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RESPONSE ENVELOPE -

A Global Description of Three-Axis Large-Angle Spacecraft Attitude Control Systems

by George Meyer

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Moffett Field, Calif.

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SUMMARY

Arbitrary three-axis, large-angle attitude control systems of highly maneuverable spacecraft capable of tracking time-varying target attitudes are considered. Such systems are inherently multivariable, multidimensional, nonlinear, and nonautonomous with infinite sets of forcing functions and initial conditions. The concept of a response envelope is introduced. It provides an analytical procedure for judging whether a proposed attitude control system of this type is fast enough for the mission of the spacecraft. It is defined as a scalar function of time which at each instant is the maximum possible three-dimensional attitude error between spacecraft and target for any admissible initial condition, time variation of target attitude, and disturbance. A simple procedure is presented for computing the response envelope approximately. The approximation is conservative and may be used as a basis for accepting a proposed system. The procedure is based on the Liapunov theory and a prototype of general attitude control systems. The proposed procedure is illustrated by a numerical example based on an Orbiting Astronomical Observatory.

INTRODUCTION

Systems for controlling spacecraft attitude vary greatly in their internal structure. Thus, for example, torque may be generated by reaction wheels, control moment gyros, reaction jets, or by some interaction with the environment (i.e., gravity gradient, solar pressure, magnetic field, etc.). Similarly, spacecraft attitude may be measured with star trackers, sun sensors, inertial gyros, or by using the earth's magnetic field. Spacecraft angular velocity may be measured directly, or it may be computed from attitude data. Finally, the feedback linking the outputs of attitude and angular velocity sensors with the inputs to the torquers may be designed using Euler angles, some property of sensors (see refs. 1 and 2), or Euler's theorem on three-dimensional rotations (see refs. 3 and 4). The design may be optimal in some sense, or it may simply be intuitively appealing. However, despite this diversity, every attitude control system is built for the single purpose of allowing the spacecraft to fulfill its mission. Consequently, there is always a stage in the design of such systems at which one must judge the quality of the proposed system relative to the mission of the spacecraft. Since all possible situations must be considered, quality must be judged on the basis of the overall (global) properties of the system. This poses no difficulty if the mission of the spacecraft presents its control system with only a small number of situations to control because then global behavior may be obtained by directly testing the system or its analytical model. If, however, the number of situations to control is so large that direct enumeration is impractical, or, indeed, impossible, tests consisting of a small sample may not be decisive: there is no certainty that every case which results in mission failure is included in the test sample.

The present note is concerned with the responsiveness of three-axis, large-angle attitude control systems of highly maneuverable spacecraft capable of tracking time-varying target attitudes. A system of this type is inherently multivariable, multidimensional, nonlinear, and nonautonomous with infinite sets of forcing functions and initial conditions. Since, on the one hand, there are infinitely many cases involved, and, on the other hand, there are no typical cases (because of nonlinearity) from which to extrapolate to any other case, the responsiveness of such systems cannot be determined by direct enumeration of cases. Another method must be found. The purpose of the present report is to present such a method.

The method involves two ideas, namely, the error angle first introduced in reference 4, and the response envelope introduced in the present note. The error angle is a distance between three-dimensional rotations. The response envelope is a function of time which at each instant is the maximum of all possible values of the error angle at that instant. The response envelope indicates the responsiveness of the system in the following sense. Under no possible circumstances is the attitude error between spacecraft and target, at any instant, greater than the value of the response envelope at that instant.

The discussion proceeds from the general to the particular in three steps. First, the response envelope is defined precisely. Second, a procedure for computing a useful approximation of the response envelope is described. Third, a numerical example is presented as an illustration of the method. One of the control laws introduced in reference 4 is assumed for an Orbiting Astronomical Observatory, and the following questions are considered: (1) What is the responsiveness of the system to step changes in the target attitude, and how well does the system follow time-varying target attitudes? (2) How significant is gyroscopic coupling? How sensitive is the system to (3) external torque disturbances, (4) variations in system parameters, (5) changes in the form of the control law, and (6) time delays in the controller? Answers are obtained by the proposed procedure.

SYMBOLS

A_{as} output matrix; actual attitude of spacecraft relative to inertial space

A_{ds} input matrix; desired attitude of spacecraft relative to inertial space

$\hat{a}$ orthonormal triplet of vectors fixed to the spacecraft

c	unit eigenvector of R; error axis
$\hat{d}$	orthonormal triplet of vectors fixed to the target
$\frac{d}{dt}$	time derivative along a system trajectory
E	set of system state equations (i.e., $\dot{x} = f[x,u_1(t),u_2(t)]$)
$f[x,u_1(t),u_2(t)]$	right-hand side of system state equations
g_i	scalar functions appearing in equation (16)
h_{max}	angular momentum storage capacity of controlling device
I	unit matrix
J_a	spacecraft coordinates of moment of inertia of main body
j_{max}	maximum eigenvalue of J_a
j_{min}	minimum eigenvalue of J_a
ℓ_{max}	torque capacity of the controlling device
$\underline{M}$	set of all motions of the system
$n(t,x)$	perturbation function
R	error matrix defined by equation (2)
$\mathcal{R}^n$	set of all real n-tuples
$S(y)$	matrix function defined by equation (A2) in appendix A
$\hat{s}$	orthonormal triplet of vectors fixed in inertial space
t	time
tr( )	trace of matrix ( )
$\underline{U}_i$	set of admissible time variations of $\underline{u}_i$
$u_i(t)$	value of $\underline{u}_i$ at t
$V(t,x)$	solution of Hamilton-Jacoby equation
$V^+(t,x)$	solution of Liapunov inequality
$V_1(t,x)$	explicit part of $V^+(t,x)$

$\underline{W}_a$	set of admissible time histories of spacecraft angular velocity
w_a	body coordinates of inertial angular velocity of spacecraft
w_{amax}	spherical bound on w_a
w_d	target coordinates of inertial angular velocity of target
w_{dmax}	spherical bound on w_d
X	state space of the system
x	element of state space
$z(x)$	control law given by equation (16)
θ	region of operation of the system
μ	real constant
$\underline{\Sigma}_a$	set of admissible time histories of spacecraft acceleration
σ_a	angular acceleration of the spacecraft
σ_{amax}	spherical bound on σ_a
ϕ	error angle
$\phi(t,x,\underline{u}_1,\underline{u}_2)$	point on a time history of error angle
$\phi^{**}(t)$	value of the response envelope at t
$\phi^{-}(t)$	value of lower estimate (of the response envelope) at t
$\phi^{+}(t)$	value of upper estimate (of the response envelope) at t
$(\underline{\ })$	time history (thus, $\underline{x} = \{[x(t),t]: \ t \geq 0\}$)
$(\dot{\ })$	time derivative of ( )
$(\ )^t$	matrix transpose of ( )

DEFINITION OF THE RESPONSE ENVELOPE

Consider in general terms a complete attitude control system. It consists of essentially three distinct parts: (1) an input generator, (2) a spacecraft, and (3) a disturbance generator. The input generator represents all admissible time variations of attitude to be followed by the spacecraft. The spacecraft consists of the spacecraft inertia, torquers, sensors, and a controller which sends commands to the torquers based on information supplied by the sensors. The disturbance generator represents all admissible time variations of undesirable but unavoidable disturbances acting on the spacecraft such as perturbations in system parameters and external torque disturbances. Thus, an attitude control system as defined in the present report is a complete spacecraft and its environment. The mathematical model to be assumed for such a system consists of the following items: (1) an n-dimensional state space X; (2) a region of operation θ in X to which the motion of the system is restricted by physical limitations of system components; (3) two sets $\underline{U}_1$ and $\underline{U}_2$ of forcing functions, where for $i = 1, 2$, $\underline{U}_i$ is a set of piecewise continuous vector functions of time $\underline{u}_i$ with values $u_i(t)$ in $U_i(t)$; and (4) a set E of state equations describing the dynamics of input generator, spacecraft, and output generator:

$$E = \{(\dot{x}_i = f_i[x,u_1(t),u_2(t)], i): \; i = 1, \ldots, n\}$$

A motion of the system is a solution of E for some initial state x in θ, some input attitude generated by a forcing function $\underline{u}_1$ in $\underline{U}_1$, and some disturbance generated by a forcing function $\underline{u}_2$ in $\underline{U}_2$. Thus, the sets E and $\theta \times \underline{U}_1 \times \underline{U}_2$ (Cartesian product) define the set $\underline{M}$ of all possible motions of the system. The central topic of this report is the description of $\underline{M}$. Those aspects of $\underline{M}$ will be considered which reflect how well the spacecraft follows any admissible time variation of the desired attitude in the presence of any possible disturbance.

To decide at any instant of time how near the actual attitude of the spacecraft is to the desired attitude, it is necessary to have a notion of a distance between three-dimensional rotations. This may be introduced as follows. Consider three right-handed orthonormal triplets of vectors, say, $\hat{s}$, $\hat{a}$, and $\hat{d}$. The attitude of $\hat{a}$ relative to $\hat{s}$ is given by the 3×3 orthogonal matrix A_{as} whose elements are the direction cosines of $\hat{a}$ relative to $\hat{s}$. Similarly, the attitude of $\hat{d}$ relative to $\hat{s}$ is given by A_{ds}. The attitude of $\hat{a}$ relative to $\hat{d}$ is given by the matrix $A_{as}A_{ds}^t$, where t denotes matrix transpose. It follows (see ref. 5) from Euler's theorem on rotations that $A_{as}A_{ds}^t$ may be interpreted as a rotation about a single axis which is the eigenvector of $A_{as}A_{ds}^t$. The angle of this rotation (according to paragraph 3 in appendix A) is the following function of A_{as} and A_{ds}:

$$\phi(A_{ds},A_{as}) = \underset{[0,\pi]}{\text{arc cos}} \left\{\frac{1}{2}\left[\text{tr}\left(A_{as}A_{ds}^t\right) - 1\right]\right\} \tag{1}$$

It is shown in appendix A that this function is a metric on the space of three-dimensional rotations. Indeed, (i) $\phi(A_{ds},A_{as}) \geq 0$; (ii) $\phi(A_{ds},A_{as}) = 0$ if and only if $A_{as} = A_{ds}$; (iii) $\phi(A_{ds},A_{as}) = \phi(A_{as},A_{ds})$; (iv) if B is any three-dimensional rotation, then $\phi(A_{ds},B) + \phi(B,A_{as}) \geq \phi(A_{ds},A_{as})$. Consequently, $\phi(A_{ds},A_{as})$ may be interpreted as representing the distance between $\hat{a}$ and $\hat{d}$. In fact, the following interpretations will be made in the remainder of this report. In inertial space $\hat{s}$ is fixed. In the spacecraft $\hat{a}$ is fixed; A_{as} is the actual attitude of the spacecraft, and is the output of the attitude control system; A_{ds} defines the desired attitude, and is the input to the system. The attitude error is the 3×3 orthogonal matrix R defined by

$$R = A_{as}A_{ds}^{t} \qquad (2)$$

It may be noted that R locates $\hat{a}$ relative to $\hat{d}$. The magnitude of attitude error is the error angle $\phi = \phi(A_{ds},A_{as})$. The direction of attitude error is the axis c (unit eigenvector) of R computed from R using equation (A5) in appendix A.

It can be shown (see appendix A) that ϕ has the following intuitively appealing properties. (1) It is the shortest unrestricted angular distance between the actual and the desired attitude. (2) It is no smaller than the angle between the ith ($i = 1, 2, 3$) vector of the $\hat{a}$-triplet and the ith vector of the $\hat{d}$-triplet. (3) In the one-dimensional case (i.e., shaft-positioning servos) in which rotation about only a single axis is allowed, the usual definition of error is $\phi_e = \phi_d - \phi_a$, where ϕ_d and ϕ_a are the input and output angles, respectively. In that case $\phi = |\phi_e|$ if $|\phi_e| \leq \pi$, and $\phi = 2\pi - |\phi_e|$ if $\pi \leq |\phi_e| \leq 2\pi$. (4) When ϕ is small, the attitude error may be represented by the vector ϕc whose components are the Euler angles of R, and whose magnitude ϕ is the square root of the sum of the squares of these angles. For these reasons it appears that the error angle ϕ is both a mathematically convenient and intuitively appealing scalar representation of three-dimensional attitude error of a spacecraft at each instant of time.

The response envelope can now be defined in terms of time variation of error angle. With each admissible motion of the system it is possible to associate a time history $\underline{\phi}$ of the error angle. For a given admissible initial state, and forcing functions $\underline{u}_1$ and $\underline{u}_2$, $\underline{\phi}$ is a curve in the first quadrant of the t-ϕ plane. The ϕ-coordinate of $\underline{\phi}$ will be denoted by $\phi(t,x,\underline{u}_1,\underline{u}_2)$. Let $\underline{\Phi}$ be the set of all such curves generated by $\underline{M}$. The response envelope, to be denoted by $\underline{\phi}^{**}$, is defined to be the curve in the first quadrant of the t-ϕ plane such that every point $\phi^{**}(t)$ of $\underline{\phi}^{**}$ is the maximum of all values of $\underline{\phi}$ in $\underline{\Phi}$ at time t. For this definition to make sense formally, it will be assumed that for all $t \geq 0$, the set $\theta \times U_1(t) \times U_2(t)$ is compact and that $f[x,u_1(t),u_2(t)]$ is uniformly bounded on this set. This assumption is sufficient to guarantee the existence of the response envelope, and represents an insignificant physical restriction. The value of the response envelope may be computed for any $t \geq 0$ by the following formula.

$$\phi^{**}(t) = \max_{x \in \theta} \left[\max_{(\underline{u}_1,\underline{u}_2) \in \underline{U}_1 \times \underline{U}_2} \phi(t,x,\underline{u}_1,\underline{u}_2) \right] \qquad (3)$$

The following diagram summarizes the discussion of the present section. The input and disturbance generators generate admissible time variations of desired attitude $\underline{A}_{ds}$ and disturbance $\underline{d}$, respectively. Both act on the spacecraft to generate the rotation $\underline{A}_{as}$. For the complete system the following relations hold: the state space $X \subseteq X_1 \times X_2 \times X_3$, the region of operation $\theta \subseteq \theta_1 \times \theta_2 \times \theta_3$, and the state equations $E \subseteq E_1 \cup E_2 \cup E_3$. By

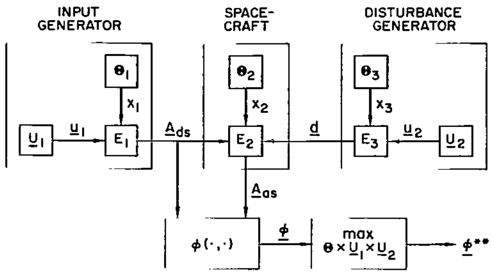

Figure 1.- Attitude control system and its response envelope.

means of equation (1), the distance between the desired and actual attitudes is computed to obtain the curve $\underline{\phi}$. The boundary of all such curves is the response envelope $\underline{\phi}^{**}$. Thus, the response envelope is a global property of an attitude control system that describes the responsiveness of the spacecraft. Regardless of where in its region of operation the system is initially and regardless of how the input attitude and disturbance vary in time within the prescribed limits, the attitude error will not at any time be greater than the value of the response envelope at that time.

APPROXIMATE COMPUTATION OF RESPONSE ENVELOPE

Consider, now, the problems involved in the computation of the response envelope whose points are defined by equation (3). The usual functional maximization techniques yield in general only local maxima. In the absence of the a priori knowledge of the number of these maxima, such methods will yield only lower estimates $\underline{\phi}^-$ of the response envelope $\underline{\phi}^{**}$, where $\phi^-(t) \leq \phi^{**}(t)$ for all $t \geq 0$. This is useful because an attitude control system may be judged unacceptable on the basis of such an estimate: there are conditions for which attitude error is at least as large as $\phi^-(t)$. In the present section a method is presented for computing an upper estimate $\underline{\phi}^+$ such that $\phi^{**}(t) \leq \phi^+(t)$ for all $t \geq 0$. This is useful because an attitude control system may be judged acceptable on the basis of such an estimate: under no circumstances can the attitude error be larger at time t than $\phi^+(t)$. The discussion proceeds in three steps. First, a general approximation procedure is set up using a state space interpretation of equation (3). Next, a prototype of arbitrary attitude control systems is formulated. Finally, the computation of an upper estimate of the prototype response envelope is stated explicitly.

State Space Interpretation of the Response Envelope and a General Procedure for Computing Its Upper Estimate

Let the multidimensional state space X and region of operation θ be represented schematically as shown in the following figure, and let one of the state coordinates be the error angle ϕ (transforming the state space if necessary). The motion of the system starting in a state x and forced by $\underline{u}_1$ and $\underline{u}_2$ may be represented by a trajectory in θ, whose ϕ-coordinate at each instant is the value $\phi(t,x,\underline{u}_1,\underline{u}_2)$ of the error angle at that instant. The same initial state but a different pair of forcing functions results in a different trajectory. Consider the bundle of such trajectories all emanating from the same initial state and generated by all admissible $\underline{u}_1$ and $\underline{u}_2$. This bundle defines a moving set of states which is reachable from the given state at a given time. The crosshatched regions in figure 2 show schematically such

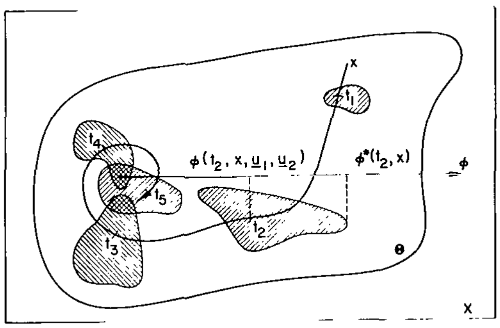

Figure 2.- Motion of the set reachable from x.

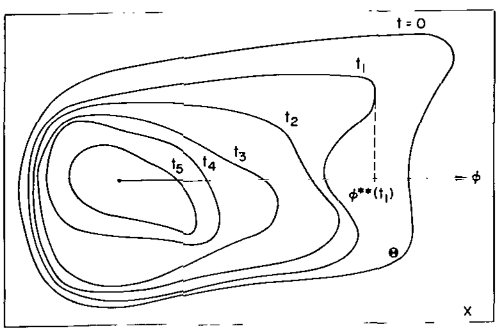

Figure 3.- Motion of the cloud of states.

a set at various times. The maximum projection $\phi^*(t,x)$ of this set on the ϕ-axis gives at each t the value of the bracketed term in equation (3). For example, in figure 2, $\phi^*(t_2,x)$ is the maximum projection of the moving set at $t = t_2$. Different initial conditions generate different moving sets. Consider at each instant the union of all such moving sets generated by all initial states in θ. This union defines a moving cloud of points shown schematically in figure 3. Initially the cloud completely fills the region of operation θ. As time progresses it changes shape in response to the forcing functions and the action of the spacecraft control. The maximum projection $\phi^{**}(t)$ of this cloud on the ϕ-axis at each t is the value of the response envelope defined by equation (3). In other words, the knowledge of the motion of the boundary of this cloud is sufficient for the computation of the response envelope.

The motion of this boundary may be described mathematically as follows. Let the boundary be defined by $V(t,x) = 0$, with $V(t,x) < 0$ inside the cloud and $V(t,x) > 0$ outside. Let $(dV/dt)[t,x,u_1(t),u_2(t)]$ denote the time rate of change of $V(t,x)$ along a trajectory of the system. According to the preceding discussion, the boundary of the cloud is characterized by two properties: (i) it is a part of the cloud, that is, at each point x on the boundary there is a pair $[u_1(t),u_2(t)]$ such that $(dV/dt)[t,x,u_1(t),u_2(t)] = 0$, and (ii) no trajectory can penetrate it outward, i.e., at each point x on the boundary and every pair $[u_1(t),u_2(t)]$ in $U_1(t) \times U_2(t)$, $(dV/dt)[t,x,u_1(t),u_2(t)] \leq 0$. Therefore, $V(t,x)$ satisfies the following equation on the boundary

$$\max_{U_1(t) \times U_2(t)} \frac{dV}{dt}[t,x,u_1(t),u_2(t)] = 0 \tag{4}$$

But, the time rate of change of $V(t,x)$ along any trajectory is $V_t + V_x\dot{x}$, and $\dot{x} = f[x,u_1(t),u_2(t)]$. Hence, the preceding equation may be expressed as follows

$$V_t + H(t,x,V_x) = 0 \tag{5}$$

where

$$H(t,x,V_x) = \max_{U_1(t) \times U_2(t)} V_x(t,x)f[x,u_1(t),u_2(t)] \tag{6}$$

It may be noted that equation (5) is formally a Hamilton-Jacobi equation. The boundary of the cloud is the solution of equation (5) such that $\{x: V(0,x) \leq 0\} = \theta$.

If a closed form solution of the Hamilton-Jacobi equation were generally available, the problem of computing the response envelope would be solved, because

$$\phi^{**}(t) = \max_{\{x:\ V(t,x) = 0\}} \phi \tag{7}$$

Unfortunately, no general solution to Hamilton-Jacobi equation is currently available. The best that can be done is to compute points on $V(t,x) = 0$ by means of the canonical equations associated with equation (5). As already noted this results in a lower estimate $\underline{\phi}^-$ of the response envelope. Another approach is, of course, to try to guess the form of solution of equation (5) on the basis of insight in much the same way as one guesses Liapunov functions. Since a guess will most likely not be a solution, it is necessary to decide what is a meaningful approximation in the present context. From the practical point of view, the next best thing to the exact response envelope is a pair $(\underline{\phi}^-,\underline{\phi}^+)$ of lower and upper estimates such that for each $t \geq 0$, $\underline{\phi^-(t)} \leq \phi^{**}(t) \leq \phi^+(t)$. Since $\underline{\phi}^-$ may be computed by means of the canonical equations, it seems reasonable to consider an approximate solution meaningful if, when substituted in equation (7), it yields an upper estimate $\underline{\phi}^+$. A function $V^+(t,x)$ will be such an approximate solution if the surface $\overline{V}^+(t,x) = 0$ encloses the cloud for all $t \geq 0$, that is, if

$$\{x: V(t,x) \leq 0\} \subseteq \{x: V^+(t,x) \leq 0\}$$

for all $t \geq 0$. However, the approximating surface need not be a part of the cloud. Hence, $V^+(t,x)$ is a solution of the following Liapunov inequality

$$V_t^+ + H(t,x,V_x^+) \leq 0 \tag{8a}$$

with the boundary condition

$$\{x: V^+(0,x) \leq 0\} \supseteq \theta \tag{8b}$$

Inequality equation (8a) must hold for all $t \geq 0$ and all x in θ such that $V^+(t,x) = 0$.

It is very easy to construct functions that solve equation (8). Simply let $V^+(t,x) = V_1(t,x) - V_2(t)$, where $V_1(t,x)$ is such that for some $a < \infty$,

$$\{x: V_1(0,x) \leq a\} \supseteq \theta \tag{9}$$

and where $V_2(t)$ is the solution of the following ordinary, first order, scalar differential equation with initial condition $V_2(0) = a$.

$$\dot{V}_2 = \max_{\{x \in \theta:\ V_1(t,x) = V_2(t)\}} [V_{1t} + H(t,x,V_{1x})] \tag{10}$$

Then $V^+(t,x)$ so defined solves equations (8) as can be tested by direct substitution. The corresponding upper estimate may be computed by means of the following natural modification of equation (7).

$$\phi^+(t) = \max_{\{x \in \theta:\ V_1(t,x) = V_2(t)\}} \phi \tag{11}$$

Thus, any function $V_1(t,x)$ that can satisfy the boundary condition (9) may always be used to compute an upper estimate. Of course, the fidelity with which $\underline{\phi}^+$ represents $\underline{\phi}^{**}$ depends directly on the choice of $V_1(t,x)$. A poor choice will result in an overly pessimistic estimate of system performance.

A Prototype of Arbitrary Attitude Control Systems

In order to construct a $V_1(t,x)$ for an arbitrary three-axis, large-angle attitude control system, $\mathcal{A}$, consider the model of such a system in somewhat greater detail than shown in figure 1. Associate with every possible motion of the system the following time histories: target attitude $\underline{A}_{ds}$, spacecraft attitude $\underline{A}_{as}$, target angular velocity $\underline{w}_d$, spacecraft angular velocity $\underline{w}_a$, and spacecraft angular acceleration $\underline{\sigma}_a$. In any practical situation, $w_d(t)$, $w_a(t)$, and $\sigma_a(t)$ will be uniformly bounded. That is, if $C^o(m)$ is defined as the set of piecewise continuous vector functions of time, which are spherically bounded by a constant m, there is a triplet $(w_{dmax}, w_{amax}, \sigma_{amax})$ of constants such that $\underline{w}_d \in \underline{C}^o(w_{dmax})$, $\underline{w}_a \in \underline{W}_a \subseteq \underline{C}^o(w_{amax})$, and $\underline{\sigma}_a \in \underline{\Sigma}_a \subseteq \underline{C}^o(\sigma_{amax})$. This means that any admissible time history $\underline{A}_{ds}$ of the desired attitude (target) is a solution of the following kinematic equation (see eq. (A1)).

$$\dot{A}_{ds} = S[w_d(t)]A_{ds} \tag{12}$$

for some orthogonal $A_{ds}(0)$ and some $\underline{w}_d \in \underline{C}^o(w_{dmax})$. Similarly, any possible time history $\underline{A}_{as}$ of spacecraft attitude is a solution of

$$\dot{A}_{as} = S[w_a(t)]A_{as} \tag{13}$$

for some orthogonal $A_{as}(0)$ and some $\underline{w}_a \in \underline{W}_a$. According to equation (A8), equation (2), and the above kinematic equations imply the following expression of the time rate of change of system attitude error R.

$$\dot{R} = S[w_a(t) - Rw_d(t)]R \tag{14}$$

This kinematic equation of attitude error will be taken as part of the set E of system state equations. Since R is always orthogonal, it could be represented using three independent coordinates. This, however, is unnecessary and inconvenient at this stage. Instead, all nine elements of R will be considered as independent with the stipulation that every $R(0)$ is orthogonal. The term $-Rw_d(t)$ in equation (14) will be interpreted as the value at time t of the forcing function $\underline{u_1}$ in figure 1. Consequently, $\underline{U_1} = \underline{C}^o(w_{dmax})$.

Because angular acceleration of the spacecraft is bounded, $\underline{W_a}$ has more structure than $\underline{C}^o(w_{amax})$. In fact, any $\underline{w_a}$ is a solution of the dynamic equation

$$\dot{w}_a = \sigma_a(t) \tag{15}$$

for some $w_a(0) \leq w_{amax}$ and some $\underline{\sigma_a}$ in the set $\underline{\Sigma_a}$ of admissible angular accelerations. Consider any such $\underline{\sigma_a}$. It is obviously possible to write $\underline{\sigma_a} = \underline{\sigma}^o + \delta\underline{\sigma}$, where $\underline{\sigma}^o$ is a specific function and $\delta\underline{\sigma} = \underline{\sigma_a} - \underline{\sigma}^o$. The interpretation to be assigned to $\underline{\sigma}^o$ is that it represents that part of the spacecraft acceleration which helps to reduce attitude error. The difference $\delta\underline{\sigma}$ will be interpreted as a disturbance. According to the previous discussion the magnitude of attitude error R may be represented by the error angle ϕ, and its direction may be represented by the unit eigenvector c of R. Hence, any part of angular acceleration $\underline{\sigma_a}$ that is antiparallel to c may be considered helpful. Another helpful part of $\underline{\sigma_a}$ is one which is antiparallel to the angular velocity w_a of the spacecraft because it provides damping. For these reasons σ^o will be defined as follows. For every possible motion of the system $\underline{\sigma}^o = \{(t, z[R(t), w_a(t)]): \ t \geq 0\}$, where

$$z(R, w_a) = -g_1(\phi)g_2(\|w_a\|)c - g_3(\phi)w_a \tag{16}$$

and where g_1, g_2, and g_3 are scalar functions. The perturbation $\delta\sigma(t)$ will be represented at every $t \geq 0$ by $n[t, R(t), w_a(t)]u_2(t)$, where $\underline{u_2} \in \underline{C}^o(1)$, and where the function n is chosen so that for every possible motion of the system and every $t \geq 0$

$$\delta\sigma(t) = \sigma_a(t) - z[R(t), w_a(t)] \in \{n[t, R(t), w_a(t)]u_2(t): \ \|u_2(t)\| \leq 1\} \tag{17}$$

The perturbation function n may be a matrix, or simply a scalar. With such a representation of spacecraft acceleration $\underline{\sigma_a}$, the dynamic equation (15) becomes the following:

$$\dot{w}_a = z[R(t), w_a] + n[t, R(t), w_a]u_2(t) \tag{18}$$

This equation will be taken as the remaining part of the set E of system state equations, and w_a will be the remaining part of the state vector x. The coordinates of w_a are restricted by the condition $\|w_a\| \leq w_{amax}$. The forcing function $\underline{u_2}$ will be identified with the forcing function in the disturbance generator as shown in figure 1. Hence $\underline{U_2} = \underline{C}^o(1)$.

The preceding discussion motivates the prototype of attitude control systems defined in the following table.

TABLE I.- PROTOTYPE $\mathcal{M}^o$ OF ARBITRARY ATTITUDE CONTROL SYSTEMS

$$X = \mathcal{R}^{12}, \quad x = (R, w_a)$$

$$\theta = \{x: \quad RR^t = I, \ \|w_a\| \leq w_{amax}\} \subset \mathcal{R}^6$$

$$E = \left\{ \begin{aligned} \dot{R} &= S[w_a + u_1(t)]R \\ \dot{w}_a &= z(x) + n(t,x)u_2(t) \end{aligned} \right\}$$

$$\underline{U_1} = \underline{C}^o(w_{dmax}), \ \underline{U_2} = \underline{C}^o(1)$$

The underlying state space X is 12-dimensional; the state variable is denoted mnemonically by $x = (R,w_a)$. The region of operation θ is defined by the orthogonality condition on the error matrix R and by the spherical bound on the angular velocity w_a of the spacecraft. Thus, θ is 6-dimensional: three for R and three for w_a. Clearly, θ is compact. The set E of state equations consists of equations (14) and (18). The control law $z(x)$ is defined by equation (16). It and the perturbation function $n(t,x)$ are bounded on θ. The two forcing functions $\underline{u_1}$ and $\underline{u_2}$ belong to the sets of piecewise continuous vector functions of time which are spherically bounded by w_{dmax} and 1, respectively. The right-hand side of the state equations is uniformly bounded on its compact domain.

In the following very specific sense $\mathcal{M}^o$ is a prototype: it generates all time histories $\underline{\phi}$ of the error angle which are possible for any attitude control system $\mathcal{A}$ having the bounds $(w_{dmax}, w_{amax}, \sigma_{amax})$ and the set $\underline{\Sigma_a}$ of admissible accelerations for which inequality (17) holds. This inequality can be satisfied for any $\mathcal{A}$, however complex, by a proper choice of $n(t,x)$. Consequently, the response envelope of $\mathcal{M}^o[w_{dmax}, w_{amax}, z(x), n(t,x)]$ is necessarily an upper estimate of the response envelope of $\mathcal{A} = (w_{dmax}, w_{amax}, \sigma_{amax}, \underline{\Sigma_a})$ for which (17) holds. If $n(t,x)$ is large, the estimate will be pessimistic.

Explicit Procedure for Computing an Upper Estimate for the Prototype

A very simple $V_1(t,x)$ is now proposed for the computation of an upper estimate of the response envelope of $\mathcal{M}^o$ by means of equations (9), (10), and (11). It is the following.

$$V_1(t,x) = \int_o^{\phi} \left[g_1(y) + \mu g_3(y)\sin\left(\frac{1}{2}\, y\right)\right] dy + \frac{1}{2}\int_o^{\|w_a\|^2} \frac{dy}{g_2(y)} + \mu \sin\left(\frac{1}{2}\,\phi\right)c^t w_a \tag{19}$$

The functions g_1, g_2, and g_3 appear in equation (16). The scalar μ is an adjustable constant. There is no explicit dependence on time. Hence, $V_{1t} \equiv 0$. The Hamiltonian $H(t,x,V_{1x})$ is given by equation (B2). A $V_1(t,x)$ of the form of equation (19) may be thought of as a natural extension to three dimensions of the Liapunov function useful for the analysis of one-dimensional servos. Thus, the first integral on the right of equation (19) depends only on the magnitude of attitude error. The second integral depends only on the magnitude of angular velocity. The last term represents a coupling which depends upon these magnitudes as well as on the angle between the error axis and the velocity vector.

The proposed procedure for computing an upper estimate of the response envelope of the prototype $\mathcal{M}^o$ is now explicit. It is summarized in the following table.

TABLE II.- COMPUTATION OF AN UPPER ESTIMATE

Step 1.	Obtain V_1 using (19)
Step 2.	Choose $V_2(0)$ to satisfy (9)
Step 3.	Solve (10) for $V_2(t)$
Step 4.	Compute $\underline{\phi}^+$ using (11)

The Hamiltonian $H(t,x,V_{1x})$ needed in step 3 is derived in appendix B.

Several points regarding this procedure are worth noting. Since V_1 defined by equation (19) depends only on the triplet $[\phi, \|w_a\|, \sin(1/2\ \phi)c^t w_a]$ of scalars, the maximization in step 4 may be performed on a two-dimensional surface in a three-dimensional space whose points are related to the states x in θ by the function $p(x) = [\phi, \|w_a\|, \sin(1/2\ \phi)c^t w_a]$.

The computation in step 3, in general, requires a maximization over a five-dimensional surface in θ. If, however, the perturbation function $n(t,x)$ depends on x only through the triplet $p(x)$, as above, then this maximization can be performed on the same two-dimensional surface as in step 4.

Steps 3 and 4 can be performed simultaneously on a digital computer. Experience has shown that the computation of an upper estimate in this manner takes about one second of computer time.

Finally, it will be noted that the perturbation enters the computation only by way of equation (B1) needed in step 3. Consequently, when $z(x)$ and $n(t,x)$ are selected for any attitude control system $\mathcal{A}$, only that component of acceleration $\underline{\sigma}_a$ need be considered in equation (17) which is in the plane of w_a and c.

EXAMPLE

The following numerical example illustrates the computation and use of upper estimates for the global description of attitude control systems. An Orbiting Astronomical Observatory (OAO) is considered. Its parameters pertinent to the discussion are summarized in appendix D. The spacecraft is controlled by means of three identical orthogonally placed motor-reaction-wheel combinations. The torque capacity of each motor is ℓ_{max}, and the angular momentum storage capacity of each wheel is h_{max}. The passive moment of inertia (moment of inertia of the spacecraft with locked wheels minus the moment of inertia of each wheel about its spin axis) is the matrix J_a. The maximum eigenvalue of J_a is j_{max}. A momentum dumping scheme maintains the total angular momentum of the system bounded by h_{smax}. The acceleration is bounded by $\sigma_{amax} = \ell_{max}/j_{max}$, and the velocity is bounded by $w_{amax} = (h_{max} - h_{smax})/j_{max}$. The torque commanded by the controller on the basis of sensor data is $J_a z(x)$, where the control law $z(x)$ is assumed to be:

$$z(x) = \frac{1}{2}\,\sigma_{amax}\left[\mathrm{sat}(\phi,\phi_s)c + \frac{1}{w_{amax}}\,w_a\right] \tag{20}$$

The function $\mathrm{sat}(\phi,\phi_s) = \phi/\phi_s$ for error angle $\phi \leq \phi_s$, and it saturates at the value 1 for $\phi \geq \phi_s$. The saturation angle ϕ_s is a dimensionless combination of the dynamic capacities of the spacecraft:

$$\phi_s = \frac{2h_{max}^2}{j_{max}\ell_{max}}$$

For the present example of the OAO, $\phi_s = 0.1$ radian. The complete model of the system is derived in appendix C. The system has three axes and is 12-dimensional. Three independent coordinates are needed for the spacecraft attitude, three for the desired attitude, three for the spacecraft angular velocity, and three for the total angular momentum of the system (spacecraft and wheels).

The stability of this system has been investigated in reference 4. There the system was shown to be asymptotically stable everywhere on its region of operation. The plots in figures 2, 3, and 4 of reference 4 show a response of the system to a particular initial condition. The questions being considered now are: (1) What is the responsiveness of the system to step changes in the desired attitude, and how well does the system follow a desired time varying attitude? (2) How significant is the gyroscopic coupling? How sensitive is the system to (3) external torque disturbances, (4) variations in system parameters, (5) changes in the form of the control law, and (6) time delays in the controller? These questions are resolved by means of upper estimates computed on a digital computer using the procedure outlined in table II for various appropriate instances of the perturbation function $n(t,x)$.

Let the system be normalized as follows: time, $t \rightarrow t/w_{amax}$, angular velocity $w \rightarrow w \cdot w_{amax}$. Then, the control law (20) becomes the following.

$$z(x) = -\frac{1}{\phi_s} \text{sat}(\phi,\phi_s)c - \frac{1}{\phi_s} w_a \tag{21}$$

This control law is a special case of (16): setting $g_1 = \text{sat}(\phi,\phi_s)$, $g_2 = 1/\phi_s$, and $g_3 = 1/\phi_s$ in equation (16) one obtains equation (21), above. With these identifications and setting the adjustable parameter $\mu = 1$, the V_1 function defined by equation (19) assumes the following completely explicit form.

$$V_1(t,x) = \int_0^{\phi} \text{sat}(y,\phi_s)dy + \frac{2}{\phi_s}\left(1 - \cos\frac{\phi}{2}\right) + \frac{1}{2}\phi_s \|w_a\|^2 + \sin\frac{\phi}{2} c^t w_a$$

The results of the computation are summarized below in the same order as the questions posed above.

(1) Responsiveness of the nominal system. The assumptions for this case are: the initial attitude of the spacecraft is arbitrary; the initial angular velocity of the spacecraft is arbitrary except that it is spherically bounded by w_{amax}; the initial step change in the desired attitude is arbitrary; the desired attitude varies in time arbitrarily except that its angular velocity is spherically bounded by w_{dmax}; there are no disturbances, i.e., $n(t,x) \equiv 0$. The result is the family of curves in figure 4. The family parameter b is the ratio w_{dmax}/w_{amax} of the two velocity bounds. The curve $b = 0$ indicates the responsiveness of the nominal system to step changes in the desired attitude. Thus, for any possible initial condition, and regardless of the size and direction of the step change of the desired attitude, the attitude error will not be greater at any time than indicated by this curve. In fact, the system is not only globally asymptotically stable, but also it is essentially ($\phi \leq 0.01$) on target no later than 5 units of time. Curves with $b > 0$ indicate how well the nominal system follows the desired attitude when that is varying in time. Thus, the curve $b = 0.2$, for example, shows that for any possible initial condition, and regardless of how the desired attitude varies in time, so long as its angular velocity remains spherically bounded by $0.2\ w_{amax}$, the attitude error will not be greater than that indicated by the curve. It is emphasized that a

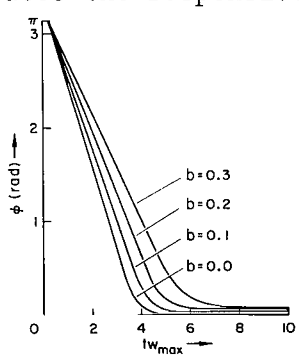

Figure 4.- Nominal system.

curve in figure 4 is not a response of the system. Rather, it is a description of all (there are infinitely many) responses of the multivariable, multidimensional, nonlinear, and nonautonomous system with infinite sets of initial conditions and forcing functions.

(2) Significance of the gyroscopic coupling. It is assumed that initial spacecraft attitude is arbitrary and its angular velocity is spherically bounded by w_{amax}. The desired attitude is stationary after an arbitrary step change. The system is preloaded with a total angular momentum which is spherically bounded by h_{smax} but is otherwise arbitrary. There are no net external torques. The appropriate perturbation term is given by equation (C6). The result is the family of curves in figure 5. The family parameter b is the ratio h_{smax}/h_{max} of the bound on the total angular momentum to the momentum capacity of the reaction wheels. The curve $b = 0$ corresponds to the case of no gyroscopic coupling. The curve $b = 0.3$ indicates that for the case of the OAO considered, gyroscopic coupling is not very significant even when the system is loaded with as much as 30 percent of its angular momentum storage capacity.

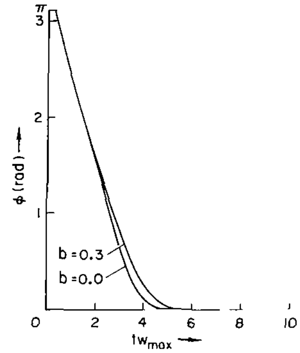

Figure 5.- Gyroscopic effects.

(3) The effects of external torque disturbances. The assumptions in this case are as in case (2) except that net external torque is allowed to be nonzero. The only restrictions are that this torque is spherically bounded by ℓ_{smax}, and that it does not overpower the momentum dumping system which maintains the total angular momentum bounded by h_{smax}. The appropriate perturbation term is given by equation (C7). The result is the pair of curves in figure 6. The lower curve is for $h_{smax} = \ell_{smax} = 0$. The upper curve is for $h_{smax} = 0.3\ h_{max}$ and $\ell_{smax} = 0.1\ \ell_{max}$. Thus, for any possible initial condition, attitude error will be less than about 10° after 6 units of time despite external torques which may be as large as 10 percent of the torque capacity of the controller.

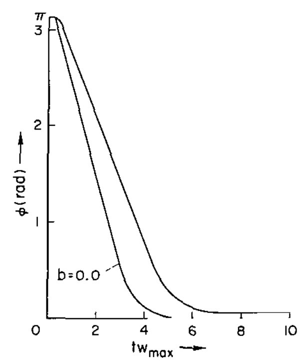

Figure 6.- External torque disturbances.

(4) Sensitivity to variations in system parameters. Until now the spacecraft was assumed to be controlled exactly by the control law (20). That is, it was assumed that the motor-reaction-wheel combinations were placed exactly on the orthogonal body axes, that the error angle ϕ, the error axis c, and the angular velocity w_a of the spacecraft were computed from sensor data without error, and that the matrix J_a was known exactly. Obviously, in practice, error will be present because motor-reaction-wheels will be slightly off axis, sensor characteristics will vary, signal and power amplifiers will drift, the computer will have roundoff errors, and the moment of inertia of the spacecraft will vary because of shifting mass caused by thermal distortion and movement onboard of instruments and personnel. The purpose of the present subsection is to investigate the effects of such errors.

Let the perturbation in the active parameters be denoted by the column δp of appropriate dimension, and let the set of all possible δp be P

which may be assumed to be infinite. The actual torque generated by the controller is a function, say $J_a^o z(x,\delta p)$, of the state x and perturbation δp. Then the effects of perturbation on the angular velocity of the spacecraft may be described by the function $m(x,\delta p) = J_a^{-1} J_a^o [z(x,0) - z(x,\delta p)]$, where J_a is exact moment of inertia of the system. For a fixed state x, let the set $M(x) = [m(x,\delta p)$ such that $\delta p \in P]$ and let the set $N(x) = [n(t,x)u_2(t)$ such that $\|u_2(t)\| \leq 1]$ where $n(t,x)$ is chosen so that for $t \geq 0$ $M(x) \subseteq N(x)$ for all x in θ, the region of operation of the system. Clearly, regardless of how δp varies in time, its effect on the angular acceleration may be accounted for by the perturbation term $n(t,x)u_2(t)$, where $\underline{u_2} \in \underline{C}^o (1)$. The significance of the perturbation in system parameters may now be estimated with an upper estimate of the response envelope. Two examples of perturbations are given next.

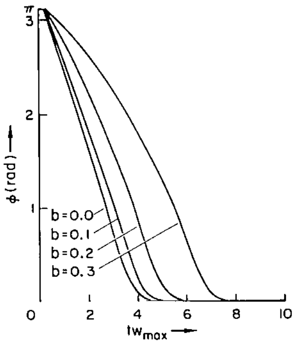

Figure 7.- Spherical error in acceleration.

(i) Spherical perturbation in angular acceleration. $n(t,x) = b\|z(x,0)\|$. The constant b is the intensity of perturbation, and $z(x,0)$ is the nominal control law $z(x)$ given by equation (20). The curves in figure 7 are upper estimates for various intensities b. The plots indicate that spherical errors in acceleration of the order of 10 percent affect the performance of the system little. This means, for example, that 10 percent drift in moment of inertia, motor and power amplifier gains, or a misalinement of the motor-reaction-wheels of about 3° is not very detrimental to the performance of the system. Even when such errors are large enough to cause 30 percent errors in acceleration, the system remains globally asymptotically stable, and it is essentially on target after about 8 units of time for any possible initial condition.

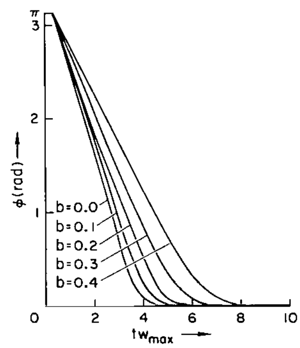

Figure 8.- Spherical error in error axis.

(ii) Spherical error in the error axis. Suppose that the difference between the true error axis c and its computed approximation c^o is spherically bounded by b, that is, suppose that $\|c - c^o\| \leq b$. Then the corresponding $n = b\,\mathrm{sat}(\phi,\phi_s)/\phi_s$. Figure 8 shows the computed upper estimates for several intensities b. The plots indicate that the system is not very sensitive to errors in the error axis. In fact, even when these errors are as large as 40 percent, the system remains asymptotically stable everywhere on its region of operation, and it is essentially on target for any possible initial condition after about 8 units of time.

(5) Sensitivity to changes in form of control law. In the preceding subsection perturbations in angular acceleration were assumed unavoidable and largely undesirable. Another point of view is possible. Suppose that all system parameters are essentially fixed, but that it is desirable for reasons of simplicity of implementation to use a control law which is different from that given by equation (20). In such a case an upper estimate to the response envelope of the modified system may be computed as in the preceding subsection if the given control law is considered a sum of equation (20) and a perturbation. The next two examples illustrate this point.

(i) A processor of signals from attitude sensors. Let the attitude of the spacecraft be measured with some arbitrary combination of star trackers, inertial gyros, sun sensors, etc. Let the outputs of all sensors be arranged in a single column matrix y of an appropriate dimension, and let the complete package of sensors be described by $y = g(A_{as}, y^o)$ where A_{as} is the attitude of the spacecraft and y^o is the set of inertial coordinates of the sensor targets (i.e., guide stars, inertial directions, sun, etc.). Let y, A_{ds}, and y^o be combined in a processor whose output is some given function $f(y, A_{ds}, y^o)$. Finally, suppose that this output is used instead of $\mathrm{sat}(\phi, \phi_s)c$ in equation (21); that is, suppose that the new control law is the following modification of equation (21).

$$z^*(x) = -\frac{1}{\phi_s}\left[f(y, A_{ds}, y^o) + w_a\right]$$

Then the perturbation in $z(x)$ is $z(x) - z^*(x) = [f - \mathrm{sat}(\phi, \phi_s)c]/\phi_s$. Consequently, the upper estimates given in figure 8 describe this modified system for constant A_{ds} and y^o, and, of course, $\phi_s = 0.1$. The intensity

$$b = \max_{\{R:\ RR^t = I\}} \left\| \frac{f[g(RA_{ds}, y^o), A_{ds}, y^o] - \mathrm{sat}(\phi, \phi_s)c}{\mathrm{sat}(\phi, \phi_s)} \right\|$$

Thus, one may conclude, for example, that if reaching the target in 8 units of time is sufficiently fast for the mission of the spacecraft, any combination of attitude sensors and processor with $b \leq 0.4$ may be considered acceptable. Two possible reasons one might be interested in a case with $b > 0$ are (1) simplification of the control law, and (2) determination of effects of failure of part of the output of the sensor package.

(ii) Control with control moment gyros. Suppose that the spacecraft is to be controlled not with reaction wheels but, rather, with a set of control moment gyros. Let the active gimbal angles of all the gyros in the package be arranged in a column y of an appropriate dimension, and let the spacecraft coordinates of the total angular momentum of all gyros be denoted by h_a^c. Assuming that the total angular momentum of each gyro may be adequately approximated by its spin momentum, we may express h_a^c as a function of y. Let $h_a^c = h(y)$, where $h(y)$ is one-to-one on Y, and $h(Y)$ is a solid sphere with radius h_{max}. Then the control torque is $-h_y\dot{y}$. Let the gyro gimbals be driven through a processor so that $\dot{y} = -F(y)J_a z(x)$; where $F(y)$ describes the processor, J_a is the moment of inertia of the spacecraft, and $z(x)$ is the control law given by equation (21). Then the new control law is given by $z^*(x,y) = J_a^{-1}h_y F(y) J_a z(x)$, and the perturbation $z(x) - z^*(x,y) = [I - J_a^{-1}h_y F(y) J_a]z(x)$. Consequently, the upper estimates given in figure 7 describe the spacecraft controlled with a set of control moment gyros when $\phi_s = 0.1$. The intensity is

$$b = \max_{Y} \| I - J_a^{-1}h_y F(y) J_a \|$$

Two possible reasons one may be interested in a case with $b > 0$ are (1) simplification of the control law, and (2) determination of effects of failure of a number of gyros in the package.

(6) Sensitivity to time delays. It is assumed that the angular velocity of the spacecraft is not measured directly but, rather, that it is computed from attitude data supplied by attitude sensors, and that this computation yields the exact angular velocity delayed by Δ units of time. The resulting control law is the following modification of equation (21).

$$z^*(x,\Delta) = -\frac{1}{\phi_s}\left[\mathrm{sat}(\phi,\phi_s)c + w_a(t - \Delta)\right]$$

The perturbation $z(x) - z^*(x,\Delta) = [w_a(t - \Delta) - w_a(t)]/\phi_s$. But,

$$w_a(t - \Delta) - w_a(t) = -\int_t^{t+\Delta} z^*[x(t),\Delta]dt$$

Consequently, the effects of small time delays are described by the curves in figure 7, where the intensity $b = \Delta/\phi_s = 10\ \Delta$. In particular, the curve $b = 0.1$ describes the OAO with a real time delay of 3.33 seconds. An estimate not restricted to small Δ may be computed using

$$n = \left(\frac{\Delta}{\phi_s}\right)\max_{\theta}\ z^*(x,\Delta)$$

CONCLUSION

The purpose of an attitude control system is to force the spacecraft to track a target attitude regardless of disturbances and initial conditions. Therefore, there is always the problem of describing how responsive the proposed system is to inputs, how unresponsive it is to disturbances, and how quickly it overcomes the initial conditions. The present note proposes the concept of response envelope as a solution to this problem. In addition, methods for computing lower and upper estimates of the response envelope are presented. These methods are useful because, although the response envelope is theoretically simple, it is difficult to compute in practice. The lower estimate is computed by means of the standard theory of functional maximization. The upper estimate is computed by means of a Liapunov inequality and a prototype of attitude control systems. A proposed attitude control system may be judged unacceptable on the basis of a lower estimate. Conversely, it may be judged acceptable on the basis of an upper estimate. The numerical example presented in the note shows that the response envelope and its estimates are useful for describing the responsiveness of complex systems whose behavior cannot be ascertained by a direct enumeration of cases.

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National Aeronautics and Space Administration
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APPENDIX A

SOME USEFUL PROPERTIES OF THREE-DIMENSIONAL ROTATIONS

(1) Let A be a 3×3 orthogonal matrix whose determinant is +1. It can be interpreted as a rotation of an orthonormal triplet $\hat{a}$ relative to an orthonormal triplet $\hat{s}$. Then, if x_s and x_a are the coordinates of an arbitrary vector $\overline{x}$ relative to the $\hat{s}$ and $\hat{a}$ triplet, respectively, $x_a = Ax_s$.

(2) Suppose that A is a function $A(\tau)$ of a real variable τ. Then it can be shown (see ref. 4) that there is a column matrix $u(\tau)$ such that

$$\frac{dA}{d\tau} = S[u(\tau)]A \tag{A1}$$

where for any 3×1 column matrix x, the skew symmetric matrix

$$S(x) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \tag{A2}$$

If τ is interpreted as time then $u(\tau)$ gives the a-coordinates of the angular velocity of $\hat{a}$ relative to $\hat{s}$.

(3) According to Euler's theorem on rotations, A may be interpreted as the result of a rotation from identity about a fixed axis. Hence, A is the solution of

$$\frac{dA}{d\tau} = S(c)A \ , \qquad A(0) = I$$

at some $\tau = \phi$, and some constant c such that $c^t c = 1$; ϕ will be referred to as the angle of A, and the eigenvector c as its axis. The solution is

$$A = e^{\phi S(c)} = I + \sin \phi S(c) + (1 - \cos \phi) S^2(c) \tag{A3}$$

The angle ϕ and the axis c may be computed from the elements of A by the following formulas:

$$\phi(A) = \underset{[0,\pi]}{\text{arc cos}} \left\{ \frac{1}{2} \left[\text{tr}(A) - 1 \right] \right\} \tag{A4}$$

$$c(A) = \frac{1}{2} \operatorname{cosec} \phi \begin{pmatrix} a_{23} & -a_{32} \\ a_{31} & -a_{13} \\ a_{12} & -a_{21} \end{pmatrix} \tag{A5}$$

It may be noted that $\phi(A^t) = \phi(A)$, $c(A^t) = -c(A)$, and, of course, $Ac = c$.

(4) Let θ_i be the angle between the ith vector of $\hat{a}$ and the ith vector of $\hat{s}$. Then $\theta_i \leq \phi(A)$. Indeed,

$$\cos \theta_i = \cos \phi(A) + [1 - \cos \phi(A)] c_i^2(A)$$

Hence, $\cos \theta_i \geq \cos \phi(A)$.

(5) Consider all paths from I to A. Each satisfies the differential equation

$$\frac{dA}{d\tau} = S[u(\tau)]A$$

with $A(0) = I$, $A(\tau_1) = A$, and some function u. Then

$$\phi(A) \leq \int_0^{\tau_1} \| u(\tau) \| d\tau$$

so that $\phi(A)$ may be considered to be the minimum angular distance between the $\hat{a}$ and the $\hat{s}$ triplet. This may be shown as follows. The Hamiltonian for the minimization problem is

$$H = \operatorname{tr}\{P^t S[u(\tau)]A\} + p_o \| u(\tau) \|$$

and

$$\dot{p}_o = 0$$

$$\dot{P} = S[u(\tau)]P$$

Thus, A and P have the same transition matrix $\Phi(\tau)$. Hence,

$$H = 2u^t(\tau)\Phi(\tau)k + p_o \| u(\tau) \|$$

where k is constant. The direction of $u(\tau)$ which minimizes H is

$$\frac{u(\tau)}{\|u(\tau)\|} = \frac{-2\Phi(\tau)k}{p_o}$$

But this means that the direction of $u(\tau)$ is fixed in the $\hat{s}$-triplet. Therefore, $u(\tau)$ is at each τ the eigenvector of $A(\tau)$, and the conclusion follows.

(6) For any three-dimensional rotations A and B,

$$\phi(AB^t) \leq \phi(A) + \phi(B) \tag{A6}$$

Suppose the contrary, and let $AB^t = C$ and $B = D^t$. Then it would be true that $\phi(C) > \phi(D) + \phi(CD^t)$. That is, the angle of the composite rotation: from I to D, followed by D to C is smaller than the angle of direct rotation from I to C. This, according to the preceding section, is impossible.

(7) Consider the set of all three-dimensional rotations. For any A and B in this set let the following function be defined

$$\phi(B,A) = \phi(AB^t) \tag{A7}$$

The function $\phi(B,A)$ so defined is a metric on the space of three-dimensional rotations. Indeed, (i) $\phi(B,A)$ is positive; (ii) $\phi(B,A) = 0$ if and only if $A = B$; (iii) $\phi(A,B) = \phi(B,A)$; (iv) $\phi(B,A) + \phi(A,C) \geq \phi(B,C)$. The triangle inequality follows from the preceding section:

$$\phi(B,C) = \phi(CB^t) = \phi(CA^tAB^t) = \phi[CA^t(BA^t)^t] \leq \phi(CA^t) + \phi(BA^t) = \phi(A,C) + \phi(B,A)$$

(8) Let $A = A(\tau)$ and $B = B(\tau)$, where τ is real. According to property (1) u and v are such that $\dot{A} = S(u)A$ and $\dot{B} = S(v)B$. Let $C = AB^t$. Then

$$\dot{C} = \dot{A}B^t + A\dot{B}^t = S(u)AB^t - AB^tS(v) = S(u)C - CS(v) = S(u)C - CS(v)C^tC = S(u)C - S(Cv)C$$

Hence

$$\frac{dC}{d\tau} = S(u - Cv)C \tag{A8}$$

(9) Let $A = e^{\phi S(c)}$, and $dA/d\tau = S(u)A$. Then it follows (e.g., appendix B of ref. 4) that

$$\frac{d\phi}{d\tau} = c^t u \tag{A9}$$

and

$$\frac{d}{d\tau}\left[\sin\left(\frac{1}{2}\,\phi\right)c\right] = \frac{1}{2}\,\sin\left(\frac{1}{2}\,\phi\right)S(u)c + \frac{1}{2}\,\cos\left(\frac{1}{2}\,\phi\right)u \tag{A10}$$

$\sin\left(\frac{1}{2}\,\phi\right)c$ is the vector part of the quaternion of A.

APPENDIX B

THE FORM OF THE HAMILTONIAN

The projection of state point velocity on the gradient V_{1x} is $V_{1x}\dot{x}$. The Hamiltonian appearing in equation (10) is according to equation (6) the maximum of $V_{1x}\dot{x}$ over $U_1(t) \times U_2(t)$. For V_1 defined by equation (19),

$$V_{1x}\dot{x} = \left[g_1(\phi) + \mu g_3(\phi)\sin\left(\frac{1}{2}\phi\right)\right]\frac{d\phi}{dt} + \frac{1}{2g_2(\|w_a\|)}\frac{d}{dt}\|w_a\|^2 + \mu\frac{d}{dt}\left[\sin\left(\frac{1}{2}\phi\right)c^t w_a\right]$$

where the derivatives are evaluated along the trajectory. They may be obtained as follows.

(i) According to equation (A9) and the kinematic equation in table I,

$$\frac{d\phi}{dt} = c^t[w_a + u_1(t)]$$

(ii) Since $\|w_a\|^2 = w_a^t w_a$,

$$\frac{d}{dt}\|w_a\|^2 = 2w_a^t\dot{w}_a$$

(iii) From equation (A10), it follows that

$$\frac{d}{dt}\left[\sin\left(\frac{1}{2}\phi\right)c^t w_a\right] = w_a^t\frac{d}{dt}\left[\sin\left(\frac{1}{2}\phi\right)c\right] + \sin\left(\frac{1}{2}\phi\right)c^t\dot{w}_a$$

$$= w_a^t\,\frac{1}{2}\left\{\sin\left(\frac{1}{2}\phi\right)S[w_a + u_1(t)]c + \cos\left(\frac{1}{2}\phi\right)[w_a + u_1(t)]\right\}$$

$$+ \sin\left(\frac{1}{2}\phi\right)c^t\dot{w}_a$$

But $\dot{w}_a = \sigma_a(t)$. Hence,

$$V_{1x}\dot{x} = \left[\mu \sin\left(\frac{1}{2}\phi\right)c + \frac{w_a}{g_2(\|w_a\|)}\right]^t \sigma_a(t) + \left[g_1(\phi) + \mu g_3(\phi)\sin\left(\frac{1}{2}\phi\right)\right] c^t w_a$$

$$+ \frac{1}{2}\mu \cos\left(\frac{1}{2}\phi\right) \|w_a\|^2 + u_1^t(t)\left[g_1(\phi)c + \mu g_3(\phi)\sin\left(\frac{1}{2}\phi\right)c\right.$$

$$\left. + \frac{1}{2}\mu \sin\left(\frac{1}{2}\phi\right)S(c)w_a\right]$$

If $\sigma_a(t) = z[R(t),w_a(t)] + n[t,R(t),w_a(t)]u_2(t)$, and $z(R,w_a)$ is given by equation (16),

$$V_{1x}\dot{x} = -\mu\, g_1(\phi)g_2(\|w_a\|)\sin\left(\frac{1}{2}\phi\right) - \left[\frac{g_3(\phi)}{g_2(\|w_a\|)} - \frac{1}{2}\mu \cos\left(\frac{1}{2}\phi\right)\right] \|w_a\|^2$$

$$+ u_1^t(t)\left[g_1(\phi)c + \mu\, g_3(\phi)\sin\left(\frac{1}{2}\phi\right)c + \frac{1}{2}\mu \cos\left(\frac{1}{2}\phi\right)w_a + \frac{1}{2}\mu \sin\left(\frac{1}{2}\phi\right)S(c)w_a\right]$$

$$+ u_2^t(t)n^t(t,R,w_a)\left[\frac{w_a}{g_2(\|w_a\|)} + \sin\left(\frac{1}{2}\phi\right)c\right] \tag{B1}$$

The Hamiltonian in equation (10) is

$$H(t,x,V_{1x}) = \max_{C^o(w_{dmax}) \times C^o(1)} V_{1x}\dot{x} \tag{B2}$$

where $V_{1x}\dot{x}$ is given by (B1), above.

For the particular control law (21) used in the example,

$$V_{1x}\dot{x} = -\frac{1}{\phi_s}\,\mathrm{sat}(\phi,\phi_s)\sin\left(\frac{1}{2}\phi\right) - \left[1 - \frac{1}{2}\cos\left(\frac{1}{2}\phi\right)\right] \|w_a\|^2$$

$$+ u_1^t(t)\left[\mathrm{sat}(\phi,\phi_s)c + \frac{1}{\phi_s}\sin\left(\frac{1}{2}\phi\right)c + \frac{1}{2}\cos\left(\frac{1}{2}\phi\right)w_a + \frac{1}{2}\sin\left(\frac{1}{2}\phi\right)S(c)w_a\right]$$

$$+ u_2^t(t)n^t(t,R,w_a)\left[\phi_s w_a + \sin\left(\frac{1}{2}\phi\right)c\right]$$

APPENDIX C

DETAILED MODEL OF SYSTEM USED IN THE EXAMPLE

The dynamic equation of a spacecraft controlled by means of an arbitrary angular momentum exchange and storage device (i.e., reaction wheels, control moment gyros) may be obtained as follows. The total angular momentum of the system is the sum of the angular momentum stored in the main body and that stored in the controlling device. Denote the inertial coordinates of the total angular momentum by h_s, the spacecraft coordinates of the portion stored in the device by h_a^c, and the spacecraft coordinates of the moment of inertia of the main body by J_a. Then,

$$A_{as}h_s = J_a w_a + h_a^c \tag{C1}$$

Taking the time derivative, using equation (13), and rearranging terms, one obtains the following equation.

$$\dot{w}_a = J_a^{-1}(-\dot{h}_a^c) + J_a^{-1}S(w_a)A_{as}h_s + J_a^{-1}A_{as}\dot{h}_s - J_a^{-1}\dot{J}_a w_a \tag{C2}$$

Thus, the angular acceleration of the spacecraft is the sum of four terms. The first will be interpreted as the control acceleration, and the control torque ℓ_a^c will be defined as follows.

$$\ell_a^c = -\dot{h}_a^c \tag{C3}$$

The second term will be interpreted as gyroscopic coupling. The third is caused by external torque. The body coordinates of the external torque will be denoted by ℓ_a^e; hence,

$$\ell_a^e = A_{as}\dot{h}_s \tag{C4}$$

The fourth term is present only when the moment of inertia of the main body varies in time.

Any practical angular momentum exchange and storage device is limited in both exchange rate and storage capacity. Thus, the torque of motors driving reaction wheels is limited as is the speed of the wheels. Similarly, the rates with which gyro gimbals can be driven are limited by available torque capacity, and the geometry defining the arrangement of gyros in the package imposes a limit on angular momentum storage. This fact will be accounted for by the following limits: $\|\ell_a^c\| \le \ell_{max}$ and $\|h_a^c\| \le h_{max}$. In addition, it will be assumed that angular momentum is dumped in such a way that $\|\ell_a^e\| \leq \ell_{emax}$ $\|h_s\| \leq h_{smax}$. Equation (C1) implies that $\|h_a^c\| \leq h_{max}$ if $\|w_a\| \leq w_{amax} = (h_{max} - h_{smax})/j_{max}$, where j_{max} is the maximum eigenvalue of J_a.

Equations (2), (12), (14), (C2), (C3), and (C4) may now be combined as follows:

TABLE III.- DETAILED MODEL OF THE SYSTEM

$$X = \mathcal{R}^{24}, \qquad x = (A_{ds}, R, w_a, h_s)$$

$$\theta = \{x: \ A_{ds}A_{ds}^t = I,\ RR^t = I,\ \|w_a\| \leq w_{amax},\ \|h_s\| \leq h_{smax}\} \subseteq \mathcal{R}^{12}$$

$$E = \left\{ \begin{aligned} \dot{A}_{ds} &= S[w_d(t)]A_{ds} \\ \dot{R} &= S[w_a - Rw_d(t)]R \\ \dot{w}_a &= J_a^{-1}(t)\ell_a^c(t) + J_a^{-1}(t)\left[S(w_a)RA_{ds}h_s + \ell_a^e(t) + \dot{J}_a(t)w_a\right] \\ \dot{h}_s &= A_{ds}^t R^t \ell_a^e(t) \end{aligned} \right\}$$

$\underline{w}_d \in \underline{W}^d \subseteq C^o(w_{dmax})$, $\underline{\ell}_a^c \in \underline{L}^c \subseteq C^o(\ell_{max})$, $\underline{\ell}_a^e \in \underline{L}^e \subseteq C^o(\ell_{emax})$, $\underline{J}_a \in \underline{J} \subseteq C^1$, and $\underline{L}^c \times \underline{L}^e \times \underline{J}$ is such that any $x(t) \in \theta$ for all $t \geq 0$.

The underlying state space of this model is 24-dimensional. However, the region of operation θ is 12-dimensional. Three dimensions are for target attitude A_{ds}, three for attitude error R, three for angular velocity of the spacecraft w_a, and three are for the total angular momentum of the system h_s. The motion of the system is given exactly by this model for any admissible initial condition and time variations of target velocity, control torque, external torque, and moment of inertia. It may be noted that in the absence of external torque, the total angular momentum is a constant of the motion, and the last state equation may be dropped. If, in addition, the total angular momentum is zero, then the gyroscopic term is absent.

In the example given in the main text, $J_a^{-1}(t)\ell_a^c(t)$ is assumed to be of the form $z(R,w_a)$ given by equation (16). Hence, the perturbation appearing on the right-hand side of the inequality (17) is the following.

$$\delta\sigma(t) = J_a^{-1}(t)\left[S(w_a)RA_{ds}h_s + \ell_a^e(t) + \dot{J}_a(t)w_a\right] \tag{C5}$$

Consider just the gyroscopic term. It is, clearly, bounded by $(h_{smax}/j_{max})\|w_a\|$. This bound could be used as the perturbation function n in figure 4. The perturbation $\delta\sigma(t) = n(t,x)u_2(t)$ enters in the computation of an upper estimate only by way of equation (B1). Therefore only the component of $\delta\sigma(t)$, which is in the plane of w_a and c, is significant. For this reason if the scalar n is replaced by the matrix N, defined as follows, a finer estimate will be obtained.

$$N(w_a) = \frac{h_{smax}}{j_{max}(1 + a^2)^{1/2}} \left[S(w_a) + a\|w_a\| I\right] \tag{C6}$$

where $a = (j_{max}/j_{min}) - 1$. This may be justified as follows. For any (A_{ds},R,w_a,h_s) in θ,

$$J_a^{-1}S(w_a)RA_{ds}h_s \in \{y: \quad y = h_{smax}J_a^{-1}S(w_a)\lambda \quad \text{and} \quad \|\lambda\| \leq 1\}$$

But that set is included in

$$\left\{ y: \quad y = \frac{h_{smax}}{j_{max}} \left[S(w_a)\lambda_1 + a\|w_a\|\lambda_2\right] \quad \text{and} \quad \|\lambda_1\| \leq 1 \quad \text{and} \quad \|\lambda_2\| \leq 1 \right\}$$

This set, in turn, is included in

$$\{y: \quad y = N(w_a)\lambda \quad \text{and} \quad \|\lambda\| \leq 1\}$$

Thus, the perturbation $N(w_a)u_2(t)$ generates all possible cases of the gyroscopic term.

The effects of external torque as well as the gyroscopic coupling can be represented by the following perturbation.

$$\delta\sigma(t) = N(w_a)u_2(t) + \frac{\ell_{emax}}{j_{min}} u_3(t) \tag{C7}$$

where both $\underline{u}_2$ and $\underline{u}_3$ belong to $C^o(1)$.

APPENDIX D

PARAMETERS OF THE SYSTEM USED IN THE EXAMPLE

maximum principal moment of inertia	$j_{max} = 1.4\times10^3$ kg-m^2
minimum principal moment of inertia	$j_{min} = 1.0\times10^3$ kg-m^2
angular momentum storage capacity	$h_{max} = 4.2$ N-m-sec
maximum torque capacity	$\ell_{max} = 0.25$ N-m

$$\phi_s = \frac{2h_{max}^2}{j_{max}\ell_{max}} = 0.1$$

$$a = \frac{j_{max} - j_{min}}{j_{min}} = 0.4$$

$$w_{max}^o = \frac{h_{max}}{j_{max}} = 3 \text{ milliradians/sec}$$

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