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AN INVESTIGATION OF THE OPTIMUM DESIGN AND FLIGHT OF ROCKETS

by John D. Bird

Langley Research Center
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AN INVESTIGATION OF THE OPTIMUM DESIGN AND FLIGHT OF ROCKETS*

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SUMMARY

An analysis using classical variational methods was made whereby significant physical characteristics of rockets were determined in an optimal sense. A two-point boundary-value problem was formulated which, when solved, provided the significant physical parameters and flight path for a two-stage rocket which would deliver a given payload to orbit with minimum total initial mass. Numerical solutions were carried out for nine cases by an iterative method. The results show how the stage masses and thrusts should be proportioned for various structural efficiencies in the fuel tanks and rocket motors. Analyses of the type carried out for this investigation should be useful in the sizing and, generally, in the design of rockets.

INTRODUCTION

Many applications of variational methods exist wherein the flight paths of rockets or aircraft are optimized. In general, a solution is found for the flight path that will enable the vehicle to deliver the maximum payload to orbit, arrive at the terminal conditions in minimum time, or accomplish the mission in some other way that is regarded as being most efficient. Another, and in some ways a more complete problem than is normally treated by variational methods, involves solution for both the flight path and the vehicle configuration which will deliver a given payload to desired terminal conditions with the object of minimizing the initial mass of the vehicle or some other quantity of significance.

The purpose of this paper is to present an investigation of this latter problem for a rocket application as completely as seems reasonable for a generalized analysis of relations between performance and structural design parameters and to illustrate the effects of substantial changes in structural efficiency as might be representative of

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various degrees of conservatism in design practice. Solutions were found for the most efficient two-stage rocket from the standpoint of minimum initial mass that would place a given payload in a circular orbit about the earth at an altitude of 370 400 meters (200 int. n. mi.) with a continuous burn of the motors. In these solutions the burn times and thrusts of the two stages were determined as well as the steering of the thrust vector.

In order to formulate this problem, a brief analysis of engineering data on the construction of rockets was made to determine the significant parameters in defining the mass of a rocket. This analysis enabled expression of the mass of a two-stage rocket in terms of certain constants representative of good construction and of the burn times and the thrusts of the two stages.

Several papers have been written that contribute to the solution of problems of this type. (See refs. 1 to 5.) Bliss, in reference 1, gives the basic multiplier rule of the calculus of variations. Denbow, in reference 2, derived necessary and sufficient conditions for a minimizing arc for problems having free corner points. His solution is restricted to the case where the state variables are continuous at the corners. Mason, Dickerson, and Smith (ref. 3) have extended the multiplier rule formulation of Denbow to the case where discontinuities at various points in the minimizing arc may be treated as functions of the state and independent variables at the ends and the points of discontinuity of the minimizing arc. This extension allows the optimization of the relative sizes of rocket stages. References 3, 4, and 5 give examples of this type of problem and its solution.

Lockheed Missiles & Space Company has had good success with problems of this type by use of the method of steepest descents. Reference 6 gives examples of solutions for problems similar to those solved in this paper. With the exception of Lockheed's work, the thrust levels in the various stages and the structural elements between stages have not been included in analyses of this type.

For this investigation the methods of the classical calculus of variations as given in reference 3 are used to derive the necessary conditions in the form of differential and algebraic equations that must be solved in order to obtain the desired result. The methods result in a two-point boundary-value problem wherein part of the conditions on the variables is known at each end of the problem. Solutions are presented for nine cases that are representative of a range of structural design practices from very heavy tanks, motors, and interstage structures to zero mass structural elements to show the influence of design practice. A comparison is made with a nominal design for the purpose of illustrating the benefit of configuration optimization. This nominal design was chosen for the beginning of the iterative solution process.

SYMBOLS

A	cross-sectional area of rocket, 37.12 meters^2
C_D	drag coefficient, 0.5
E	admissible solution
$\bar{F}$	force vector
F,G,g	functions of given arguments
g_e	gravitational acceleration at earth surface, 9.807 meters per second^2
H	H-function of Bliss and Pontryagin
h	altitude, $r - r_e$, meters
I	specific impulse of rocket fuel, 400 seconds
$\hat{i},\hat{j}$	unit vectors of polar coordinate system
J	functional of given form
k_M	rocket motor structural mass fraction
k_S	rocket interstage structural mass fraction
k_T	rocket fuel tank structural mass fraction
l_μ	constant multipliers
m	mass, kilograms
m_p	mass of payload, 4207 kilograms
n,m,q,r	designates number of variables or functions in series, appendix B
q_{max}	maximum dynamic pressure, newtons per meter^2

$\bar{r}$	radius vector
r	magnitude of radius vector, $\|\bar{r}\|$, meters
r_e	radius of earth, 6 371 203 meters
t	time, seconds
t_a	independent variable, appendix B
t_c	value of t at a corner, appendix B
t_1	time at initiation of rocket flight
t_2	time of first-stage burnout, staging time
t_3	time of second-stage burnout at orbital injection
t_1^+	refers to instant after t_1
t_2^-	refers to instant prior to t_2
t_2^+	refers to instant after t_2
t_3^-	refers to instant prior to t_3
$\bar{u}$	control force vector
u	magnitude of control force vector, $\|\bar{u}\|$, newtons
$\hat{u}$	unit control force vector
x_1	magnitude of radius vector, r, meters
x_2	time rate of change of magnitude of radius vector, $\dot{r}$, meters per second
x_3	angular displacement of radius vector, θ, degrees
x_4	time rate of change of angular displacement of radius vector, $\dot{\theta}$

x_5 rocket mass, m

x_6 inclination of thrust vector to local horizontal, β, degrees

x_7 magnitude of thrust vector, u

$x_{20},x_{21},x_{22},x_{23}$ functions defined by equations (15)

α angle of attack, degrees

β inclination of thrust vector to local horizontal, x_6, degrees

γ flight-path angle relative to local horizontal, degrees

θ angular displacement of radius vector, x_3, degrees

λ_j Lagrange multipliers, functions of time

λ_0 constant multiplier

ν constant for atmospheric density exponential function, 1/6705 per meter

$\rho(h)$ mass density of atmosphere as a function of altitude

ρ_e mass density of atmosphere at earth surface, taken as 1.752 kilograms per meter^3

ϕ_j first-order differential function

ψ_μ algebraic function

A dot over a symbol denotes a time derivative; the form F_{x_i} denotes a partial derivative; a repeated subscript indicates a summation over that index.

The notation $|\;|$ indicates the magnitude of a vector.

ANALYSIS AND PROBLEM FORMULATION

Assumptions

In the solution of this problem a number of assumptions were made in an effort to reduce a rather complex problem to a form for which numerical solutions could be made with a minimum of programing difficulties. More complete problems are substantially more difficult in programing, checkout, and solution. The assumptions made for this investigation are summarized as follows for convenience. Some of these assumptions are discussed later.

(1) The initial mass of the rocket is taken as the significant parameter for optimization.

(2) Point mass equations of motion are used.

(3) A planar trajectory is used.

(4) The rocket is assumed to have a constant cross-sectional area.

(5) The aerodynamic drag coefficient of the rocket is assumed to be independent of Mach number and angle of attack, and the lift-curve slope of the rocket is assumed to be zero.

(6) The earth rotational velocity is assumed to be zero.

(7) An exponential function of altitude is assumed to represent adequately the atmospheric density.

(8) The effect of the atmospheric pressure on the rocket motor thrust is assumed to be zero.

In this investigation the assumption is made that the initial mass of the rocket is a significant parameter for minimization, and the problem formulation and numerical solutions are carried out on this basis. To some extent this assumption is valid but actually the problem is more complex because of the fact that rocket motor development is a long and expensive process whereas fuel tank enlargement is generally a simpler process. As a result, when the payload capacity of a rocket must be increased the least expensive step is to increase the fuel capacity and move into a less efficient mode of operation insofar as fuel consumption is concerned. Much additional thought needs to be given to the choice of an appropriate payoff quantity or quantity for optimization. A criterion based on cost would certainly seem to be most significant.

Point mass equations of motion are employed in this analysis. The necessary attitude changes should be slow and of little consequence in the overall trajectory determination. This assumption is common in trajectory analysis. Moreover for computational simplicity only motion in the vertical plane is considered which reduces the motion to one in two degrees of freedom in a plane passing through the center of the earth. The equations of motion are derived in appendix A. The rocket is assumed to have a constant cross-sectional area throughout the numerical solution process. Physically, this assumption requires that the fuel tank size be changed in the solution for the optimum by varying the length of the tank rather than by changing its diameter. An exponential functional representation was employed for the atmospheric density. This representation depends on the altitude and a single density gradient constant and has been shown to be reasonably accurate. (See ref. 7.)

Rocket Structural Mass

An analysis of available literature on liquid fuel rockets was made in order to establish what is presently considered to be good rocket design practice and to formulate rocket mass as a function of the significant parameters of the problem. As a result of this analysis the following relationships were established:

$$\left.\begin{aligned} m_{\text{rocket motor hardware}} &= k_M u \\ m_{\text{fuel tank}} &= k_T m_{\text{fuel}} \\ m_{\text{interstage structure}} &= k_S m_{\text{elements above}} \end{aligned}\right\} \qquad (1)$$

where k_M, k_T, and k_S are constants determined by design practice. The first relationship indicates that the mass of all hardware directly associated with a liquid fuel rocket motor is directly proportional to the magnitude of the vacuum thrust for which the motor was designed. The second relationship is generally well known among rocket designers and indicates that the mass of all hardware directly associated with the containment and transfer of liquid fuel or propellant is directly proportional to the mass of the fuel to be contained. This mass includes tanks, bulkheads, baffles, and major piping. The third relationship indicates that the mass of the interstage structure is directly proportional to the mass of all components mounted above it. This relationship indicates that the mass of a supporting structure is directly dependent on the mass to be supported as might well be assumed.

An element of mass not associated with rocket motor thrust or fuel mass, but rather more mission dependent, is associated with each stage of a rocket. This element of mass is associated with mission control and telemetry, in addition to other factors. No effort was made to account for this mass in the analysis although it is significant.

These relationships are naturally only valid for rockets constructed and erected in the earth gravitational and atmospheric environment. The interstage and tank structure might be appreciably less if the rocket could be built and erected in a lunar environment. No dependence on rocket acceleration is shown in these relationships. This is undoubtedly associated with the rather substantial structural mass needed for satisfactory handling and erection of a rocket under the earth gravitational and atmospheric conditions and to the approximate nature of the analysis and the limited amount of data readily available for analysis. Exponential relationships seem to give a more exact representation of the structural weights, but the linear relationships used here were felt to be adequate.

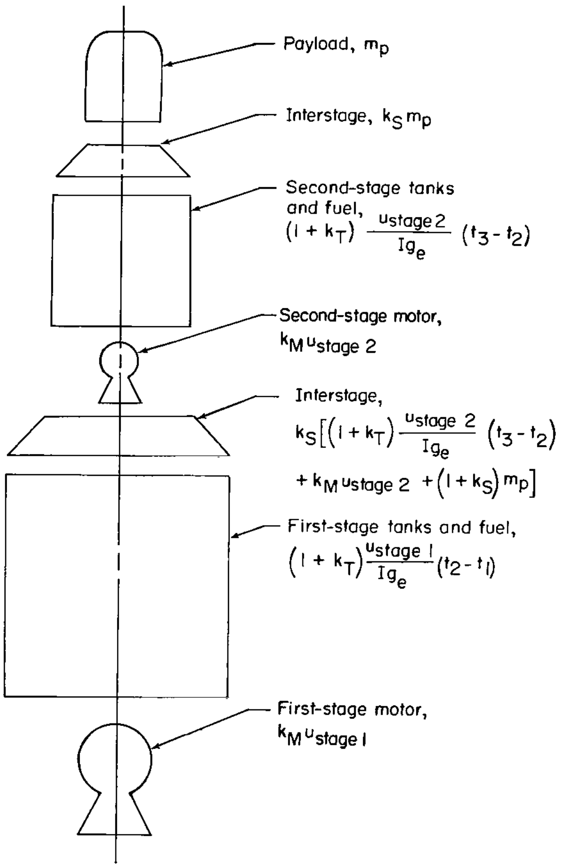

Figure 1.- Rocket components and masses at ignition time.

By use of these three relationships (eqs. (1)) and the knowledge that the thrust of a rocket depends on the specific impulse of the fuel used and the time rate of fuel consumption according to

$$u = -Ig_e \frac{dm}{dt} \tag{2}$$

an expression can be derived for the mass of a two-stage liquid fuel rocket as a function of time and certain parameters such as the constant thrust of each stage, the burning time of each stage, the payload mass, and the mass fractions determined here. This relation between thrust and mass-flow rate does not include a correction for atmospheric pressure. The relationship between rocket mass and parameters is based on the two-stage rocket structure shown schematically in figure 1. In this figure the mass of all elements of the rocket is given at ignition time t_1 in terms of the significant parameters of the problem. The general relationship for the mass of the rocket for any time during burn is

$$
\left.
\begin{aligned}
m(t) &= \left(1 + k_S\right)\left[\left(1 + k_S\right)m_p + \left(1 + k_T\right)\frac{u_{stage\ 2}}{Ig_e}\left(t_3 - t_2\right) + k_M u_{stage\ 2}\right] \\
&\quad + \left[\left(1 + k_T\right)\frac{u_{stage\ 1}}{Ig_e}\left(t_2 - t_1\right) + k_M u_{stage\ 1}\right] - \frac{u_{stage\ 1}}{Ig_e}\left(t - t_1\right) \qquad \left(t_1 < t < t_2\right) \\
m(t) &= \left(1 + k_S\right)m_p + \left(1 + k_T\right)\frac{u_{stage\ 2}}{Ig_e}\left(t_3 - t_2\right) + k_M u_{stage\ 2} - \frac{u_{stage\ 2}}{Ig_e}\left(t - t_2\right) \\
&\qquad \left(t_2 < t < t_3\right)
\end{aligned}
\right\} \tag{3}
$$

where m_p is the payload mass, t_1 is the ignition time, t_2 is the staging time, and t_3 is the time of burnout. The thrust magnitudes u are labeled for the appropriate stage. This formulation for the mass of the rocket is used in the derivation of the necessary conditions for a minimizing arc.

Formulation of the Two-Point Boundary-Value Problem

The problem formulation of concern here involves the determination of the minimum mass, two-stage rocket capable of placing a specified payload in a circular orbit at an altitude of 370 400 meters. The essential parameters of the rocket such as thrusts and stage burn times are to be determined along with the trajectory in space. The rocket coordinate geometry is shown in figure 2. The necessary conditions that must be satisfied in order to assure an extremal solution and, hopefully, a minimum solution to the variational problem at hand may be derived from the multiplier rule stated in appendix B.

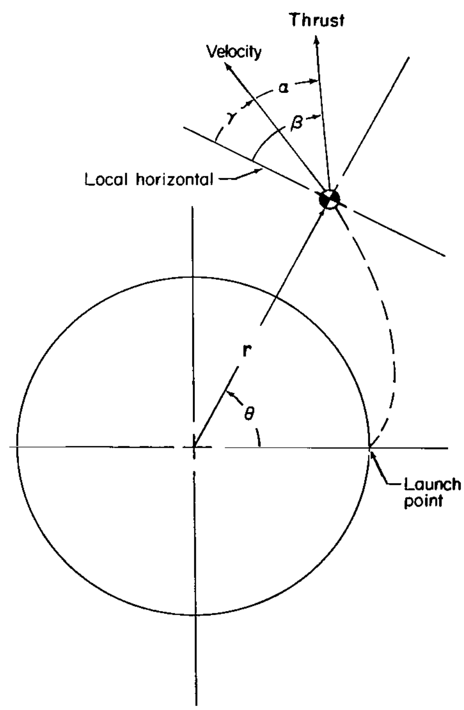

Figure 2.- Coordinate system.

The differential constraints of the problem may be put in first-order or state-vector form by appropriate substitutions. To accomplish this, the following terms are defined:

$$x_1 = r(t)$$

$$x_2 = \dot{r}(t)$$

$$x_3 = \theta(t)$$

$$x_4 = \dot{\theta}(t)$$

$$x_5 = m(t)$$

$$x_6 = \beta(t)$$

$$x_7 = u$$

where, also, x_6 has been introduced as the direction of the thrust vector $\bar{u}$ as given in figure 2 and x_7 has been introduced as the magnitude of the thrust. The magnitude of the thrust is fixed during any one stage; hence, its differential equation assumes a simple or null form, specifically $\dot{x}_7 = 0$. In line with the definitions,

$$\dot{x}_1 = x_2$$

$$\dot{x}_3 = x_4$$

Utilizing these definitions together with the equations of motion derived in appendix A and the formulation developed for the rocket mass (eq. (3)) enables the development of the following differential constraint functions for use in the formulation of the necessary conditions. These functions correspond to ϕ_j of appendix B.

$$\left.\begin{aligned}
&\phi_1(t,x,\dot{x}) = \dot{x}_1 - x_2 = 0 \\
&\phi_2(t,x,\dot{x}) = \dot{x}_2 - x_1 x_4^2 - \frac{x_7}{x_5}\sin x_6 + g_e r_e^2\left(\frac{1}{x_1^2}\right) + C_D A\,\frac{1}{2}\,\rho(h)\,\frac{x_2}{x_5}\left(x_1^2 x_4^2 + x_2^2\right)^{1/2} = 0 \\
&\phi_3(t,x,\dot{x}) = \dot{x}_3 - x_4 = 0 \\
&\phi_4(t,x,\dot{x}) = \dot{x}_4 + \frac{2x_2x_4}{x_1} - \frac{x_7}{x_1x_5}\cos x_6 + C_D A\,\frac{1}{2}\,\rho(h)\,\frac{x_4}{x_5}\left(x_1^2 x_4^2 + x_2^2\right)^{1/2} = 0 \\
&\phi_5(t,x,\dot{x}) = \dot{x}_5 + \frac{x_7}{Ig_e} = 0 \\
&\phi_7(t,x,\dot{x}) = \dot{x}_7 = 0
\end{aligned}\right\} \tag{4}$$

In terms of the state vector x_i the rocket mass as derived previously becomes

$$\left.\begin{aligned}
x_5 &= (1 + k_S)\left[(1 + k_S)m_p + (1 + k_T)\frac{x_7(t_2^+)}{Ig_e}(t_3 - t_2) + k_M x_7(t_2^+)\right] \\
&\quad + \left[(1 + k_T)\frac{x_7(t_1^+)}{Ig_e}(t_2 - t_1) + k_M x_7(t_1^+)\right] - \frac{x_7(t_1^+)}{Ig_e}(t - t_1) \qquad (t_1 < t < t_2) \\
x_5 &= (1 + k_S)m_p + (1 + k_T)\frac{x_7(t_2^+)}{Ig_e}(t_3 - t_2) + k_M x_7(t_2^+) - \frac{x_7(t_2^+)}{Ig_e}(t - t_2) \\
&\qquad (t_2 < t < t_3)
\end{aligned}\right\} \tag{5}$$

The function F defined in the multiplier rule of appendix B is given as

$$\begin{aligned}
F &= \lambda_j \phi_j \\
&= \lambda_1\left[\dot{x}_1 - x_2\right] \\
&\quad + \lambda_2\left[\dot{x}_2 - x_1 x_4^2 - \frac{x_7}{x_5}\sin x_6 + g_e r_e^2 \frac{1}{x_1^2} + C_D A \frac{1}{2}\rho(h)\frac{x_2}{x_5}\left(x_1^2 x_4^2 + x_2^2\right)^{1/2}\right] \\
&\quad + \lambda_3\left[\dot{x}_3 - x_4\right] + \lambda_4\left[\dot{x}_4 + \frac{2x_2x_4}{x_1} - \frac{x_7}{x_1x_5}\cos x_6 + C_D A \frac{1}{2}\rho(h)\frac{x_4}{x_5}\left(x_1^2 x_4^2 + x_2^2\right)^{1/2}\right] \\
&\quad + \lambda_5\left[\dot{x}_5 + \frac{x_7}{Ig_e}\right] + \lambda_6\left[0\right] + \lambda_7\left[\dot{x}_7\right]
\end{aligned} \tag{6}$$

Then from Euler's equation (eq. (B1)), $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = \frac{\partial F}{\partial x_i}$, the following equations may be obtained for arcs of continuous $\dot{x}_i$:

$$\begin{aligned}
\dot{\lambda}_1 &= -\lambda_2\left[x_4^2 + 2g_e r_e^2\left(\frac{I}{x_1^3}\right) - C_D A \frac{\rho(h)}{2}\frac{x_2}{x_5}\left(x_1^2x_4^2 + x_2^2\right)^{-1/2} x_1 x_4^2\right. \\
&\quad \left. - C_D A \frac{1}{2}\frac{\partial\rho(h)}{\partial x_1}\frac{x_2}{x_5}\left(x_1^2x_4^2 + x_2^2\right)^{1/2}\right] - \lambda_4\left[\frac{2x_2x_4}{x_1^2} - \frac{x_7}{x_1^2x_5}\cos x_6\right. \\
&\quad \left. - C_D A \frac{1}{2}\rho(h)\frac{x_4}{x_5}\left(x_1^2x_4^2 + x_2^2\right)^{-1/2} x_1x_4^2 - C_D A \frac{1}{2}\frac{\partial\rho(h)}{\partial x_1}\frac{x_4}{x_5}\left(x_1^2x_4^2 + x_2^2\right)^{1/2}\right]
\end{aligned} \tag{7a}$$

$$\dot{\lambda}_2 = -\lambda_1[1] - \lambda_2\left[-C_DA\,\frac{1}{2}\,\rho(h)\,\frac{1}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{1/2} - C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_2}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{-1/2} x_2\right]$$

$$- \lambda_4\left[-\frac{2x_4}{x_1} - C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_4}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{-1/2} x_2\right] \tag{7b}$$

$$\dot{\lambda}_3 = 0 \tag{7c}$$

$$\dot{\lambda}_4 = -\lambda_2\left[2x_1x_4 - C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_2}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{-1/2} x_1{}^2x_4\right] - \lambda_3[1]$$

$$-\lambda_4\left[-\frac{2x_2}{x_1} - C_DA\,\frac{1}{2}\,\rho(h)\,\frac{1}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{1/2} - C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_4}{x_5}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{1/2} x_1{}^2x_4\right] \tag{7d}$$

$$\dot{\lambda}_5 = -\lambda_2\left[-\frac{x_7}{x_5{}^2}\sin x_6 + C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_2}{x_5{}^2}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{1/2}\right] - \lambda_4\left[-\frac{x_7}{x_1x_5{}^2}\cos x_6\right.$$

$$\left. + C_DA\,\frac{1}{2}\,\rho(h)\,\frac{x_4}{x_5{}^2}\left(x_1{}^2x_4{}^2 + x_2{}^2\right)^{1/2}\right] \tag{7e}$$

$$\dot{\lambda}_7 = -\lambda_2\,\frac{\sin x_6}{x_5} - \lambda_4\,\frac{\cos x_6}{x_1x_5} + \frac{\lambda_5}{Ig_e} \tag{7f}$$

The Euler equation associated with the component x_6 serves to define the thrust direction for an extremal solution in terms of two of the multipliers. This result is

$$0 = \lambda_2\left[-\frac{x_7}{x_5}\cos x_6\right] + \lambda_4\left[\frac{x_7}{x_1x_5}\sin x_6\right]$$

which can be written

$$\tan x_6 = \frac{\lambda_2}{\lambda_4}\,x_1 \tag{8}$$

This result allows for two choices for x_6. This ambiguity is resolved by the necessary condition on the Weierstrass E-function which is equivalent to requiring that the function $H = \lambda_i\dot{x}_i$ be a maximum.

Also since H is explicitly independent of t and since from the multiplier rule

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{9}$$

H is a constant along the arcs of continuous $\dot{x}_i$ of the problem.

For application of the end and corner conditions of the multiplier rule (eqs. (B2) and (B3)), the function G is written and contains the optimized quantity for this problem, the initial mass, and, in addition, the necessary end and corner constraints of the problem. The function G is given as

$$G = \lambda_0 g + l_\mu \psi_\mu$$

which, in terms of the problem, becomes

$$G = \lambda_0 x_5\left(t_1^+\right) + l_\mu \psi_\mu \tag{10}$$

where the quantity to be minimized g is the initial mass of the rocket and where the constraining relations $l_\mu \psi_\mu$ are given as

$$l_1\psi_1 = l_1\left\{x_5\left(t_3^-\right) - \left(1 + k_S\right)m_p - k_T \frac{x_7\left(t_2^+\right)}{Ig_e}\left(t_3 - t_2\right) - k_M x_7\left(t_2^+\right)\right\} \tag{11a}$$

$$l_2\psi_2 = l_2\left\{x_5\left(t_2^-\right) - x_5\left(t_2^+\right) - k_S\left[\left(1 + k_S\right)m_p + \left(1 + k_T\right)\frac{x_7\left(t_2^+\right)}{Ig_e}\left(t_3 - t_2\right) + k_M x_7\left(t_2^+\right)\right] - k_T \frac{x_7\left(t_1^+\right)}{Ig_e}\left(t_2 - t_1\right) - k_M x_7\left(t_1^+\right)\right\} \tag{11b}$$

$$l_3\psi_3 = l_3\left\{t_1 - 0\right\} \tag{11c}$$

$$l_4\psi_4 = l_4\left\{x_1\left(t_1^+\right) - r_{initial}\right\} \tag{11d}$$

$$l_5\psi_5 = l_5\left\{x_2\left(t_1^+\right) - \dot{r}_{initial}\right\} \tag{11e}$$

$$l_6\psi_6 = l_6\left\{x_3\left(t_1^+\right) - \theta_{initial}\right\} \tag{11f}$$

$$l_7\psi_7 = l_7\left\{x_4\left(t_1^+\right) - \dot{\theta}_{initial}\right\} \tag{11g}$$

$$l_8\psi_8 = l_8\left\{x_1\left(t_2^-\right) - x_1\left(t_2^+\right)\right\} \tag{11h}$$

$$l_9\psi_9 = l_9\left\{x_2\left(t_2^-\right) - x_2\left(t_2^+\right)\right\} \tag{11i}$$

$$l_{10}\psi_{10} = l_{10}\left\{x_3\left(t_2^-\right) - x_3\left(t_2^+\right)\right\} \tag{11j}$$

$$l_{11}\psi_{11} = l_{11}\left\{x_4\left(t_2^-\right) - x_4\left(t_2^+\right)\right\} \tag{11k}$$

$$l_{12}\psi_{12} = l_{12}\left\{x_1\left(t_3^-\right) - r_{final}\right\} \tag{11l}$$

$$l_{13}\psi_{13} = l_{13}\left\{x_2\left(t_3^-\right) - \dot{r}_{final}\right\} \tag{11m}$$

$$l_{14}\psi_{14} = l_{14}\left\{x_4\left(t_3^-\right) - \dot{\theta}_{final}\right\} \tag{11n}$$

The constraining relation ψ_1 is an expression for the final mass of the rocket in terms of the defining parameters and is obtained from the expression for rocket mass as a function of time. The constraining relation ψ_2 is an expression for the mass discontinuity at the staging point t_2 and is also obtained from the expression for rocket mass. The constraining relations ψ_3, ψ_4, ψ_5, ψ_6, and ψ_7 express the known initial conditions of the problem. The relations ψ_{12}, ψ_{13}, and ψ_{14} express the known final conditions of the problem. The relations ψ_8, ψ_9, ψ_{10}, and ψ_{11} express the fact that the components of the state vectors x_1, x_2, x_3, and x_4 are continuous at the staging point t_2.

The end and corner conditions given in equations (B2) and (B3) require that certain expressions be zero at the points t_1^+, t_2^-, t_2^+, and t_3^-, where the superscripts + and - indicate conditions immediately after and prior to the designated point in time. These expressions give relations between the variables of the problem at these points and enable the end conditions of the problem to be established. Figure 3 gives the conditions associated with the present problem. The variables that are subject to solution in this problem are listed for the points t_1^+, t_2^-, t_2^+, and t_3^-. The component x_7, the thrust magnitude, is free at all four of these points subject to the differential constraint $\dot{x}_7 = 0$ because x_7 is held constant on each continuous arc or rocket stage. The values of certain components are fixed at the launch and orbital conditions, of course.

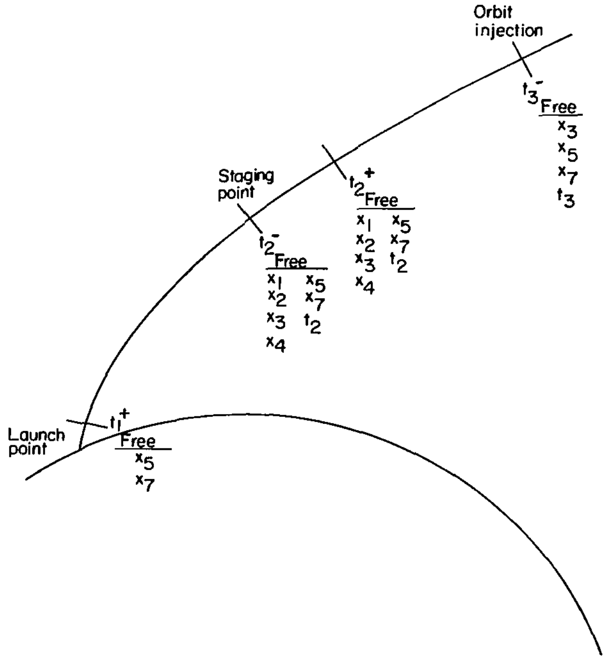

Figure 3.- Free variables at various points in the trajectory.

Substituting the values of F and G of this problem (eqs. (6) and (10)) into the end and corner relations (eqs. (B2) and (B3)) gives the following relations that must be satisfied. The multiplier λ_0 is taken to be 1 since the relations are homogeneous in the multipliers. For the point t_1,

$$\left.\begin{aligned}
l_4 - \lambda_1\left(t_1^+\right) &= 0 \\
l_5 - \lambda_2\left(t_1^+\right) &= 0 \\
l_6 - \lambda_3\left(t_1^+\right) &= 0 \\
l_7 - \lambda_4\left(t_1^+\right) &= 0 \\
1 - \lambda_5\left(t_1^+\right) &= 0 \\
l_3 + H\left(t_1^+\right) &= 0
\end{aligned}\right\} \tag{12}$$

for the point t_2,

$$l_8 + \lambda_1\left(t_2^-\right) = 0 \tag{13a}$$

$$-l_8 - \lambda_1\left(t_2^+\right) = 0 \tag{13b}$$

$$l_9 + \lambda_2\left(t_2^-\right) = 0 \tag{13c}$$

$$-l_9 - \lambda_2\left(t_2^+\right) = 0 \tag{13d}$$

$$l_{10} + \lambda_3\left(t_2^-\right) = 0 \tag{13e}$$

$$-l_{10} - \lambda_3\left(t_2^+\right) = 0 \tag{13f}$$

$$l_{11} + \lambda_4\left(t_2^-\right) = 0 \tag{13g}$$

$$-l_{11} - \lambda_4\left(t_2^-\right) = 0 \tag{13h}$$

$$l_2 + \lambda_5\left(t_2^-\right) = 0 \tag{13i}$$

$$-l_2 - \lambda_5\left(t_2^+\right) = 0 \tag{13j}$$

$$\lambda_7\left(t_2^-\right) = 0 \tag{13k}$$

$$l_1\left\{-\frac{k_T}{Ig_e}\left(t_3 - t_2\right) - k_M\right\} + l_2\left\{-k_S\left[\frac{1 + k_T}{Ig_e}\left(t_3 - t_2\right) + k_M\right]\right\} - \lambda\left(t_2^+\right) = 0 \tag{13l}$$

$$l_1\left\{k_T\,\frac{x_7\left(t_2^+\right)}{Ig_e}\right\} + l_2\left\{k_S\left(1 + k_T\right)\frac{x_7\left(t_2^+\right)}{Ig_e} - k_T\,\frac{x_7\left(t_1^+\right)}{Ig_e}\right\} + H\left(t_2^+\right) - H\left(t_2^-\right) = 0 \tag{13m}$$

and, for the point t_3,

$$\left.\begin{aligned}
&l_{12} + \lambda_1\left(t_3^-\right) = 0\\
&l_{13} + \lambda_2\left(t_3^-\right) = 0\\
&\lambda_3\left(t_3^-\right) = 0\\
&l_{14} + \lambda_4\left(t_3^-\right) = 0\\
&l_1 + \lambda_5\left(t_3^-\right) = 0\\
&\lambda_7\left(t_3^-\right) = 0\\
&l_1\left\{-k_T\,\frac{x_7\left(t_2^+\right)}{Ig_e}\right\} + l_2\left\{-k_S\left[\left(1 + k_T\right)\frac{x_7\left(t_2^+\right)}{Ig_e}\right]\right\} - H\left(t_3^-\right) = 0
\end{aligned}\right\} \tag{14}$$

The relations involving l_8, l_9, l_{10}, l_{11}, and l_2 and the values of λ_j on each side of t_2 show that the multipliers λ_1, λ_2, λ_3, λ_4, and λ_5 are continuous at t_2. The relations involving l_4, l_5, l_6, l_7, l_3, l_{12}, l_{13}, and l_{14} reflect the initial and final conditions on the state variables of the problem.

Substituting to eliminate l_1 and l_2 in the appropriate relations and defining four new quantities x_{20}, x_{21}, x_{22}, and x_{23} give

$$\left.\begin{aligned}
x_{20} &= \lambda_5\left(t_3^-\right)\left\{k_T \frac{x_7\left(t_2^+\right)}{Ig_e}\right\} + \lambda_5\left(t_2^-\right)\left\{k_S\left(1 + k_T\right)\frac{x_7\left(t_2^+\right)}{Ig_e}\right\} - H\left(t_2^+\right) = 0 \\
x_{21} &= \lambda_5\left(t_2^-\right)\left\{k_T \frac{x_7\left(t_1^+\right)}{Ig_e}\right\} - H\left(t_1^+\right) = 0 \\
x_{22} &= \lambda_5\left(t_2^-\right)\left\{\frac{k_T}{Ig_e}\left(t_2 - t_1\right) + k_M\right\} - \lambda_7\left(t_1^+\right) = 0 \\
x_{23} &= \lambda_5\left(t_3^-\right)\left\{k_T \frac{t_3 - t_2}{Ig_e} + k_M\right\} + \lambda_5\left(t_2^-\right)\left\{k_S \frac{1 + k_T}{Ig_e}\left(t_3 - t_2\right) + k_S k_M\right\} \\
&\quad - \lambda_7\left(t_2^+\right) = 0
\end{aligned}\right\} \qquad (15)$$

The end and corner conditions are summarized in brief form in table I, with the exception of the conditions on x_{20}, x_{21}, x_{22}, and x_{23}; numerical values for the end conditions of the launch trajectory are given in table II. Table I shows the conditions that are fixed at given values and the conditions that are free to be determined by the problem solution.

In integrating from t_3^- to t_1^+ it can be seen that $x_1\left(t_3^-\right)$, $x_2\left(t_3^-\right)$, $x_4\left(t_3^-\right)$, $\lambda_3\left(t_3^-\right)$, $\lambda_7\left(t_2^-\right)$, and $\lambda_7\left(t_3^-\right)$ are specified and that $x_3\left(t_3^-\right)$, $\lambda_1\left(t_3^-\right)$, $\lambda_2\left(t_3^-\right)$, $\lambda_4\left(t_3^-\right)$, $\lambda_5\left(t_3^-\right)$, $x_7\left(t_1^+\right)$, $x_7\left(t_2^+\right)$, t_2^-, and t_3^- are free to be chosen to satisfy conditions on $x_1\left(t_1^+\right)$, $x_2\left(t_1^+\right)$, $x_3\left(t_1^+\right)$, $x_4\left(t_1^+\right)$, $\lambda_5\left(t_1^+\right)$, x_{20}, x_{21}, x_{22}, and x_{23}. The value of x_5 is determined at all points in time by the specification of t_2^-, t_3^-, $x_7\left(t_1^+\right)$, and $x_7\left(t_2^+\right)$. Note that the number of free conditions corresponds to the number of conditions to be satisfied. The quantities t_2^- and t_3^- occur in the equations in the forms $t_2 - t_1$ and $t_3 - t_2$ which are equivalent in information content because t_1 is zero.

TABLE I.- END AND CORNER CONDITIONS ON TRAJECTORIES

Variable	Values of variable at specific times			
	t_1^+	t_2^-	t_2^+	t_3^-
t	0	Free	t_2^-	Free
x_1	$r_{initial}$	Free	$x_1(t_2^-)$	r_{final}
x_2	$\dot{r}_{initial}$	Free	$x_2(t_2^-)$	$\dot{r}_{final}$
x_3	$\theta_{initial}$	Free	$x_3(t_2^-)$	Free
x_4	$\dot{\theta}_{initial}$	Free	$x_4(t_2^-)$	$\dot{\theta}_{final}$
$^*x_5$				
x_7	Free	$x_7(t_1^+)$	Free	$x_7(t_2^+)$
λ_1	Free	Free	$\lambda_1(t_2^-)$	Free
λ_2	Free	Free	$\lambda_2(t_2^-)$	Free
λ_3	Free	Free	$\lambda_3(t_2^-)$	0
λ_4	Free	Free	$\lambda_4(t_2^-)$	Free
λ_5	1.0	Free	$\lambda_5(t_2^-)$	Free
λ_7	Free	0	Free	0

$^*x_5(t)$ is determined explicitly by a choice of t_2^-, t_3^-, $x_7(t_1^+)$, and $x_7(t_2^+)$.

TABLE II.- NUMERICAL VALUES OF END CONDITIONS OF TRAJECTORIES

$x_1(t_1^+)$ or $r_{initial}$ . . .	6 371 203 m
$x_2(t_1^+)$ or $\dot{r}_{initial}$ . . .	0
$x_3(t_1^+)$ or $\theta_{initial}$ . . .	0
$x_4(t_1^+)$ or $\dot{\theta}_{initial}$ . .	0
$x_1(t_3^-)$ or $r_{orbital}$ . .	6 741 712 m
$x_2(t_3^-)$ or $\dot{r}_{orbital}$ . .	0
$x_3(t_3^-)$ or $\theta_{orbital}$ . .	Free
$x_4(t_3^-)$ or $\dot{\theta}_{orbital}$ . .	0.00114023 rad/sec

In the numerical solution of a problem, values of the quantities $x_3(t_3^-)$, $\lambda_1(t_3^-)$, $\lambda_2(t_3^-)$, $\lambda_4(t_3^-)$, $\lambda_5(t_3^-)$, $t_2 - t_1$, $t_3 - t_2$, $x_7(t_1^+)$, and $x_7(t_2^+)$ are chosen in an effort to cause the quantities $x_1(t_1^+)$, $x_2(t_1^+)$, $x_3(t_1^+)$, $x_4(t_1^+)$, $\lambda_5(t_1^+)$, x_{20},

x_{21}, x_{22}, and x_{23} to achieve required values. These last quantities are obtained by integrating the differential equations of constraint and Euler equations in reverse and assembling the necessary quantities from the results. The quantity $x_1(t_1^+)$ must be equal to the radius of the launch point, $\lambda_5(t_1^+)$ must be equal to 1.0, and the other quantities must be zero. The inclusion of $x_3(t_3^-)$ and $x_3(t_1^+)$ is trivial in that the remainder of the solution is independent of these quantities.

Method of Solution

A differential correction procedure of simple form, the Newton-Raphson method, was used to obtain numerical solutions to the two-point boundary-value problem. In this procedure the differential equations that define the evolution of the state and multiplier vectors in time are integrated along a nominal path starting with some known and some assumed conditions at one end of the problem, and then along adjacent paths corresponding to perturbed values of each of the assumed parameters. The incremental changes in the boundary and other required conditions are obtained, and a matrix of partial derivatives is constructed from which a system of linear equations is obtained that relate changes in the boundary conditions to changes in the parameters free for variation. These equations are solved by matrix inversion, and corrections are made to the parameters free for variation so that a better match of required conditions results. Because of the nonlinearity of the problem, a diagonal weighting matrix was employed to limit the degree to which the various boundary conditions were corrected in one iteration. The necessity for employing this weighting matrix to avoid instability of the process of iteration to the optimum solution was recognized by Robert E. Smith, of the Langley Research Center, who carried out the programing for the problem solution.

The correction procedure was repeated until all conditions pertinent to the problem were satisfied. Frequent stops were required for adjusting the weighting matrix to avoid instabilities in the iteration and to adjust the size of the perturbations so that excursions were kept from becoming too large and nonlinear or too small and, hence, inaccurate. In general, large weightings were required to hold the initial altitude and velocities close to the required values and, hence, avoid divergence of the iteration routine, and small weightings were required on the other parameters.

RESULTS

Numerical Calculations

Numerical calculations were made to determine the minimum initial mass, two-stage rocket vehicle capable of delivering a payload of 4207 kg to a circular orbit about the earth at an altitude of 370 400 m (200 int. n. mi.). (See table II for these conditions.)

These calculations were made for various values of the mass fraction constants as shown in table III to show the changes that occur in booster proportions for a range of structural weights. Case 1 is a nominal design used as a starting point for the calculations. Cases 1, 2, 3, and 4 correspond to rather heavy construction. Cases 5, 6, and 7 are representative of present design practice. Cases 8 and 9 correspond to very light and nonexistent structure, respectively.

TABLE III.- STRUCTURAL MASS FRACTIONS FOR NUMERICAL CALCULATIONS

Case	k_T	$k_M g_e$	k_S
1	0.10	0.10	0
2	.10	.10	.008
3	.08	.10	.008
4	.08	.06	.008
5	.06	.06	.008
6	.06	.03	.008
7	.03	.03	.008
8	.03	0	.008
9	0	0	0

The calculations of the present investigation were carried out without regard to computational efficiency or sophistication in that a simple differential correction procedure was employed that made use of numerically determined partial derivatives for iteration to the point where the necessary conditions were satisfied. Once the first case was run and the procedure was established, each succeeding case required about 1 hr of computation time. As the structural mass fractions were reduced, the problem became more sensitive to the choice of the weighting matrix and somewhat more computation time was required. One case required in excess of 2 hr of computer time. The adoption of a more sophisticated convergence routine including, perhaps, analytically determined partial derivatives should reduce the machine time.

All calculations were made for a planar or two-dimensional motion. See the equations of appendix B. The rocket was permitted to assume any necessary thrust direction for optimization of the vehicle proportions and trajectory. The results of these calculations are given in tables IV, V, and VI and in figures 4 to 7. Table IV is a summary of the significant results of all of the calculations made. Table V gives the free-space velocity increments of the optimized rockets for the various cases. Table VI gives the angle of attack of the rocket at maximum dynamic pressure for each case calculated. Mass, aerodynamic drag, and velocity are plotted against time in figures 4, 5, and 6, respectively, and altitude is plotted against the range angle θ in figure 7. Case 9 of table III which corresponds to zero structural mass is not included in these plots.

Discussion of Results

Effects on vehicle parameters.- A summary of the results of the numerical calculations of optimum two-stage launch vehicles for a 4207-kg payload and a series of structural mass fractions are given in table IV. This table gives the burn time and thrust for each of the two stages and, in addition, the initial and final booster and payload mass. The nominal or starting trial trajectory for these calculations is given for reference.

TABLE IV.- SUMMARY OF RESULTS OF CALCULATIONS OF OPTIMUM LAUNCH VEHICLES FOR A 4207-KILOGRAM PAYLOAD DELIVERED TO A CIRCULAR EARTH ORBIT AT AN ALTITUDE OF 370 400 METERS

Case	Structural mass fraction			Initial mass, $m(t_1^+)$, kg	Mass before staging, $m(t_2^-)$, kg	Mass after staging, $m(t_2^+)$, kg	Burnout mass, $m(t_3^-)$, kg	First-stage burn, $t_2 - t_1$, sec	First-stage thrust, $x_7(t_1^+)$, N	Second-stage burn, $t_3 - t_2$, sec	Second-stage thrust, $x_7(t_2^+)$, N
	k_T	$k_M g_e$	k_S								
Nominal	0.10	0.10	0	646 889	238 989	125 463	23 496	225.0	7 116 800	450.0	889 600
1	.10	.10	0	377 150	103 908	32 529	8 479	248.8	4 311 215	512.6	184 133
2	.10	.10	.008	381 411	105 572	33 113	8 610	248.1	4 364 475	508.5	189 111
3	.08	.10	.008	310 383	84 659	31 260	7 982	258.1	3 433 464	495.8	184 253
4	.08	.06	.008	183 693	46 510	22 124	6 289	248.8	2 164 383	480.3	129 419
5	.06	.06	.008	148 507	45 722	25 597	6 377	179.3	2 249 767	462.6	163 032
6	.06	.03	.008	107 177	27 407	17 089	5 268	176.5	1 776 393	435.6	106 787
7	.03	.03	.008	91 649	25 670	19 424	5 034	192.8	1 343 064	459.8	122 818
8	.03	0	.008	73 115	25 714	24 094	4 815	136.8	1 359 322	441.7	171 345
9	0	0	0	62 841	23 977	23 977	4 203	112.9	1 350 835	445.4	174 259

An examination of table IV shows a progressive decrease in initial mass of the launch vehicle as the structural mass fractions associated with the fuel tanks and rocket motors are decreased as was expected. A comparison between the nominal case and the rest of the optimized cases indicates that a major effect was the reduction of the thrust levels between the nominal and optimized cases, particularly in the second stage. For case 1, which has the same structural mass fractions as the nominal case, the second-stage thrust was reduced from 889 600 N to 184 133 N. This difference indicates that the nominal case was substantially nonoptimal for this particular mission in the sense of minimum booster mass.

A major effect that may be noted as the structural mass fractions are decreased in the successive cases calculated (cases 2 to 9) is that the first-stage weight and thrust are reduced as the structural mass is reduced but the second-stage thrust and mass are held more nearly constant.

A detailed examination of the results of table IV indicates that the effects on the burn times and thrusts resulting from changes in the structural mass fractions are not always consistent. For instance, a reduction in k_M between cases 3 and 4, as well as 5 and 6, causes a reduction in the second-stage thrust $x_7(t_2^+)$ for the optimum vehicle; yet a reduction in k_M between cases 7 and 8 causes an increase in the second-stage thrust $x_7(t_2^+)$. This inconsistency also is evident for the first-stage thrust level $x_7(t_1^+)$. The burn times $t_2 - t_1$ and $t_3 - t_2$ are more consistent in respect to this change in k_M in that they decrease in each case. Some inconsistencies in this same regard may be noted for changes in the structural mass factor k_T. A complete analysis has not been made of this point, but some of these inconsistencies may be associated with the fact that the vehicle burn time and thrust parameters were not in a sense strongly unique near the optimum; that is, changes in the parameters produced little change in the initial mass at this point, and continued convergence toward satisfying the necessary conditions for an optimum became difficult. Such behavior is characteristic of many optimum solutions, of course, in that the variations of the optimized quantity with respect to the state variables approach zero as the optimum is neared. Because of differing sensitivities and nonlinearities, the necessary conditions of each case calculated were not satisifed to the same degree. Usually, iteration was continued until little improvement in the initial mass could be obtained on successive iterations and until the stability of the iteration process became a problem.

The interstage structural mass fraction k_S was changed from 0 to 0.008 between cases 1 and 2, and an increase in initial mass was obtained, as would be expected; however, the effect of this change was not large.

Table V gives the velocity increments of the optimized rockets computed on the basis of free space (no gravity and no atmosphere) from the appropriate logarithmic expressions in mass ratio. The optimized rockets all had a total velocity of between

TABLE V.- FREE-SPACE VELOCITY INCREMENTS OF ROCKETS FOR THE VARIOUS CASES

Case	Velocity of stage 1, m/sec	Velocity of stage 2, m/sec	Total velocity, m/sec
Nominal	3909	6576	10 485
1	5060	5278	10 339
2	5042	5287	10 330
3	5100	5358	10 458
4	5392	4937	10 329
5	4624	5455	10 080
6	5353	4619	9 973
7	4995	5300	10 296
8	4102	6320	10 422
9	3785	6831	10 616

9973 and 10 616 m/sec. The optimized rockets, with the exception of cases 8 and 9, all had approximately equal velocity increments in each of the two stages. The nominal case and cases 8 and 9 had considerably more of the velocity increment in the second stage.

Effects on trajectories.- Significant trajectory parameters are plotted in figures 4 to 7 for a number of the cases calculated for this investigation. (See table III.)

An examination of figure 4 shows a much diminished first-stage mass and fuel-consumption rate as the structural mass fractions are decreased. The staging time is appreciably lower for the last four cases than for the first three cases shown. A gradual consistent trend in the masses and staging times is not evident from these results. Possible reasons for this situation have been touched upon in the discussion of the overall **results** of these cases.

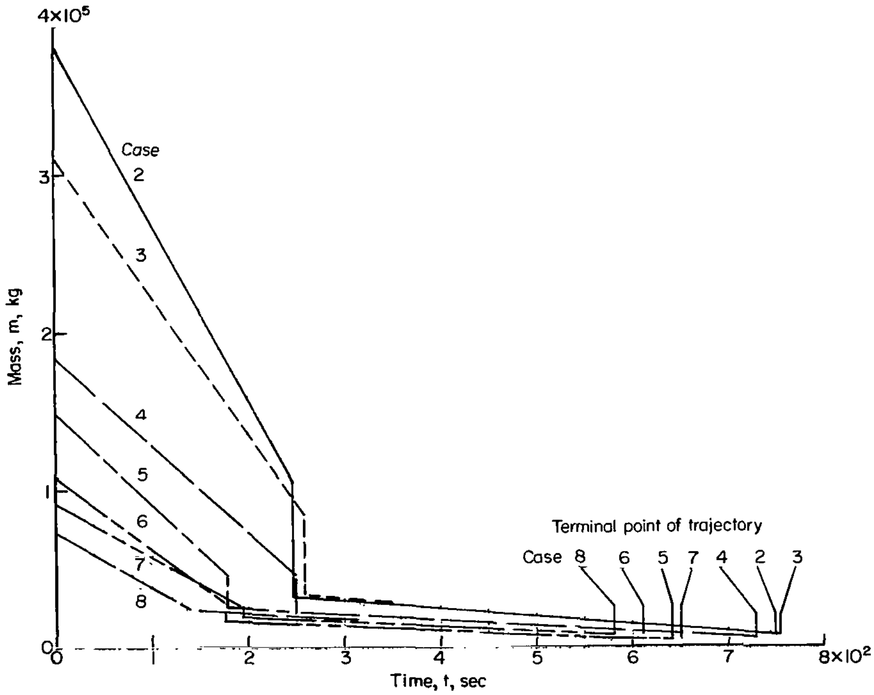

Figure 4.- Mass, m, as a function of time for cases 2, 3, 4, 5, 6, 7, and 8.

An examination of the aerodynamic drag (fig. 5) for these cases shows a similar lack of an even or consistent trend as was noted for the mass and staging points in figure 4. The maximum drag occurs at various times for the different cases shown, with the largest drags occurring for cases 5, 6, and 8. For these cases, this maximum drag occurs at an early point in the trajectory where presumably the altitude is somewhat low and thus the air density relatively large. The drag impulse or the integral of the drag over the time would appear to be more nearly the same for the various cases than the individual details of the drag time history, but even this quantity seems to be considerably different for the cases shown.

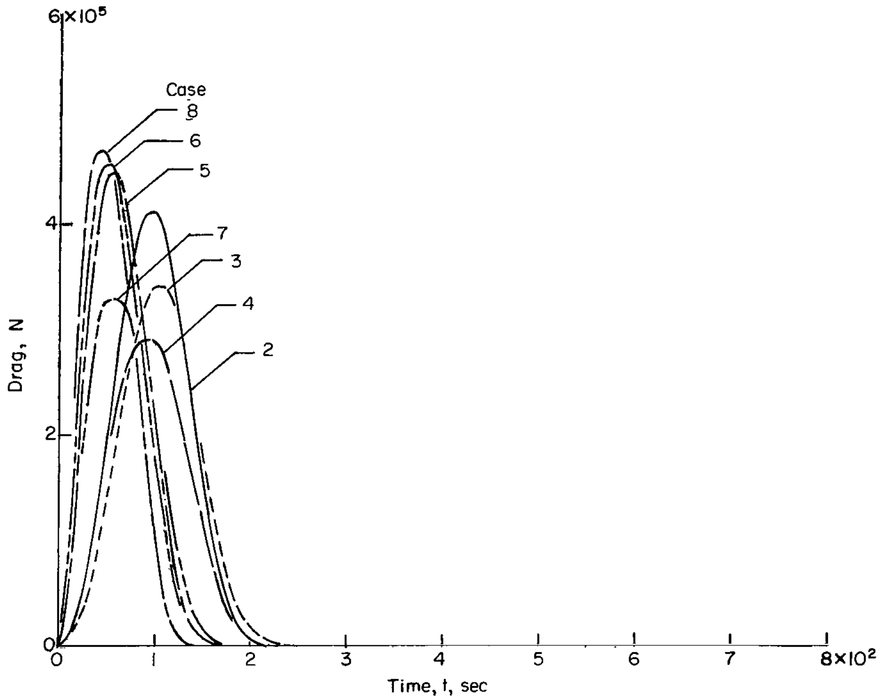

Figure 5.- Aerodynamic drag as a function of time for cases 2, 3, 4, 5, 6, 7, and 8.

The velocity time histories for the various cases show a considerable spread and, again, give no obvious indication of a consistent trend with structural mass reduction (fig. 6). Case 8 shows a larger acceleration rate near the end of its trajectory than do the other cases. In this case, the motor structural mass fraction was zero, giving no penalty in mass for large thrusts of the rocket motor. This factor may have been significant in this particular solution.

The altitude—range-angle variations for the cases considered show a similar character overall (fig. 7). However, case 8 shows a considerably greater loft or steepness than the other cases; also, the range angle at orbit injection was considerably less for case 8 than for the other cases. This fact seems to be associated, as was noted previously, with the greater thrust level and, hence, rate of acceleration for this case than for the remainder of the cases.

An examination of the trajectory printout for case 6 showed that the most severe environment insofar as structural bending is concerned occurred at about 70 sec at an altitude of 14 420 m, where the aerodynamic angle of attack was 2.61° and the aerodynamic drag was 386 500 N. In this case the product of the dynamic pressure and the angle of attack was a maximum; hence, presumably, the aerodynamic bending moment

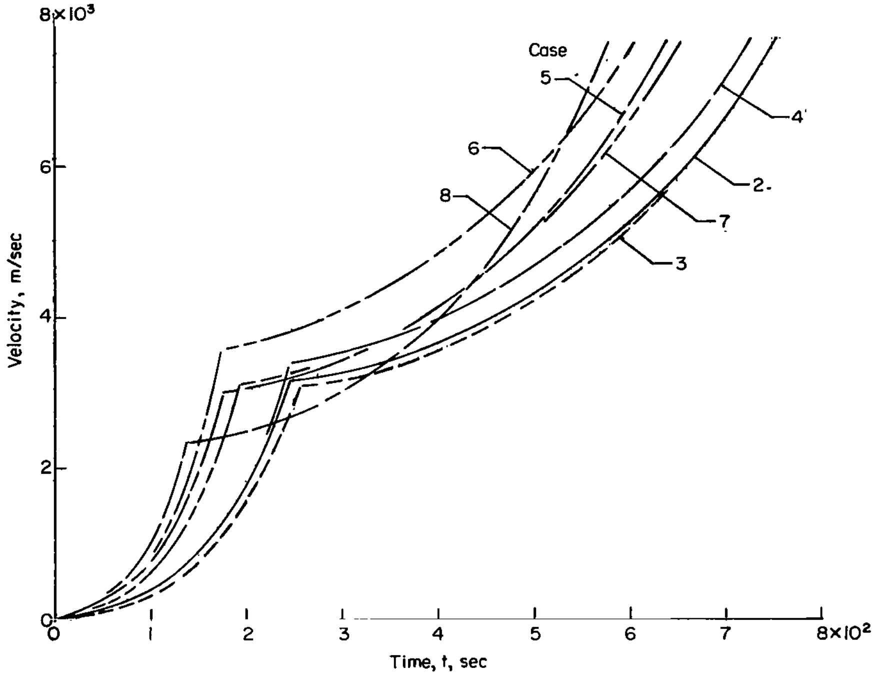

Figure 6.- Velocity as a function of time for cases 2, 3, 4, 5, 6, 7, and 8.

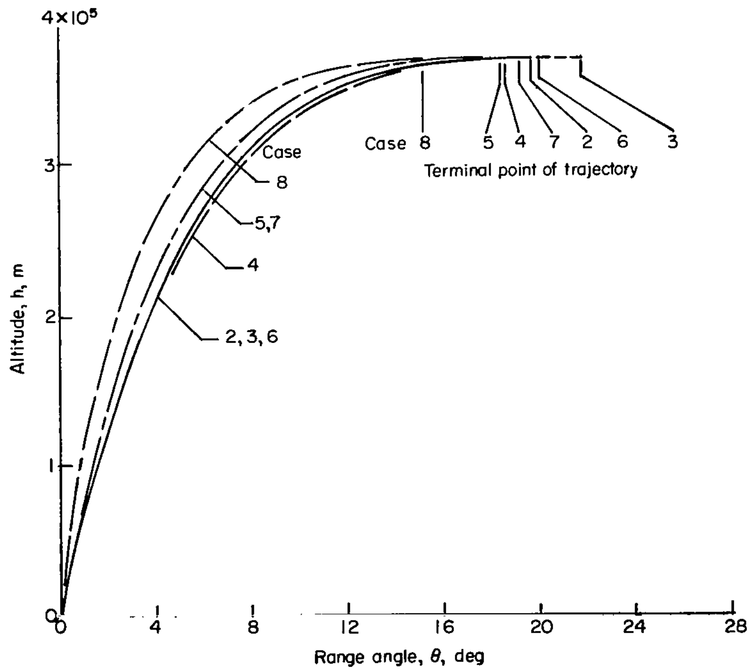

Figure 7.- Altitude as a function of range angle for cases 2, 3, 4, 5, 6, 7, and 8.

would be a maximum. The aerodynamic angle of attack is given by the difference between the flight-path angle and the rocket thrust direction (fig. 2). The angle of attack at maximum dynamic pressure is given in table VI for all cases. For the heavier structural masses, cases 1, 2, and 3, the angle of attack is as high as 16.9° which is excessive. No effort was made to apply a constraint on angle of attack for these calculations.

Many other points of greater or lesser significance could be extracted from the cases calculated, but the primary purpose of this investigation was the optimization of the significant physical parameters of a rocket; therefore, a detailed discussion of the trajectories has not been undertaken.

TABLE VI.- ANGLE OF ATTACK AT MAXIMUM DYNAMIC PRESSURE FOR THE VARIOUS CASES

Case	q_{max}, N/m^2	α, deg
1	2 176 241	16.8
2	2 203 054	16.9
3	1 830 021	15.9
4	1 571 325	8.6
5	2 407 406	1.9
6	2 446 668	-.8
7	1 787 647	.2
8	2 521 025	-1.3
9	2 952 999	-2.5

Extensions to other problems.- A problem such as the optimization of an airplane configuration and its flight path may prove to be considerably more difficult to formulate than the rocket problem solved in this investigation because the relationships between the significant parameters of an airplane and the stage variables and other constants of the problem may not be easily determined. No effort has been made to examine an application of this type; however, a few of the more easily established effects dealing with the most appropriate or optimum size and sweep of the wings for a given airplane configuration could likely be determined with a reasonable problem formulation. Such an analysis may prove fruitful in that variational methods have not generally been applied to such problems.

CONCLUDING REMARKS

An analysis using classical variational methods was made whereby both the flight path and certain significant physical characteristics of a two-stage rocket were determined in an optimal sense. Results were obtained for the most efficient two-stage rocket from the standpoint of minimum initial mass that would place a given payload in a circular orbit about the earth at an altitude of 370 400 meters (200 int. n. mi.) with a continuous burn of the motors. Nine cases were calculated that were representative of a range of structural design practices – from very heavy tanks, motors, and interstage structures to zero mass structural elements – to show the influence of structural mass. The most significant result from these numerical calculations was that the second-stage thrust was considerably lower than was expected. Analyses of the type carried out for this investigation should be useful in the sizing and, generally, in the design of rockets.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., July 31, 1968,
125-17-05-04-23.

APPENDIX A

EQUATIONS OF MOTION

The equations of motion of a rocket-powered vehicle operating in the vicinity of the earth and in its atmosphere can be derived directly from Newton's equation of motion

$$\bar{F} = m \frac{d^2\bar{r}}{dt^2}$$

where the applied forces $\bar{F}$ are taken to be the gravitational force of the earth, the rocket thrust, and the aerodynamic drag, and are given by

$$\bar{F}_{gravitational} = -mg_e \frac{r_e{}^2\bar{r}}{r^3}$$

$$\bar{F}_{rocket} = -\frac{dm}{dt} I\hat{u} = \bar{u}$$

$$\bar{F}_{drag} = -\frac{1}{2}\rho(h)AC_D\left|\dot{\bar{r}}\right|\dot{\bar{r}}$$

These quantities are given in relation to the axes shown in figure 8, $\bar{r}$ being the position vector. The drag is assumed to be independent of the angle of attack of the rocket. No lift force is included in this analysis in that the calculations are intended to be illustrative in nature and thus have been kept as simple in physical representation as possible.

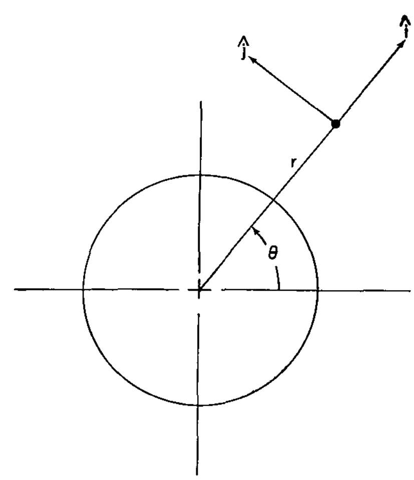

Figure 8.- Polar coordinate system.

The equation of motion is then

$$m(t)\ddot{\bar{r}} = \bar{u} - m(t)g_e \frac{r_e{}^2\bar{r}}{r^3} - \frac{1}{2}\rho(h)AC_D\left|\dot{\bar{r}}\right|\dot{\bar{r}}$$

where m is indicated as a function of time and ρ is a function of altitude.

The problem is taken to be a planar one that can be adequately represented in polar coordinates, and the unit vectors are as shown by $\hat{i}$ and $\hat{j}$ in figure 8. Resolving this equation in terms of these components gives

$$\left.\begin{aligned} m(t)\left(\ddot{r} - r\dot{\theta}^2\right) &= u \sin\beta - m(t)g_e \frac{r_e^2}{r^2} - \frac{1}{2}\rho(h)AC_D\left[\dot{r}^2 + (r\dot{\theta})^2\right]^{1/2}\dot{r} \\ m(t)(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= u \cos\beta - \frac{1}{2}\rho(h)AC_D\left[\dot{r}^2 + (r\dot{\theta})^2\right]^{1/2} r\dot{\theta} \end{aligned}\right\} \quad \text{(A1)}$$

where β refers to the inclination of the thrust to the local horizontal (fig. 2). In this problem an exponential representation of the earth atmosphere is used as given by

$$\rho(h) = \rho\left(r - r_e\right) = \rho_e e^{-\nu\left(r - r_e\right)}$$

rather than a tabular form in order to simplify the determination of ρ. In this representation ν is a constant and ρ_e is the mass density of the atmosphere at the earth surface (ref. 7). The flight-path angle γ is given as

$$\gamma = \tan^{-1} \frac{\dot{r}}{r\dot{\theta}}$$

APPENDIX B

THE NECESSARY MULTIPLIER RULE

In reference 2, a multiplier rule is formulated for a Bolza type variational problem that was later extended in reference 3. This multiplier rule as extended may be applied to the present problem to derive necessary conditions for a minimizing arc. This rule corresponds to the multiplier rule studied by Bliss (ref. 1) and others except that the function to be minimized depends not only on the coordinates of the end points of an admissible curve, but also on the coordinates and discontinuities at variable intermediate points on this curve. The coordinates of the end points and the discontinuities in coordinates or the coordinates at intermediate points are further restricted to satisfy certain side conditions. A short description of this rule is given here for completeness along with a brief comment on the use of the necessary condition on the Weierstrass E-function.

This multiplier rule concerns itself with finding in a class of sets $\left[t_a, x(t)\right]$ which satisfies the differential equations

$$\phi_j\left[t,x,\dot{x}\right] = 0 \qquad (j = 1, \ldots, m < n)$$

and the end and corner conditions

$$\psi_\mu\left[t_a,x_i\left(t_a^+\right),x_i\left(t_a^-\right)\right] = 0 \qquad (\mu = 1, \ldots, r \leqq (n+1)(q+1))$$

(where the superscripts + and - indicate conditions immediately after and prior to the designated point in time), a set which minimizes a functional J of the form

$$J = g\left[t_a,x_i\left(t_a^+\right),x_i\left(t_a^-\right)\right]$$

where t_a denotes the set of variables $\left(t_1, t_2, \ldots, t_q\right)$ which is always understood to satisfy the inequalities $t_1 < t_2 < \ldots t_q$ and $x_i(t)$, or simply $x(t)$, denotes the set of functions

$$\left[x_1(t), x_2(t), \ldots, x_{n+1}(t)\right] \qquad \left(t_1 \leqq t \leqq t_q\right)$$

In these expressions, $x_i\left(t_a^-\right)$ and $x_i\left(t_a^+\right)$ denote the values of x_i immediately on each side of t_a where a discontinuity or corner may exist, and $\dot{x}_i(t) = \frac{dx_i(t)}{dt}$ $(i = 1, \ldots, n + 1)$. For the present problem, t_a will be taken as the time.

APPENDIX B

An admissible solution E of the equations $\phi_j = 0$, defined on an interval $\left[t_1, t_q\right]$, is said to satisfy the multiplier rule if there exists a function

$$G\left[t_a, x_i(t_a^-), x_i(t_a^+), \lambda_0, l_\mu\right] = \lambda_0 g\left[t_a, x_i(t_a^-), x_i(t_a^+)\right] + l_\mu \psi_\mu \left[t_a, x_i(t_a^-), x_i(t_a^+)\right]$$

with constant λ_0, l_μ not all equal to zero, and a function

$$F(t,x,\dot{x},\lambda) = \lambda_j(t)\phi_j$$

with multipliers $\lambda_j(t)$ continuous on t_1, t_q except possibly at corners of E where they have well-defined right and left limits, such that the following equations are satisfied:

$$\left\{\frac{d}{dt}\left(F_{\dot{x}_i}\right) - F_{x_i}\right\} = 0 \tag{B1}$$

$$\left.\begin{aligned} &\left\{G_{t_1} + \left(\dot{x}_i F_{\dot{x}_i}\right)^{t_1}\right\} = 0 \\ &\left\{G_{t_c} + \left(\dot{x}_i F_{\dot{x}_i}\right)_{t_c^-}^{t_c^+}\right\} = 0 \\ &\left\{G_{t_q} + \left(\dot{x}_i F_{\dot{x}_i}\right)_{t_q}\right\} = 0 \end{aligned}\right\} \tag{B2}$$

$$\left.\begin{aligned} &\left\{G_{x_i(t_1)} - \left(F_{\dot{x}_i}\right)^{t_1}\right\} = 0 \\ &\left\{G_{x_i(t_c)} - \left(F_{\dot{x}_i}\right)_{t_c^-}\right\} = 0 \\ &\left\{G_{x_i(t_c^+)} - \left(F_{\dot{x}_i}\right)^{t_c^+}\right\} = 0 \\ &\left\{G_{x_i(t_q)} - \left(F_{\dot{x}_i}\right)_{t_q}\right\} = 0 \end{aligned}\right\} \tag{B3}$$

where the subscript c designates a corner or point of discontinuity; a number of these may exist. For an arc E satisfying the multiplier rule, the multipliers λ_0 and $\lambda_j(t)$

do not vanish simultaneously at any set of associated points on E. Every minimizing arc must satisfy the multiplier rule.

The necessary condition on the E function as given by Bliss (ref. 1), which is equivalent to requiring that the function $H = \lambda_i \dot{x}_i$ be a maximum (ref. 8), may be used to choose the minimizing arc from among those satisfying the multiplier rule.

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