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Technical Report 32-1281

*Effect of Interference on a Binary Communication
Channel Using Known Signals*

M. A. Koerner

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

December 1, 1968

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Approved by:



R. P. Mathison, Manager
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TECHNICAL REPORT 32-1281

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Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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Abstract

This report presents the results of an analysis of the effects of sinusoidal and gaussian interference on the performance of the maximum-likelihood receiver for extracting binary data from a sequence of messages in white gaussian noise when each signal has duration T and is chosen with equal a priori probability from a dictionary of two messages. The report presents equations for the receiver error probability and the receiver degradation as a function of the parameters λ and η or ξ . Graphs are included which show the behavior of the receiver error probability and the receiver degradation as a function of $10 \log \lambda$ and $10 \log \eta$ or $10 \log \xi$. Equations are presented which relate the parameters λ , η , and ξ to the basic parameters of the signal, interference, and noise. Finally, a comparison is made of the effect of a sinusoidal interfering signal with that of a gaussian interfering signal. The comparison shows that for small values of λ and η or ξ , the degradation produced by sinusoidal interference is close to that produced by gaussian interference. However, the approximation is not valid for large λ or large η or ξ .

Effect of Interference on a Binary Communication Channel Using Known Signals

I. Introduction

Many communication systems are the aggregate of one or more communication channels multiplexed to operate over the same radio link. The receivers for these communication channels are usually designed to extract information from a signal observed in white gaussian noise. In such systems, interfering signals may seriously degrade the performance of these receivers. In some cases, the interfering signal may be generated within the communication system itself. The distortion signals generated in frequency-multiplexed, PM communication systems are of this type. In other cases, the interfering signal may be generated by a second communication system operating on an adjacent frequency band. The problem common to both cases is one of evaluating the effect of the interfering signal on the performance of a receiver.

This report examines the effect of sinusoidal or gaussian interfering signals on the probability of error for a maximum-likelihood receiver for extracting binary data from a sequence of messages in white gaussian noise when each signal has duration T and is chosen randomly, with equal a priori probability, from a dictionary of two messages. The report first derives equations for the form

of the receiver and the probability of error for the receiver when no interfering signal is present. The effect of sinusoidal and gaussian interference on the probability of error for the receiver is then evaluated.

II. Summary

A. Maximum Likelihood Receiver

Figure 1 shows a block diagram of the maximum-likelihood receiver for extracting binary data from a sequence of signals in white gaussian noise when each signal has duration T and is chosen randomly, with equal a priori probability, from a dictionary of two signals.

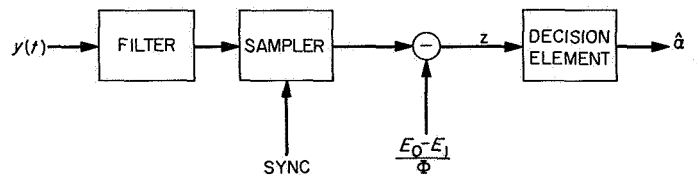


Fig. 1. Maximum-likelihood receiver functional block diagram

If $s(0; t)$ and $s(1; t)$ are the two signals which can be received and Φ is the one-sided power spectral density of the white gaussian noise, the filter F has impulse response.

$$h_F(\tau) = \begin{cases} \frac{2}{\Phi} [s(0; t) - s(1; t)], & 0 \leq \tau \leq T \\ 0, & \tau > T \end{cases} \quad (1)$$

At the end of each received signal, the output of the filter F is sampled and a bias of $(E_0 - E_1)/\Phi$ is removed.

$$E_\alpha = \int_0^T s^2(\alpha; t) dt, \quad \alpha = 0, 1 \quad (2)$$

is the received signal energy. A decision element determines whether the resulting statistic z is positive or negative and sets $\hat{\alpha}$, the maximum-likelihood receiver output, to zero or one. If $z > 0$, $\hat{\alpha} = 0$, and if $z < 0$, $\hat{\alpha} = 1$.

When no interfering signal is present, the bit error probability for this receiver is

$$P_E = p(\lambda) = \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2})] \quad (3)$$

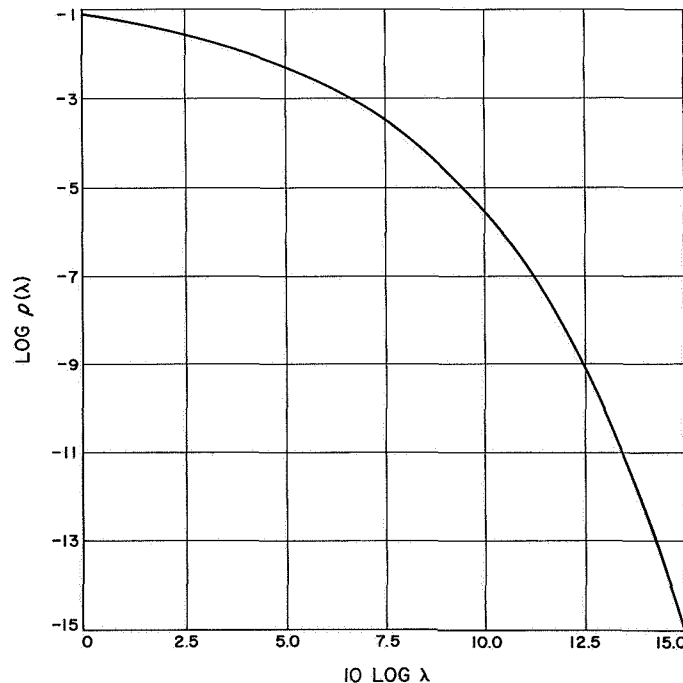


Fig. 2. $\text{Log } p(\lambda)$ as a function of $10 \log \lambda$

where

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (4)$$

and

$$\lambda = \frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{4\Phi} \quad (5)$$

The parameter ρ in Eq. (5) is the crosscorrelation between $s(0; t)$ and $s(1; t)$. In Fig. 2, $\log p(\lambda)$ is plotted as a function of $10 \log \lambda$.

B. Receiver Error Probability as a Function of the Interference-to-Signal Ratio

When either a sinusoidal or a gaussian interfering signal is present in addition to the white gaussian receiver noise, the receiver performance will be degraded. When a sinusoidal interfering signal having power P_i and angular frequency ω_i is present, the bit error probability for the receiver is

$$P_E = p_s(\lambda; \eta) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}\{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]\}] du \quad (6)$$

where, if $A_F(\omega)$ is the amplitude response of the filter F , the interference-to-signal ratio at the input to the decision element is

$$\eta = \frac{P_i A_F^2(\omega_i)}{(4\lambda)^2} \quad (7)$$

The function $\log p_s(\lambda; \eta)$ is plotted as a function of $10 \log \lambda$ for selected values of $10 \log \eta$ in Fig. 3 and as a function of $10 \log \eta$ for selected values of $10 \log \lambda$ in Fig. 4.

When a gaussian interfering signal having two-sided power spectral density $G_i(f)$ is present

$$P_E = p_G(\lambda; \eta) = \frac{1}{2} \{1 - \text{Erf}[\lambda^{1/2} (1 + 2\eta\lambda)^{-1/2}]\} \quad (8)$$

where

$$\eta = (4\lambda)^{-2} \int_{-\infty}^{\infty} G_i(f) A_F^2(\omega) df \quad (9)$$

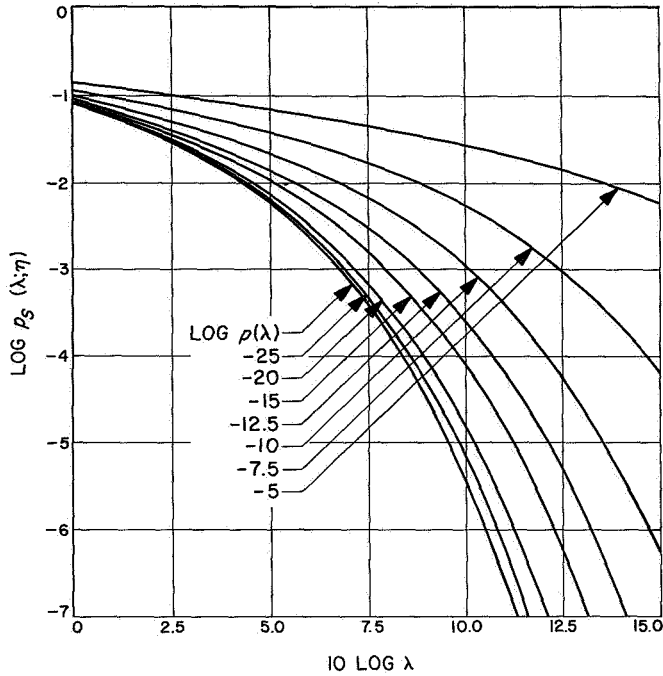


Fig. 3. $\text{Log } p_S(\lambda; \eta)$ as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \eta$

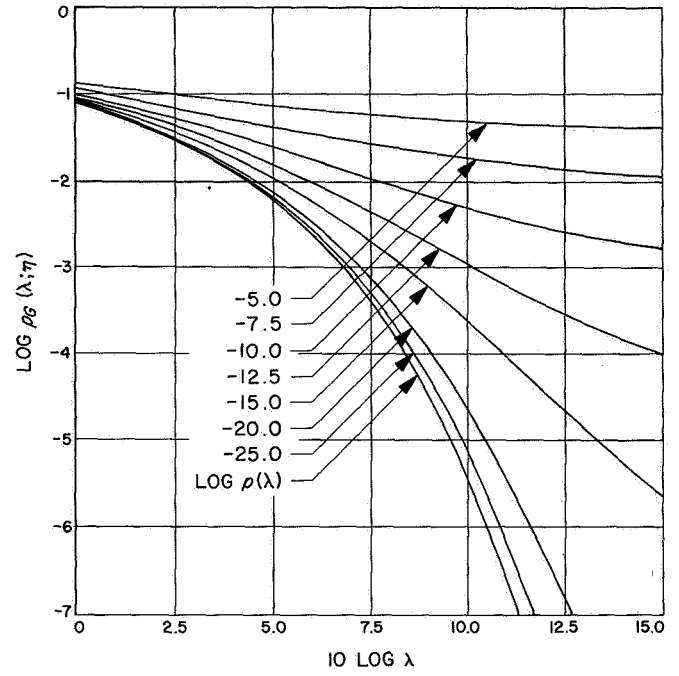


Fig. 5. $\text{Log } p_G(\lambda; \eta)$ as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \eta$

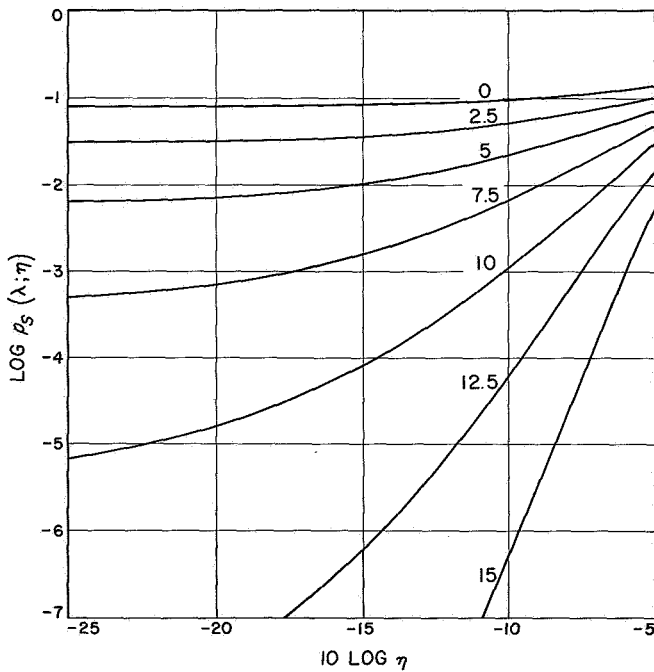


Fig. 4. $\text{Log } p_S(\lambda; \eta)$ as a function of $10 \text{ log } \eta$ for selected values of $10 \text{ log } \lambda$

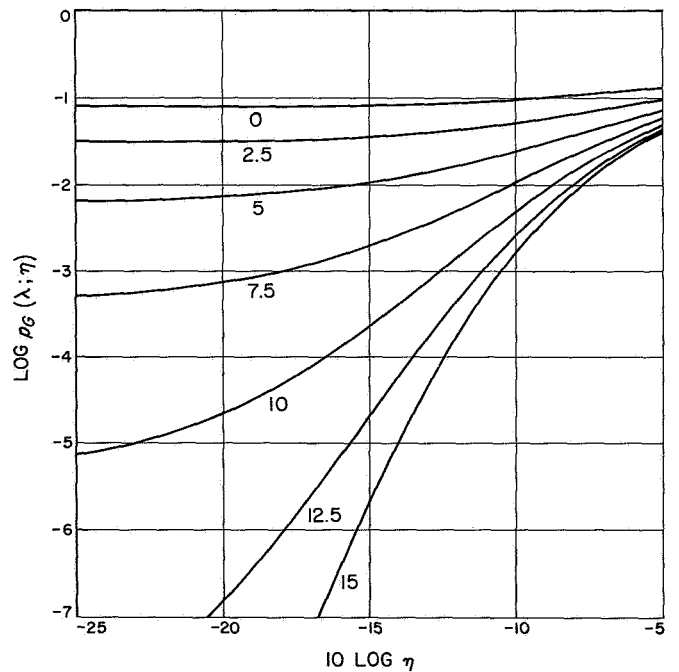


Fig. 6. $\text{Log } p_G(\lambda; \eta)$ as a function of $10 \text{ log } \eta$ for selected values of $10 \text{ log } \lambda$

The function $\text{log } p_G(\lambda; \eta)$ is plotted as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \eta$ in Fig. 5 and as a function of $10 \text{ log } \eta$ for selected values of $10 \text{ log } \lambda$ in Fig. 6.

C. Interference-To-Signal and Interference-To-Noise Ratios

In evaluating the effect of an interfering signal on the performance of this receiver, one finds that change in

receiver bit error probability is an inconvenient measure of the receiver degradation caused by the interfering signal. Hence, we shall introduce the parameter δ , the factor by which λ must be increased to make the receiver bit error probability, when an interfering signal is present, equal to what it would be were the interfering signal absent. In most cases, δ will be a more convenient measure of receiver degradation than the change in receiver bit error probability.

Using δ as a measure of receiver degradation has the disadvantage that the value of δ depends not only on the initial values of λ and η , but also on the relationship between η and λ as the latter parameter is increased to compensate for the presence of the interfering signal. To illustrate this point let us examine the special case where $s(0; t)$ and $s(1; t)$ are antipodal, binary-valued signals. In this case

$$s(\alpha; t) = (-1)^\alpha P_s^{1/2} \quad (10)$$

$$E_0 = E_1 = P_s T \quad (11)$$

where P_s is the received signal power, and

$$\lambda = \frac{P_s T}{\Phi} \quad (12)$$

Then, for a sinusoidal interfering signal

$$\eta = \frac{P_i}{P_s} \frac{\sin^2(\pi f_i T)}{(\pi f_i T)^2} \quad (13)$$

while, for a gaussian interfering signal

$$\eta = \frac{P_i}{P_s} \int_{-\infty}^{\infty} \frac{G_i(f)}{P_i} \frac{\sin^2(\pi f T)}{(\pi f T)^2} df \quad (14)$$

In Fig. 7 the function $10 \log \frac{\sin^2(\pi f_i T)}{(\pi f_i T)^2}$ is plotted as a function of $f_i T$.

Examining Eqs. (12) through (14), one notes that η may either remain constant or vary as λ is increased, depending on the source of the interfering signal and which parameters of the communication system are changed to compensate for the degradation produced by the interfering signal. In frequency-multiplexed PM communication systems, interfering signals are generated in the process of phase-modulating an RF carrier. In this case the ratio of P_i to P_s is fixed and η will remain constant.

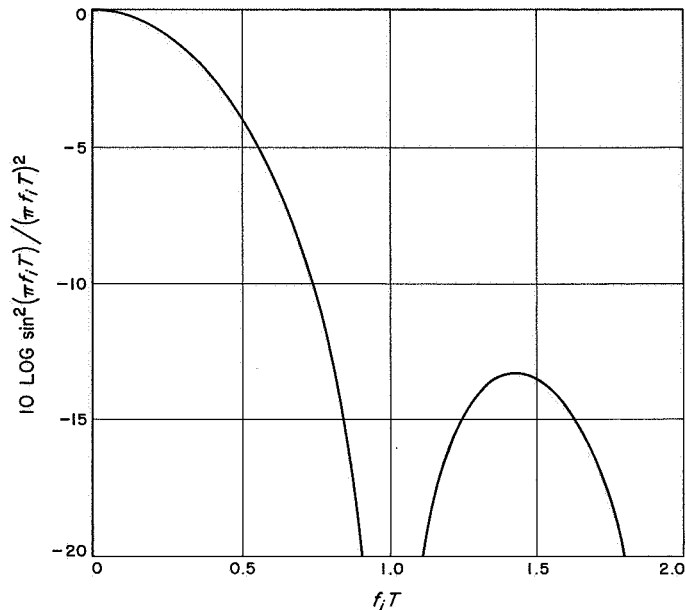


Fig. 7. $10 \log \sin^2(\pi f_i T)/(\pi f_i T)^2$ as a function of $f_i T$

When signals are received from the transmitters for two communication systems operating on adjacent frequency bands, a portion of the signal from one transmitter may fall into the frequency band used by the other communication system. If one compensates for the degradation caused by this interfering signal by changing the receiving system parameters of the communication system, P_i/P_s , and therefore η , will remain constant. However, if one compensates for the degradation caused by this interfering signal by changing the transmitting system parameters of this communication system, P_i/P_s , and therefore η , will decrease as λ is increased. In the latter case the parameter remaining constant is ξ , the interference-to-noise ratio at the input of the decision element. In a communication system using antipodal, binary-valued signals,

$$\xi = 2 \frac{P_i T}{\Phi} \frac{\sin^2(\pi f_i T)}{(\pi f_i T)^2} \quad (15)$$

for sinusoidal interfering signals, while for gaussian interfering signals

$$\xi = 2 \frac{P_i T}{\Phi} \int_{-\infty}^{\infty} \frac{G_i(f)}{P_i} \frac{\sin^2(\pi f T)}{(\pi f T)^2} df \quad (16)$$

Hence, in evaluating δ , we must consider both the case where η remains constant and the case where ξ remains constant.

For arbitrary signal waveforms

$$\xi = \frac{P_i A_F^2(\omega_i)}{8\lambda} \quad (17)$$

for sinusoidal interfering signals, and

$$\xi = \frac{P_i}{8\lambda} \int_{-\infty}^{\infty} \frac{G_i(f)}{P_i} A_F^2(\omega) df \quad (18)$$

for gaussian interfering signals. Examining Eqs. (7), (9), (17), and (18), as well as Eqs. (13) through (16), one notes that

$$\xi = 2\lambda\eta \quad (19)$$

D. Receiver Error Probability as a Function of the Interference-to-Noise Ratio

Expressing the receiver bit error probability as a function of λ and ξ , for sinusoidal interfering signals the bit error probability is

$$\begin{aligned} P_E &= p_S \left(\lambda; \frac{\xi}{2\lambda} \right) \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2} + \xi^{1/2} \sin u)] du \end{aligned} \quad (20)$$

The function $\log p_S(\lambda; \xi/2\lambda)$ is plotted as a function of $10 \log \lambda$ for selected values of $10 \log \xi$ in Fig. 8 and as a function of $10 \log \xi$ for selected values of $10 \log \lambda$ in Fig. 9.

For gaussian interfering signals

$$\begin{aligned} P_E &= p_G \left(\lambda; \frac{\xi}{2\lambda} \right) \\ &= \frac{1}{2} \{1 - \text{Erf}[\lambda^{1/2} (1 + \xi)^{-1/2}]\} \end{aligned} \quad (21)$$

The function $\log p_G(\lambda; \xi/2\lambda)$ is plotted as a function of

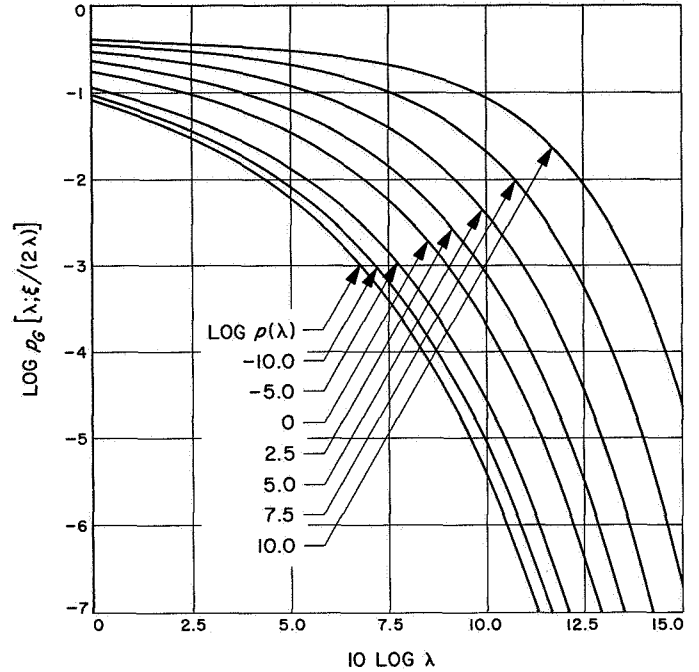


Fig. 8. $\log p_S [\lambda; \xi/(2\lambda)]$ as a function of $10 \log \lambda$ for selected values of $10 \log \xi$

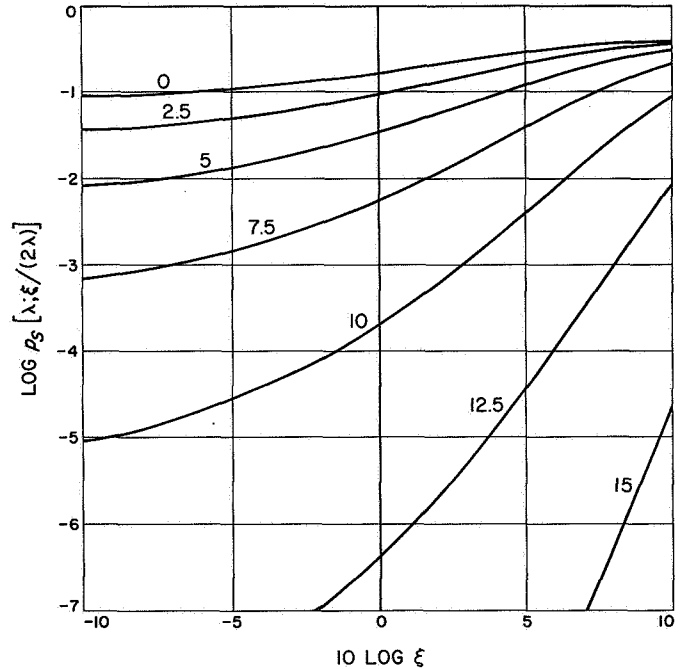


Fig. 9. $\log p_S [\lambda; \xi/(2\lambda)]$ as a function of $10 \log \xi$ for selected values of $10 \log \lambda$

$10 \log \lambda$ for selected values of $10 \log \xi$ in Fig. 10 and as a function of $10 \log \xi$ for selected values of $10 \log \lambda$ in Fig. 11.

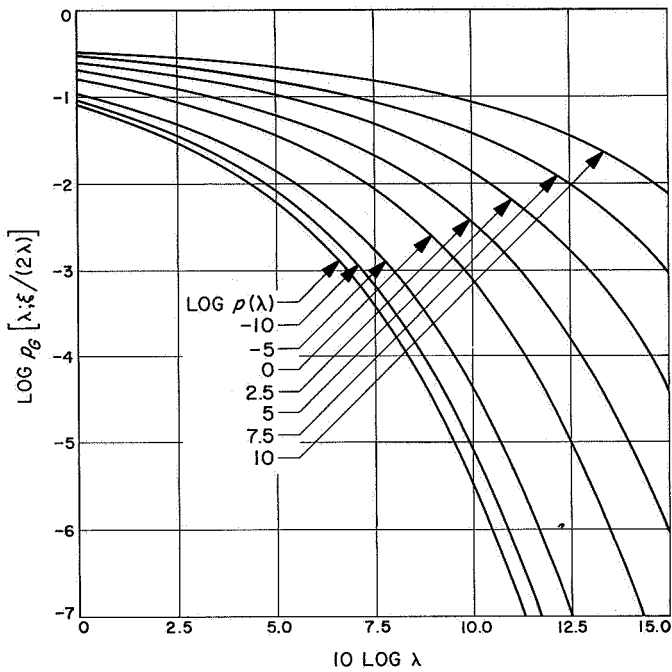


Fig. 10. $\text{Log } p_g [\lambda; \xi/(2\lambda)]$ as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \xi$

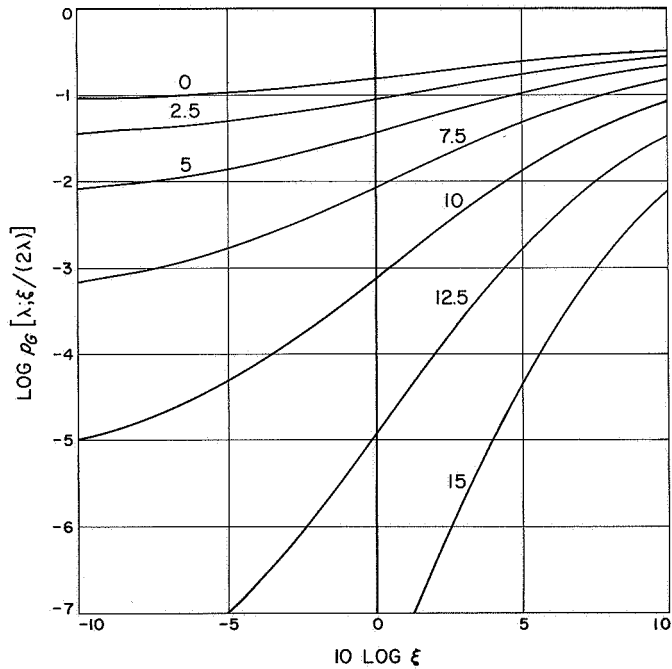


Fig. 11. $\text{Log } p_g [\lambda; \xi/(2\lambda)]$ as a function of $10 \text{ log } \xi$ for selected values of $10 \text{ log } \lambda$

E. Receiver Degradation

1. *Sinusoidal interference, constant interference-to-signal ratio.* Since the factor δ is the amount λ must be

increased to compensate for the presence of the interfering signal, when η is fixed and the interference is sinusoidal, δ is the solution of the equation

$$p_s(\delta\lambda; \eta) = p(\lambda) \tag{22}$$

or, using Eqs. (3) and (6),

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{ \delta^{1/2} \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \\ = \frac{1}{2} [1 - \text{Erf} (\lambda^{1/2})] \end{aligned} \tag{23}$$

for sinusoidal signals. $10 \text{ log } \delta$ is plotted as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \eta$ in Fig. 12 and as a function of $10 \text{ log } \eta$ for selected values of $10 \text{ log } \lambda$ in Fig. 13. Since

$$\lim_{\lambda \rightarrow \infty} p_s(\lambda; \eta) = \begin{cases} 0, & \eta \leq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{\pi} \sin^{-1} [(2\eta)^{-1/2}], & \eta > \frac{1}{2} \end{cases} \tag{24}$$

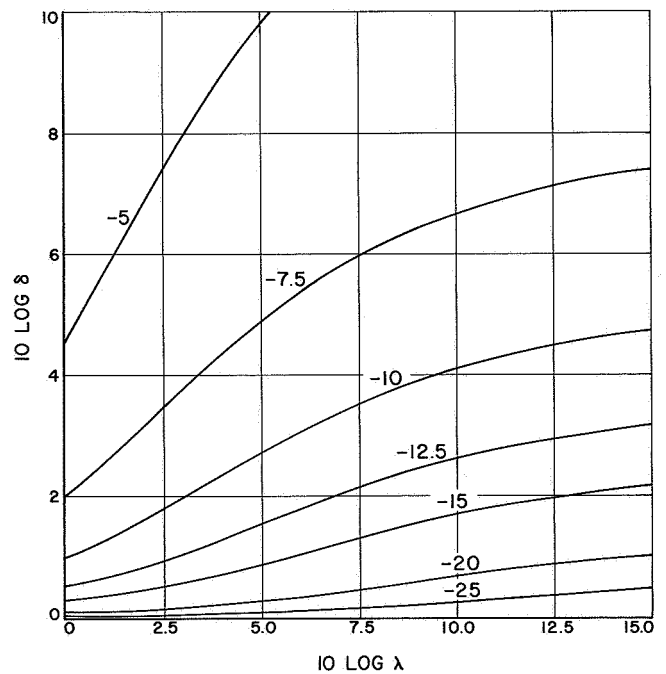


Fig. 12. $10 \text{ log } \delta$ for sinusoidal interference and constant η as a function of $10 \text{ log } \lambda$ for selected values of $10 \text{ log } \eta$

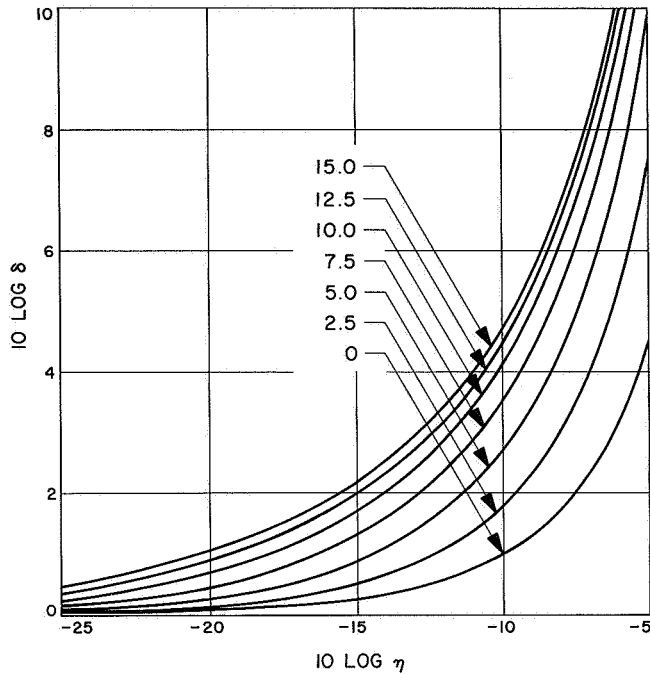


Fig. 13. 10 log δ for sinusoidal interference and constant η as a function of 10 log λ for selected values of 10 log η

a finite solution of Eq. (23) for δ will exist for all values of λ when $\eta < 1/2$ and for $\lambda < \lambda_0$, where

$$\text{Erf}(\lambda_0^{1/2}) = \frac{2}{\pi} \sin^{-1} [(2\eta)^{-1/2}] \quad (25)$$

when $\eta \geq 1/2$. For cases where a solution of Eq. (23) does not exist ($\eta \geq 1/2, \lambda > \lambda_0$), δ is infinite. In Fig. 14, 10 log λ_0 is plotted as a function of 10 log η .

2. Gaussian interference, constant interference-to-signal ratio. When η is fixed and a gaussian interfering signal is present, δ is the solution of the equation

$$p_G(\delta\lambda; \eta) = p(\lambda) \quad (26)$$

Since

$$\lim_{\lambda \rightarrow \infty} p_G(\lambda; \eta) = \frac{1}{2} \{1 - \text{Erf} [(2\eta)^{-1/2}]\} \quad (27)$$

for $\lambda > \lambda_0 = (2\eta)^{-1}$, δ is infinite, while

$$\delta = (1 - 2\eta\lambda)^{-1}, \quad \lambda < \lambda_0 = (2\eta)^{-1} \quad (28)$$

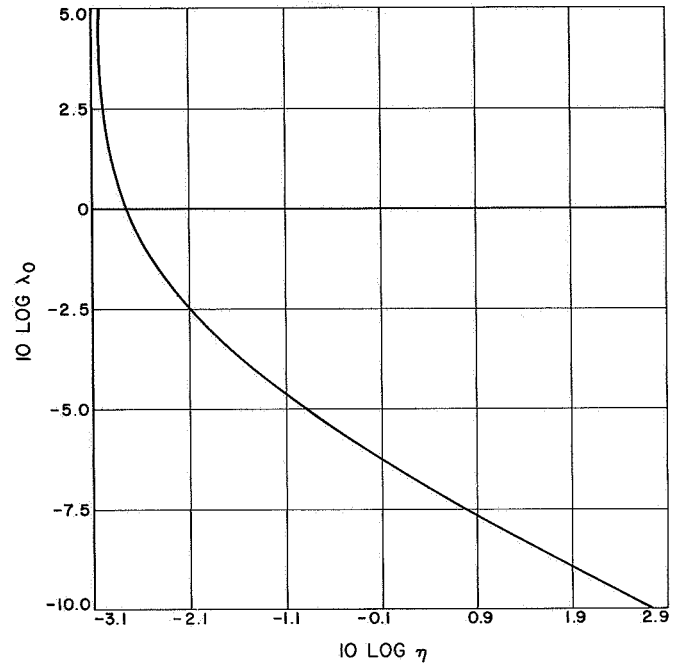


Fig. 14. 10 log λ_0 for sinusoidal interference and constant η as a function of 10 log η

10 log δ is plotted as a function of 10 log λ for selected values of 10 log η in Fig. 15 and as a function of 10 log η for selected values of 10 log λ in Fig. 16.

3. Sinusoidal interference, constant interference-to-noise ratio. When ξ is fixed, for sinusoidal interfering signals δ is the solution of the equation

$$p_S\left(\delta\lambda; \frac{\xi}{2\delta\lambda}\right) = p(\lambda) \quad (29)$$

or, using Eqs (3) and (20),

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}(\delta^{1/2}\lambda^{1/2} + \xi^{1/2} \sin u)] du \\ = \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2})] \end{aligned} \quad (30)$$

10 log δ is plotted as a function of 10 log λ for selected values of 10 log ξ in Fig. 17 and as a function of 10 log ξ for selected values of 10 log λ in Fig. 18.

4. Gaussian interference, constant interference-to-noise ratio. When ξ is fixed, for gaussian interfering signals, δ is the solution of the equation

$$p_G\left(\delta\lambda; \frac{\xi}{2\delta\lambda}\right) = p(\lambda) \quad (31)$$

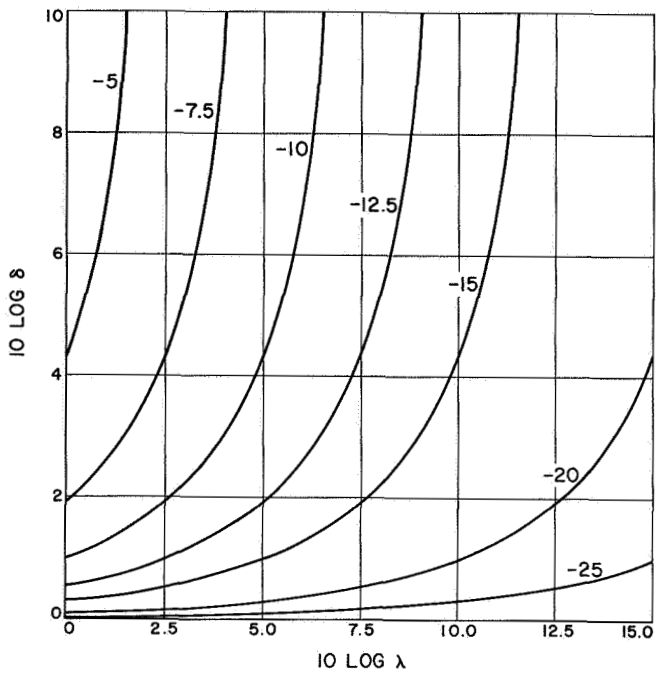


Fig. 15. $10 \log \delta$ for gaussian interference and constant η as a function of $10 \log \lambda$ for selected values of $10 \log \eta$

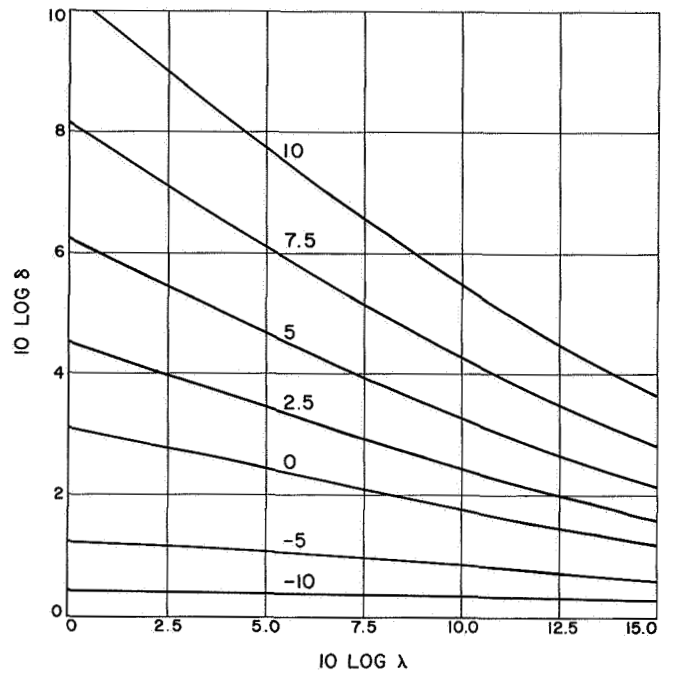


Fig. 17. $10 \log \delta$ for sinusoidal interference and constant ξ as a function of $10 \log \lambda$ for selected values of $10 \log \xi$

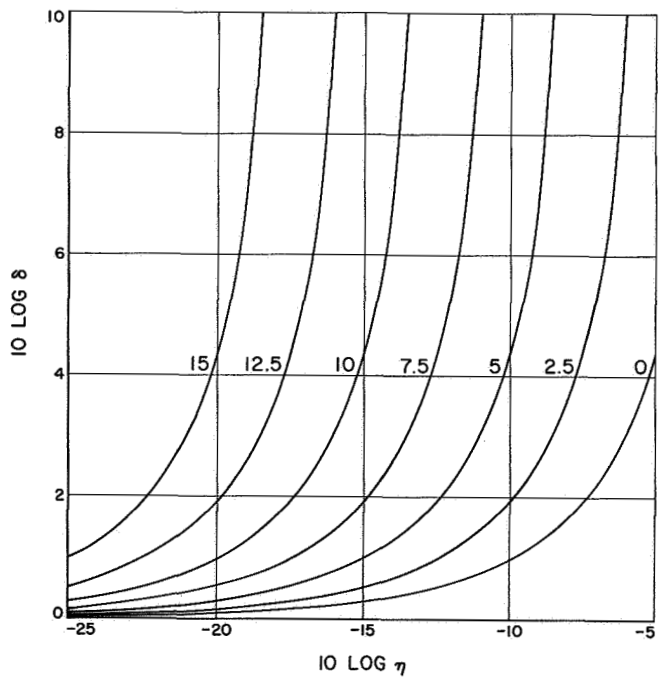


Fig. 16. $10 \log \delta$ for gaussian interference and constant λ as a function of $10 \log \eta$ for selected values of $10 \log \lambda$

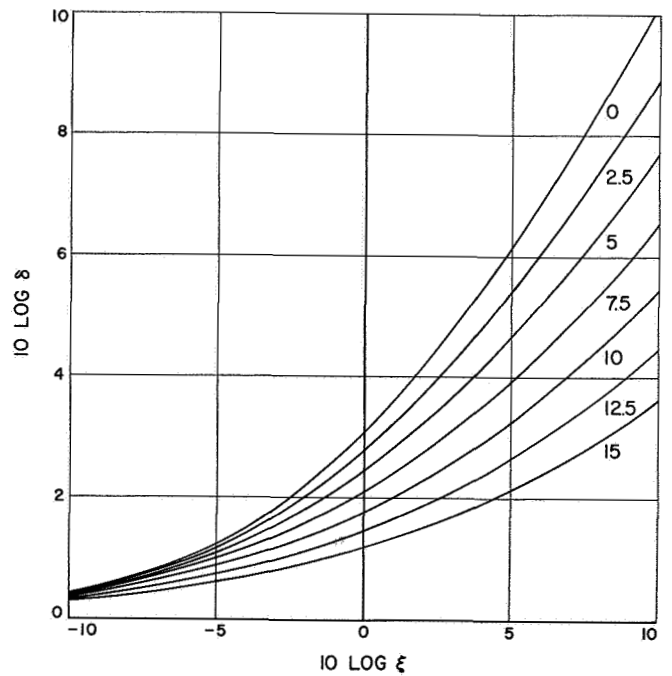


Fig. 18. $10 \log \delta$ for sinusoidal interference and constant λ as a function of $10 \log \xi$ for selected values of $10 \log \lambda$

or, using Eqs. (3) and (21),

$$\delta = 1 + \xi \quad (32)$$

In Fig. 19, $10 \log \delta$ is plotted as a function of $10 \log \xi$. One should note that in this case δ is not dependent on λ .

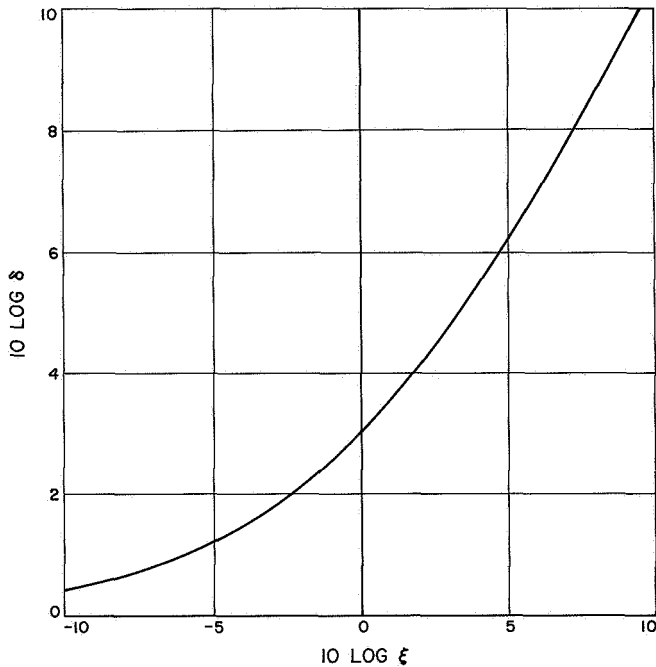


Fig. 19. $10 \log \delta$ for gaussian interference and constant ξ as a function of $10 \log \xi$

Since

$$\lim_{\lambda \rightarrow \infty} p_s \left(\lambda; \frac{\xi}{2\lambda} \right) = \lim_{\lambda \rightarrow \infty} p_g \left(\lambda; \frac{\xi}{2\lambda} \right) = 0 \quad (33)$$

where ξ is fixed, δ is finite for all values of λ .

F. Comparison of the Effect of Sinusoidal and Gaussian Interference

A convenient approximation often used to evaluate the effect of a nongaussian interfering signal on the performance of a receiver is to assume that the effect of the interfering signal is the same as that of a gaussian process which produces equal power at the receiver output. In Figs. 20 through 26, we compare the behavior of the receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for values of $10 \log \lambda$ in the 0.0 to 15.0-dB range. The obvious conclusion is that for sinusoidal interference the gaussian approximation is satisfactory for small λ and η or ξ but breaks down for large λ and large η or ξ .

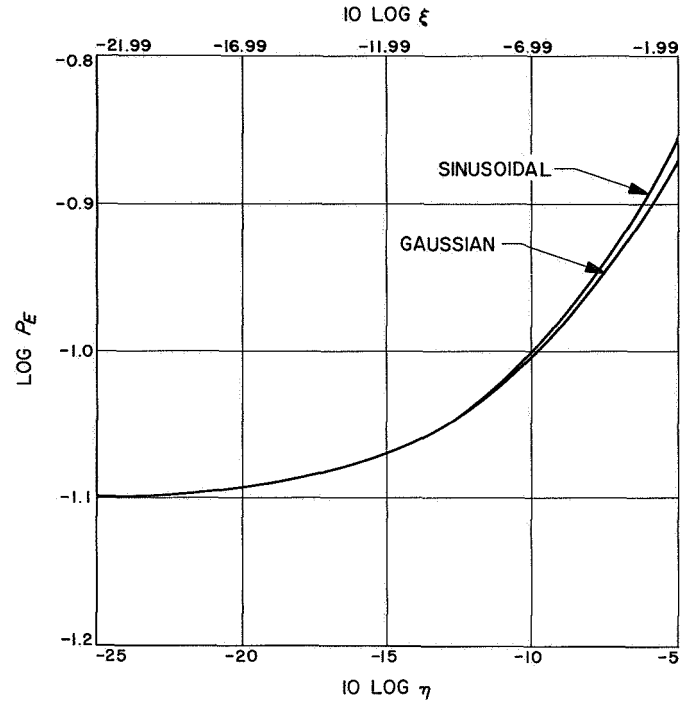


Fig. 20. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 0.0$

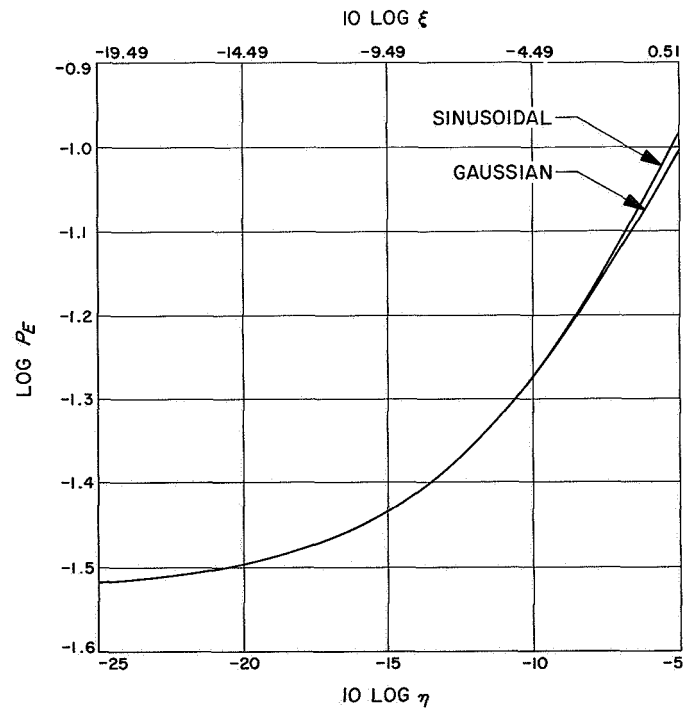


Fig. 21. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 2.5$

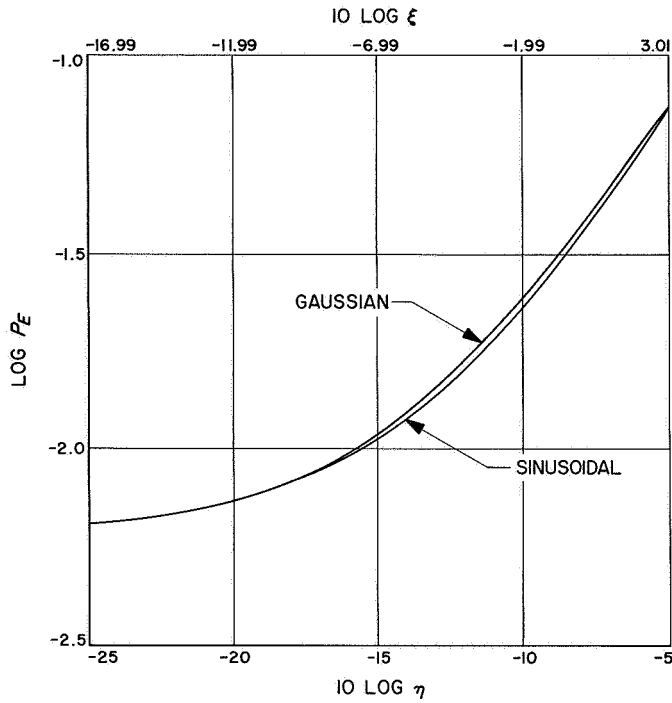


Fig. 22. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 5.0$

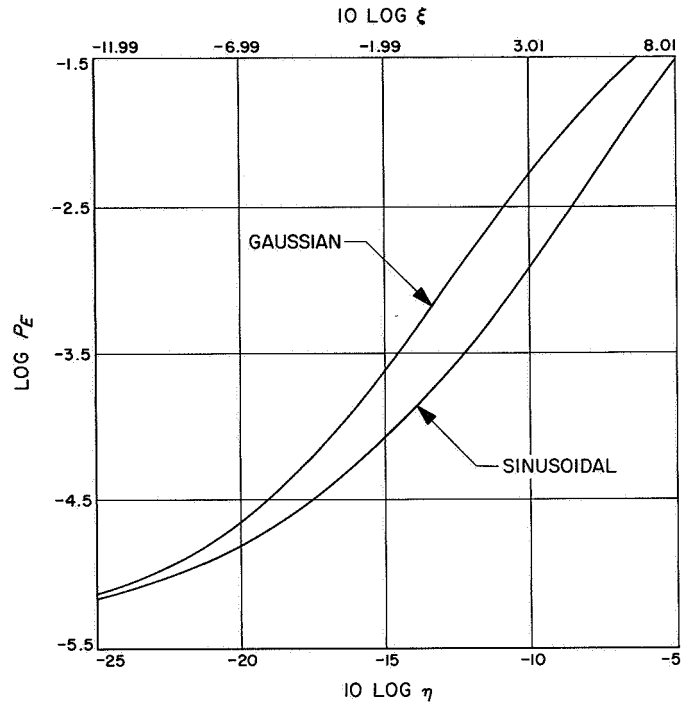


Fig. 24. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 10.0$

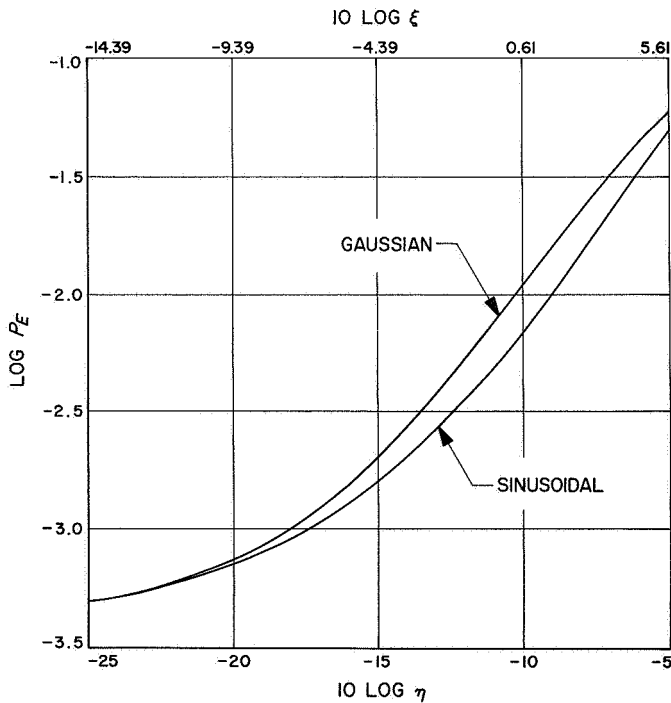


Fig. 23. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 7.5$

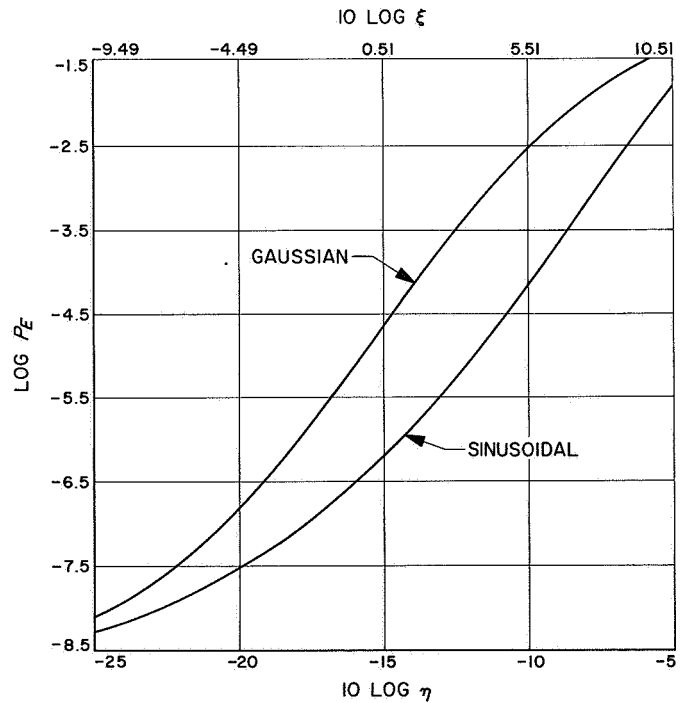


Fig. 25. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 12.5$

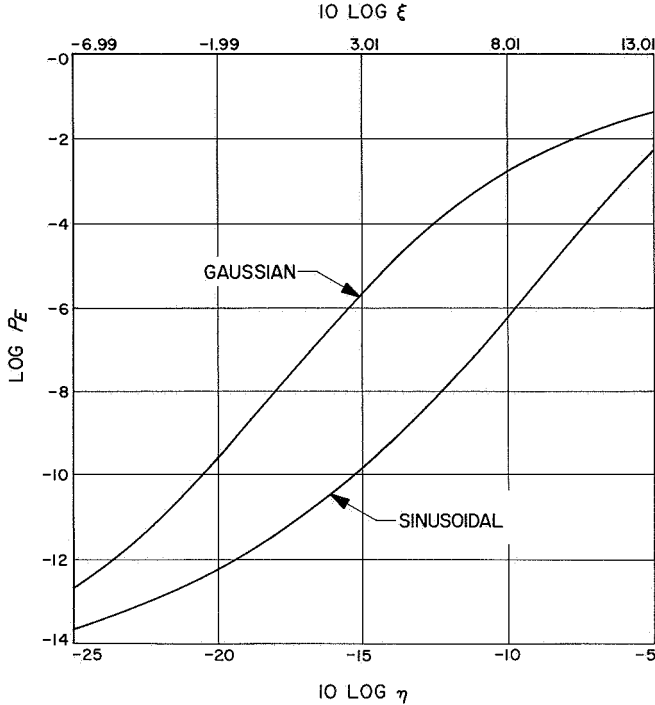


Fig. 26. Comparison of receiver error probability for sinusoidal and gaussian interference as a function of $10 \log \eta$ and $10 \log \xi$ for $10 \log \lambda = 15.0$

III. Analysis

A. The Maximum-Likelihood Receiver

If each of the two messages has duration T , during the time interval $(0, T)$ the receiver input is

$$y(t) = s(\alpha; t) + n(t) \quad (34)$$

where either $\alpha = 0$ or $\alpha = 1$ and where $n(t)$ is gaussian noise with mean

$$\mu_n = E[n(t)] = 0 \quad (35)$$

autocorrelation function

$$R_n(\tau) = E[n(t) n(t + \tau)] = \frac{\Phi}{2} \delta(\tau) \quad (36)$$

and power spectral density

$$\begin{aligned} G_n(f) &= \int_{-\infty}^{\infty} R_n(\tau) \exp(-i\omega\tau) d\tau \\ &= \frac{\Phi}{2}, \quad -\infty < f < \infty \end{aligned} \quad (37)$$

where $\delta(\tau)$ is the Dirac delta function.

The function of the maximum-likelihood receiver is to determine the most probable value of α , after observing $y(t)$ for $0 \leq t \leq T$, and set an estimate $\hat{\alpha}$ equal to this value. From Refs. 1 and 2 the a posteriori probability of α , given $y(t)$, $0 \leq t \leq T$, is

$$p(\alpha | y) = \frac{p(\alpha) \exp \left\{ -\frac{1}{\Phi} \int_0^T [y(t) - s(\alpha; t)]^2 dt \right\}}{\sum_{l=0}^1 p(l) \exp \left\{ -\frac{1}{\Phi} \int_0^T [y(t) - s(l; t)]^2 dt \right\}} \quad (38)$$

As we have assumed the messages are chosen randomly with equal a priori probabilities,

$$p(0) = p(1) = \frac{1}{2} \quad (39)$$

and

$$p(\alpha | y) = \frac{\exp \left\{ -\frac{1}{\Phi} \int_0^T [y(t) - s(\alpha; t)]^2 dt \right\}}{\sum_{l=0}^1 \exp \left\{ -\frac{1}{\Phi} \int_0^T [y(t) - s(l; t)] dt \right\}} \quad (40)$$

Expanding and cancelling factors common to both the numerator and the denominator of Eq. (40),

$$p(\alpha | y) = \frac{\exp \left[\frac{2}{\Phi} \int_0^T y(t) s(\alpha; t) dt - \frac{1}{\Phi} \int_0^T s^2(\alpha; t) dt \right]}{\sum_{l=0}^1 \exp \left[+ \frac{2}{\Phi} \int_0^T y(t) s(l; t) dt - \frac{1}{\Phi} \int_0^T s^2(l; t) dt \right]} \quad (41)$$

Since

$$\int_0^T s^2(\alpha; t) dt = E_\alpha \quad (42)$$

the received signal energy when the α th message is transmitted,

$$p(\alpha | y) = \frac{\exp \left[-\frac{E_\alpha}{\Phi} + \frac{2}{\Phi} \int_0^T y(t) s(\alpha; t) dt \right]}{\sum_{l=0}^1 \exp \left[-\frac{E_l}{\Phi} + \frac{2}{\Phi} \int_0^T y(t) s(l; t) dt \right]} \quad (43)$$

Thus

$$p(0|y) = \frac{1}{1 + \exp \left\{ \frac{E_0 - E_1}{\Phi} - \frac{2}{\Phi} \int_0^T y(t) [s(0;t) - s(1;t)] dt \right\}} \quad (44)$$

and

$$p(1|y) = \frac{1}{1 + \exp \left\{ -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T y(t) [s(0;t) - s(1;t)] dt \right\}} \quad (45)$$

Introducing

$$z = \ln \left[\frac{p(0|y)}{p(1|y)} \right] = \ln [p(0|y)] - \ln [p(1|y)] \quad (46)$$

or, using Eq. (43)

$$z = -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T y(t) [s(0;t) - s(1;t)] dt \quad (47)$$

$$p(0|y) = [1 + \exp(-z)]^{-1} \quad (48)$$

and

$$p(1|y) = [1 + \exp(z)]^{-1} \quad (49)$$

Since z is positive when $p(0|y) > p(1|y)$ and negative when $p(0|y) < p(1|y)$, the maximum-likelihood receiver requires only the equipment to compute z and a decision element which sets $\hat{\alpha}$, the receiver output, to zero when z is positive and to 1 when z is negative. The required

equipment is shown in Fig. 1. Substituting $t = T - \tau$ in Eq. (47),

$$z = -\frac{E_0 - E_1}{\Phi} + \int_0^T \frac{2}{\Phi} [s(0;T-\tau) - s(1;T-\tau)] y(T-\tau) d\tau \quad (50)$$

The sampler in Fig. 1 samples the output of the filter F at time T . Hence, if $h_F(\tau)$ is the impulse response of F ,

$$z = -\frac{E_0 - E_1}{\Phi} + \int_0^{\infty} h_F(\tau) y(T-\tau) dt \quad (51)$$

and for a maximum-likelihood receiver F must have impulse response

$$h_F(\tau) = \begin{cases} \frac{2}{\Phi} [s(0;T-\tau) - s(1;T-\tau)], & 0 \leq \tau \leq T \\ 0, & \tau < 0, \tau > T \end{cases} \quad (52)$$

B. Probability of Error for the Maximum-Likelihood Receiver

Since the receiver output will be

$$\hat{\alpha} = \begin{cases} 0, & z > 0 \\ 1, & z < 0 \end{cases} \quad (53)$$

the receiver error probability is

$$P_E = p(0) \int_{-\infty}^0 p(z|0) dz + p(1) \int_0^{\infty} p(z|1) dz \quad (54)$$

where $p(z|\alpha)$ is the conditional probability density of z given α .

Using Eq. (34) in Eq. (47),

$$z = -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T s(\alpha;t) [s(0;t) - s(1;t)] dt + \frac{2}{\Phi} \int_0^T n(t) [s(0;t) - s(1;t)] dt \quad (55)$$

or

$$z = z_s + z_n \quad (56)$$

where

$$z_s = -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T s(\alpha;t) [s(0;t) - s(1;t)] dt \quad (57)$$

and

$$z_n = \frac{2}{\Phi} \int_0^T n(t) [s(0;t) - s(1;t)] dt \quad (58)$$

are the signal and noise components of z .

If $\alpha = 0$,

$$z_s = -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T s^2(0;t) dt - \frac{2}{\Phi} \int_0^T s(0;t) s(1;t) dt \quad (59)$$

Introducing

$$\rho = \frac{\int_0^T s(0;t) s(1;t) dt}{\left[\int_0^T s^2(0;t) dt \right]^{1/2} \left[\int_0^T s^2(1;t) dt \right]^{1/2}} \quad (60)$$

the crosscorrelation between $s(0;t)$ and $s(1;t)$, or using Eq. (42),

$$\rho = (E_0 E_1)^{-1/2} \int_0^T s(0;t) s(1;t) dt \quad (61)$$

$$z_s = \frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{\Phi} \quad (62)$$

If $\alpha = 1$,

$$z_s = -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T s(1;t) s(0;t) dt - \frac{2}{\Phi} \int_0^T s^2(1;t) dt \quad (63)$$

or, using Eqs. (42) and (61),

$$z_s = -\frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{\Phi} \quad (64)$$

Combining (62) and (64)

$$z_s = (-1)^\alpha \frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{\Phi} \quad (65)$$

or, introducing the positive-valued parameter

$$\begin{aligned} \lambda &= \frac{1}{4\Phi} \int_0^T [s(0;t) - s(1;t)]^2 dt \\ &= \frac{1}{4\Phi} \left[\int_0^T s^2(0;t) dt - 2 \int_0^T s(0;t) s(1;t) dt \right. \\ &\quad \left. + \int_0^T s^2(1;t) dt \right] \\ &= \frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{4\Phi} \end{aligned} \quad (66)$$

$$z_s = (-1)^\alpha 4\lambda \quad (67)$$

Examining Eq. (58), we note that z_n is a gaussian random variable with mean

$$\begin{aligned} \mu_{z_n} &= \frac{2}{\Phi} \int_0^T E[n(t)] [s(0;t) - s(1;t)] dt \\ &= 0 \end{aligned} \quad (68)$$

and variance

$$\begin{aligned} \sigma_{z_n}^2 &= \left(\frac{2}{\Phi} \right)^2 \int_0^T \int_0^T E[n(t_1) n(t_2)] [s(0;t_1) - s(1;t_1)] [s(0;t_2) \\ &\quad - s(1;t_2)] dt_1 dt_2 \end{aligned} \quad (69)$$

Using Eq. (36),

$$\begin{aligned} \sigma_{z_n}^2 &= \left(\frac{2}{\Phi} \right)^2 \int_0^T \int_0^T \frac{\Phi}{2} \delta(t_1 - t_2) [s(0;t_1) - s(1;t_1)] [s(0;t_2) \\ &\quad - s(1;t_2)] dt_1 dt_2 \\ &= \frac{2}{\Phi} \int_0^T [s(0;t) - s(1;t)]^2 dt \\ &= \frac{2}{\Phi} \left\{ \int_0^T s^2(0;t) dt - 2 \int_0^T s(0;t) s(1;t) dt \right. \\ &\quad \left. + \int_0^T s^2(1;t) dt \right\} \end{aligned} \quad (70)$$

Using Eqs. (42) and (61)

$$\sigma_{z_n}^2 = 2 \frac{E_0 + E_1 - 2\rho(E_0 E_1)^{1/2}}{\Phi} \quad (71)$$

or, using Eq. (66),

$$\sigma_{z_n}^2 = 8\lambda \quad (72)$$

Thus, given α , z is a gaussian random variable with mean $(-1)^\alpha 4\lambda$ and variance 8λ . The probability that a gaussian random variable x with mean μ and variance σ^2 is negative is

$$\begin{aligned} \int_{-\infty}^0 p(x) dx &= \int_{-\infty}^0 (2\pi\sigma^2)^{-1/2} \exp [-(2\sigma^2)^{-1} (x - \mu)^2] \\ &= \pi^{-1/2} \int_{-\infty}^{-\mu/\sqrt{2}\sigma} \exp(-t^2) dt \\ &= \frac{1}{2} \left[1 - \operatorname{Erf} \left(\frac{\mu}{\sqrt{2}\sigma} \right) \right] \end{aligned} \quad (73)$$

where

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (74)$$

Similarly, the probability that x is positive is

$$\begin{aligned} \int_0^\infty p(x) dx &= \int_0^\infty (2\pi\sigma^2)^{-1/2} \exp [-(2\sigma^2)^{-1} (x - \mu)^2] dx \\ &= \frac{1}{2} \left[1 + \operatorname{Erf} \left(\frac{\mu}{\sqrt{2}\sigma} \right) \right] \end{aligned} \quad (75)$$

Therefore

$$\int_{-\infty}^0 p(z|0) dz = \frac{1}{2} [1 - \operatorname{Erf}(\lambda^{1/2})] \quad (76)$$

$$\int_0^\infty p(z|1) dz = \frac{1}{2} [1 - \operatorname{Erf}(\lambda^{1/2})] \quad (77)$$

and, using Eqs. (76) and (77) in Eq. (54), in the absence of any interfering signal,

$$\begin{aligned} P_E &= p(\lambda) \\ &= [p(0) + p(1)] \frac{1}{2} [1 - \operatorname{Erf}(\lambda^{1/2})] \\ &= \frac{1}{2} [1 - \operatorname{Erf}(\lambda^{1/2})] \end{aligned} \quad (78)$$

C. Interference-to-Signal and Interference-to-Noise Ratios

The first step in evaluating the effect of sinusoidal and gaussian interfering signals on the performance of the receiver is calculation of η , the interference-to-signal ratio,

and ξ , the interference-to-noise ratio, for the random variable z . Assuming that an interfering signal $i(t)$ is present at the receiver input in addition to the message and white gaussian noise for which the receiver was designed,

$$y(t) = s(\alpha; t) + i(t) + n(t) \quad (79)$$

and

$$\begin{aligned} z &= -\frac{E_0 - E_1}{\Phi} + \frac{2}{\Phi} \int_0^T s(\alpha; t) [s(0; t) - s(1; t)] dt \\ &\quad + \frac{2}{\Phi} \int_0^T i(t) [s(0; t) - s(1; t)] dt \\ &\quad + \frac{2}{\Phi} \int_0^T n(t) [s(0; t) - s(1; t)] dt \end{aligned} \quad (80)$$

or

$$z = z_s + z_i + z_n \quad (81)$$

where

$$z_i = \frac{2}{\Phi} \int_0^T i(t) [s(0; t) - s(1; t)] dt \quad (82)$$

is the component of z produced by the interfering signal.

Using Eqs. (39) and (67)

$$\begin{aligned} \mu_{z_s} &= (4\lambda) p(0) + (-4\lambda) p(1) \\ &= 0 \end{aligned} \quad (83)$$

and therefore

$$\begin{aligned} \sigma_{z_s}^2 &= (4\lambda)^2 p(0) + (-4\lambda)^2 p(1) \\ &= (4\lambda)^2 \end{aligned} \quad (84)$$

If we restrict ourselves to interfering signals with mean

$$\mu_i = E[i(t)] = 0 \quad (85)$$

$$\begin{aligned} \mu_{z_i} &= E(z_i) \\ &= \frac{2}{\Phi} \int_0^T E[i(t)] [s(0; t) - s(1; t)] dt \\ &= 0 \end{aligned} \quad (86)$$

and

$$\begin{aligned} \sigma_{z_i}^2 &= \left(\frac{2}{\Phi}\right)^2 \int_0^T \int_0^T E[i(t_1) i(t_2)] [s(0; t_1) - s(1; t_1)] \\ &\quad \times [s(0; t_2) - s(1; t_2)] dt_1 dt_2 \end{aligned} \quad (87)$$

Then, if $i(t)$ has autocorrelation function

$$R_i(\tau) = E[i(t) i(t + \tau)] \quad (88)$$

and power spectral density $G_i(f)$, where

$$R_i(\tau) = \int_{-\infty}^{\infty} G_i(f) \exp(i\omega\tau) df \quad (89)$$

$$\begin{aligned} \sigma_{z_i}^2 &= \left(\frac{2}{\Phi}\right)^2 \int_0^T \int_0^T R_i(t_1 - t_2) [s(0; t_1) - s(1; t_1)] [s(0; t_2) \\ &\quad - s(1; t_2)] dt_1 dt_2 \end{aligned} \quad (90)$$

Using Eq. (89),

$$\begin{aligned} \sigma_{z_i}^2 &= \int_{-\infty}^{\infty} G_i(f) \int_0^T \frac{2}{\Phi} [s(0; t_1) - s(1; t_1)] \exp(i\omega t_1) dt_1 \\ &\quad \times \int_0^T \frac{2}{\Phi} [s(0; t_2) - s(1; t_2)] \exp(-i\omega t_2) dt_2 df \end{aligned} \quad (91)$$

or substituting $t_1 = T - \tau$ and $t_2 = T - \tau$,

$$\begin{aligned} \sigma_{z_i}^2 &= \int_{-\infty}^{\infty} G_i(f) \int_0^T \frac{2}{\Phi} [s(0; T - \tau) - s(1; T - \tau)] \exp(-i\omega\tau) d\tau \\ &\quad \times \int_0^T \frac{2}{\Phi} [s(0; T - \tau) - s(1; T - \tau)] \exp(i\omega\tau) d\tau df \end{aligned} \quad (92)$$

Since

$$H_F(i\omega) = \int_0^{\infty} h_F(\tau) \exp(-i\omega\tau) d\tau \quad (93)$$

or, using Eq. (52),

$$H_F(i\omega) = \int_0^T \frac{2}{\Phi} [s(0; T - \tau) - s(1; T - \tau)] \exp(-i\omega\tau) d\tau \quad (94)$$

$$\sigma_{z_i}^2 = \int_{-\infty}^{\infty} G_i(f) H_F(i\omega) H_F(-i\omega) df \quad (95)$$

Introducing polar representation for $H_F(i\omega)$,

$$H_F(i\omega) = A_F(\omega) \exp[i\phi_F(\omega)] \quad (96)$$

where both $A_F(\omega)$ and $\phi_F(\omega)$ are real-valued functions of ω . Since

$$H_F(-i\omega) = H_F^*[i(-\omega)] \quad (97)$$

and, therefore,

$$A_F(\omega) \exp[-i\phi_F(\omega)] = A_F(-\omega) \exp[i\phi_F(-\omega)] \quad (98)$$

the functions $A_F(\omega)$ and $\phi_F(\omega)$ have the properties

$$A_F(-\omega) = A_F(\omega) \quad (99)$$

and

$$\phi_F(-\omega) = -\phi_F(\omega) \quad (100)$$

Then

$$\sigma_{z_i}^2 = \int_{-\infty}^{\infty} G_i(f) A_F^2(\omega) df \quad (101)$$

Hence, the interference-to-signal ratio for z is

$$\begin{aligned} \eta &= \frac{\sigma_{z_i}^2}{\sigma_{z_s}^2} \\ &= (4\lambda)^{-2} \int_{-\infty}^{\infty} G_i(f) A_F^2(\omega) df \end{aligned} \quad (102)$$

and the interference-to-noise ratio for z is

$$\begin{aligned} \xi &= \frac{\sigma_{z_i}^2}{\sigma_{z_n}^2} \\ &= (8\lambda)^{-1} \int_{-\infty}^{\infty} G_i(f) A_F^2(\omega) df \end{aligned} \quad (103)$$

Examining Eqs. (102) and (103) we note that

$$\xi = 2\lambda\eta \quad (104)$$

Thus for any particular value of λ , the principal independent variable in this problem, specification of a value for η determines ξ or specification of ξ determines η . In the subsequent analysis we shall find that either λ and η or λ and ξ are sufficient to define the receiver error probability. However, the factor δ by which λ must be

increased to compensate for the presence of the interfering signal will depend on whether η or ξ remains fixed. Usually the interference-to-signal ratio η is fixed in systems where the interfering signal is internally generated, while the interference-to-noise ratio ξ is fixed in systems where the interfering signal is externally generated.

The case of antipodal, binary-valued signals is of particular interest. In this case, if P_s is the average received signal power

$$s(\alpha; t) = (-1)^\alpha P_s^{1/2} \quad (105)$$

Using (105) in Eqs. (42) and (61),

$$E_0 = E_1 = P_s T \quad (106)$$

and

$$\rho = -1 \quad (107)$$

Using Eqs. (106) and (107) in Eq. (66)

$$\lambda = \frac{P_s T}{\Phi} \quad (108)$$

Moreover, using Eq. (105) in Eq. (52)

$$h_F(\tau) = \begin{cases} \frac{4P_s^{1/2}}{\Phi}, & 0 \leq \tau \leq T \\ 0, & \tau < 0, \quad \tau > T \end{cases} \quad (109)$$

Thus, using Eq. (93)

$$\begin{aligned} H_F(i\omega) &= \frac{4P_s^{1/2}}{\Phi} \int_0^T \exp(-i\omega t) dt \\ &= \frac{4P_s^{1/2} T}{\Phi} \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \exp\left(\frac{-i\omega T}{2}\right) \end{aligned} \quad (110)$$

and

$$\begin{aligned} A_F^2(\omega) &= H_F(i\omega) H_F(-i\omega) \\ &= \frac{16P_s T^2}{\Phi^2} \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2} \end{aligned} \quad (111)$$

or, using (108),

$$A_F^2(\omega) = (4\lambda)^2 P_s^{-1} \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2} \quad (112)$$

Therefore, in this case

$$\eta = P_s^{-1} \int_{-\infty}^{\infty} G_i(f) \frac{\sin^2(\pi f T)}{(\pi f T)^2} df \quad (113)$$

and

$$\begin{aligned} \xi &= 2\lambda P_s^{-1} \int_{-\infty}^{\infty} G_i(f) \frac{\sin^2(\pi f T)}{(\pi f T)^2} df \\ &= \frac{2T}{\Phi} \int_{-\infty}^{\infty} G_i(f) \frac{\sin^2(\pi f T)}{(\pi f T)^2} df \end{aligned} \quad (114)$$

D. Effect of Sinusoidal Interference

1. Interference-to-signal and interference-to-noise ratios. If the interfering signal is a sine wave of power P_i and frequency ω_i ,

$$i(t) = (2P_i)^{1/2} \sin(\omega_i t + \phi_i) \quad (115)$$

where the phase is a time-invariant, random variable with probability density

$$p(\phi_i) = \begin{cases} (2\pi)^{-1}, & |\phi_i| \leq \pi \\ 0, & |\phi_i| > \pi \end{cases} \quad (116)$$

Then

$$\begin{aligned} R_i(\tau) &= P_i E[2 \sin(\omega_i t + \phi_i) \sin(\omega_i t + \omega_i \tau + \phi_i)] \\ &= P_i \{E[\cos(\omega_i \tau)] + E[\cos(2\omega_i t + \omega_i \tau + 2\phi_i)]\} \\ &= P_i \cos(\omega_i \tau) \end{aligned} \quad (117)$$

and

$$\begin{aligned} G_i(f) &= \int_{-\infty}^{\infty} P_i \cos(\omega_i \tau) \exp(-i\omega \tau) d\tau \\ &= \frac{P_i}{2} \int_{-\infty}^{\infty} \{\exp[i(\omega_i - \omega)\tau] + \exp[-i(\omega_i + \omega)\tau]\} d\tau \\ &= \frac{P_i}{2} [\delta(\omega + \omega_i) + \delta(\omega - \omega_i)] \end{aligned} \quad (118)$$

Using Eq. (118) in Eq. (102)

$$\begin{aligned}\eta &= (4\lambda)^{-2} \int_{-\infty}^{\infty} \frac{P_i}{2} [\delta(\omega + \omega_i) + \delta(\omega - \omega_i)] A_F^2(\omega) d\omega \\ &= (4\lambda)^{-2} \frac{P_i}{2} [A_F^2(-\omega_i) + A_F^2(\omega_i)]\end{aligned}\quad (119)$$

or, using Eq. (99),

$$\eta = \frac{P_i A_F^2(\omega_i)}{(4\lambda)^2} \quad (120)$$

Similarly

$$\xi = \frac{P_i A_F^2(\omega_i)}{8\lambda} \quad (121)$$

Using Eq. (112), for the case of antipodal, binary-valued message signals,

$$\eta = \frac{P_i}{P_s} \frac{\sin^2(\pi f_i T)}{(\pi f_i T)^2} \quad (122)$$

and

$$\xi = 2 \frac{P_i T}{\Phi} \frac{\sin^2(\pi f_i T)}{(\pi f_i T)^2} \quad (123)$$

2. Receiver error probability. Having simplified the expressions for η and ξ , the next step is to evaluate the receiver error probability. Using Eq. (115) in Eq. (82),

$$z_i = \frac{2}{\Phi} \int_0^T (2P_i)^{1/2} \sin(\omega_i t + \phi_i) [s(0;t) - s(1;t)] dt \quad (124)$$

or setting $t = T - \tau$ and using Eq. (52)

$$\begin{aligned}z_i &= \int_0^T \frac{2}{\Phi} [s(0;T - \tau) - s(1;T - \tau)] \\ &\quad \times (2P_i)^{1/2} \sin[\omega_i(T - \tau) + \phi_i] d\tau \\ &= \int_0^\infty h_F(\tau) (2P_i)^{1/2} \sin[\omega_i(T - \tau) + \phi_i] d\tau\end{aligned}\quad (125)$$

Using $Re(z)$ and $Im(z)$ to denote the real and imaginary parts of a complex variable z ,

$$\begin{aligned}z_i &= Im \left\{ \int_0^\infty h_F(\tau) (2P_i)^{1/2} \exp[i(\omega_i T - \omega_i \tau + \phi_i)] d\tau \right\} \\ &= (2P_i)^{1/2} Im \left\{ \exp[i(\omega_i T + \phi_i)] \int_0^\infty h_F(\tau) \exp(-\omega_i \tau) d\tau \right\}\end{aligned}\quad (126)$$

Using Eqs. (93) and (96),

$$\begin{aligned}z_i &= (2P_i)^{1/2} Im \{ \exp[i(\omega_i T + \phi_i)] H_F(+i\omega_i) \} \\ &= (2P_i)^{1/2} A_F(\omega_i) Im \{ \exp[i(\omega_i T + \phi_i + \phi_F(\omega_i))] \} \\ &= (2P_i)^{1/2} A_F(\omega_i) \sin[\omega_i T + \phi_i + \phi_F(\omega_i)]\end{aligned}\quad (127)$$

Using Eq. (120) and defining

$$u = \omega_i T + \phi_i + \phi_F(\omega_i) \quad (128)$$

$$z_i = 4\lambda (2\eta)^{1/2} \sin(u) \quad (129)$$

where

$p(u) =$

$$\begin{cases} (2\pi)^{-1}, \omega_i T - \pi + \phi_F(\omega_i) \leq \phi_i \leq \omega_i T + \pi + \phi_F(\omega_i) \\ 0, \phi_i < \omega_i T - \pi + \phi_F(\omega_i), \phi_i > \omega_i T + \pi + \phi_F(\omega_i) \end{cases} \quad (130)$$

Thus, using Eqs. (130) and (67) in Eq. (81),

$$z = 4\lambda [(-1)^\alpha + (2\eta)^{1/2} \sin u] + z_n \quad (131)$$

Since z_n is a gaussian random variable with mean zero and variance 8λ ,

$$\begin{aligned}p(z | \alpha, u) &= (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\ &\quad + (2\eta)^{1/2} \sin u]\}^2]\end{aligned}\quad (132)$$

Then

$$\begin{aligned}
p(z|\alpha) &= \int_{-\infty}^{\infty} p(z|\alpha, u) p(u) du \\
&= (2\pi)^{-1} \int_{\omega_i T - \pi + \phi_F(\omega_i)}^{\omega_i T + \pi + \phi_F(\omega_i)} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda \\
&\quad \times [(-1)^\alpha + (2\eta)^{1/2} \sin u]\}^2] du
\end{aligned} \tag{133}$$

or

$$\begin{aligned}
p(z|\alpha) &= (2\pi)^{-1} \int_{+\omega_i T - \pi + \phi_F(\omega_i)}^{\pi} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du \\
&\quad + (2\pi)^{-1} \int_{\pi}^{\omega_i T + \pi + \phi_F(\omega_i)} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda \\
&\quad \times [(-1)^\alpha + (2\eta)^{1/2} \sin u]\}^2] du
\end{aligned} \tag{134}$$

Observing that

$$\sin(u \pm 2\pi) = \sin u \tag{135}$$

and substituting $u - 2\pi$ for u in the second integral of Eq. (101) yields

$$\begin{aligned}
p(z|\alpha) &= (2\pi)^{-1} \int_{-\pi}^{\pi} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du
\end{aligned} \tag{136}$$

Expanding again

$$\begin{aligned}
p(z|\alpha) &= (2\pi)^{-1} \int_{-\pi}^{-\pi/2} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du \\
&\quad + (2\pi)^{-1} \int_{-\pi/2}^{\pi/2} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du \\
&\quad + (2\pi)^{-1} \int_{\pi/2}^{\pi} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du
\end{aligned} \tag{137}$$

Observing that

$$\sin(-\pi - u) = \sin(-\pi) \cos u - \cos(-\pi) \sin u = \sin u \tag{138}$$

and

$$\sin(\pi - u) = \sin(\pi) \cos u - \cos(\pi) \sin u = \sin u \tag{139}$$

and substituting $-\pi - u$ for u in the first integral of Eq. (137) and $\pi - u$ for u in the third integral of Eq. (137),

$$\begin{aligned}
p(z|\alpha) &= \pi^{-1} \int_{-\pi/2}^{\pi/2} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \{z - 4\lambda [(-1)^\alpha \\
&\quad + (2\eta)^{1/2} \sin u]\}^2] du
\end{aligned} \tag{140}$$

Then

$$\begin{aligned}
\int_{-\infty}^0 p(z|0) dz &= (\pi)^{-1} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^0 (16\pi\lambda)^{-1/2} \\
&\quad \exp[-(16\lambda)^{-1} \{z - 4\lambda [1 + (2\eta)^{1/2} \sin u]\}^2] dz du
\end{aligned} \tag{141}$$

or, setting

$$t = \frac{z - 4\lambda [1 + (2\eta)^{1/2} \sin u]}{4\lambda^{1/2}} \tag{142}$$

$$\begin{aligned}
\int_{-\infty}^0 p(z|0) dz &= \\
&\quad \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]} \exp(-t^2) dt
\end{aligned} \tag{143}$$

Comparing Eq. (143) with Eq. (73),

$$\begin{aligned}
\int_{-\infty}^0 p(z|0) dz &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}\{\lambda^{1/2} \\
&\quad \times [1 + (2\eta)^{1/2} \sin u]\}] du
\end{aligned} \tag{144}$$

Similarly,

$$\begin{aligned}
\int_0^{\infty} p(z|1) dz &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^{\infty} (16\pi\lambda)^{-1/2} \exp[-(16\lambda)^{-1} \\
&\quad \times \{z - 4\lambda [-1 + (2\eta)^{1/2} \sin u]\}^2] dz du
\end{aligned} \tag{145}$$

or setting

$$t = \frac{-z' + 4\lambda [-1 + (2\eta)^{1/2} \sin u]}{4\lambda^{1/2}} \quad (146)$$

$$\begin{aligned} \int_0^\infty p(z|1) dz &= \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{-\lambda^{1/2} [1 - (2\eta)^{1/2} \sin u]} \exp(-t^2) dt du \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{\lambda^{1/2} \\ &\quad \times [1 - (2\eta)^{1/2} \sin u]\}] du \quad (147) \end{aligned}$$

Substituting $-u$ for u in Eq. (147) and observing that

$$\sin(-u) = -\sin u \quad (148)$$

$$\begin{aligned} \int_0^\infty p(z|1) dz &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} \\ &[1 - \text{Erf} \{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]\}] du \quad (149) \end{aligned}$$

Therefore, substituting Eqs. (144) and (149) into Eq. (54),

$$\begin{aligned} P_E &= p_S(\lambda; \eta) \\ &= p(0) \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]\}] du \\ &\quad + p(1) \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]\}] du \quad (150) \end{aligned}$$

Thus, for sinusoidal interfering signals,

$$\begin{aligned} P_E &= p_S(\lambda; \eta) \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]\}] du \quad (151) \end{aligned}$$

and using Eq. (104),

$$\begin{aligned} P_E &= p_S\left(\lambda; \frac{\xi}{2\lambda}\right) \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2} + \xi^{1/2} \sin u)] du \quad (152) \end{aligned}$$

3. Asymptotic results. Examining Eq. (151), we note that

$$\lim_{\lambda \rightarrow \infty} \{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin(u)]\} = \begin{cases} \infty, \eta < \frac{1}{2}, \\ \infty, \eta \geq \frac{1}{2}, u > -\sin^{-1} [(2\eta)^{-1/2}] \\ -\infty, \eta \geq \frac{1}{2}, u < -\sin^{-1} [(2\eta)^{-1/2}] \end{cases} \quad (153)$$

Since

$$\text{Erf}(\pm\infty) = \pm 1 \quad (154)$$

for sinusoidal interfering signals, when η is constant,

$$\lim_{\lambda \rightarrow \infty} P_E = \lim_{\lambda \rightarrow \infty} p_S(\lambda; \eta) = 0, \eta < \frac{1}{2} \quad (155)$$

and

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} P_E &= \lim_{\lambda \rightarrow \infty} p_S(\lambda; \eta) = \frac{1}{\pi} \int_{-\pi/2}^{-\sin^{-1} [(2\eta)^{-1/2}]} du \\ &= \frac{1}{2} - \frac{1}{\pi} \sin^{-1} [(2\eta)^{-1/2}], \eta \geq \frac{1}{2} \quad (156) \end{aligned}$$

Examining Eq. (152), we note that

$$\lim_{\lambda \rightarrow \infty} (\lambda^{1/2} + \xi^{1/2} \sin u) = \infty \quad (157)$$

Thus, for sinusoidal interfering signals, when ξ is constant

$$\lim_{\lambda \rightarrow \infty} P_E = \lim_{\lambda \rightarrow \infty} p_S\left(\lambda; \frac{\xi}{2\lambda}\right) = 0 \quad (158)$$

4. Receiver degradation. The effect of an interfering signal on the receiver may be measured in terms of the factor δ by which the parameter λ must be increased to compensate for the presence of the interfering signal.

For sinusoidal interfering signals, when η is constant, provided

$$\lim_{\delta \rightarrow \infty} p_S(\delta\lambda; \eta) = \lim_{\lambda \rightarrow \infty} p_S(\lambda; \eta) < p(\lambda) \quad (159)$$

or, using Eqs. (78) and (151)

$$\frac{1}{2} - \frac{1}{\pi} \sin^{-1} [(2\eta)^{-1/2}] < \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2})], \quad \eta > \frac{1}{2} \quad (160)$$

the receiver degradation is finite and the solution of the equation

$$p_s(\delta\lambda; \eta) = p(\lambda) \quad (161)$$

Thus for $\eta > 1/2$, if λ_0 is the solution of

$$\text{Erf}(\lambda_0^{1/2}) = \frac{2}{\pi} \sin^{-1} [(2\eta)^{-1/2}], \quad \eta > \frac{1}{2} \quad (162)$$

or, asymptotically,

$$\lambda_0 = (2\pi\eta)^{-1}, \quad \eta \gg 1 \quad (163)$$

δ is finite for $\lambda < \lambda_0$ and infinite for $\lambda \geq \lambda_0$. For $\eta < 1/2$, δ is finite for all values of λ . Using Eqs. (78) and (151) in Eq. (161), the receiver degradation δ is the solution of

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf} \{ \delta^{1/2} \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \\ = \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2})] \end{aligned} \quad (164)$$

whenever $\eta < 1/2$ or $\eta \geq 1/2$, $\lambda < \lambda_0$.

For sinusoidal interfering signals, when ξ is constant,

$$\lim_{\delta \rightarrow \infty} p_s \left(\delta\lambda; \frac{\xi}{2\delta\lambda} \right) = \lim_{\lambda \rightarrow \infty} p_s \left(\lambda; \frac{\xi}{2\lambda} \right) = 0 \leq p(\lambda) \quad (165)$$

for all values of λ . Hence δ is finite for all values of λ . Thus, δ is the solution of

$$p_s \left(\delta\lambda; \frac{\xi}{2\delta\lambda} \right) = p(\lambda) \quad (166)$$

or, using Eqs. (78) and (152),

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} [1 - \text{Erf}(\delta^{1/2} \lambda^{1/2} + \xi^{1/2} \sin u)] du \\ = \frac{1}{2} [1 - \text{Erf}(\lambda^{1/2})] \end{aligned} \quad (167)$$

5. Bounds on receiver degradation. For sinusoidal interfering signals, when η is constant, the receiver degradation δ is the solution of Eq. (164). By introducing the function

$$A_1(\delta) = \int_{-\pi/2}^{\pi/2} [\text{Erf}(\lambda^{1/2}) - \text{Erf} \{ \delta^{1/2} \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \quad (168)$$

we may write Eq. (164) in the form

$$A_1(\delta) = 0 \quad (169)$$

Then

$$\begin{aligned} A_1(1) = \int_{-\pi/2}^0 [\text{Erf}(\lambda^{1/2}) - \text{Erf} \{ \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \\ + \int_0^{\pi/2} [\text{Erf}(\lambda^{1/2}) - \text{Erf} \{ \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \end{aligned} \quad (170)$$

Substituting $-u$ for u in the first integral of Eq. (170) and using Eq. (148),

$$\begin{aligned} A_1(1) = \int_0^{\pi/2} [2 \text{Erf}(\lambda^{1/2}) - \text{Erf} \{ \lambda^{1/2} [1 - (2\eta)^{1/2} \sin u] \} \\ - \text{Erf} \{ \lambda^{1/2} [1 + (2\eta)^{1/2} \sin u] \}] du \end{aligned} \quad (171)$$

or, using Eq.(74),

$$\begin{aligned} A_1(1) = \int_0^{\pi/2} \left[\int_{\lambda^{1/2} [1 - (2\eta)^{1/2} \sin u]}^{\lambda^{1/2}} \exp(-t^2) dt \right. \\ \left. - \int_{\lambda^{1/2}}^{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]} \exp(-t^2) dt \right] du \end{aligned} \quad (172)$$

Since λ and η are positive,

$$\begin{aligned} \int_{\lambda^{1/2} [1 - (2\eta)^{1/2} \sin u]}^{\lambda^{1/2}} \exp(-t^2) dt \\ - \int_{\lambda^{1/2}}^{\lambda^{1/2} [1 + (2\eta)^{1/2} \sin u]} \exp(-t^2) dt \geq 0, \\ 0 \leq u \leq \frac{\pi}{2} \end{aligned} \quad (173)$$

and hence

$$A_1(1) \geq 0 \quad (174)$$

For $\eta < \frac{1}{2}$,

$$A_1([1 - (2\eta)^{1/2}]^{-2}) = \int_{-\pi/2}^{\pi/2} \left\{ \text{Erf}(\lambda^{1/2}) - \text{Erf} \left[\lambda^{1/2} \frac{1 + (2\eta)^{1/2} \sin u}{1 - (2\eta)^{1/2}} \right] \right\} du \quad (175)$$

Since

$$1 + (2\eta)^{1/2} \sin u > 1 - (2\eta)^{1/2}, \quad \eta < \frac{1}{2}, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \quad (176)$$

$$\text{Erf}(\lambda^{1/2}) - \text{Erf} \left[\lambda^{1/2} \frac{1 + (2\eta)^{1/2} \sin u}{1 - (2\eta)^{1/2}} \right] \leq 0, \quad \eta < \frac{1}{2}, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \quad (177)$$

and therefore

$$A_1([1 - (2\eta)^{1/2}]^{-2}) \leq 0, \quad \eta < \frac{1}{2} \quad (178)$$

Since $A_1(\delta)$ is positive for $\delta = 1$ and negative for $\delta = [1 - (2\eta)^{1/2}]^{-2}$, for $\eta < 1/2$, Eq. (169) must have a solution for δ on the interval $\{1, [1 - (2\eta)^{1/2}]^{-2}\}$. Thus, for sinusoidal interfering signals, when η is constant,

$$1 \leq \delta \leq [1 - (2\eta)^{1/2}]^{-2}, \quad \eta < \frac{1}{2} \quad (179)$$

For sinusoidal interfering signals, when ξ is constant, the receiver degradation δ is the solution of Eq. (167). By introducing the function

$$A_2(\delta) = \int_{-\pi/2}^{\pi/2} [\text{Erf}(\lambda^{1/2}) - \text{Erf}(\delta^{1/2} \lambda^{1/2} + \xi^{1/2} \sin u)] du \quad (180)$$

we may write Eq. (167) in the form

$$A_2(\delta) = 0 \quad (181)$$

However,

$$A_2(1) = \int_{-\pi/2}^0 [\text{Erf}(\lambda^{1/2}) - \text{Erf}(\lambda^{1/2} + \xi^{1/2} \sin u)] du + \int_0^{\pi/2} [\text{Erf}(\lambda^{1/2}) - \text{Erf}(\lambda^{1/2} + \xi^{1/2} \sin u)] du \quad (182)$$

Substituting $-u$ for u in the first integral of Eq. (182) and using Eq. (148),

$$A_2(1) = \int_0^{\pi/2} [2 \text{Erf}(\lambda^{1/2}) - \text{Erf}(\lambda^{1/2} - \xi^{1/2} \sin u) - \text{Erf}(\lambda^{1/2} + \xi^{1/2} \sin u)] du \quad (183)$$

or, using Eq. (74),

$$A_2(1) = \int_0^{\pi/2} \left[\int_{\lambda^{1/2} - \xi^{1/2} \sin u}^{\lambda^{1/2}} \exp(-t^2) dt - \int_{\lambda^{1/2}}^{\lambda^{1/2} + \xi^{1/2} \sin u} \exp(-t^2) dt \right] du \quad (184)$$

Since λ and ξ are positive,

$$\int_{\lambda^{1/2} - \xi^{1/2} \sin u}^{\lambda^{1/2}} \exp(-t^2) dt - \int_{\lambda^{1/2}}^{\lambda^{1/2} + \xi^{1/2} \sin u} \exp(-t^2) dt \geq 0, \quad 0 \leq u \leq \frac{\pi}{2} \quad (185)$$

and hence

$$A_2(1) \geq 0 \quad (186)$$

Moreover,

$$A_2 \left(\left[1 + \left(\frac{\xi}{\lambda} \right)^{1/2} \right]^2 \right) = \int_{-\pi/2}^{\pi/2} \left[\text{Erf}(\lambda^{1/2}) - \text{Erf} \left\{ \lambda^{1/2} \left[1 + \left(\frac{\xi}{\lambda} \right)^{1/2} \right] + \xi^{1/2} \sin u \right\} \right] du = \int_{-\pi/2}^{\pi/2} \{ \text{Erf}(\lambda^{1/2}) - \text{Erf}[\lambda^{1/2} + \xi^{1/2}(1 + \sin u)] \} du \quad (187)$$

Since

$$1 + \sin u \geq 0, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad (188)$$

$$\begin{aligned} \text{Erf}(\lambda^{1/2}) - \text{Erf}[\lambda^{1/2} + \xi^{1/2}(1 + \sin u)] &\leq 0, \\ -\frac{\pi}{2} &\leq u \leq \frac{\pi}{2} \end{aligned} \quad (189)$$

and hence

$$A_2 \left(\left[1 + \left(\frac{\xi}{\lambda} \right)^{1/2} \right]^2 \right) \leq 0 \quad (190)$$

Since $A_2(\delta)$ is positive for $\delta = 1$ and negative for $\delta = [1 + (\xi/\lambda)^{1/2}]^2$, Eq. (181) has a solution for δ on the interval $\{1, [1 + (\xi/\lambda)^{1/2}]^2\}$. Thus, for sinusoidal interfering signals, when ξ is constant,

$$1 \leq \delta \leq \left[1 + \left(\frac{\xi}{\lambda} \right)^{1/2} \right]^2 \quad (191)$$

E. Effect of Gaussian Interference

1. Receiver error probability. If $i(t)$ is a gaussian random process, z_i is a gaussian random variable with zero mean and, from Eqs. (101) and (102), variance

$$\sigma_{z_i}^2 = \eta(4\lambda)^2 \quad (192)$$

Since z_n is a gaussian random variable with mean zero and variance 8λ , if $i(t)$ and $n(t)$ are statistically independent,

$$z = (-1)^\alpha 4\lambda + z_i + z_n \quad (193)$$

is, given α , a gaussian random variable with mean $(-1)^\alpha 4\lambda$ and variance

$$\begin{aligned} \sigma_{z_i}^2 + \sigma_{z_n}^2 &= \eta(4\lambda)^2 + 8\lambda \\ &= 8\lambda(1 + 2\eta\lambda) \end{aligned} \quad (194)$$

Thus

$$\begin{aligned} p(z | \alpha) &= (16\pi\lambda)^{-1/2} (1 + 2\eta\lambda)^{-1/2} \\ \exp \{ -[16\lambda(1 + 2\eta\lambda)]^{-1} [z - (-1)^\alpha 4\lambda]^2 \} \end{aligned} \quad (195)$$

Thus, using Eq. (74),

$$\begin{aligned} \int_{-\infty}^0 p(z | 0) dz &= \frac{1}{2} \left\{ 1 - \text{Erf} \left[\frac{4\lambda}{4\lambda^{1/2}(1 + 2\eta\lambda)^{1/2}} \right] \right\} \\ &= \frac{1}{2} \{ 1 - \text{Erf} [\lambda^{1/2}(1 + 2\eta\lambda)^{-1/2}] \} \end{aligned} \quad (196)$$

and, using Eq. (75)

$$\begin{aligned} \int_0^\infty p(z | 1) dz &= \frac{1}{2} \left\{ 1 + \text{Erf} \left[\frac{-4\lambda}{4\lambda^{1/2}(1 + 2\eta\lambda)^{1/2}} \right] \right\} \\ &= \frac{1}{2} \{ 1 - \text{Erf} [\lambda^{1/2}(1 + 2\eta\lambda)^{-1/2}] \} \end{aligned} \quad (197)$$

Thus, using Eqs. (196) and (197) in Eq. (54) for internally generated, gaussian interfering signals,

$$\begin{aligned} P_E &= p_G(\lambda; \eta) \\ &= \frac{1}{2} \{ 1 - \text{Erf} [\lambda^{1/2}(1 + 2\eta\lambda)^{-1/2}] \} \end{aligned} \quad (198)$$

and using Eq. (104),

$$\begin{aligned} P_E &= p_G \left(\lambda; \frac{\xi}{2\lambda} \right) \\ &= \frac{1}{2} \{ 1 - \text{Erf} [\lambda^{1/2}(1 + \xi)^{-1/2}] \} \end{aligned} \quad (199)$$

2. Asymptotic results. Having written Eq. (198) in the form

$$\begin{aligned} P_E &= p_G(\lambda; \eta) \\ &= \frac{1}{2} \{ 1 - \text{Erf} [(\lambda^{-1} + 2\eta)^{-1/2}] \} \end{aligned} \quad (200)$$

it is clear that, for gaussian interfering signals, when η is constant,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} P_E &= \lim_{\lambda \rightarrow \infty} p_G(\lambda; \eta) \\ &= \frac{1}{2} \{ 1 - \text{Erf} [(2\eta)^{-1/2}] \} \end{aligned} \quad (201)$$

Examining Eq. (199), we note that for gaussian interfering signals, when ξ is constant,

$$\lim_{\lambda \rightarrow \infty} P_E = \lim_{\lambda \rightarrow \infty} p_G \left(\lambda; \frac{\xi}{2\lambda} \right) = 0 \quad (202)$$

3. Receiver degradation. For gaussian interfering signals, when η is constant, provided

$$\lim_{\delta \rightarrow \infty} p_G(\delta\lambda; \eta) = \lim_{\lambda \rightarrow \infty} p_G(\lambda; \eta) < p(\lambda) \quad (203)$$

or, using Eqs. (78) and (201),

$$\frac{1}{2} \{1 - \text{Erf} [(2\eta)^{-1/2}]\} < \frac{1}{2} [1 - \text{Erf} (\lambda^{1/2})] \quad (204)$$

the receiver degradation δ is finite and the solution of the equation

$$p_G(\delta\lambda; \eta) = p(\lambda) \quad (205)$$

Thus, if

$$\lambda_0 = (2\eta)^{-1} \quad (206)$$

the receiver degradation is finite for $\lambda < \lambda_0$ and infinite for $\lambda > \lambda_0$.

Using Eqs. (78) and (198) in Eq. (205), for $\lambda < \lambda_0$, δ is the solution of the equation

$$\frac{1}{2} \{1 - \text{Erf} [(\delta\lambda)^{1/2} (1 + 2\eta\delta\lambda)^{-1/2}]\} = \frac{1}{2} [1 - \text{Erf} (\lambda^{1/2})] \quad (207)$$

or, equivalently, the equation

$$\frac{\delta}{1 + 2\eta\delta\lambda} = 1 \quad (208)$$

Thus, provided $\lambda < \lambda_0$, for gaussian interfering signals, when η is constant,

$$\delta = (1 - 2\eta\lambda)^{-1} \quad (209)$$

For gaussian interfering signals, when ξ is constant,

$$\lim_{\delta \rightarrow \infty} p_G \left(\delta\lambda; \frac{\xi}{2\delta\lambda} \right) = \lim_{\lambda \rightarrow \infty} p_G \left(\lambda; \frac{\xi}{2\lambda} \right) = 0 \leq p(\lambda) \quad (210)$$

for all values of λ , and thus δ is finite and the solution of the equation

$$p_G \left(\delta\lambda; \frac{\xi}{2\delta\lambda} \right) = p(\lambda) \quad (211)$$

for all values of λ . Using Eqs. (78) and (199), we may write Eq. (211) in the form

$$\frac{1}{2} \{1 - \text{Erf} [\delta^{1/2} \lambda^{1/2} (1 + \xi)^{-1/2}]\} = \frac{1}{2} [1 - \text{Erf} (\lambda^{1/2})] \quad (212)$$

Thus, for gaussian interfering signals, when ξ is constant,

$$\delta = 1 + \xi \quad (213)$$

References

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