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OPTIMIZED COMPUTATION WITH RECURSIVE POWER SERIES INTEGRATION

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ABSTRACT

The power series technique is applied to the restricted three-body problem and recursion formulas are developed for the coefficients of the power series which reduce significantly the computation time required by the recursion formulas developed by previous authors. Van Flandern's method for maximizing the step size per unit of computer time is discussed and extended on the basis of relative error control. A method for controlling the magnitudes of the series coefficients is also presented.

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INTRODUCTION

The method of solution of second degree systems of differential equations by recurrent power series is well known [6]. This technique is especially useful when variable step sizes are needed in the integration, and has the bonus property of automatically choosing the correct step size relative to some preset error tolerance. As pointed out by Fehlberg [2], a substantial saving in computing time is realizable under the above conditions by choosing a power series method over a Runge-Kutta or interpolation procedure.

In this paper the power series technique will be applied to the restricted three-body problem and a few "tricks" will be presented which reduce significantly the computation time required by the recursion formulas presented in the papers we have seen [2, 5, 6]. Van Flandern's method [7] for maximizing the step size per unit of computer time is discussed and extended on the basis of relative error control. A method for controlling the magnitudes of the series coefficients is also presented.

THE RESTRICTED THREE-BODY PROBLEM

Second degree systems of differential equations are not always met in practice. However, upon introducing appropriate auxiliary functions a given system may in many instances be transformed into one of the second degree. It is to be noted that in general for a particular set of equations there are no unique auxiliary functions which will transform the system to the above form, but rather one must search for the functional substitutions which lead to more symmetric recursion formulas and fewer computational operations. As an illustration, consider the restricted problem of three bodies.

In a rotating coordinate system the equations of motion are

$$\frac{d^2x}{dt^2} = x + 2\frac{dy}{dt} - (1-\mu)\frac{x+\mu}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \mu\frac{x-1+\mu}{\left[(x-1+\mu)^2+y^2\right]^{3/2}}$$

$$\frac{d^2y}{dt^2} = y - 2\frac{dx}{dt} - (1-\mu)\frac{y}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \mu\frac{y}{\left[(x-1+\mu)^2+y^2\right]^{3/2}} \qquad (1)$$

where μ is the mass of the lighter body and $1-\mu$ is the mass of the heavier body. The commonly used auxiliary functions

$$r^2 = (x + \mu)^2 + y^2$$

$$s^2 = (x - 1 + \mu)^2 + y^2 \qquad (2)$$

$$u = \frac{1 - \mu}{r^3}$$

$$v = \frac{\mu}{s^3}$$

upon introduction into the equations of motion yield the following second degree system:

$$\frac{d^2x}{dt^2} = x + 2\frac{dy}{dt} - u(x + \mu) - v(x - 1 + \mu)$$

$$\frac{d^2y}{dt^2} = y - 2\frac{dy}{dt} - uy - vy$$

$$r\frac{du}{dt} + 3u\frac{dr}{dt} = 0 \qquad (3)$$

$$s\frac{dv}{dt} + 3v\frac{ds}{dt} = 0$$

$$r^2 = (x + \mu)^2 + y^2$$

$$s^2 = (x - 1 + \mu)^2 + y^2 \quad .$$

Letting

$$x = \sum_{i=0}^{\infty} x_i t^i$$

$$y = \sum_{i=0}^{\infty} y_i t^i$$

$$u = \sum_{i=0}^{\infty} u_i t^i$$

$$v = \sum_{i=0}^{\infty} v_i t^i \qquad (4)$$

$$r = \sum_{i=0}^{\infty} r_i t^i$$

$$s = \sum_{i=0}^{\infty} s_i t^i$$

and substituting into (3) we obtain, after equating coefficients of like powers,

$$2r_0 r_n = \sum_{i=0}^{n} x_i x_{n-i} + 2\mu x_n + \sum_{i=0}^{n} y_i y_{n-i} - \sum_{i=1}^{n-1} r_i r_{n-i}$$

$$2s_0 s_n = \sum_{i=0}^{n} x_i x_{n-i} - 2(1-\mu)x_n + \sum_{i=0}^{n} y_i y_{n-i} - \sum_{i=1}^{n-1} s_i s_{n-i}$$

$$n r_0 u_n = -3 \sum_{i=1}^{n} i r_i u_{n-i} - \sum_{i=1}^{n-1} i u_i r_{n-i}$$

$$n s_0 v_n = -3 \sum_{i=1}^{n} i s_i v_{n-i} - \sum_{i=1}^{n-1} i v_i s_{n-i} \qquad (5)$$

$$(n+1)n x_{n+1} = x_{n-1} + 2n y_n - \mu u_{n-1} + (1-\mu)v_{n-1} - \sum_{i=0}^{n-1} (u_i + v_i) x_{n-1-i}$$

$$(n+1)n y_{n+1} = y_{n-1} - 2n x_n - \sum_{i=0}^{n-1} (u_i + v_i)\, y_{n-1-i}$$

for $n \geq 1$.

The initial conditions of the problem yield x_0, x_1, y_0, and y_1, and r_0, s_0, u_0, v_0 are obtained from (2), the auxiliary functions. It may be easily seen that proceeding through this algorithan to calculate the (n + 1) st term, i.e., x_{n+1} and y_{n+1}, there are required

$14n + 4$ additions,

$16n + 16$ multiplications,

and 6 divisions.

Further, $26n + 4$ quantities are addressed.

Let us return now to (1) and consider instead the following auxiliary functions:

$$s = x + \mu$$

$$\alpha = s^2 + y^2$$

$$\beta = \alpha - 2s + 1 \tag{6}$$

$$p = \frac{1 - \mu}{\alpha^{3/2}}$$

$$q = \frac{\mu}{\beta^{3/2}} .$$

Substituting (6) into (1) then yields the second degree system:

$$\frac{d^2s}{dt^2} = s + 2\frac{dy}{dt} - sp - (s-1)q - \mu \quad ,$$

$$\frac{d^2y}{dt^2} = y - 2\frac{ds}{dt} - py - qy \quad ,$$

$$3p\frac{d\alpha}{dt} + 2\alpha\frac{dp}{dt} = 0 \quad ,$$

$$3q\frac{d\beta}{dt} + 2\beta\frac{dq}{dt} = 0 \quad , \tag{7}$$

$$\beta = \alpha - 2s + 1 \quad ,$$

$$\alpha = s^2 + y^2 \quad .$$

Letting

$$s = \sum_{i=0}^{\infty} s_i t^i$$

$$y = \sum_{i=0}^{\infty} y_i t^i$$

$$\alpha = \sum_{i=0}^{\infty} \alpha_i t^i$$

$$\beta = \sum_{i=0}^{\infty} \beta_i t^i \tag{8}$$

$$p = \sum_{i=0}^{\infty} p_i t^i$$

$$q = \sum_{i=0}^{\infty} q_i t^i$$

and substituting into (7) there results

$$\alpha_n = \begin{cases} 2\sum_{v=0}^{k} (s_v s_{n-v} + y_v y_{n-v}) , & n = 2k+1 \\ 2\sum_{v=0}^{k} (s_v s_{n-v} + y_v y_{n-v}) + s_k^2 + y_k^2 , & n = 2k \end{cases}$$

$$\beta_n = \alpha_n - 2s_n$$

$$2n\alpha_0 p_n = -3n\alpha_n p_0 - \sum_{v=1}^{n-1} (2n+v)\, \alpha_n p_{n-v} \qquad (9)$$

$$2n\beta_0 q_n = -3n\beta_n q_0 - \sum_{v=1}^{n-1} (2n+v)\, \beta_n q_{n-v}$$

$$(n+1)n s_{n+1} = s_{n-1} + 2n y_n + q_{n-1} - \sum_{v=0}^{n-1} (p_v + q_v)\, s_{n-1-v}$$

$$(n+1)n y_{n+1} = y_{n-1} - 2n s_n - \sum_{v=0}^{n-1} (p_v + q_v)\, y_{n-1-v}$$

for $n \geq 1$. Again, the initial conditions yield s_0, s_1, y_0, and y_1; and α_0, β_0, p_0, q_0 are determined from (6). However, the calculation of the (n + 1) st term now involves

	$9n + 5$	additions
	$7n + 20$	multiplications
and	4	divisions.

Also, only $12n + 12$ quantities are addressed. Thus, by simply choosing the auxiliary functions (6) instead of (2) the computation time for high order terms will be approximately halved. It is to be noted that symmetries such as in the first of equations (9) occur frequently in practice and full advantage of these symmetries should be taken in a computational scheme.

MINIMIZATION OF COMPUTER TIME

In a power series integration method, the solution is represented as a power series, e.g.,

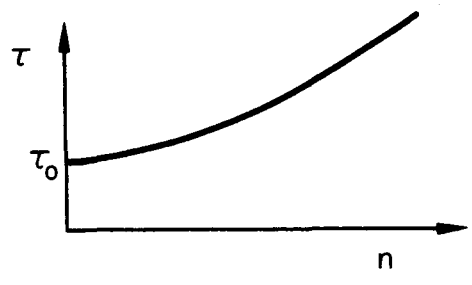
$$x = \sum_{n=0}^{\infty} a_n t^n \quad . \tag{10}$$

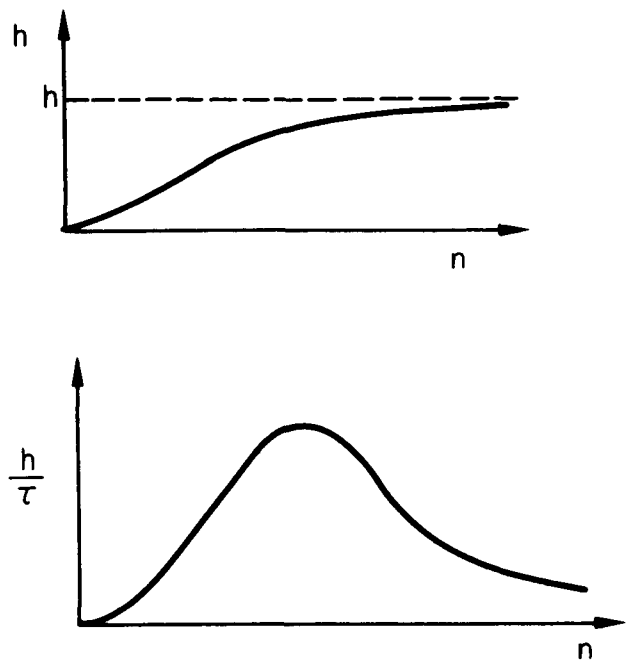
For a given number of terms n, t is chosen so that x has a desired accuracy. By using the solutions of the previous step as initial conditions for a new step and computing a new power series, the solution is extended by analytic continuation. The following technique, due to Thomas Van Flandern,[7] minimizes the computer time required to integrate over a given interval. Let τ designate total computation time for a step and assume that a fixed time, τ_0, is the initialization time for each step. Since the computation time required for the (n + 1) st. term is greater than that for the n^{th} term, $\tau = \tau(n)$ is an increasing monotone function. Further, with the addition of more terms in the series the integration step size h increases for fixed accuracy and approaches h_{max}, the radius of convergence. Thus h = h(n) is also an increasing monotone function.

We wish to make h as large as possible for a given amount of computer time. Thus we are led to choose the value of n such that the quantity h/τ is a maximum. That this ratio has at least one maximum is obvious from the fact that h/τ is zero for n = 0, approaches zero as n gets large, and has only positive values for n> 0.

The ratio h/τ should be calculated for each new term of the series, and a check made for its maximum value. The step size h may be estimated from the newly calculated term and a preset error tolerance, while τ may be obtained from an accurate computer clock or from operation counting for the particular recursive formulas used.

The following graphs illustrate the above procedure:





In practice, convergence of the series will usually insure that the sum of neglected terms will be smaller in magnitude than the first term omitted. In such cases we may calculate the step size for each new term from

$$h = \left(\frac{\epsilon}{|a_n|} \right)^{1/n} , \tag{11}$$

where a_n is the n^{th} coefficient of the series and ϵ is the absolute error tolerance. Many times with this procedure it is found to be true that the magnitudes of the calculated series coefficients are such that $h = h(n)$ is not an increasing monotone function, i.e., it has small oscillations in violation of the criteria of applicability for Van Flandern's procedure. However, the authors have found in general that such coefficients which disrupt the smoothly increasing sequence of the magnitude of h are isolated and so the requirement that the ratio h/τ of three consecutive terms form a strictly decreasing sequence yields excellent results as a test for the maximum value of h/τ.

In order to avoid unnecessary labor in computation it is desirable that all errors committed at any integration step contribute uniformly to the entire integration error. Thus the error tolerance ϵ of equation (11) must be a relative, rather than an absolute error. The authors have found it convenient to form the absolute error, ϵ_a, for each integration step by selecting a relative error ϵ_r for the entire integration and from the initial conditions a_0 at each step, forming

$$\epsilon_a = a_0 \epsilon_r \quad . \tag{12}$$

When a_0 is small, however, ϵ_a is replaced by a preset value.

A major problem which occurs in practice is that the series coefficients a_n may become quite large for increasing n. This results in a loss of accuracy when the truncated series is evaluated at a value of the independent variable for a given integration step. However, the following convenient property of recursive power series may be used to easily overcome this difficulty.

Given the series (10), i.e.

$$x = \sum_{n=0}^{\infty} a_n t^n$$

we may scale t such that $t = hs$ so that the above series becomes

$$x = \sum_{n=0}^{\infty} a_n (hs)^n$$

$$= \sum_{v=0}^{\infty} b_n h^n \quad ,$$

where the coefficients b_n are calculated with the recursive relations which determine a_n slightly changed and with simple changes in the initial conditions [1], [4]. By assuming that the step size for a given integration step will be nearly equal to that of the previous step, we set s equal to the previous step size so that $h \approx 1$ and the coefficients b_n are well behaved.

In the interest of reducing roundoff error over the course of the integration the authors have found it useful to set the scale factor equal to a power of two so that the operations involving the scale factor are merely shifts within a binary digital computer.

We have programmed the above techniques for the restricted problem of three bodies and applied the same initial conditions as those of Fehlberg [3] for the integration of a periodic orbit. Our inability to obtain a replica of the operating program used by this author has led to limited comparison. However, our own results indicate that the reduced computational procedures outlined in this paper produce a 40%-50% time saving over a conventional power series integration of this periodic orbit with each step constrained to sixteen terms, where both methods were performed in double precision arithmetic on an IBM

7094 computer with a preset error tolerance of 10^{-17}. We conclude from this that the above techniques lead to the same high precision in comparable computing time as the high order Runge-Kutta tranformation type formulas of Fehlberg[3] with the advantage of simpler and more compact programming.

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