[redacted]

AN ACCURACY STUDY OF [redacted] METHODS IN STRUCTURAL ANALYSIS

By Nancy Jane Cyrus and Robert E. Fulton

NASA Langley Research Center
Langley Station, Hampton, Va.

Presented to the Fourth Conference on Electronic Computation
American Society of Civil Engineers

Los Angeles, California
September 7-9, 1966

[redacted]

AN ACCURACY STUDY OF FINITE DIFFERENCE METHODS

IN STRUCTURAL ANALYSIS†

By Nancy Jane Cyrus* and Robert E. Fulton**

NASA Langley Research Center

SUMMARY

An accuracy study is made of central finite difference methods for solving boundary value problems in structural analysis which are governed by equations with variable coefficients leading to odd order derivatives. Two methods are studied through application to beam-columns with nonuniform inplane loads and nonuniform stiffness. Definitive expressions for the error in each method are obtained by using Taylor series to derive the differential equations which exactly represent the finite difference approximations. The resulting differential equations are accurately solved by a perturbation technique which yields the error directly. A "half station" method, which corresponds to making finite difference approximations before expanding derivatives of function products in the beam-column differential equations, was found clearly superior to a "whole station" method which corresponds to expanding such products first.

---

†The material included herein was carried out by the first author in partial fulfillment of the requirements for a degree of Master of Science in Mathematics at Virginia Polytechnic Institute.

*Mathematician, NASA Langley Research Center.

**Aerospace Engineer, NASA Langley Research Center, Member, ASCE.

L-4910

[redacted]

INTRODUCTION

The differential equations governing the behavior of beams, plates, and shells are often solved by approximating the derivatives by finite differences and solving the resulting algebraic equations on a digital computer. In design analyses of complicated structures, such as civil engineering shell structures or aerospace vehicle structures, the number of simultaneous equations resulting from finite difference approximations can be sufficiently large to exceed the capacity of the computer or introduce round-off error. For such problems, the accuracy of the difference procedure can be a critical item in obtaining meaningful design results. In reference 1, for example, it was found that accurate answers for the stress in a shell could not be obtained by using certain finite difference approximations unless the mesh spacing was smaller than machine capacity permitted.

The most popular difference approximations are the so-called central differences which are given in textbooks on numerical methods. There are alternate formulations of central differences which can be used when odd order derivatives occur in the differential equations. Such a situation results in structural problems, for example, when inplane loads are not uniform (a column loaded by its own weight or a shell of revolution) or where the stiffness of the structure is nonuniform (a tapered beam or a variable thickness shell).

The purpose of this paper is to investigate the accuracy of two alternate forms of central finite difference approximations used in the solution of structural problems. A new approach for studying the accuracy of finite difference or finite element methods is presented and utilized. The study is confined to beam-column problems; however, the approach and conclusions are applicable to a wide class of plate and shell problems.

SYMBOLS

$EI(x)$	bending stiffness of beam
$f(x)$	nondimensional tension in beam or string
$g(x)$	nondimensional stiffness of beam
h	finite difference spacing
$L(y)$	linear differential operator
$N(x)$	tension in beam or string
$p(x)$	nondimensional lateral load
$q(x)$	lateral load
x	axial coordinate of beam or string
y	deflection of beam or string
Y	deflection function in perturbation series (see eq. (12))

STATEMENT OF THE PROBLEM

Consider a general beam-column (fig. 1) with nonuniform stiffness EI and nonuniform inplane load N (taken positive in tension). The well-known differential equation governing the lateral deflection y of the beam is

$$(EIy'')'' - (Ny')' - q(x) = 0 \qquad (1)$$

where $q(x)$ is the distributed lateral load and where primes indicate differentiation with respect to x. This equation can be solved by finite differences by dividing the beam into stations of equal spacing h. The quantities EI and N are known, and finite difference equations are written in terms of the displacements at the ith station $(i = 1,2,3 \ldots)$.

In the present paper, two different finite difference approximations are considered. For convenience one formulation is called the "Half Station" method and the other the "Whole Station" method. For the term $(Ny')'$ in equation (1) these two methods lead to the following finite difference expressions:

1. Half Station Method

$$(Ny')_i' = \frac{1}{h}\left[-(Ny')_{i-1/2} + (Ny')_{i+1/2}\right]$$

$$= \frac{1}{h^2}\left[N_{i-1/2}y_{i-1} - (N_{i-1/2} + N_{i+1/2})y_i + N_{i+1/2}y_{i+1}\right] \tag{2}$$

or

2. Whole Station Method

$$\left[Ny'' + N'y'\right]_i = \frac{N_i}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{N_i'}{2h}(-y_{i-1} + y_{i+1})$$

$$= \frac{1}{h^2}\left[\left(N_i - \frac{hN_i'}{2}\right)y_{i-1} - 2N_iy_i + \left(N_i + \frac{hN_i'}{2}\right)y_{i+1}\right] \tag{3}$$

Note that the half station method is the natural result of making the finite difference approximation before expanding the derivatives while the whole station method results from making the approximation after the expansion. The latter type of approximation is widely used (see, for example, refs. 2 and 3). Corresponding choices for the term $(EIy'')''$ are:

1. Half Station Method

$$(EIy'')_i'' = \frac{1}{h^2}\left[(EIy'')_{i-1} - 2(EIy'')_i + (EIy'')_{i+1}\right]$$

$$= \frac{1}{h^4}\left\{(EI)_{i-1}y_{i-2} - 2\left[(EI)_{i-1} + (EI)_i\right]y_{i-1} + \left[(EI)_{i-1} + 4(EI)_i + (EI)_{i+1}\right]y_i - 2\left[(EI)_i + (EI)_{i+1}\right]y_{i+1} + (EI)_{i+1}y_{i+2}\right\} \tag{4}$$

2. Whole Station Method

$$\left[EIy^{iv} + 2(EI)'y''' + (EI)''y''\right]_i = \frac{(EI)_i}{h^4}\left[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}\right]$$

$$+ \frac{2(EI)'_i}{2h^3}\left[-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}\right]$$

$$+ \frac{(EI)''_i}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

$$= \frac{1}{h^4}\left\{\left[(EI)_i - h(EI)'_i\right]y_{i-2} + \left[-4(EI)_i + 2h(EI)'_i + h^2(EI)''_i\right]y_{i-1} + \left[6(EI)_i - 2h^2(EI)''_i\right]y_i + \left[-4(EI)_i - 2h(EI)'_i + h^2(EI)''_i\right]y_{i+1} + \left[(EI)_i + h(EI)'_i\right]y_{i+2}\right\} \qquad (5)$$

While the preceding two sets of finite difference approximations are both of order h^2, they clearly lead to different coefficients for the simultaneous equations in terms of the displacements at the ith station. Of concern here are the relative magnitudes of the errors in these different approximations.

ERROR ANALYSIS AND RESULTS

The usual approach in a finite difference accuracy study is to carry out the numerical solution to a number of problems for which exact solutions can be obtained and to compare the resulting numerical answers at each station with the exact answers. Such a procedure has the liability that comparisons can only be made for each problem at specific stations and the calculations must be redone each time the mesh size changes.

The approach used in this paper is one which has not been reported previously in the literature. The finite difference approximations are expanded in Taylor series. This procedure results in differential equations which are exactly equivalent to the finite difference approximations. The resulting differential equations are then solved by a perturbation technique and yield analytical expressions for the largest error term. These expressions are independent of mesh spacing, are directly comparable, and give a clear indication of the relative accuracy of the difference approximations not just at discrete points but over the length of the beam.

There are two terms in the beam-column equation which are approximated by finite differences: (1) the nonuniform tension effect and (2) the nonuniform stiffness effect. It is convenient to consider these two effects separately.

Effect of Nonuniform Tension

To study the effect of the inplane load term in equation (1) let $EI = 0$. The resulting equation describes the behavior of a laterally loaded string supported at each end and subjected to nonuniform tension. For convenience, the variables are nondimensionalized so that the length of the string is 1 and tension is 1 at the left end. This leads to the following problem:

$$-(f(x)y')' - p(x) = 0 \tag{6}$$

$$y(x_o) = 0 \qquad\qquad y(x_o + 1) = 0$$

where $f(x)$ now represents the nondimensional tension in the string, $p(x)$ is a nondimensional lateral load, and x_o is the coordinate of the left end of the string. Application of the two difference patterns, equations (2) and (3), to equation (6) yields:

1. Half Station Method

$$-\frac{1}{h^2}\left[f_{i-1/2}y_{i-1} - (f_{i-1/2} + f_{i+1/2})y_i + f_{i+1/2}y_{i+1}\right] - p_i = 0 \tag{7}$$

2. Whole Station Method

$$-\frac{1}{h^2}\left[\left(f_i - \frac{hf_i'}{2}\right)y_{i-1} - 2f_iy_i + \left(f_i + \frac{hf_i'}{2}\right)y_{i+1}\right] - p_i = 0 \tag{8}$$

Expand the finite difference recursion formula equations (7) and (8) about the ith point using such Taylor series expansions as:

$$y_{i\pm1} = y_i \pm hy_i' + \frac{h^2}{2!}y_i'' \pm \ldots$$

$$f_{i\pm1} = f_i \pm hf_i' + \frac{h^2}{2!}f_i'' \pm \ldots$$

For both the half station and whole station method this procedure leads to a differential equation of the form

$$L_o(y_i) - p_i + h^2L_1(y_i) + h^4L_2(y_i) + \ldots = 0 \tag{9}$$

subject to the boundary conditions

$$y_i = 0 \quad \text{at} \quad x = x_o$$

$$y_i = 0 \quad \text{at} \quad x = x_o + 1$$

The symbols L_o, L_1, and L_2 are linear differential operators given by

$$L_o(y_i) = -(f_iy_i')' \tag{10}$$

and

1. Half Station Method

$$\left.\begin{aligned} L_1(y_i) &= -\left(\frac{f_i y_i^{iv}}{12} + \frac{f_i' y_i'''}{6} + \frac{f_i'' y_i''}{8} + \frac{f_i''' y_i'}{24}\right) \\ L_2(y_i) &= -\left(\frac{f_i y_i^{vi}}{360} + \frac{f_i' y_i^{v}}{120} + \frac{f_i'' y_i^{iv}}{96} + \frac{f_i''' y_i'''}{144} + \frac{f_i^{iv} y_i''}{384} + \frac{f_i^{v} y_i'}{1920}\right) \\ \cdots \quad & \quad \cdots \end{aligned}\right\} \tag{11a}$$

2. Whole Station Method

$$\left.\begin{aligned} L_1(y_i) &= -\left(\frac{f_i y_i^{iv}}{12} + \frac{f_i' y_i'''}{6}\right) \\ L_2(y_i) &= -\left(\frac{f_i y_i^{vi}}{360} + \frac{f_i' y_i^{v}}{120}\right) \\ \cdots \quad & \quad \cdots \end{aligned}\right\} \tag{11b}$$

Equations (9) and (10) together with either (11a) or (11b) are clearly differential equations which represent exactly the finite difference recursion formulas. As h goes to zero, equation (9) approaches equation (6). The solution to equation (9), satisfying the appropriate boundary conditions, gives an analytical representation of the numerical finite difference answers. Unfortunately a closed form solution to equation (9) does not appear feasible because it contains an infinite number of terms. For a practical problem, however, h is perhaps 0.1 or 0.01 or even smaller. This suggests that equation (9) can be solved by a perturbation method with the perturbation parameter taken to be h^2.

Let the solution y_i to equation (9) be taken in the form

$$y_i = Y_o + h^2 Y_1 + \ldots \tag{12}$$

Substituting equation (12) into equation (9) leads to

$$L_o(Y_o) - p_i + h^2\left[L_o(Y_1) + L_1(Y_o)\right] + \ . \ . \ . = 0 \tag{13}$$

subject to

$$Y_o(x_o) + h^2\left[Y_1(x_o)\right] + \ . \ . \ . = 0$$

$$Y_o(x_o + 1) + h^2\left[Y_1(x_o + 1)\right] + \ . \ . \ . = 0$$

If each order of error term is solved in sequence, the following series of problems result:

(1) $L_o(Y_o) - p_i = 0$ $\quad Y_o(x_o) = 0, \quad Y_o(x_o + 1) = 0$ (14)

(2) $L_o(Y_1) + L_1(Y_o) = 0$ $\quad Y_1(x_o) = 0, \quad Y_1(x_o + 1) = 0$ (15)

(3) . . . $\quad$ . . . $\quad$ . . .

Note that since equation (6) is linear Y_o given by equations (14) is in fact the exact solution. From the form of y_i it is seen that Y_1 can be interpreted as the first order error term in the finite difference results. The magnitude of Y_1 is therefore a measure of the error in the finite difference results as compared to the exact answer to the problem. A comparison of the error terms Y_1 resulting from two different finite difference approximations indicates the relative accuracy of the two approximations when the node point spacing is the same.

Using this method, the error functions Y_1 corresponding to the half station and whole station finite difference approximations have been obtained for a family of problems. These problems are a string having a lateral load which is distributed uniformly and a tension force $f(x)$ which varies as follows:

(1) $f(x) = \frac{1}{x^n}$ for $1 \leqq n \leqq 6$

subject to the boundary conditions

$$y(1) = 0$$
$$y(2) = 0$$

and

(2) $f(x) = 1 + x^n$ for $2 \leqq n \leqq 6$

subject to the boundary conditions

$$y(0) = 0$$

$$y(1) = 0$$

For the case where $f(x)$ is linear (corresponding to $f(x) = 1, x,$ or $1 + x$) the results for the half station and whole station finite difference approximations are exactly the same. In fact for $f(x) = 1$, both difference answers are the exact answer. For all other cases, however, the two difference methods lead to different results. It is useful to compare the results for the case $f(x) = \frac{1}{x^3}$ in detail as a typical example.

For $f(x) = \frac{1}{x^3}$ and $y(1) = y(2) = 0$

$$Y_o = -\frac{x^5}{5} + \frac{31}{75} x^4 - \frac{16}{75} \tag{16}$$

1. Half Station Method

$$Y_1 = -\frac{41}{1125} x^4 + \frac{x^3}{6} - \frac{31}{150} x^2 + \frac{86}{1125} \tag{17}$$

2. Whole Station Method

$$Y_1 = -\frac{187}{450}x^4 + \frac{4}{3}x^3 - \frac{31}{30}x^2 + \frac{26}{225} \tag{18}$$

A plot of the two error terms Y_1 over the length of the string is given in figure 2(a). Solutions were also obtained for the error terms in deflection for all of the remaining load functions $f(x)$ noted previously; an additional plot of results, for the case $f(x) = 1 + x^3$, is shown in figure 2(b). Detailed plots of the remaining solutions are not shown because figure 2 serves to illustrate the character of the results; an overall measure of the relative errors in the two methods will be shown later for all the solutions obtained.

While errors in the deflections of the string are important, errors in numerically obtained derivatives should also be considered for a thorough error analysis. Therefore, results were obtained by using the finite difference answers for approximate curvatures (second derivatives). The second difference operator was applied to the difference results followed by Taylor and perturbation series expansions to yield:

$$\begin{aligned} y_i'' &= \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \\ &= Y_o'' + h^2Y_1'' + \frac{h^2}{12}(Y_o^{iv} + h^2Y_1^{iv} + \ldots) + \ldots \end{aligned}$$

or

$$y_i'' = Y_o'' + h^2\left(Y_1'' + \frac{Y_o^{iv}}{12}\right) + \ldots \tag{19}$$

The h^2 error terms in the curvatures for the two methods and for the case $f(x) = \frac{1}{x^3}$ are as follows:

1. Half Station Method

$$Y_1'' + \frac{Y_0^{iv}}{12} = -\frac{164}{375}x^2 - x + \frac{31}{75} \tag{20}$$

2. Whole Station Method

$$Y_1'' + \frac{Y_0^{iv}}{12} = -\frac{374}{75}x^2 + 6x - \frac{31}{25} \tag{21}$$

A plot of the error in the curvature for each of the two methods is also given in figure 2(a) for this case and in figure 2(b) for the case $f(x) = 1 + x^3$. Again, results for the remaining load functions will be shown later in the form of an overall measure of the relative error.

Numerical calculations were also carried out for the deflections and curvatures for the problems cited to determine if the analytical errors adequately represented the numerical errors. The data are not included here; however, for h less than about 0.1 all analytical errors agree with calculated numerical errors to within 1 percent.

Effect of Nonuniform Stiffness

To study the effect of nonuniform stiffness on the numerical results for the behavior of a beam-column, the tension N is set equal to zero and the difference approximations given by equations (4) and (5) are compared. Results are obtained for a simply supported beam having a uniformly distributed load. Here again the variables have been nondimensionalized to make the length of the beam and the bending stiffness at the left end each equal to 1. This leads to the following problem:

$$\left[g(x)y''\right]'' = 1 \tag{22}$$

$$y(x_o) = 0 \qquad y(x_o + 1) = 0$$

$$y''(x_o) = 0 \qquad y''(x_o + 1) = 0$$

where $g(x)$ now represents the stiffness of the beam and the distributed load is 1.

From equations (4) and (5) the two difference equations resulting from equation (22) are

1. Half Station Method

$$\frac{1}{h^4}\Big[g_{i-1}y_{i-2} - 2(g_{i-1} + g_i)y_{i-1} + (g_{i-1} + 4g_i + g_{i+1})y_i - 2\left(g_i + g_{i+1}\right)y_{i+1} + g_{i+1}y_{i+2}\Big] = 1 \tag{23}$$

2. Whole Station Method

$$\frac{1}{h^4}\Big[\left(g_i - hg_i'\right)y_{i-2} + \left(-4g_i + 2hg_i' + h^2g_i''\right)y_{i-1} + \left(6g_i - 2h^2g_i''\right)y_i + \left(-4g_i - 2hg_i' + h^2g_i''\right)y_{i+1} + \left(g_i + hg_i'\right)y_{i+2}\Big] = 1 \tag{24}$$

As before, expanding y_i and g_i about the ith point leads to the differential equation

$$L_o(y_i) + h^2L_1(y_i) + h^4L_2(y_i) + \ldots\ldots = 1 \tag{25}$$

where, now

$$L_o = \left[g(x)_i y_i''\right]'' \tag{26}$$

and

1. Half Station Method

$$\left.\begin{aligned} L_1(y_i) &= \frac{g_i y_i^{vi}}{6} + \frac{g_i' y_i^{v}}{2} + \frac{7}{12} g_i'' y_i^{iv} + \frac{g_i''' y_i'''}{3} + \frac{g_i^{iv} y_i''}{12} \\ L_2(y_i) &= \frac{g_i y_i^{viii}}{80} + \frac{g_i' y_i^{vii}}{20} + \frac{31}{360} g_i'' y_i^{vi} + \frac{g_i''' y_i^{v}}{12} + \frac{7}{144} g_i^{iv} y_i^{iv} \\ &\quad + \frac{g_i^{v} y_i'''}{60} + \frac{g_i^{vi} y_i''}{360} \end{aligned}\right\} \quad (27a)$$

2. Whole Station Method

$$\left.\begin{aligned} L_1(y_i) &= \frac{g_i y_i^{vi}}{6} + \frac{g_i' y_i^{v}}{2} + \frac{g_i'' y_i^{iv}}{12} \\ L_2(y_i) &= \frac{g_i y_i^{viii}}{80} + \frac{g_i' y_i^{vii}}{20} + \frac{g_i'' y_i^{vi}}{360} \end{aligned}\right\} \quad (27b)$$

If solutions to equation (25), taking into account (26) and either (27a) or (27b), are again taken in the form (12), the series of simpler equations (14) and (15) are again obtained (with $p = 1$). However, since the beam equation is fourth order rather than second, a boundary condition on bending moment must also be considered. The moment is taken to be zero at the ends of the beam; this leads to

$$Y_0'' = 0 \qquad \text{at} \qquad x = x_0 \qquad \text{and} \qquad x = x_0 + 1 \quad (28)$$

and

$$Y_1'' + \frac{Y_0^{iv}}{12} = 0 \qquad \text{at} \qquad x = x_0 \qquad \text{and} \qquad x = x_0 + 1$$

for the zeroth and first order error problems, respectively (see eq. (19)).

Results have been obtained for

$$g(x) = x^n \qquad n = 2, 3, 4$$

and

$$1 \leqq x \leqq 2$$

for both the half station and whole station methods of approximating the derivatives. The error terms for both deflections and curvatures are shown in figure 3 for the case $g(x) = x^3$ corresponding to the case of a linearly tapered beam. An overall measure of the relative error in the half and whole station methods is given below for all three cases. The analytical error results for both deflection and curvature also agree with numerical error calculations within 1 percent for h less than about 0.1.

Relative Errors of the Half and Whole Station Methods

While results such as those given in figures 2 and 3 are usually sufficient to identify which of the two methods is superior for a given problem, identification of the superior method for specific results is sometimes difficult (see, for example, the curvature errors of fig. 2(b)). Moreover, a quantitative measure of the relative accuracy of the methods is desirable. Probably the fairest comparison of their overall merit can be made by examining the root-mean-square values of the errors for the whole structure; that is:

$$\overline{Y}_1 = \sqrt{\int_{x_o}^{x_o+1} Y_1^2 dx}$$

for the error in deflection and

$$\overline{Y}_1'' = \sqrt{\int_{x_o}^{x_o+1} \left(Y_1'' + \frac{Y_o^{iv}}{12}\right)^2 dx}$$

for the error in curvature, where the integration is over the (unit) length of the string or beam. Thus, to assess quantitatively the relative merits of the half station and whole station methods for the various problems solved, the ratios

$$\frac{\bar{Y}_{1,half}}{\bar{Y}_{1,whole}}$$

and

$$\frac{\bar{Y}''_{1,half}}{\bar{Y}''_{1,whole}}$$

have been calculated for each problem. The results are shown in figure 4.

DISCUSSION OF RESULTS

The results given in figure 4(a) show that for all problems studied, the error in the deflection resulting from use of the half station method is less than the error due to the whole station method - in some cases, by an order of magnitude. The investigation of the accuracy of the curvature approximations gives the same result in general. Thus, the half station method is generally superior for calculation of both deflections and bending curvature for the problems studied.

While the results are a clear victory for the half station method, one exception occurs: for the case of the string with the load $f(x) = 1 + x^2$, the error in the curvature is 25 percent greater with the half station method. Curiously, the difference between the two methods is seen to be generally less in calculating the second derivatives of deflections than in calculating the deflections themselves; moreover, differences in the comparative error from problem to problem are noticeably less with the second derivatives than with the deflections. Both of these results are unexpected.

It should be noted that the analytical representation of errors in the present paper shows clearly the danger of using numerical data at a single station or a few points to characterize the error in a problem. A typical case is shown in figure 2(a) for $f(x) = \frac{1}{x^3}$. If comparisons are made of the curvature near the end $x = 1$, the whole station method appears much more accurate than the half station method; whereas figure 4(b) shows clearly that the average error with the whole station method is over twice as great.

It should be noted also that the present approach to error assessment may also be useful for comparison of different finite element structural approximations. In fact, the recursion formulas given by the half station method (eqs. (2) and (4)) are the same recursion formulas which occur for a finite element model consisting of rigid bars connected by rotational springs, which often is used to replace the beam-column of figure 1 (see, for example, ref. 4). Thus, the results of the present paper verify that the finite element model of reference 4 is a good representation of beam-column behavior.

Reasons for the superiority of the half station method are not altogether clear, but may include the symmetry of the matrix of coefficients in this method. By contrast, the matrix of coefficients associated with whole stations is not symmetric. Matrix symmetry can be of great value for many numerical procedures associated with eigenvalue routines and simultaneous equation solving routines and, in some cases, is required for an efficient numerical solution of a large order system.

CONCLUDING REMARKS

A new procedure has been developed to determine an analytical representation of the error in a finite difference solution and to allow a direct comparison between two difference methods which is independent of mesh size. This procedure appears to have considerable merit for assessment of the relative accuracy of finite difference and finite element numerical techniques of structural analysis.

Using this procedure, a comparison has been made of the accuracy of two different finite difference methods for solving structural problems through applications to a spectrum of beam and string problems having the characteristics of nonuniform stiffness and inplane load. The methods investigated were a "half station" method which corresponds to making the finite difference approximation before expanding the derivatives of function products and a "whole station" method which corresponds to expanding such products first; both methods are in use. It was found that, for the same number of stations, the average error in calculated deflection resulting from use of half station difference approximations was always less than the error which would result from the use of whole station difference approximations. In some cases this error is reduced by an order of magnitude. The investigation of the accuracy of the curvature approximations gave similar results in general. Thus, the half station method is indicated to be clearly superior to the whole station method and its use in finite difference solution of structural problems is recommended.

REFERENCES

1. Chuang, K. P.; and Veletsos, A. S: A Study of Two Approximate Methods of Analyzing Cylindrical Shell Roofs. Civil Engineering Studies, Structural Research Series No. 258, University of Illinois, October 1962.
2. Sepetoski, W. K.; Pearson, C. E.; Dingwell, I. W.; and Adkins, A. W.: A Digital Computer Program for the General Axially Symmetric Thin-Shell Problem. Journal of Applied Mechanics, December 1962.
3. Budiansky, Bernard; and Radkowski, Peter P.: Numerical Analysis of Unsymmetrical Bending of Shells of Revolution. AIAA Journal, August 1963.
4. Newmark, N. M.: Numerical Methods of Analysis of Bars, Plates, and Elastic Bodies. Numerical Methods of Analysis in Engineering, Ed. by L. E. Grinter, MacMillan Co., New York, 1949.

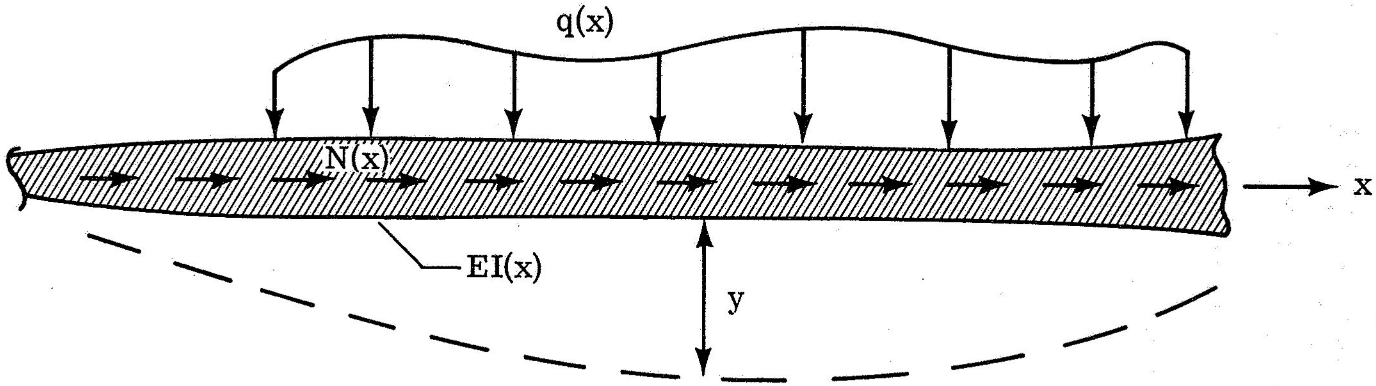
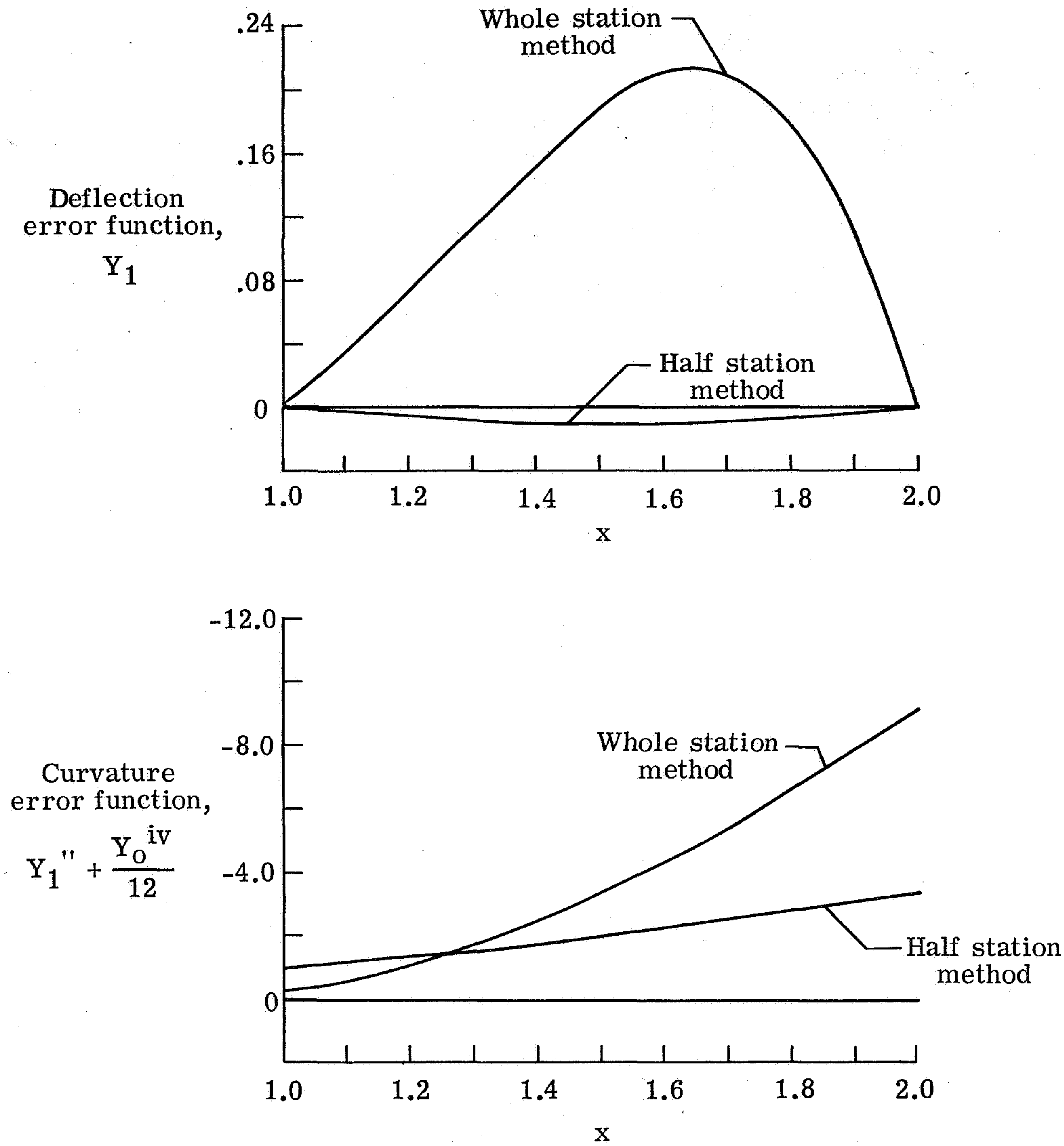
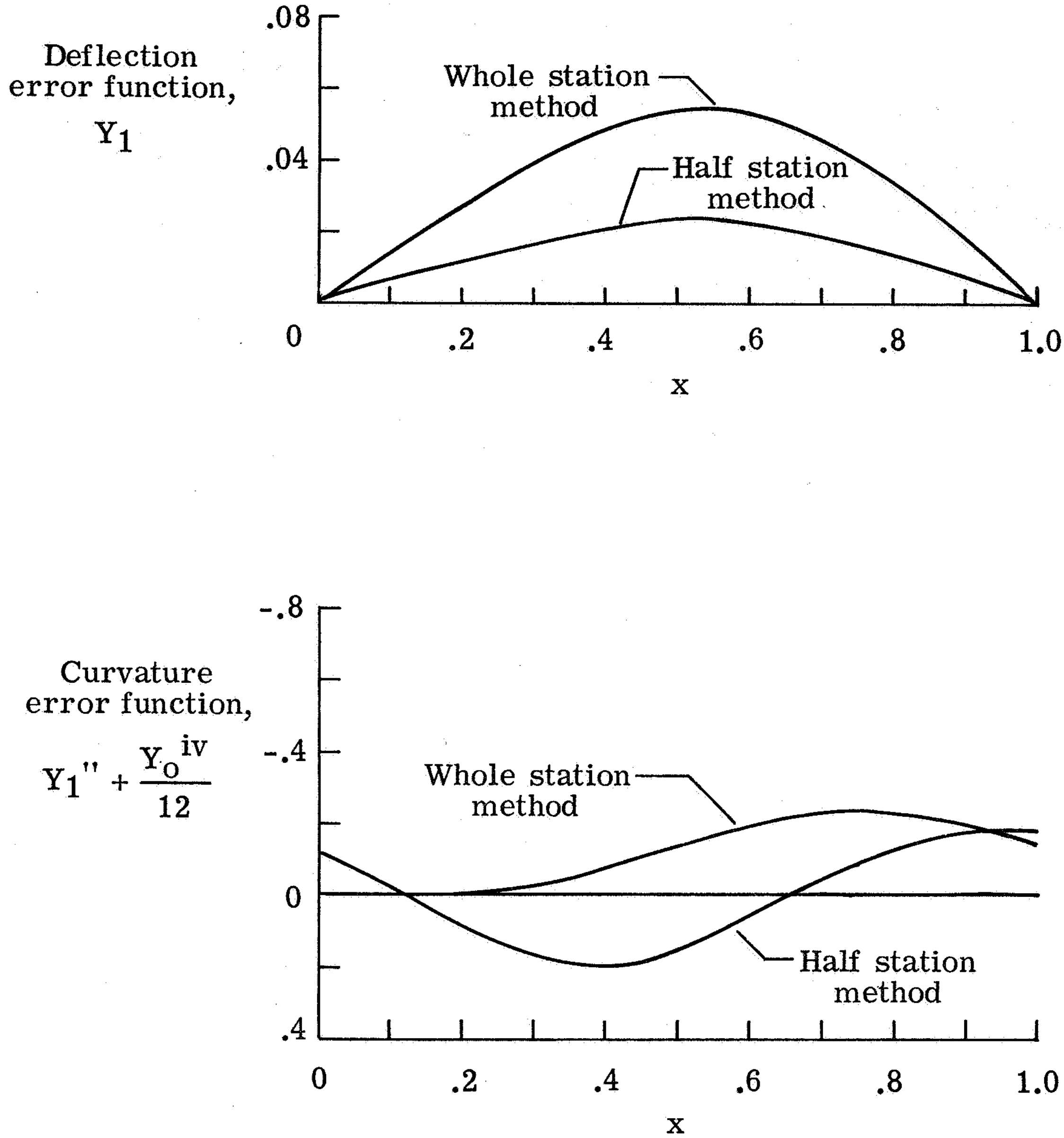


Figure 1.- General beam column.



(a) $f(x) = 1/x^3$.

Figure 2.- Finite difference error in deflection and curvature for a uniformly loaded string with nonuniform tension, $f(x)$.



(b) $f(x) = 1 + x^3$.

Figure 2.- Concluded.

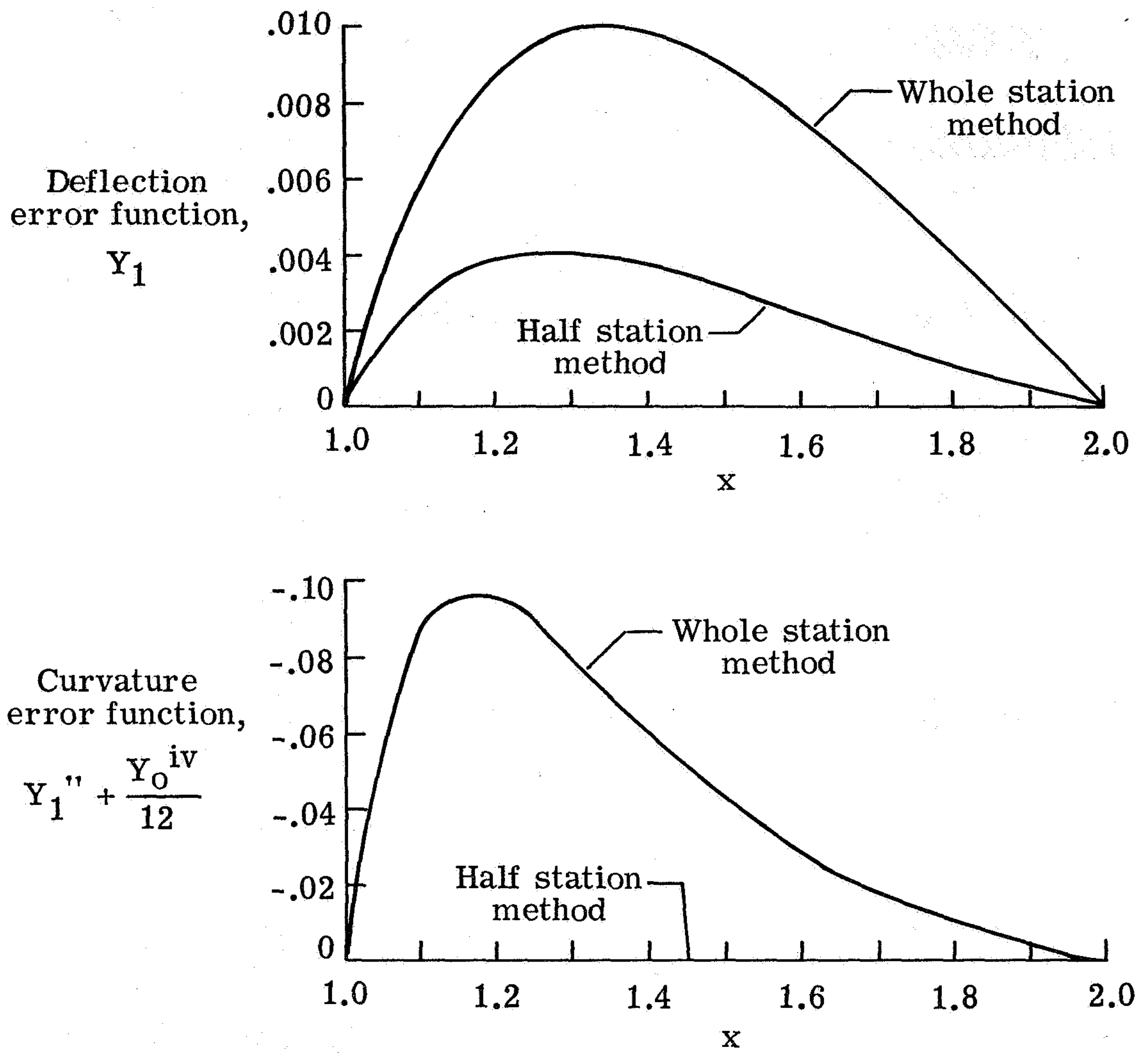
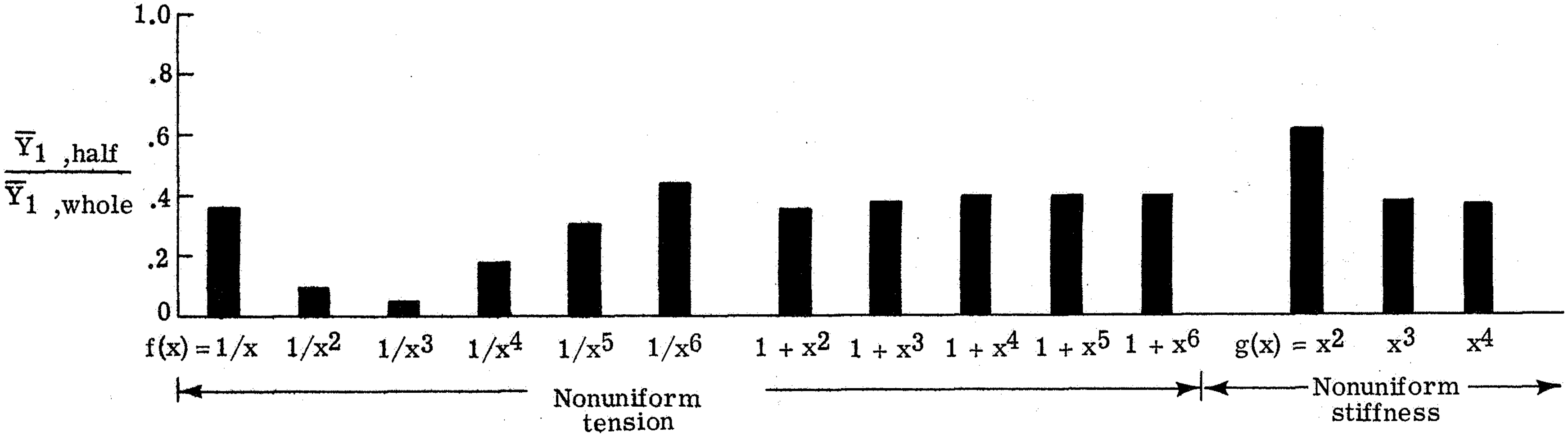
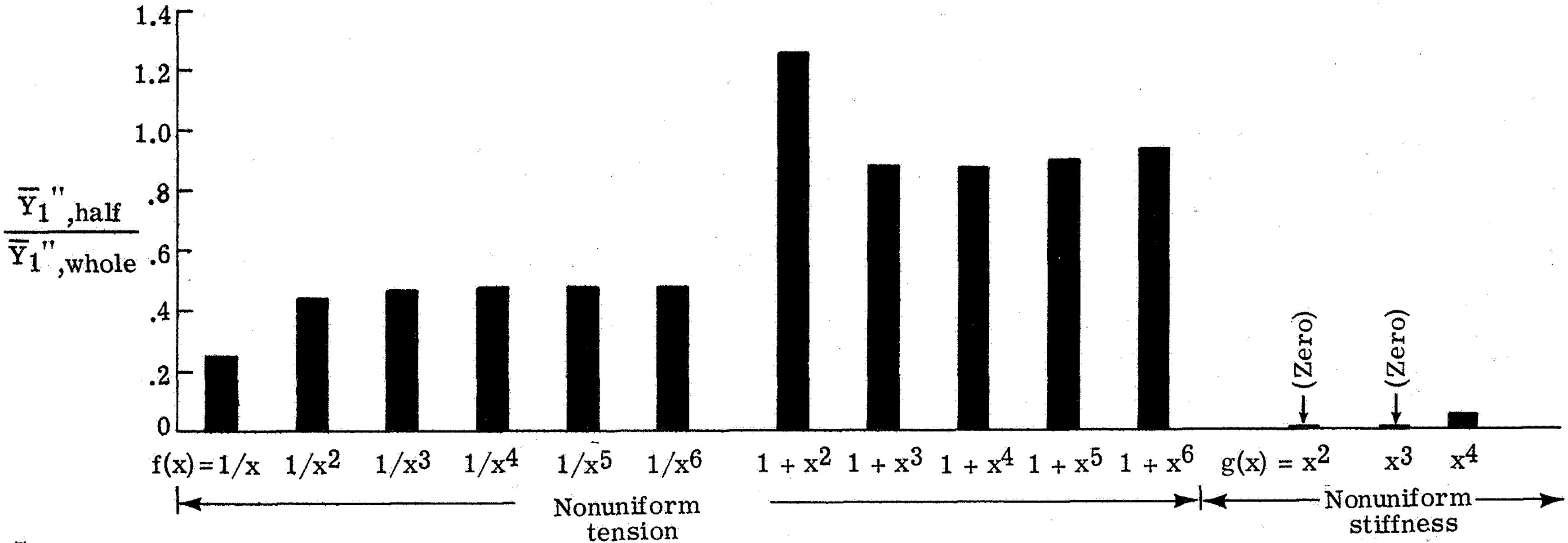


Figure 3.- Finite difference error in deflection and curvature of a uniformly loaded simply supported beam with nonuniform stiffness, $g(x) = x^3$.



(a) Deflection.



(b) Curvature.

Figure 4.- Ratio of root-mean-square errors for half and whole station methods.

NASA-Langley, 1966