solutions (Hyperbolic Cases; enlarged scale)

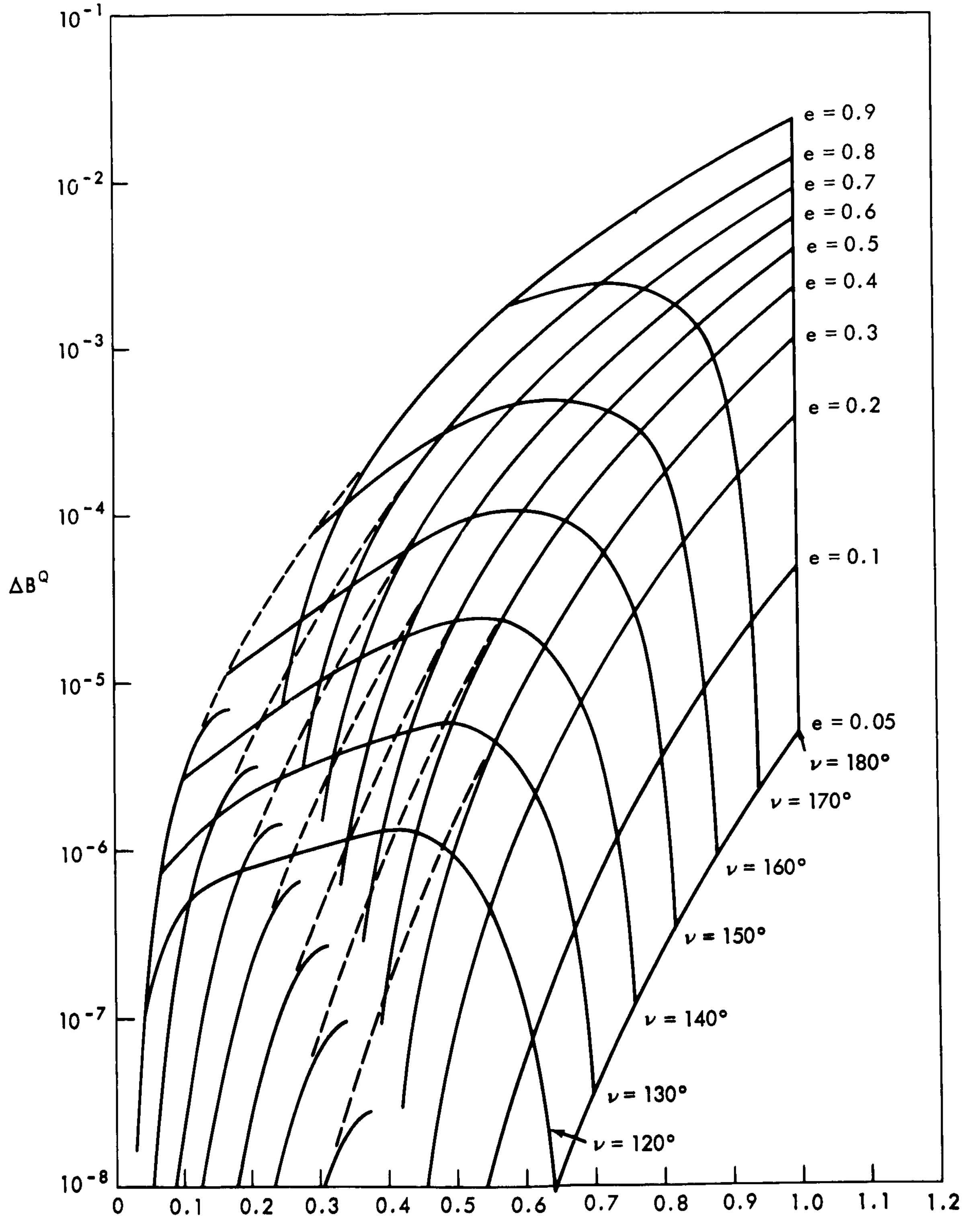

Figure 3a. Differences between approximate and true solutions after quadratic correction given by formula (4.7) (Elliptic Cases)

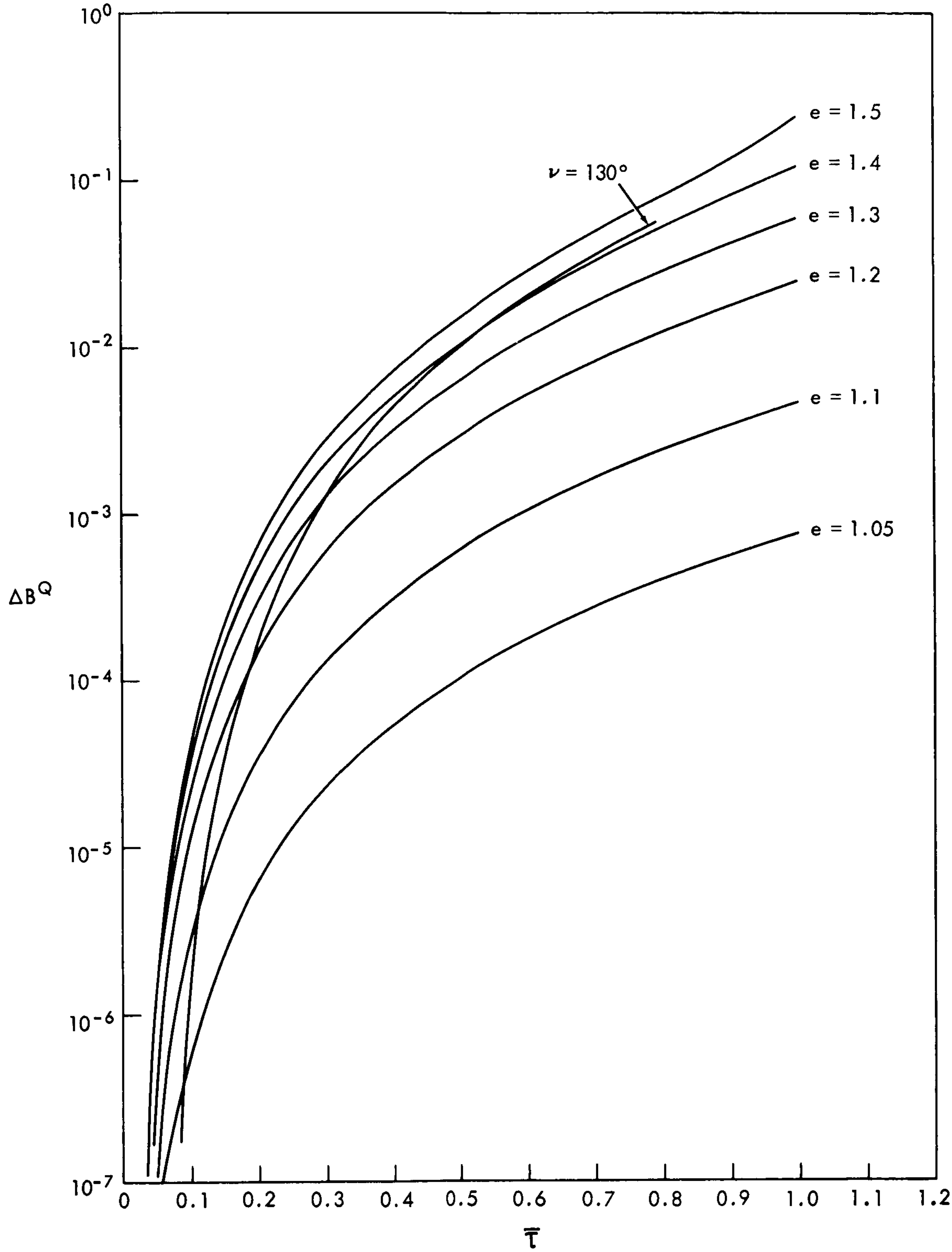

Figure 3b. Differences between approximate and true solutions after quadratic correction given by formula (4.7) (Hyperbolic Cases)

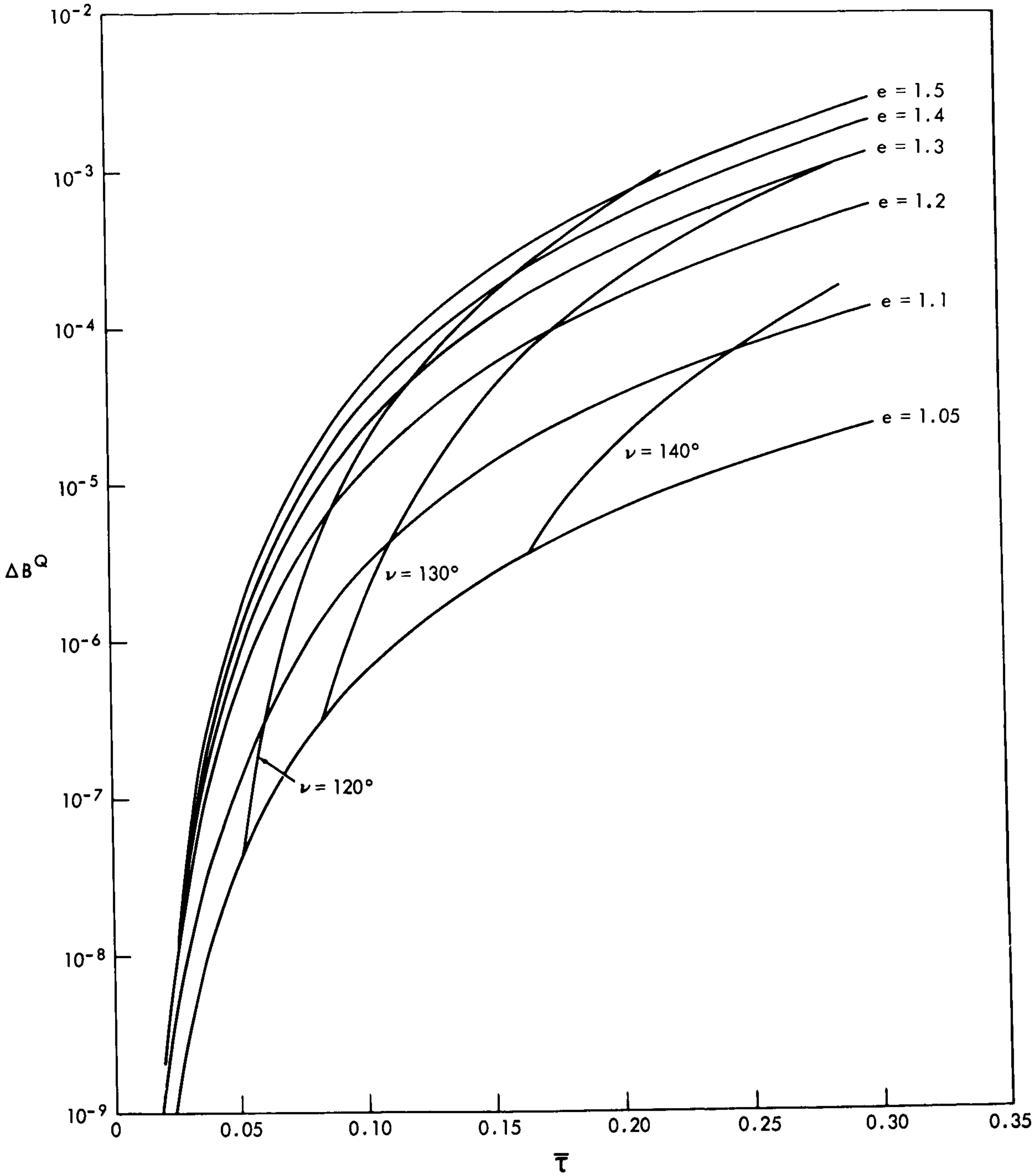

Figure 3c. Differences between approximate and true solutions after quadratic correction given by formula (4.7) (Hyperbolic Cases, enlarged scale)

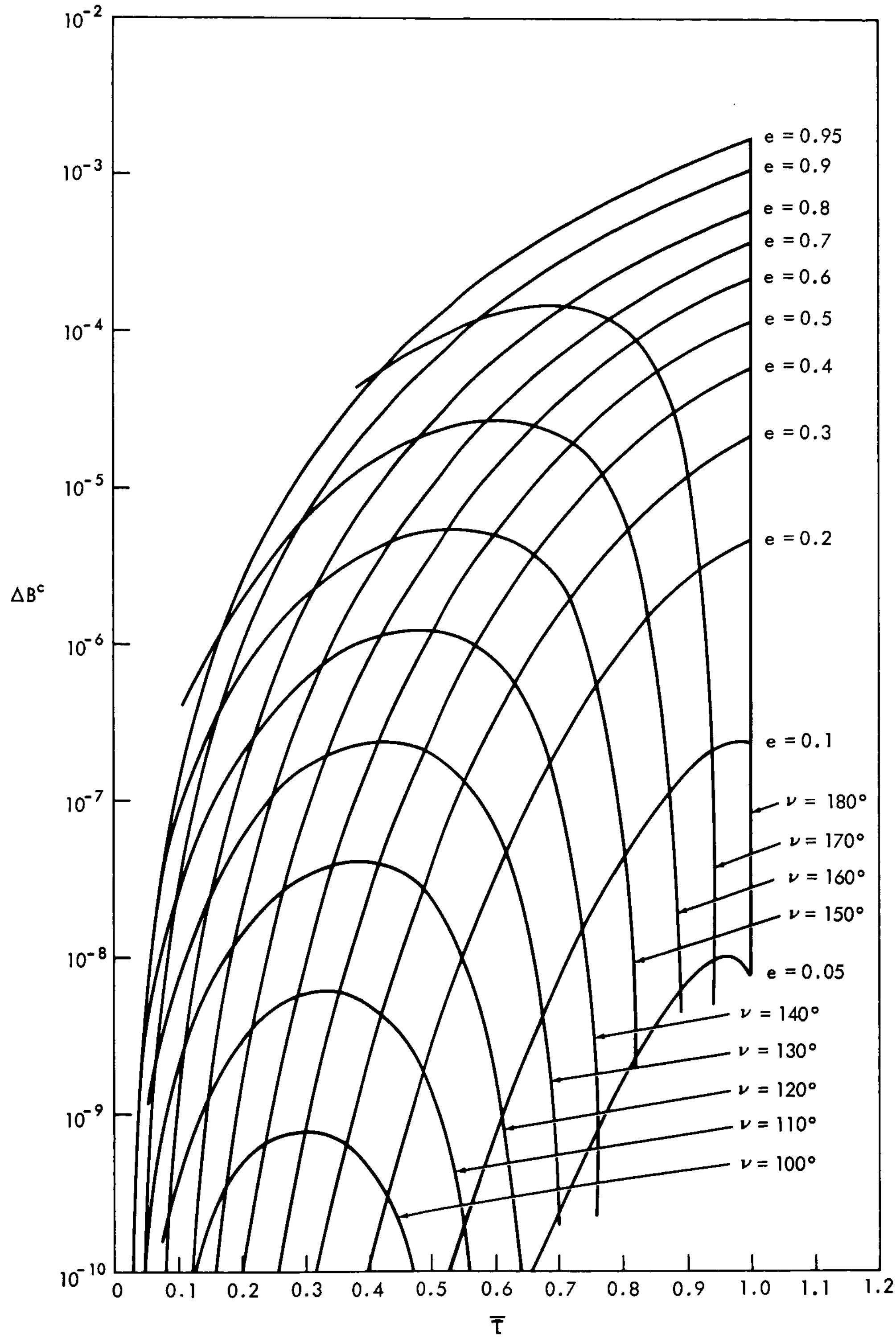

Figure 4a. Differences between approximate and true solutions after cubic correction given by formula (4.9) (Elliptic Cases)

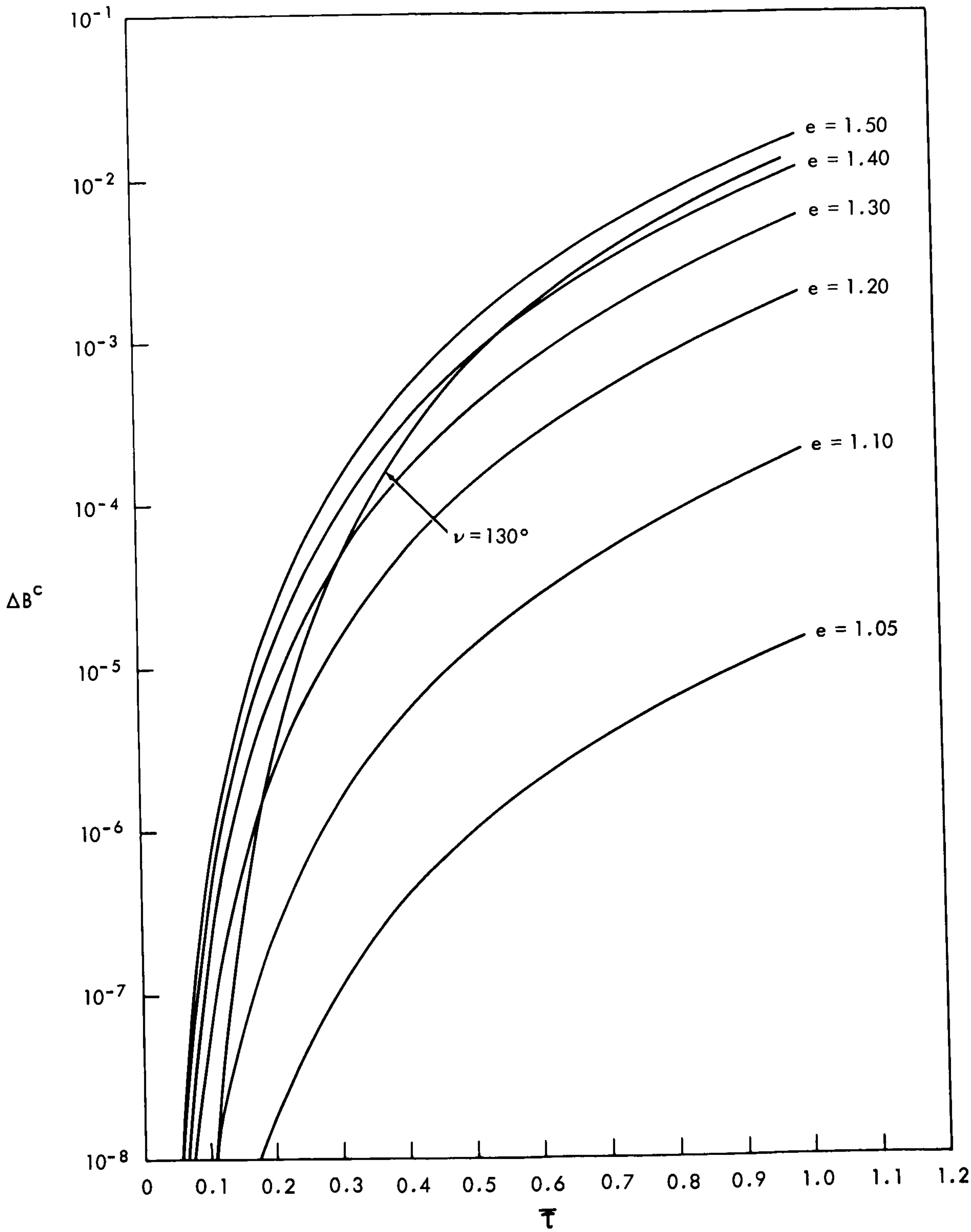

Figure 4b. Differences between approximate and true solutions after cubic correction given by formula (4.9) (Hyperbolic Cases)

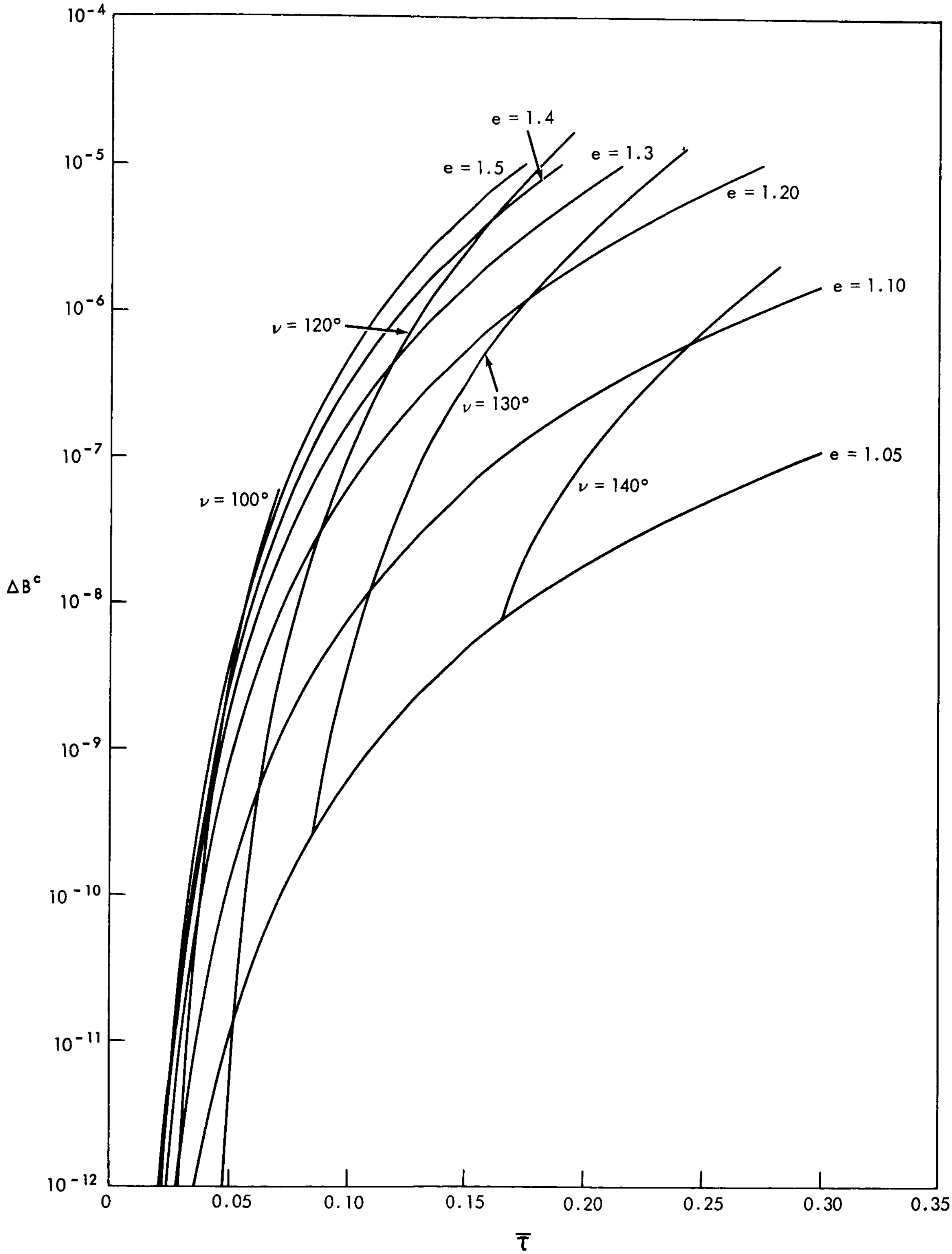

Figure 4c. Differences between approximate and true solutions after correction given by formula (4.9) (Hyperbolic Cases, enlarged scale)

APPENDIX A

PROPERTIES OF THE Z -FUNCTIONS

The universal Z-Functions used in this study are defined as follows;

$$Z_i = e B^i \sum_{n=0}^{\infty} \frac{\zeta^n}{(2n+i)!} = e B^i F_i$$

where

$$\zeta = -B^2 (1-e)$$

I. For any $i > 0$, $Z_i > 0$. This follows from the fact that e and B are positive and for $\zeta > 0$ the result is obvious; for $\zeta < 0$, function F_i is an alternate series where terms decrease in absolute value and the first term is $1/i! > 0$.

II. $dZ_i/dB = Z_{i-1}$. This expression is obtained by differentiating Z_i after replacing ζ in terms B.

III. Let us write

$$F_i = \frac{1}{i!} + \sum_{n=0}^{\infty} \frac{\zeta^n}{(2n+i)!}$$

then rearranging and noting that $\zeta/B^2 = e - 1$ we obtain the recursive formula

$$Z_i = \frac{eB^i}{i!} + (e-1) Z_{i+2} \quad .$$

This result means that for calculating the Z-functions it is necessary only to apply the definition formula for those of the largest order of even and odd indices and the rest of the lower order Z-functions are calculated by the recursive formula. Also, reduction formulas exist, (see Blanchard and Wolf (1967)) which allow the infinite series presented above to be calculated with only a relatively few number of terms. Herron, et al. (1967) also suggests some numerical techniques for efficiently calculating series of this type.

APPENDIX B

A "CLOSED" FORM FOR CORRECTING THE FIRST APPROXIMATION

Stumpff (1959), in his book on celestial mechanics (Vol. I, p. 220-222), has proposed a rearrangement of the series (4.2) into a "closed" form that in the context of our approach to the problem may be described as follows. The expansion (4.2) may be written briefly as

$$\phi\left(B + \Delta B\right) = \phi(B) + \frac{\phi'(B)}{1!}\Delta B + \frac{\phi''(B)}{2!}\Delta B^2 + \frac{\phi'''(B)}{3!}\Delta B^3 + \cdots \qquad \text{(B-1)}$$

We have

$$\phi(B) = B + Z_3(B) - \tau \qquad \text{(B-2)}$$

and applying Properties II and III of Appendix A we have

$$\left.\begin{aligned} \phi'(B) &= 1 + Z_2 \\ \phi''(B) &= Z_1 = eB + (e-1)Z_3 = (e-1)\phi(B) + B + (e-1)\tau \\ \phi'''(B) &= (e-1)\phi'(B) + 1 \\ \phi''''(B) &= (e-1)\phi''(B) \end{aligned}\right\} \qquad \text{(B-3)}$$

and in general

$$\phi^{(p+2)}(B) - (e-1)\phi^{(p)}(B) = 0 \qquad \text{for } p \geq 2 \qquad \text{(B-4)}$$

Let us now recall the recursion formula for the function F_i

$$F_i(z) - zF_{i+2}(z) = \frac{1}{i!}, \qquad i = 0, 1, 2, \cdots \qquad \text{(B-5)}$$

valid for any argument z. In particular we shall call γ_i the function F_i for the argument $z = (e - 1)\Delta B^2$ in which case

$$\gamma_i = \sum_{n=0}^{\infty} \frac{\left[(e-1)\Delta B^2\right]^n}{(2n+i)!} \tag{B-6}$$

and we have also the recursion formula

$$\gamma_i - (e-1)\Delta B^2 \gamma_{i+2} = \frac{1}{i!}, \qquad i = 0, 1, 2, \cdots \tag{B-7}$$

Applying this expression for the factorials in (B-1) and after a simple rearrangement we obtain

$$\phi(B + \Delta B) = \gamma_0 \phi(B) + \gamma_1 \phi'(B)\Delta B + \gamma_2 \left[\phi''(B) - (e-1)\phi(B)\right]\Delta B^2$$

$$+ \gamma_3\left[\phi'''(B) - (e-1)\phi'(B)\right]\Delta B^3 + \sum_{p=2}^{\infty} \gamma_{p+2}\left[\phi^{(p+2)}(B) - (e-1)\phi^{(p)}(B)\right]\Delta B^{p+2} \tag{B-8}$$

The summation in the last term must vanish because of property (B-4); then this is formally a closed form for the expansion of $\phi(B + \Delta B)$ but, of course, the higher powers of ΔB are still contained in the function γ_i. However, based on this result, it is possible to obtain an iteration formula to calculate the correction ΔB when a first approximation B_0 is known. In fact we must have

$$\phi(B_0 + \Delta B) = \gamma_0 \phi(B_0) + \gamma_1 \phi'(B_0)\Delta B + \gamma_2\left[\phi''(B_0) - (e-1)\phi(B_0)\right]\Delta B^2$$

$$+ \gamma_3\left[\phi'''(B_0) - (e-1)\phi'(B_0)\right]\Delta B^3 = 0 . \tag{B-9}$$

which suggests the iterative formula

$$\Delta B_{\nu+1} = -\frac{1}{\gamma_1 \phi'(B_\nu)}\left\{\gamma_0 \phi(B_0) + \gamma_2\left[B_0 + (e-1)\tau\right]\Delta B_\nu + \gamma_3 \Delta B_\nu^{\,3}\right\} \tag{B-10}$$

The first approximation may be obtained by the formula

$$\Delta B_0 \cong -\frac{\gamma_0 \phi(B_0)}{\gamma_1 \phi'(B_0)} \cong \frac{\phi(B_0)}{\phi'(B_0)} \qquad \text{(B-11)}$$

which again is Newton-Raphson's formula. By neglecting in the expansions (B-6) all but the first terms one obtains $\gamma_2 = 1/2$ and $\gamma_3 = 1/6$. It is then possible to show by a simple algebraic manipulation that equation (B-9) becomes identical to what one could obtain directly by truncating (B-1) after the fourth term, as we have done in Section 4.