NASA TECHNICAL NOTE

NASA TN D-4296

NASA TN D-4296

DEVELOPMENT OF A GENERAL FORMULA EXPANDING THE HIGHER-ORDER DERIVATIVES OF THE FUNCTION tanh z IN POWERS OF tanh z AND A-NUMBERS

by Hans H. Hosenthien

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1968

DEVELOPMENT OF A GENERAL FORMULA EXPANDING THE HIGHER-ORDER DERIVATIVES OF THE FUNCTION tanh z IN POWERS OF tanh z AND A-NUMBERS

By Hans H. Hosenthien

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

TABLE OF CONTENTS

LIST OF TABLES

DEVELOPMENT OF A GENERAL FORMULA EXPANDING THE HIGHER-ORDER DERIVATIVES OF THE FUNCTION tanh z IN POWERS OF tanh z AND A-NUMBERS

SUMMARY

A general formula for the rth derivative of the function tanh z with respect to z is developed. This formula is a finite polynominal in powers of tanh z where the coefficients are of simple structure containing A-numbers. The A-numbers, $A_r^{(m)}$, of order m and degree r are introduced as an abbreviation for an expression containing C-numbers which are related to Euler's numbers. A recursion formula for the A-numbers and their special properties are derived. The paper contains a tabulation of derivatives of tanh z from the first to the sixth order as well as a table of A-numbers covering all combinations of order and degree from 0 to 10.

METHOD OF DEVELOPMENT

While deriving general transfer relations for electrical networks employing synchronous commutation (modulation and demodulation), it was found desirable to obtain closed-form expressions for the rth derivative of the function tanh z with respect to z.*

Considering the Taylor series expansion of tanh (z + y) in powers of y,

$$\tanh(z+y) = \sum_{r=0}^{\infty} \frac{d^r}{dz^r}(\tanh z)\frac{y^r}{r!}, \quad \left(|y| < \frac{\pi}{2}\right) \tag{1}$$

we note that the rth derivative of tanh z is contained in the coefficient of y^r. Since it is desired to express $\frac{d^r}{dz^r}(\tanh z)$ in powers of tanh z, we use a

---

* The author is indebted to Prof. Dr. Richard F. Arenstorf of MSFC's Computation Laboratory who suggested the basic approach as well as rigorous treatment of the infinite series involved.

second approach to develop tanh $(z + y)$ into a power series in y through the identity

$$\tanh(z + y) = (\tanh z + \tanh y)(1 + \tanh z \cdot \tanh y)^{-1} \tag{2}$$

and the binomial expansion,

$$(1 + \tanh z \cdot \tanh y)^{-1} = \sum_{m=0}^{\infty} (-1)^m \tanh^m z \cdot \tanh^m y \tag{3}$$

which converges for $|\tanh^m z \cdot \tanh^m y| < 1$. Combining equations (2) and (3) we obtain

$$\tanh(z + y) = \tanh z + \sum_{m=1}^{\infty} (-1)^m \left[\tanh^{m+1} z - \tanh^{m-1} z\right] \tanh^m y \tag{4}$$

$$(|\tanh z| < 1, \quad |\tanh y| < 1) \quad .$$

Now, for every fixed real value of z, the infinite series on the right of equation (4) is uniformly convergent in a closed circular region of radius ρ around the origin of the y - plane defined by

$$|y| \leq \rho < \frac{\pi}{4} \quad .$$

For, abbreviating the general terms of the series by

$$f_m(y) = (-1)^m \left[\tanh^{m+1} z - \tanh^{m-1} z\right] \tanh^m y$$

we have, for every fixed real z, $|\tanh z| < 1$ and

$$|f_m(y)| < 2\,|\tanh y|^m \quad .$$

To satisfy Weierstrass's M - test [1] we require that

$$|\tanh y| \leq K < 1$$

and find that this condition can be satisfied by the restriction

$$|y| \leq \rho < \frac{\pi}{4}$$

which insures that any closed circular region of radius $\rho < \frac{\pi}{4}$ about the origin of the y - plane is mapped into a closed region in the interior of the unit circle about the origin of the w - plane through the transformation

$$w = \tanh y = \frac{e^y - 1}{e^y + 1} \quad .$$

Now we substitute in equation (4) the power series for $\tanh^m y$, developed below,

$$\tanh^m y = \sum_{r=1}^{\infty} A_r^{(m)} \frac{y^r}{r!} \quad (m > 0,\ |y| < \frac{\pi}{2},\ A_r^{(m)} = 0 \text{ for } m > r) \tag{5}$$

and obtain

$$\tanh(z + y) = \tanh z + \sum_{m=1}^{\infty} (-1)^m \left[\tanh^{m+1} z - \tanh^{m-1} z\right] \sum_{r=1}^{\infty} A_r^{(m)} \frac{y^r}{r!} \quad . \tag{6}$$

Since the series (4) converges uniformly for $|y| \leq \rho < \frac{\pi}{4}$ for every $\rho < \frac{\pi}{4}$ and since the series (5) converges at least for $|y| < \frac{\pi}{4}$, we can, by Weierstrass's double-series theorem [1], reverse the order of summation over r and m in equation (6) and obtain the desired power series of $\tanh(z + y)$,

$$\tanh(z + y) = \tanh z + \sum_{r=1}^{\infty} \left\{ \sum_{m=1}^{\infty} (-1)^m \left[\tanh^{m+1} z - \tanh^{m-1} z\right] A_r^{(m)} \right\} \frac{y^r}{r!}$$

$$(z \text{ real},\ |y| < \frac{\pi}{4}) \quad . \tag{7}$$

Since both power series for $\tanh(z + y)$, equations (1) and (7), have the region of convergence $|y| < \frac{\pi}{4}$ in common and since their sums are equal in this region, both power series are identical by the identity theorem for power series [1]. Therefore we can equate corresponding coefficients of equations (1) and (7) and, considering that $A_r^{(m)} = 0$ for $m > r$, we obtain

$$\frac{d^r}{dz^r}(\tanh z) = \sum_{m=1}^{r} (-1)^m A_r^{(m)} \left[\tanh^{m+1} z - \tanh^{m-1} z\right] \tag{8}$$

$$(z \text{ real}, \ r > 0) \quad .$$

We note that the function tanh z, its derivative, and integral powers are analytic functions for all complex z except for the poles of tanh z and that equation (8) holds along the real z-axis. Therefore, by the identity theorem for analytic functions [1], equation (8) holds for all complex z except for the poles of tanh z, and we obtain the desired expansion for the rth derivative of tanh z with respect to z in powers of tanh z:

$$\frac{d^r}{dz^r}(\tanh z) = \sum_{m=1}^{r} (-1)^{(m)} A_r^{(m)} \left[\tanh^{m+1} z - \tanh^{m-1} z\right] \tag{9}$$

$$r > 0$$

The A-numbers, $A_r^{(m)}$, are obtained from the recursion relation

$$A_{r+1}^{(m+1)} = (m+1)\left(A_r^{(m)} - A_r^{(m+2)}\right) \tag{10}$$

$$(r = 0, 1, \ldots, \infty; \ m = 0, 1, \ldots, r; \ m, r \geq 0)$$

where

$$\begin{aligned}
&A_m^{(m)} = m! \quad (m \geq 0) \\
&A_r^{(o)} = 0 \quad (r \geq 1) \\
&A_r^{(m)} = 0 \quad (m > r \geq 0) \\
&A_r^{(m)} = 0, \text{ when } (m - r) \text{ is an odd integer in the range } 1 \leq m < r \\
&A_r^{(m)} \neq 0, \text{ when } (m - r) \text{ is an even integer in the range } 1 < m \leq r.
\end{aligned} \tag{11}$$

Now we develop the power-series expansion of $\tanh^m y$, stated in equation (5), which led to the introduction of the A-numbers, $A_r^{(m)}$, with the recursion relation (10) and the properties (11). Consider that the well known power series of tanh y begins with y and contains only odd integral powers of y. Factoring y out of this series, we obtain tanh y in the form

$$\tanh y = y \{1 + a_2 y^2 + a_4 y^4 + \ldots\} . \tag{12}$$

Raising both sides of equation (12) to the mth power, we have

$$\tanh^m y = y^m \{1 + b_2 y^2 + b_4 y^4 \ldots\} . \tag{13}$$

Note that the infinite series in brackets in equations (12) and (13) both contain only even powers of y, since the multinomial expansion of any finite order m, $\{1 + a_2 y^2 + a_4 y^4 \ldots\}^m$, must again contain only even powers of y. Writing equation (13) as

$$\tanh^m y = y^m + b_2 y^{m+2} + b_4 y^{m+4} + \ldots , \tag{14}$$

we see that the power series of $\tanh^m y$ starts with y^m and contains only even powers of y when m is even and only odd powers of y when m is an odd integer.

To determine the coefficients of equation (14), we start with the identity

$$\tanh y = 1 - \frac{2}{(e^{2y} + 1)} . \tag{15}$$

Raising equation (15) to the mth power and using binomial expansion, we obtain

$$\tanh^m y = \left[1 - \frac{2}{(e^{2y} + 1)}\right]^m = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{2^n}{(e^{2y} + 1)^n} . \tag{16}$$

By the generating function for the C-numbers, $C_r^{(n)}$, from Milne-Thomson [2]

$$\frac{2^n}{(e^t + 1)^n} = \sum_{r=0}^{\infty} \frac{t^r}{r!\, 2^r} C_r^{(n)} \tag{17}$$

and after putting t = 2y, equation (17) becomes

$$\tanh^m y, = \sum_{r=0}^{\infty} \left\{ \sum_{n=0}^{m} (-1)^n \binom{m}{n} C_r^{(n)} \right\} \frac{y^r}{r!} \tag{18}$$

where the C-numbers of order n and degree r are given by the recursion relation

$$C_r^{(n)} = n\left(C_{r-1}^{(n+1)} - 2C_{r-1}^{(n)}\right) \tag{19}$$

$$C_0^{(n)} = 1 \text{ and } C_r^{(0)} = 0 \text{ when } r \geq 1.$$

The C-numbers are related to Euler's numbers.

Since, by equation (14), the lowest power in the series of $\tanh^m y$ is y^m and since, by equation (19), $C_r^{(0)} = 0$ when $r \geq 1$, the summation over r must start with $r = m$ and the summation over n must start with $n = 1$ in equation (18). Thus, equation (18) becomes

$$\tanh^m y = \sum_{r=m}^{\infty} \left\{ \sum_{n=1}^{m} (-1)^n \binom{m}{n} C_r^{(n)} \right\} \frac{y^r}{r!} . \tag{20}$$

Introducing the A-numbers, $A_r^{(m)}$, by

$$A_r^{(m)} = \sum_{n=1}^{m} (-1)^n \binom{m}{n} C_r^{(n)} \tag{21}$$

and substituting equation (21) into equation (20), we have

$$\tanh^m y = \sum_{r=m}^{\infty} A_r^{(m)} \frac{y^r}{r!}$$

which was stated in equation (5).

Equation (21) expresses the A-numbers in terms of the C-numbers where $A_r^{(m)}$ is an A-number of order m and degree r. The A-numbers have the property stated in equation (11) which results from the discussion of equation (14).

The recursion relation (10) for the A-number can be developed from equation (5) considering the properties stated in equation (11). Differentiating equation (5) r-times and letting $y = 0$, we obtain

$$A_r^{(m)} = \left[\frac{d^r(\tanh^m y)}{dy^r}\right]_{y=0} . \tag{22}$$

Increasing m to m + 1 in equation (5) yields

$$\tanh^{m+1} y = \sum_{r'=m+1}^{\infty} A_{r'}^{(m+1)} \frac{y^{r'}}{r'!} \qquad (23)$$

Differentiating equation (23) (r + 1)-times and letting y = 0, we have

$$A_{r+1}^{(m+1)} = \left[\frac{d^{r+1} (\tanh^{m+1} y)}{dy^{r+1}}\right]_{y=0}$$

$$A_{r+1}^{(m+1)} = \left[\frac{d^r}{dy^r} \frac{d(\tanh^{m+1} y)}{dy}\right]_{y=0}$$

$$= (m+1) \left\{\frac{d^r}{dy^r} (\tanh^m y) - \frac{d^r}{dy^r} (\tanh^{m+2} y)\right\}_{y=0} . \qquad (24)$$

By equation (22), equation (24) yields the desired recursion formula

$$A_{r+1}^{(m+1)} = (m+1) \left(A_r^{(m)} - A_r^{(m+2)}\right)$$

which was stated in equation (10).

Now, to further evaluate equation (9), we rewrite it as follows:

$$\frac{d^r \tanh z}{dz^r} = \sum_{m=1}^{r} (-1)^m A_r^{(m)} \tanh^{m+1} z - \sum_{m=1}^{r} (-1)^m A_r^{(m)} \tanh^{m-1} z .$$

After replacing m by m-2 in the first right hand summation and taking into account that, by equation (11), $A_r^{(r+2)} = A_r^{(r+1)} = 0$ and $A_r^{(0)} = 0$, when $r > 0$, we obtain

$$\frac{d^r \tanh z}{dz^r} = \sum_{m=2}^{r+2} (-1)^m \left(A_r^{(m-2)} - A_r^{(m)}\right) \tanh^{m-1} z + A_r^{(1)} . \qquad (25)$$

$$r > 0$$

Replacing m by m-2 in the recursion relation (10) for A-numbers results in

$$\left(A_r^{(m-2)} - A_r^{(m)}\right) = \frac{A_{r+1}^{(m-1)}}{(m-1)} \quad . \tag{26}$$

Then substitution of equation (26) into equation (25) yields

$$\frac{d^r}{dz^r} (\tanh z) = A_r^{(1)} + \sum_{m=2}^{r+2} (-1)^m \frac{A_{r+1}^{(m-1)}}{(m-1)} \tanh^{m-1} z \, . \qquad r > 0 \tag{27}$$

Since, by equation (11), $A_r^{(m)} = 0$ when $(m-r)$ is an odd integer $(1 \leq m < r)$, we obtain the following formulas for m, r odd and m, r even, respectively:

$$\frac{d^r}{dz^r} (\tanh z) = A_r^{(1)} - \sum_{m=3}^{r+2} \frac{A_{r+1}^{(m-1)}}{(m-1)} \tanh^{m-1} z; \text{ m, r odd}$$

$$r > 0 \tag{28}$$

$$\frac{d^r}{dz^r} (\tanh z) = \sum_{m=2}^{r+2} \frac{A_{r+1}^{(m-1)}}{(m-1)} \tanh^{m-1} z; \text{ m, r even.}$$

Derivatives of tanh z up to the sixth order are listed in Table I. The A-numbers, $A_r^{(m)}$, for r, m from 0 to 10 are listed in Table II.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, August 10, 1967
933-50-02-0062

TABLE I. DERIVATIVES OF tanh z FROM FIRST TO SIXTH ORDER

$$\frac{d}{dz}(\tanh z) = 1 - \tanh^2 z$$

$$\frac{d^2}{dz^2}(\tanh z) = -2 \tanh z + 2 \tanh^3 z$$

$$\frac{d^3}{dz^3}(\tanh z) = -2 + 8 \tanh^2 z - 6 \tanh^4 z$$

$$\frac{d^4}{dz^4}(\tanh z) = 16 \tanh z - 40 \tanh^3 z + 24 \tanh^5 z$$

$$\frac{d^5}{dz^5}(\tanh z) = 16 - 136 \tanh^2 z + 240 \tanh^4 z - 120 \tanh^6 z$$

$$\frac{d^6}{dz^6}(\tanh z) = -272 \tanh z + 1232 \tanh^3 z - 1680 \tanh^5 z + 720 \tanh^7 z$$

TABLE II. THE A-NUMBERS, $A_r^{(m)}$, FOR r, m FROM 0 TO 10

$A_r^{(m)}$							m					
		0	1	2	3	4	5	6	7	8	9	10
	0	1	0	0	0	0	0	0	0	0	0	0
	1	0	1	0	0	0	0	0	0	0	0	0
	2	0	0	2	0	0	0	0	0	0	0	0
	3	0	-2	0	6	0	0	0	0	0	0	0
	4	0	0	-16	0	24	0	0	0	0	0	0
r	5	0	16	0	-120	0	120	0	0	0	0	0
	6	0	0	272	0	-960	0	720	0	0	0	0
	7	0	-272	0	3696	0	-8400	0	5040	0	0	0
	8	0	0	-7936	0	48384	0	-80640	0	40320	0	0
	9	0	7936	0	-168960	0	645120	0	-846720	0	362880	0
	10	0	0	353792	0	-3256320	0	8951040	0	-9676800	0	3628800

REFERENCES

1. Knopp, Konrad: Theory of Functions, Part I, Elements of the General Theory of Analytic Functions. Dover Publications, New York, 1945, pp. 73, 81, 83, 85.

2. Milne-Thomson, L. M., C. B. E.: The Calculus of Finite Differences. MacMillan & Co., Ltd., London, 1966, pp. 124-153.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546