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APPROXIMATE SOLUTION

OF A TWO-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT

Approximation formulas are found for $\dot{x}(0)$ and $\dot{x}(1)$, where $x(t)$ satisfies

$$\ddot{x} = f(x, t) + A(t)\dot{x} ,$$

$$x(0) = x_0, \; x(1) = x_1 .$$

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INTRODUCTION

Consider the following boundary-value problem:

$$\begin{aligned} \ddot{x} &= f(x, t) + A(t)\dot{x} , \\ x(0) &= x_0, \quad x(1) = x_1 , \end{aligned} \tag{1}$$

where t is a scalar, x is a column matrix, $A(t)$ is a matrix function of t, and the overdots indicate differentiation with respect to t.

Formulas will be found for $\dot{x}_0$ and $\dot{x}_1$ such that

$$\dot{x}_0 = \dot{\phi}(0), \quad \dot{x}_1 = \dot{\phi}(1) ,$$

where $\phi(t)$ satisfies

$$\phi(0) = x(0), \ \ddot{\phi}(0) = \ddot{x}(0), \ \dddot{\phi}(0) = \dddot{x}(0) ,$$

$$\phi(1) = x(1), \ \ddot{\phi}(1) = \ddot{x}(1), \ \dddot{\phi}(1) = \dddot{x}(1) .$$

It will not be necessary to find $\phi(t)$ but only to make certain assumptions as to its form.

The author has found the method useful in the calculation of transfer trajectories for space vehicles and in preliminary orbit determination from observations of the position of a spacecraft at two times.

AN APPROXIMATE SOLUTION

Let α be any component of x, $\alpha(0) = \alpha_0, \alpha(1) = \alpha_1$, and assume approximations of the form

$$\dot{\alpha}_0 = a_0\alpha_0 + b_0\alpha_1 + a_2\ddot{\alpha}_0 + b_2\ddot{\alpha}_1 + a_3\dddot{\alpha}_0 + b_3\dddot{\alpha}_1 , \tag{2}$$

$$\dot{\alpha}_1 = c_0\alpha_0 + d_0\alpha_1 + c_2\ddot{\alpha}_0 + d_2\ddot{\alpha}_1 + c_3\dddot{\alpha}_0 + d_3\dddot{\alpha}_1 . \tag{3}$$

The scalars a_0, b_0, a_2, b_2, a_3, and b_3 are determined by assuming (2) to be exact when $\alpha = \phi_i(t)$, $i = 0, \cdots, 5$, where the ϕ_i's are linearly independent over the interval $[0, 1]$ with derivatives through the third order at $t = 0$ and $t = 1$. The coefficients in (3) are determined in a similar way. If we use the same set of ϕ_i's for each component of x, we can write

$$\dot{x}_0 = a_0x_0 + b_0x_1 + a_2\ddot{x}_0 + b_2\ddot{x}_1 + a_3\dddot{x}_0 + b_3\dddot{x}_1 , \tag{4}$$

$$\dot{x}_1 = c_0x_0 + d_0x_1 + c_2\ddot{x}_0 + d_2\ddot{x}_1 + c_3\dddot{x}_0 + d_3\dddot{x}_1 . \tag{5}$$

We eliminate the third derivatives in (4) and (5) by using

$$\dddot{x} = (P + \dot{A})\ddot{x} + A\dddot{x} + w , \tag{6}$$

obtained from (1), where P is a matrix with element in the ith row and jth column equal to the value of $\partial f^i/\partial x^j$, and w is a column matrix with ith element equal to the value of $\partial f^i/\partial t$, f^i and x^j being respectively the ith component of f and the jth component of x. Substituting (6) into (4) and (5), we obtain

$$\left(I - a_3Q_0\right)\dot{x}_0 - b_3Q_1\dot{x}_1 = \beta , \tag{7}$$

$$-c_3Q_0\dot{x}_0 + \left(I - d_3Q_1\right)\dot{x}_1 = \gamma , \tag{8}$$

where I is the unit matrix and

$$Q = P + \dot{A} ,$$

$$\beta = a_0x_0 + b_0x_1 + \left(a_2I + a_3A_0\right)\ddot{x}_0 + \left(b_2I + b_3A_1\right)\ddot{x}_1 + a_3w_0 + b_3w_1 ,$$

$$\gamma = c_0x_0 + d_0x_1 + \left(c_2I + c_3A_0\right)\ddot{x}_0 + \left(d_2I + d_3A_1\right)\ddot{x}_1 + c_3w_0 + d_3w_1 .$$

Solving (7) and (8) we obtain

$$\left(B + bQ_1Q_0\right)\dot{x}_0 = \beta + Q_1\left(b_3\gamma - d_3\beta\right), \tag{9}$$

$$\left(B + bQ_0Q_1\right)\dot{x}_1 = \gamma + Q_0\left(c_3\beta - a_3\gamma\right). \tag{10}$$

$$B = I - a_3Q_0 - d_3Q_1 ,$$

$$b = a_3d_3 - b_3c_3 .$$

A POLYNOMIAL APPROXIMATION

Substituting successively

$$\alpha = \phi_i(t) = t^i, \quad i = 0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

into equation (2), we obtain

$$a_0 + b_0 = 0 ,$$

$$b_0 = 1 ,$$

$$b_0 + 2a_2 + 2b_2 = 0 ,$$

$$b_0 + 6b_2 + 6a_3 + 6b_3 = 0 ,$$

$$b_0 + 12b_2 + 24b_3 = 0 ,$$

$$b_0 + 20b_2 + 60b_3 = 0 .$$

The solution of this set of equations is

$$a_0 = -1, \quad b_0 = 1, \quad a_2 = -\frac{7}{20}, \quad b_2 = -\frac{3}{20}, \quad a_3 = -\frac{1}{20}, \quad b_3 = \frac{1}{30}.$$

In a similar way we find

$$c_0 = -1, \quad d_0 = 1, \quad c_2 = \frac{3}{20}, \quad d_2 = \frac{7}{20}, \quad c_3 = \frac{1}{30}, \quad d_3 = -\frac{1}{20}.$$