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FORMULAS FOR LONG PERIOD RADIATION PRESSURE, LUNAR AND SOLAR GRAVITATIONAL EFFECTS ON THE MOTION OF ARTIFICIAL SATELLITES

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SUMMARY

Long period formulations are given for satellites in which the perturbations due to solar radiation pressure, lunar and solar gravitational forces are of first order in magnitude. This occurs for the classes of scientific satellites whose ratio of reflecting surface to mass are .07 cm^2 per gram or larger, and for satellites whose semi-major axes are between 12,000 and 55,000 kilometers such as communications satellites and satellites having eccentricities as high as 0.85.

The method of canonical transformations is applied to obtain analytic formulations which may be added to existing long period perturbation formulas given by Brouwer or Kozai for the earth's oblateness.

A useful application of these results would be to incorporate them into existing programs to improve the accuracy of orbit determination.

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FORMULAS FOR LONG PERIOD RADIATION PRESSURE, LUNAR AND SOLAR GRAVITATIONAL EFFECTS ON THE MOTION OF ARTIFICIAL SATELLITES

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INTRODUCTION

As a result of examining the data derived from satellite observations it has been established that the sun and the moon perturb artificial satellites to a significant extent even when the satellites are relatively close to the earth. In Reference 1, for example, it is shown that the order of magnitude of the disturbing functions for satellites whose major axes are only 12,000 kilometers are of the same order of magnitude as those of the disturbing functions due to the earth's oblateness.

An unexpected effect on the orbits of satellites was that due to radiation pressure. In Reference 1 it is shown that the perturbations due to radiation pressure were significant on a satellite of the Telstar II type with surface area to mass ratio of 0.075 cm^2 per gram. A ratio of this magnitude is often obtained in scientific satellites. Balloon type satellites such as Echo 1 have had ratios as great as 125 cm^2 per gram.

An analysis of several distant satellites given in Reference 5 indicates that the expansions for the solar and lunar disturbing forces are valid at least for satellites whose major axes are 55,000 kilometers. This covers the range of the U. S. and Soviet communications satellites and includes satellites of eccentricity as high as 0.85. In this report the long period perturbations are developed using the same disturbing forces as those in References 1 and 5 and consequently the results should have the same range of validity both for solar and lunar gravitational forces and solar radiation pressure.

For the accurate prediction of the orbits of satellites these solar and lunar perturbing forces should be taken into account. It is the purpose of this report to provide analytic perturbations in the elements of satellites. These may be readily added to the perturbations in the elements of satellites computed by analytic theories which take into account only the earth's oblateness such as those given by Brouwer or Kozai, Reference 3.

In this report the method of canonical transformations, Reference 2, is adopted in order to derive the perturbations in the required form.

GENERAL FORM OF THE DISTURBING FUNCTIONS

A useful tool in finding perturbations by the method of canonical variables is to obtain a determining function S, which in turn is derived from a disturbing function R. The function R, Reference 1, is a Fourier series of the form.

$$R = R_S + Q(a) \sum_i A_i \cos \alpha_i \; . \tag{1}$$

Q(a) is different for each type of perturbing force and is given in Table 1. All symbols are defined in the appendix.

Table 1

Q Functions.

Type of Perturbation	Q(a)
Long Period Solar Radiation Pressure	$-\frac{3}{2} F \frac{A}{m} a$
Short Period Solar Gravitational	$\frac{3}{16} n_\odot^2 m_\odot' a^2$
Long Period Solar Gravitational	$\frac{3}{16} \frac{n_\odot^2 m_\odot'}{(1 - e_\odot^2)^{3/2}} a^2$
Long Period Lunar Gravitational (independent of moon's node)	$\frac{3}{16} \frac{n_☾^2 m_☾'}{(1 - e_☾^2)^{3/2}} a^2$
Long Period Lunar Gravitational (dependent on moon's node)	$\frac{3}{16} \frac{n_☾^2 m_☾'}{(1 - e_☾^2)^{3/2}} a^2$

R_S and A_i are functions of L, G, H, ϵ, and i_{ec}, where L, G, and H are Delaunay variables, ϵ and i_{ec} are the obliquity of the ecliptic and the angle of the moon's orbit with the ecliptic respectively.

L, ϵ, and i_{ec} are considered as constants. The arguments α_i are defined by

$$\alpha_i = j_i \omega + k_i \Omega + \ell_i \lambda_\odot \; .$$

j_i, k_i, and ℓ_i are integers which may take on values of 0, ±1, ±2.

ω is the argument of perigee, Ω the right ascension of the node, and $\lambda_\odot$ the longitude of the sun. The functions A_i and the arguments α_i for the Radiation Pressure, Short Period Solar Gravitational, Long Period Solar Gravitational and Long Period Lunar Gravitational effects of the moon are given in Tables 2, 3, and 4.

Table 2

Radiation Pressure Functions.

i	$\bar{A}_i$	$\bar{\alpha}_i$	$\bar{\lambda}_i$	$\bar{\mu}_i$	$\bar{\nu}_i$
1	$e \cos^2 \frac{i}{2} \sin^2 \frac{\epsilon}{2}$	$\omega + \Omega + \lambda_\odot$	$\frac{e^2 - 2\cos^2 \frac{i}{2}}{2e^2 \cos^2 \frac{i}{2}}$	$\frac{1}{2\cos^2 \frac{i}{2}}$	$\frac{1 + e^2}{na^2 e^2}$
2	$e \cos^2 \frac{i}{2} \cos^2 \frac{\epsilon}{2}$	$\omega + \Omega - \lambda_\odot$	$\frac{e^2 - 2\cos^2 \frac{i}{2}}{2e^2 \cos^2 \frac{i}{2}}$	$\frac{1}{2\cos^2 \frac{i}{2}}$	$\frac{1 + e^2}{na^2 e^2}$
3	$e \sin^2 \frac{i}{2} \cos^2 \frac{\epsilon}{2}$	$\omega - \Omega + \lambda_\odot$	$\frac{e^2 - 2\sin^2 \frac{i}{2}}{2e^2 \sin^2 \frac{i}{2}}$	$-\frac{1}{2\sin^2 \frac{i}{2}}$	$\frac{1 + e^2}{na^2 e^2}$
4	$e \sin^2 \frac{i}{2} \sin^2 \frac{\epsilon}{2}$	$\omega - \Omega - \lambda_\odot$	$\frac{e^2 - 2\sin^2 \frac{i}{2}}{2e^2 \sin^2 \frac{i}{2}}$	$-\frac{1}{2\sin^2 \frac{i}{2}}$	$\frac{1 + e^2}{na^2 e^2}$
5	$-\frac{1}{2} e \sin i \sin \epsilon$	$\omega + \lambda_\odot$	$\frac{e^2 - \sin^2 i}{e^2 \sin^2 i}$	$-\frac{\cos i}{\sin i}$	$\frac{1 + e^2}{na^2 e^2}$
6	$\frac{1}{2} e \sin i \sin \epsilon$	$\omega - \lambda_\odot$	$\frac{e^2 - \sin^2 i}{e^2 \sin^2 i}$	$-\frac{\cos i}{\sin i}$	$\frac{1 + e^2}{na^2 e^2}$

Table 3

Solar Gravitational Fourier Coefficients.

i	$A'_{\odot i}$	$\alpha'_{\odot i}$	$A_{\odot i}$	α_i
1	$10\, e^2 \cos^4 \frac{i}{2} \cos^4 \frac{\epsilon}{2}$	$2\lambda_\odot - 2\omega - 2\Omega$	$5\, e^2 \cos^4 \frac{i}{2} \sin^2 \epsilon$	$-2\omega - 2\Omega$
2	$10\, e^2 \sin i \cos^2 \frac{i}{2} \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$2\lambda_\odot - 2\omega - \Omega$	$-5\, e^2 \sin i \cos^2 \frac{i}{2} \sin 2\epsilon$	$-2\omega - \Omega$
3	$\frac{15}{4} e^2 \sin^2 i \sin^2 \epsilon$	$2\lambda_\odot - 2\omega$	$5\, e^2 \sin^2 i \left(1 - \frac{3}{2} \sin^2 \epsilon\right)$	-2ω
4	$2\left(1 + \frac{3}{2} e^2\right) \sin^2 i \cos^4 \frac{\epsilon}{2}$	$2\lambda_\odot - 2\Omega$	$\left(1 + \frac{3}{2} e^2\right) \sin^2 i \sin^2 \epsilon$	-2Ω
5	$-2\left(1 + \frac{3}{2} e^2\right) \sin 2i \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$2\lambda_\odot - \Omega$	$\left(1 + \frac{3}{2} e^2\right) \sin 2i \sin 2\epsilon$	$-\Omega$
6	$2\left(1 + \frac{3}{2} e^2\right)\left(1 - \frac{3}{2} \sin^2 i\right) \sin^2 \epsilon$	$2\lambda_\odot$	—	—
7	$10\, e^2 \sin^4 \frac{i}{2} \cos^4 \frac{\epsilon}{2}$	$2\lambda_\odot + 2\omega - 2\Omega$	$5\, e^2 \sin^4 \frac{i}{2} \sin^2 \epsilon$	$2\omega - 2\Omega$
8	$-10\, e^2 \sin i \sin^2 \frac{i}{2} \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$2\lambda_\odot + 2\omega - \Omega$	$5\, e^2 \sin i \sin^2 \frac{i}{2} \sin 2\epsilon$	$2\omega - \Omega$
9	$\frac{15}{4} e^2 \sin^2 i \sin^2 \epsilon$	$2\lambda_\odot + 2\omega$	—	—

Table 4

Lunar Gravitational Fourier Coefficients.

m	i	$A_{m,i}$	$\alpha_{m,i}$	A_{ei}	α_i
1	1	$5e^2 \cos^4 \frac{i}{2} \sin 2i_{ec} \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$-2\omega - 2\Omega + \Omega_{ec}$	$5e^2 \cos^4 \frac{i}{2}\left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin^2 \epsilon$	$-2\omega - 2\Omega$
2	1	$5e^2 \cos^4 \frac{i}{2} \sin^2 i_{ec} \cos^4 \frac{\epsilon}{2}$	$-2\omega - 2\Omega + 2\Omega_{ec}$		
3	2	$5e^2 \sin i \cos^2 \frac{i}{2} \sin 2i_{ec} (1 - 2\cos\epsilon) \cos^2 \frac{\epsilon}{2}$	$-2\omega - \Omega + \Omega_{ec}$	$-5e^2 \sin i \cos^2 \frac{i}{2}\left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin 2\epsilon$	$-2\omega - \Omega$
4	3	$-\frac{15}{8} e^2 \sin^2 i \sin 2i_{ec} \sin 2\epsilon$	$-2\omega + \Omega_{ec}$	$5e^2 \sin^2 i \left(1 - \frac{3}{2} \sin^2 i_{ec}\right)\left(1 - \frac{3}{2} \sin^2 \epsilon\right)$	-2ω
5	3	$-\frac{15}{8} e^2 \sin^2 i \sin 2i_{ec} \sin 2\epsilon$	$-2\omega - \Omega_{ec}$		
6	4	$\left(1 + \frac{3}{2} e^2\right) \sin^2 i \sin 2i_{ec} \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$-2\Omega + \Omega_{ec}$	$\left(1 + \frac{3}{2} e^2\right) \sin^2 i \left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin^2 \epsilon$	-2Ω
7	4	$\left(1 + \frac{3}{2} e^2\right) \sin^2 i \sin^2 i_{ec} \cos^4 \frac{\epsilon}{2}$	$-2\Omega + 2\Omega_{ec}$		
8	5	$-\left(1 + \frac{3}{2} e^2\right) \sin 2i \sin 2i_{ec} \cos^2 \frac{\epsilon}{2} (1 - 2\cos\epsilon)$	$-\Omega + \Omega_{ec}$	$\left(1 + \frac{3}{2} e^2\right) \sin 2i \left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin 2\epsilon$	$-\Omega$
9	6	$-\left(1 + \frac{3}{2} e^2\right)\left(1 - \frac{3}{2} \sin^2 i\right) \sin 2i_{ec} \sin 2\epsilon$	Ω_{ec}	—	—
10	7	$5e^2 \sin^4 \frac{i}{2} \sin 2i_{ec} \sin \epsilon \cos^2 \frac{\epsilon}{2}$	$2\omega - 2\Omega + \Omega_{ec}$	$5e^2 \sin^4 \frac{i}{2}\left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin^2 \epsilon$	$2\omega - 2\Omega$
11	7	$5e^2 \sin^4 \frac{i}{2} \sin^2 i_{ec} \cos^4 \frac{\epsilon}{2}$	$2\omega - 2\Omega + 2\Omega_{ec}$		
12	8	$-5e^2 \sin i \sin^2 \frac{i}{2} \sin 2i_{ec} \cos^2 \frac{\epsilon}{2} (1 - 2\cos\epsilon)$	$2\omega - \Omega + \Omega_{ec}$	$5e^2 \sin i \sin^2 \frac{i}{2}\left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \sin 2\epsilon$	$2\omega - \Omega$

When the disturbing function R is also dependent on the node of the moon Ω_{ec}, R takes the form

$$R = Q(a) \sum_{m} A_{m,i} \cos \alpha_{m,i} \cdot \qquad (2)$$

The functions $A_{m,i}$, and the arguments $\alpha_{m,i}$ are given in Table 4.

The arguments $\alpha_{m,i}$ are of the form

$$\alpha_{m,i} = j_i \omega + k_i \Omega + q_{m,i} \Omega_{ec} \cdot$$

The quantities j_i and k_i are integers which take on values of 0, ±1, or ±2 as in Equation 1. The quantities $q_{m,i}$ take the values ±1, or ±2.

In selecting terms for the solar and lunar gravitational functions the following two criteria were adopted

(i) Terms only to order $\sin^3 \epsilon$ were kept

(ii) Terms involving the eccentricity of the sun or moon were not kept

THE RADIATION PRESSURE FUNCTION

The disturbing function for Radiation Pressure is given by

$$R = F \frac{A}{m} r \cos \tilde{S} \qquad (3)$$

where

F = const = -4.63×10^{-5} dyne/cm^2

A = presentation area of satellite

m = mass of satellite

r = radius of satellite = $a(1 - e \cos E)$

$\tilde{S}$ = angle between the radius to the satellite (r) and the radius to the sun (r')

Thus

$$\cos \tilde{S} = \frac{xx' + yy' + zz'}{rr'}$$

where by Reference 4 we have

$$\cos\tilde{S} = \cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos\left(f+\omega+\Omega+\lambda_\odot\right) + \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos\left(f+\omega+\Omega-\lambda_\odot\right)$$

$$+\sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos\left(f+\omega-\Omega+\lambda_\odot\right) + \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos\left(f+\omega-\Omega-\lambda_\odot\right)$$

$$-\frac{i}{2}\sin i\sin\epsilon\cos\left(f+\omega+\lambda_\odot\right) + \frac{i}{2}\sin i\sin\epsilon\cos\left(f+\omega-\lambda_\odot\right) \quad . \quad (4)$$

The long period part of R denoted by R_p is found by integration with respect to the mean anomaly ℓ, thus

$$R_p = \frac{1}{2\pi} F \frac{A}{m} \int_0^{2\pi} r\cos\tilde{S}\, d\ell \quad . \quad (5)$$

If we recall that

$$d\ell = (1 - e\cos E)\, dE$$

$$r\cos f = a(\cos E - e)$$

$$r\sin f = a\sqrt{1-e^2}\sin E$$

we can readily derive the useful result that

$$\frac{1}{2\pi}\int_0^{2\pi} r\cos\left(f+\alpha_i\right) d\ell = -\frac{3}{2}ae\cos\alpha_i \quad , \quad (6)$$

where α_i is an arbitrary argument independent of f.

Applying this formula to the expression for R_p we find that R_p can be expressed by the Fourier series

$$R_p = -F\frac{A}{m}a\sum \bar{A}_i\cos\bar{\alpha}_i \quad (7)$$

where the functions $\bar{A}_i$ and arguments $\bar{\alpha}_i$ are given in Table 2.

SHORT PERIOD SOLAR GRAVITATION DISTURBING FUNCTION

The short period solar disturbing function is so called because it contains the argument $\lambda_\odot$, the true longitude of the sun. However, no short period argument of the satellite itself appears.

If we apply the criterion of keeping terms only up to order $\sin^3 \epsilon$ we find from References 1 or 4 that

$$R_\odot' = \frac{3}{16} n_\odot^2 m_\odot' a^2 \sum A_{\odot i}' \cos \alpha_i' . \tag{8}$$

Both $A_{\odot i}'$ and α_i' are given in Table 3. The argument α_i' has the form

$$\alpha_i' = 2\lambda_\odot + j_i \omega + k_i \Omega$$

$$j_i = 0, \pm 2, \qquad k_i = 0, -1, -2 .$$

An important property of $A_{\odot i}'$ is that it can be separated into the product of three functions. Thus

$$A_{\odot i} \equiv A_{\odot i}(\epsilon) A_i(e) A_i(i) . \tag{9}$$

The independent variables ϵ, e, and i are the obliquity, eccentricity, and inclination respectively. The independent variable for the inclination i is of course not to be confused with the subscript i used as a summation index.

THE SOLAR AND LUNAR SECULAR AND LONG PERIOD DISTURBING FUNCTIONS

The solar and lunar disturbing function, representing the gravitational effect of distant bodies on the earth satellite have of course the same general form. Figure 1 shows a sketch of the earth, moon, sun configuration in order to present some of the elements appearing in the disturbing functions.

We start by writing a representation of the disturbing function of the moon, $R_{\mathbb{C}}$, as obtained from Reference 1 or 4. The disturbing function for the sun is readily obtained from $R_{\mathbb{C}}$. Then $R_{\mathbb{C}}$ finally is separated into two parts in order to obtain the long period lunar perturbations analytically. From Reference 1 or

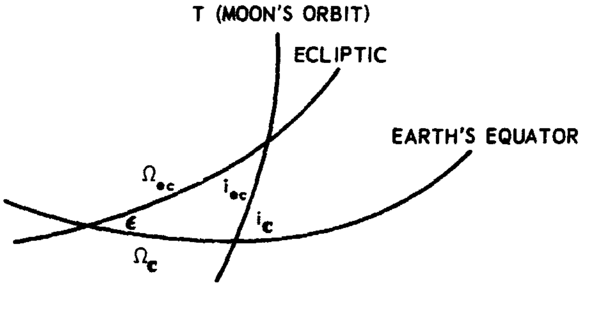

Figure 1

4 we have

$$R_{\mathbb{C}} = \frac{1}{4}\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{\left(1-e_{\mathbb{C}}^2\right)^{3/2}}\left(1-\frac{3}{2}\sin^2 i_{\mathbb{C}}\right)\left(1-\frac{3}{2}\sin^2 i\right)$$

$$+\frac{3}{16}\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{\left(1-e_{\mathbb{C}}^2\right)^{3/2}}\Bigg\{5e^2\cos^4\frac{i}{2}\sin^2 i_{\mathbb{C}}\cos\left(2\Omega_{\mathbb{C}}-2\omega-2\Omega\right)$$

$$-5e^2\sin i\cos^2\frac{i}{2}\sin 2i_{\mathbb{C}}\cos\left(\Omega_{\mathbb{C}}-2\omega-\Omega\right)+5e^2\sin^2 i\left(1-\frac{3}{2}\sin^2 i_{\mathbb{C}}\right)\cos 2\omega$$

$$+\left(1+\frac{3}{2}e^2\right)\sin^2 i\sin^2 i_{\mathbb{C}}\cos\left(2\Omega_{\mathbb{C}}-2\Omega\right)+\left(1+\frac{3}{2}e^2\right)\sin 2i\sin 2i_{\mathbb{C}}\cos\left(\Omega_{\mathbb{C}}-\Omega\right)$$

$$+5e^2\sin^4\frac{i}{2}\sin^2 i_{\mathbb{C}}\cos\left(2\Omega_{\mathbb{C}}+2\omega-2\Omega\right)+5e^2\sin i\sin^2\frac{i}{2}\sin 2i_{\mathbb{C}}\cos\left(\Omega_{\mathbb{C}}+2\omega-\Omega\right)\Bigg\} \quad (10)$$

This expression gives both the secular and trigonometric terms of the lunar disturbing function.

THE SOLAR SECULAR AND LONG PERIOD DISTURBING FUNCTION

From Figure 1 it is seen that by sliding the moon's trajectory T to a position coinciding with the ecliptic, the configuration of the sun will be obtained. Consequently the long period disturbing function for the sun is obtained by replacing $i_{\mathbb{C}}$ with ϵ, setting $\Omega_{\mathbb{C}}$ equal to zero, and replacing the parameters of the moon by that of the sun in the expression for $R_{\mathbb{C}}$, Equation 10. When this is done the expression for $R_{\odot}$ is given by

$$R_{\odot} = \frac{1}{4}\frac{n_{\odot}^2 m_{\odot}' a^2}{\left(1-e_{\odot}^2\right)^{3/2}}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left(1-\frac{3}{2}\sin^2 i\right)+\frac{3}{16}\frac{n_{\odot}^2 m_{\odot}' a^2}{\left(1-e_{\odot}^2\right)^{3/2}}\sum A_{\odot i}\cos\alpha_i \; . \quad (11)$$

The first term of this disturbing function is a function of the non angular elements only. From this term the contribution of the sun to the secular motion of the satellite is derived below. The remaining terms of the expansion contain the arguments α_i which in turn are linear combinations of ω and Ω.

The functions $A_{\odot i}$ and the arguments α_i are given in Table 3.

The functions $A_{\odot i}$ can in turn be written as the product

$$A_{\odot i} = A_{\odot i}(\epsilon)\,A_i(e)\,A_i(i) \; . \quad (12)$$

The functions $A_i(e)$ and $A_i(i)$ involving the satellite elements are the same as those in the short period solar disturbing function.

In addition the arguments α_i of the long period disturbing function contain the same linear combinations of ω and Ω as do the corresponding arguments α_i' of the short period disturbing function.

Thus

$$\alpha_i = j_i \omega + k_i \Omega$$

where $j_i = 0, \pm 2$, $k_i = 0, -1, -2$ exactly as in Equation 8.

THE LUNAR LONG PERIOD AND SECULAR DISTURBING FUNCTIONS

The form for $R_{☾}$ given in Equation 10 is not suitable for ready analysis since the elements of the moon $i_{☾}$ and $\Omega_{☾}$ are periodic functions in the variable Ω_{ec}. However by referring to Figure 1 we obtain the necessary relations between $i_{☾}$, $\Omega_{☾}$ and Ω_{ec} to facilitate analytic formulations. Thus from spherical trigonometry it follows that

$$\cos i_{☾} = \cos \epsilon \cos i_{ec} - \sin \epsilon \sin i_{ec} \cos \Omega_{ec}$$

$$\sin i_{☾} \sin \Omega_{☾} = \sin i_{ec} \sin \Omega_{ec} \quad . \qquad (13)$$

In the disturbing function $R_{☾}$ we find terms of the following three types

$$\left(1 - \frac{3}{2} \sin^2 i_{☾}\right) \cos \tau \ , \qquad \sin^2 i_{☾} \cos\left(2\Omega_{☾} + \tau\right) \ ,$$

and $\sin 2i_{☾} \cos\left(\Omega_{☾} + \tau\right)$, where τ is a linear combination of ω and Ω.

If we now apply formulas (13) we find after some algebraic manipulations that

$$\left(1 - \frac{3}{2} \sin^2 i_{☾}\right) \cos \tau = \left(1 - \frac{3}{2} \sin^2 i_{ec}\right)\left(1 - \frac{3}{2} \sin^2 \epsilon\right) - \frac{3}{8} \sin 2\epsilon \sin 2i_{ec} \cos\left(\tau + \Omega_{ec}\right)$$

$$+ \frac{3}{8} \sin^2 \epsilon \sin^2 i_{ec} \cos\left(\tau + 2\Omega_{ec}\right) - \frac{3}{8} \sin 2\epsilon \sin 2i_{ec} \cos\left(\tau - \Omega_{ec}\right)$$

$$+ \frac{3}{8} \sin^2 \epsilon \sin^2 i_{ec} \cos\left(\tau - 2\Omega_{ec}\right) \ ,$$

$$\sin^2 i_{☾} \cos\left(2\Omega_{☾} + \tau\right) = \left(1 - \frac{3}{2}\sin^2 i_{ec}\right)\sin^2\epsilon + \sin 2i_{ec} \sin\epsilon \cos^2\frac{\epsilon}{2}\cos\left(\tau + \Omega_{ec}\right)$$

$$+ \sin^2 i_{ec} \cos^4\frac{\epsilon}{2}\cos\left(\tau + 2\Omega_{ec}\right) - \sin 2i_{ec} \sin\epsilon \sin^2\frac{\epsilon}{2}\cos\left(\tau - \Omega_{ec}\right)$$

$$+ \sin^2 i_{ec} \sin^4\frac{\epsilon}{2}\cos\left(\tau - 2\Omega_{ec}\right),$$

$$\sin 2i_{☾} \cos\left(\Omega_{☾} + \tau\right) = \left(1 - \frac{3}{2}\sin^2 i_{ec}\right)\sin 2\epsilon + \sin 2i_{ec} \cos^2\frac{\epsilon}{2}(-1 + 2\cos\epsilon)\cos\left(\tau + \Omega_{ec}\right)$$

$$- \sin^2 i_{ec} \sin\epsilon \cos^2\frac{\epsilon}{2}\cos\left(\tau + 2\Omega_{ec}\right) - \sin 2i_{ec} \sin^2\frac{\epsilon}{2}(1 + 2\cos\epsilon)\cos\left(\tau - \Omega_{ec}\right)$$

$$+ \sin^2 i_{ec} \sin\epsilon \sin^2\frac{\epsilon}{2}\cos\left(\tau - 2\Omega_{ec}\right). \quad (14)$$

The next step is to apply Equations 14 to the expression for $R_{☾}$, Equation 10, given above. Where this is done the expansion for $R_{☾}$, keeping terms only to order $\sin^3\epsilon$, takes the form

$$R_{☾} = \frac{n_{☾}^2 m_{☾}'}{4\left(1 - e_{☾}^2\right)^{3/2}}\left(1 - \frac{3}{2}\sin^2\epsilon\right)\left(1 - \frac{3}{2}\sin^2 i_{ec}\right)\left(1 - \frac{3}{2}\sin^2 i\right)$$

$$+ \frac{3}{16}\frac{n_{☾}^2 m_{☾}' a^2}{\left(1 - e_{☾}^2\right)^{3/2}}\sum A_{☾ i}\cos\alpha_i + \frac{3}{16}\frac{n_{☾}^2 m_{☾}' a^2}{\left(1 - e_{☾}^2\right)^{3/2}}\sum_m A_{m,i}\cos\alpha_{m,i}. \quad (15)$$

The first term on the right-hand side of Equation 15 is a function of the non angular variables only. Consequently from this term the contribution of the moon to the mean motion of the satellite is derived.

The second term is a Fourier series whose arguments α_i are independent of the node of the moon Ω_{ec} and are consequently identical with the arguments of the long period solar gravitational function. The arguments α_i and the function $A_{☾ i}$ are given in Table 4. It is interesting to note that the relationship

$$A_{☾ i} = \left(1 - \frac{3}{2}\sin^2 i_{ec}\right)A_{\odot i} \quad (16)$$

holds.

The last Fourier series in Equation 15 contains terms with the argument $\alpha_{m,i}$ where

$$\alpha_{m,i} = j_i \omega + k_i \Omega + q_{m,i} \Omega_{ec} .$$

We also have the separability relation

$$A_{m,i} = A_{m,i} \left(\epsilon, i_{ec}\right) A_i (e) A_i (i) . \qquad (17)$$

We have kept the subscript i to preserve the correspondence between the Fourier series of the solar and lunar gravitational expansions of the disturbing functions.

The functions $A_{m,i}$ and the arguments $\alpha_{m,i}$ are given in Table 4.

THE MEAN MOTIONS

If R_S is the secular part of the disturbing function of the sun or moon then the contribution to the mean motions by the sun or the moon are given by the formulas

$$\dot{\ell} = -\frac{\partial R_S}{\partial L}, \qquad \dot{\omega} = -\frac{\partial R_S}{\partial G}, \qquad \dot{\Omega} = -\frac{\partial K_S}{\partial H} .$$

Performing the differentiations indicated above the combining with the secular motion due to the earth's oblateness given in Reference 1, we have

$$\dot{\ell} = n\left\{1 - \frac{3}{4}\frac{\mu^2 J_2 a_e^2}{L}\left(\frac{L^3}{G^3}\right)\left(1 - 3\frac{H^2}{G^2}\right) + \frac{3}{128}\frac{\mu^4 J_2^2 a_e^4}{L}\left(\frac{L^7}{G^7}\right)\left[-15 + 16\frac{G}{L} + 25\frac{G^2}{L^2}\right.\right.$$

$$\left. + \left(30 - 96\frac{G}{L} - 90\frac{G^2}{L^2}\right)\frac{H^2}{G^2} + \left(105 + 144\frac{G}{L} + 25\frac{G^2}{L^2}\right)\frac{H^4}{G^4}\right]$$

$$\left. - \frac{45}{128}\frac{\mu^4 J_4 a_e^4 e^2}{L}\left(\frac{L^7}{G^7}\right)\left(3 - 30\frac{H^2}{G^2} + 35\frac{H^4}{G^4}\right)\right\}$$

$$+ \frac{1}{8n}\left(1 - \frac{3}{2}\sin^2\epsilon\right)\left(10 - 3\frac{G^2}{L^2}\right)\left(1 - 3\frac{H^2}{G^2}\right)\left[\frac{n_\odot^2 m_\odot'}{\left(1 - e_\odot^2\right)^{3/2}} + \left(1 - \frac{3}{2}\sin^2 i_{ec}\right)\frac{n_{☾}^2 m_{☾}'}{\left(1 - e_{☾}^2\right)^{3/2}}\right]$$

$$\dot{\omega} = n\left\{\frac{3}{4}\frac{\mu^2 J_2 a_e^2}{G^4}\left(-1+5\frac{H^2}{G^2}\right)+\frac{3}{128}\frac{\mu^4 J_2^2 a_e^4}{G^8}\left[-35+24\frac{G}{L}+25\frac{G^2}{L^2}\right.\right.$$

$$\left.+\left(90-192\frac{G}{L}-126\frac{G^2}{L^2}\right)\frac{H^2}{G^2}+\left(385+360\frac{G}{L}+45\frac{G^2}{L^2}\right)\frac{H^4}{G^4}\right]$$

$$\left.-\frac{15}{128}\frac{\mu^4 J_4 a_e^4}{G^8}\left[21-96\frac{G^2}{L^2}+\left(-270+126\frac{G^2}{L^2}\right)\frac{H^2}{G^2}+\left(385-189\frac{G^2}{L^2}\right)\frac{H^4}{G^4}\right]\right\}$$

$$-\frac{3}{8}\frac{a^2}{G}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left(\frac{G^2}{L^2}-5\frac{H^2}{G^2}\right)\left[\frac{n_\odot^2 m_\odot'}{(1-e_\odot^2)^{3/2}}+\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{(1-e_{\mathbb{C}}^2)^{3/2}}\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\right],$$

$$\dot{\Omega} = n\left\{-\frac{3}{2}\frac{\mu^2 J_2 a_e^2}{G^4}\left(\frac{H}{G}\right)+\frac{3}{32}\frac{\mu^4 J_2^2 a_e^4}{G^8}\left[\left(-5+12\frac{G}{L}+9\frac{G^2}{L^2}\right)\frac{H}{G}\right.\right.$$

$$\left.\left.-\left(35+36\frac{G}{L}+5\frac{G^2}{L^2}\right)\left(\frac{H^3}{G^3}\right)\right]-\frac{15}{32}\frac{\mu^4 J_4 a_e^4}{G^8}\left[\left(5-3\frac{G^2}{L^2}\right)\left(3-7\frac{H^2}{G^2}\right)\frac{H}{G}\right]\right\}$$

$$-\frac{3}{8}\frac{a^2}{G}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left(5-3\frac{G^2}{L^2}\right)\frac{H}{G}\left[\frac{n_\odot^2 m_\odot'}{(1-e_\odot^2)^{3/2}}+\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{(1-e_{\mathbb{C}}^2)^{3/2}}\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\right].$$

The symbols employed here are defined in the Appendix. It is shown below that differential coefficients derived from the mean motions, $\partial\dot{\omega}/\partial L$, $\partial\dot{\Omega}/\partial L$, $G(\partial\dot{\omega}/\partial G)$, $G(\partial\dot{\Omega}/\partial G)$, $G(\partial\dot{\omega}/\partial H)$, and $G(\partial\dot{\Omega}/\partial H)$ are useful in forming analytic perturbations. Performing the indicated differentiations we now give the formulas for the above differential coefficients.

Formulas for $\partial\dot{\omega}/\partial L$ and $\partial\dot{\Omega}/\partial L$

$$\frac{\partial\dot{\omega}}{\partial L} = \frac{1}{a^2}\left\{\frac{9}{4}\frac{J_2 a_e^2}{a^4}\frac{L^4}{G^4}\left(1-5\cos^2 i\right)+\frac{3}{128}\frac{J_2^2 a_e^4}{a^6}\frac{L^8}{G^8}\left[\left(105-96\frac{G}{L}-125\frac{G^2}{L^2}\right)\right.\right.$$

$$\left.+\left(-270+768\frac{G}{L}+630\frac{G^2}{L^2}\right)\cos^2 i-\left(1155+1440\frac{G}{L}+225\frac{G^2}{L^2}\right)\cos^4 i\right]$$

$$\left.+\frac{15}{128}\frac{J_4 a_e^4}{a^6}\frac{L^8}{G^8}\left[\left(63-45\frac{G^2}{L^2}\right)+\left(-810+630\frac{G^2}{L^2}\right)\cos^2 i+\left(1155-945\frac{G^2}{L^2}\right)\cos^4 i\right]\right\}$$

$$-\frac{3}{4nG}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left[\frac{n_\odot^2 m_\odot'}{(1-e_\odot^2)^{3/2}}+\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{(1-e_{\mathbb{C}}^2)^{3/2}}\right]\left(1-e^2-10\cos^2 i\right),$$

$$\frac{\partial \dot{\Omega}}{\partial L} = \frac{1}{a^2} \left\{ \frac{9}{2} \frac{J_2 a_e^2}{a^4} \frac{L^4}{G^4} \cos i + \frac{3}{32} \frac{J_2^2 a_e^4}{a^6} \frac{L^8}{G^8} \left[\left(15 - 48 \frac{G}{L} - 45 \frac{G^2}{L^2} \right) \cos i \right.\right.$$

$$\left.\left. + \left(105 + 144 \frac{G}{L} + 25 \frac{G^2}{L^2} \right) \cos^3 i \right] + \frac{225}{32} \frac{J_4 a_e^4}{a^6} \frac{L^8}{G^8} e^2 \left(3 - 7 \cos^2 i \right) \cos i \right\}$$

$$- \frac{3}{4n\,G} \left(1 - \frac{3}{2} \sin^2 \epsilon \right) \left[\frac{n_\odot^2 m_\odot'}{\left(1 - e_\odot^2\right)^{3/2}} + \left(1 - \frac{3}{2} \sin^2 i_{ec} \right) \frac{n_☾^2 m_☾'}{\left(1 - e_☾^2\right)^{3/2}} \right] \left(7 + 3 e^2 \right) \cos i \; .$$

Formulas for $\dot{\omega}_G$, $\dot{\Omega}_G$, $\dot{\omega}_H$, $\dot{\Omega}_H$

$$\omega_G = n \left\{ \frac{3}{2} \frac{J_2 a_e^2}{a^2} \frac{L^4}{G^4} \left(2 - 15 \cos^2 i \right) + \frac{3}{128} \frac{J_2^2 a_e^4}{a^4} \frac{L^8}{G^8} \left[\left(280 - 168 \frac{G}{L} - 150 \frac{G^2}{L^2} \right) \right.\right.$$

$$\left. + 36 \left(-25 + 48 \frac{G}{L} + 28 \frac{G^2}{L^2} \right) \cos^2 i - 30 \left(154 + 132 \frac{G}{L} + 15 \frac{G^2}{L^2} \right) \cos^4 i \right]$$

$$\left. - \frac{15}{128} \frac{J_4 a_e^4}{a^4} \frac{L^8}{G^8} \left[\left(-168 + 54 \frac{G^2}{L^2} \right) + 36 \left(75 - 28 \frac{G^2}{L^2} \right) \cos^2 i + 30 \left(-154 + 63 \frac{G^2}{L^2} \right) \cos^4 i \right] \right\}$$

$$- \frac{3}{8} \frac{a^2}{G} \left(1 - \frac{3}{2} \sin^2 \epsilon \right) \left[\frac{n_\odot^2 m_\odot'}{\left(1 - e_\odot^2\right)^{3/2}} + \left(1 - \frac{3}{2} \sin^2 i_{ec} \right) \frac{n_☾^2 m_☾'}{\left(1 - e_☾^2\right)^{3/2}} \right] \left(\frac{G^2}{L^2} + 15 \cos^2 i \right) ,$$

$$\dot{\Omega}_G = n \left\{ \frac{15}{2} \frac{J_2 a_e^2}{a^2} \frac{L^4}{G^4} \cos i + \frac{3}{32} \frac{J_2^2 a_e^4}{a^4} \frac{L^8}{G^8} \left[\left(45 - 96 \frac{G}{L} - 63 \frac{G^2}{L^2} \right) \cos i \right.\right.$$

$$\left. + \left(385 + 360 \frac{G}{L} + 45 \frac{G^2}{L^2} \right) \cos^3 i \right]$$

$$\left. - \frac{15}{32} \frac{J_4 a_e^4}{a^4} \frac{L^8}{G^8} \left[9 \left(-15 + 7 \frac{G^2}{L^2} \right) \cos i + 7 \left(55 - 27 \frac{G^2}{L^2} \right) \cos^3 i \right] \right\}$$

$$+ \frac{15}{4} \frac{a^2}{G} \left(1 - \frac{3}{2} \sin^2 \epsilon \right) \left[\frac{n_\odot^2 m_\odot'}{\left(1 - e_\odot^2\right)^{3/2}} + \left(1 - \frac{3}{2} \sin^2 i_{ec} \right) \frac{n_☾^2 m_☾'}{\left(1 - e_☾^2\right)^{3/2}} \right] \cos i \;\; ,$$

$$\dot{\Omega}_H = n\left\{-\frac{3}{2}\frac{J_2 a_e^2}{a^2}\frac{L^4}{G^4} + \frac{3}{32}\frac{J_2^2 a_e^4}{a^4}\frac{L^8}{G^8}\left[\left(-5+12\frac{G}{L}+9\frac{G^2}{L^2}\right)\right.\right.$$

$$\left. - 3\left(35+36\frac{G}{L}+5\frac{G^2}{L^2}\right)\cos^2 i\right]$$

$$\left. - \frac{15}{32}\frac{J_4 a_e^4}{a^4}\frac{L^8}{G^8}\left[\left(15-9\frac{G^2}{L^2}\right)+3\left(-35+21\frac{G^2}{L^2}\right)\cos^2 i\right]\right\}$$

$$- \frac{3}{8}\frac{a^2}{G}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left[\frac{n_\odot^2 m_\odot'}{(1-e_\odot^2)^{3/2}} + \frac{n_☾^2 m_☾'}{(1-e_☾^2)^{3/2}}\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\right](2+3e^2) \ .$$

$$\dot{\omega}_H = \dot{\Omega}_G \ .$$

THE DETERMINING FUNCTION

In this report the method of canonical transformations is applied to obtain perturbations in the elements. In the notation of Reference 5 we wish to determine expressions for the unknown variables L', G', H', ℓ', g', h' in terms of the variables L'', G'', H'', ℓ'', g'', h'' which are known (usually derived from experimental data). L'', G'', H'' are constants while ℓ'', g'', h'' are linear functions of time. The disturbing functions described above should be considered to be expressed in terms of the single primed elements.

By extending the method given in References 2 and 5 we may write the determining function S as

$$S = L''\ell' + G''g' + H''h' + Q(a)\sum \frac{A_i \sin\alpha_i}{\dot{\alpha}_i}$$

The functions $Q(a)$, A_i, $\dot{\alpha}_i$ are functions of the double primed elements while the α_i are functions of the single primed elements. The expressions for $Q(a)$, A_i, and α_i come from the expressions for the disturbing functions and are given in Tables 1-5. The quantities $\dot{\alpha}_i$ are linear combinations of the secular motions which have been discussed above.

THE PERTURBATIONS IN THE NON ANGULAR ELEMENTS

From the definition of the determining function we find

$$\delta L = \frac{\partial S_T}{\partial \ell'} \ ,$$

$$\delta G = \frac{\partial S_T}{\partial g'} ,$$

$$\delta H = \frac{\partial S_T}{\partial h'} ,$$

when

$$\delta L = L' - L''$$

$$\delta G = G' - G''$$

$$\delta H = H' - H''$$

$$S_T = \text{periodic part of } S .$$

Table 5

Auxiliary Lunar and Solar Gravitational Functions.

i	ν_i	λ_i	μ_i
1	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} - \frac{\cos i}{\cos^2 \frac{i}{2}}$	$\frac{1}{\cos^2 \frac{i}{2}}$
2	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} + \frac{\cos i}{\sin^2 i}(2\cos i - 1)$	$\frac{1 - 2\cos i}{\sin^2 i}$
3	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} + 2\frac{\cos^2 i}{\sin^2 i}$	$-\frac{2\cos i}{\sin^2 i}$
4	$\frac{7+3e^2}{na^2\left(1+\frac{3}{2}e^2\right)}$	$-\frac{3(1-e^2)}{1+\frac{3}{2}e^2} + \frac{2\cos^2 i}{\sin^2 i}$	$-\frac{2\cos i}{\sin^2 i}$
5	$\frac{7+3e^2}{na^2\left(1+\frac{3}{2}e^2\right)}$	$-\frac{3(1-e^2)}{1+\frac{3}{2}e^2} + 2\cot i \cot 2i$	$-\frac{2\cot 2i}{\sin i}$
6	$\frac{7+3e^2}{na^2\left(1+\frac{3}{2}e^2\right)}$	$-\frac{3(1-e^2)}{1+\frac{3}{2}e^2} - \frac{3\cos^2 i}{1-\frac{3}{2}\sin^2 i}$	$\frac{3\cos i}{1-\frac{3}{2}\sin^2 i}$
7	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} + \frac{\cos i}{\sin^2 \frac{i}{2}}$	$-\frac{1}{\sin^2 \frac{i}{2}}$
8	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} + \frac{\cos i}{\sin^2 i}(2\cos i + 1)$	$-\frac{(2\cos i + 1)}{\sin^2 i}$
9	$\frac{2(1+e^2)}{na^2 e^2}$	$-\frac{2(1-e^2)}{e^2} + 2\frac{\cos^2 i}{\sin^2 i}$	$-\frac{2\cos i}{\sin^2 i}$

To simplify the nomenclature we replace g by ω, h by Ω, and drop the primed and double primed notation for the remainder of the analysis.

From the representation of the determining function given above we find

$$\delta L = 0$$

$$\delta G = Q(a) \sum j_i \frac{A_i}{\dot{\alpha}_i} \cos \alpha_i$$

$$\delta H = Q(a) \sum k_i \frac{A_i}{\dot{\alpha}_i} \cos \alpha_i \ .$$

The relation between the Keplerian elements and the Delaunay variables is given by the formulas

$$G = \sqrt{\mu a \left(1 - e^2\right)}$$

$$H = G \cos i \ .$$

Consequently we find

$$\delta e = - \frac{\left(1 - e^2\right)}{eG} \delta G$$

$$\delta i = \frac{1}{G \sin i} \left[\cos i \ \delta G - \delta H\right] \ .$$

THE PERTURBATIONS IN THE ANGULAR ELEMENTS

From the theory of canonical transformations we have

$$\delta \ell = - \frac{\partial S_T}{\partial L}$$

$$\delta \omega = - \frac{\partial S_T}{\partial G}$$

$$\delta \Omega = - \frac{\partial S_T}{\partial H} \ .$$

Where $\delta \ell = \ell' - \ell''$, $\delta \omega = \omega' - \omega''$, $\delta \Omega = \Omega' - \Omega''$ and

$$S_T = Q(a) \sum \frac{A_i}{\dot{\alpha}_i} \sin \alpha_i \ .$$

In order to find the perturbation in the mean anomaly it is convenient to replace $Q(a)$ by its factors

$$Q(a) = Q' f(a)$$

where Q' is a constant.

The $\partial S_T/\partial L$ is then

$$\frac{\partial S_T}{\partial L} = Q' \sum \left[\frac{1}{\dot{\alpha}_i} \frac{\partial \left(f(a) A_i \right)}{\partial L} - \frac{\left(f(a) A_i \right)}{\dot{\alpha}_i^2} \frac{\partial \dot{\alpha}_i}{\partial L} \right] \sin \alpha_i \ .$$

It then follows that

$$\delta \ell = -Q(a) \sum \frac{A_i}{\dot{\alpha}_i} \left(\nu_i - \frac{1}{\dot{\alpha}_i} \frac{\partial \dot{\alpha}_i}{\partial L} \right) \sin \alpha_i$$

where

$$\nu_i = \frac{1}{f(a) A_i} \frac{\partial \left(f(a) A_i \right)}{\partial L}, \qquad \frac{\partial \dot{\alpha}_i}{\partial L} = j_i \frac{\partial \dot{\omega}}{\partial L} + k_i \frac{\partial \dot{\Omega}}{\partial L}$$

where for example in the case of radiation pressure

$$f(a) A_i \text{ x factor independent of } L$$

for all i so that

$$\nu_i = \frac{1 + e^2}{e^2 L} = \frac{1 + e^2}{n a^2 e^2}$$

for all i.

In a similar manner it is found that

$$\delta \omega = -\frac{Q(a)}{G} \sum \frac{A_i}{\dot{\alpha}_i} \left(\lambda_i - \frac{1}{\dot{\alpha}_i} \dot{\alpha}_{iG} \right) \sin \alpha_i$$

$$\lambda_i = \frac{G}{A_i}\frac{\partial A_i}{\partial G}$$

$$\dot{\alpha}_{iG} = G\frac{\partial \dot{\alpha}_i}{\partial G} = G\left(j_i\frac{\partial \dot{\omega}}{\partial G} + k_i\frac{\partial \dot{\Omega}}{\partial G}\right)$$

$$\delta\Omega = -\frac{Q(a)}{G}\sum\frac{A_i}{\dot{\alpha}_j}\left(\mu_i - \frac{1}{\dot{\alpha}_i}\dot{\alpha}_{iH}\right)\sin\alpha_i$$

$$\mu_i = \frac{G}{A_i}\frac{\partial A_i}{\partial H}$$

$$\alpha_{iH} = G\frac{\partial \dot{\alpha}_i}{\partial H} = G\left(j_i\frac{\partial \dot{\omega}}{\partial H} + k_i\frac{\partial \dot{\Omega}}{\partial H}\right).$$

The quantities appearing in the formulas for the perturbations are given in Tables 1-5 and in the section on mean motions.

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Appendix

Nomenclature

A - presentation area of a satellite

A_i - coefficients of a Fourier series describing the periodic part of a disturbing function

$\bar{A}_i$ - coefficients of the Fourier series in the solar radiation pressure disturbing function

$A_{m,i}$ - coefficients of the Fourier series in the long period lunar disturbing function dependent on the moon's node

$A_i(e)$ - factors of coefficients of the Fourier series which depend on eccentricity of the satellite

$A_i(i)$ - factors of coefficients of the Fourier series which depend on the inclination of the satellite's orbit

$A_{\odot i}$ - coefficients of Fourier series in the long period solar gravitational disturbing function

$A_{☾ i}$ - coefficients of Fourier series in the long period lunar gravitational disturbing function independent of the moon's node

$A'_{\odot i}$ - coefficients of Fourier series in the short period solar gravitational disturbing function

a - semi-major axis of a satellite

a_e - mean value of earth's radius—(6378.166 km currently in use)

E - eccentric anomaly of a satellite

e - eccentricity of a satellite

$e_{\odot}$ - eccentricity of sun's orbit (0.01675184)

$e_{☾}$ - eccentricity of moon's orbit (0.054900489)

F - radiation pressure constant -4.63×10^{-5} dynes per cm^2

f - true anomaly of satellite

G - Delaunay variable $= \sqrt{\mu a (1 - e^2)}$

G' - that part of Dalaunay variable G which consists of a constant plus long period terms

G'' - mean value of G (constant)

H - Delaunay variable = $G \cos i$

H' - that part of Delaunay variable H which consists of a constant plus long period terms

H'' - mean value of H (constant)

i - inclination of satellite orbit plane to earth's equator as a subscript-i indicates a summation index

i_{ec} - inclination of moon's orbit plane to ecliptic ~5°1453964

$i_{\mathbb{C}}$ - inclination of moon's orbit plane to earth's equator

J_2 - earth's second zonal harmonic (1.0822×10^{-3})

J_3 - earth's third zonal harmonic (-2.285×10^{-6})

J_4 - earth's zonal harmonic (-2.123×10^{-6})

j_i - integer associated with argument of perigee

k_i - integer associated with right ascension of node

L - Delaunay veriable = $\sqrt{\mu a}$

L' - that part of L which contains no periodic terms

L'' - equal to L'

ℓ - mean anomaly cf satellite

$\dot{\ell}$ - mean motion of mean anomaly ℓ

ℓ' - that part of mean anomaly consisting of a linear function of time ℓ'' plus long period terms

ℓ'' - that part of ℓ which is a linear function of time

m - mass of a satellite

$m_{\odot}'$ - ratio of mass of sun to combined mass of sun and earth (0.999997)

$m_{\mathbb{C}}'$ - ratio of mass of moon to combined mass of moon and earth (0.012150668)

$n_{\odot}$ - mean motion of sun 0.98560027 degrees per day

$n_{\mathbb{C}}$ - mean motion of moon 13.064999 degrees per day

$Q(a)$ - function of semi-major axis a containing also the disturbing force constant

Q' - disturbing force constant (e.g. for Radiation Pressure $Q' = F(A/m)$)

R - disturbing function

R_p - long period part of Radiation Pressure disturbing function

R_S - secular part of a disturbing function

$R_{\odot}'$ - short period solar gravitational disturbing function

$R_{☾}$ - lunar gravitational disturbing function

$R_{\odot}$ - long period solar disturbing function

r - radius vector from center of earth to satellite

r' - radius vector from center of earth to sun

S - determining function

S_T - periodic part of S

$\tilde{S}$ - angle between r and r'

T - trace of moon's orbit on celestial sphere

x, y, z - rectangular coordinates of satellite

x', y', z' - rectangular coordinates of sun

α_i - arguments of trigonometric terms of disturbing function

$\dot{\alpha}_i$ - mean motion of α_i

$\bar{\alpha}_i$ - arguments appearing in long period radiation pressure disturbing function

α_i' - arguments appearing in short period solar gravitation disturbing function

ϵ - obliquity of the ecliptic

$\lambda_{\odot}$ - longitude of the sun

μ - $k^2 M$ k = Gaussian constant M = mass of the earth

ω - argument of perigee of satellite

$\dot{\omega}$ - mean motion of ω

ω'' - linear function of time appearing in ω

$\omega' = \omega''$ + long period terms

Ω - right ascension of node of satellite

Ω'' - linear function of time appearing in Ω

$\Omega' = \Omega''$ + long period terms

τ - linear combination of ω and Ω