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SUMMARY

The Michell strain criterion is used to develop minimum weight structures to equilibrate some interesting force systems of significance to aeronautics and the space technology. Michell prescribed a strain criterion for the members of a minimum volume structure (i.e., a structure having the minimum amount of material) to equilibrate a given force system.

The research work contributory to this paper is divided into three independent sections. In the first section, a theorem is presented on the superposition of Michell structures and applied to develop optimum structures for two basic force systems. With this theorem a force system that does not lend itself to a unique solution (i.e., a unique optimum structure) may be treated by resolving it into two or more systems, determining the solutions to the resolutes, and then superposing them to give the solution for the total force system. The theorem shows that the solution obtained by this superposition method satisfies the Michell optimum structure.

In the second section a solution is obtained to the governing equations developed by Hemp for the form of an optimum structure satisfying the Michell strain criterion condition in two dimensions. This solution gives a unique system of coordinate curves for the member layout of the Michell structure. This system can be employed to offer optimum structures for a variety of force systems acting on plane rectangular and curvilinear domains.

In the third section two well-known orthogonal systems of curves - the dipolar and the spherical - are selected for member layout of Michell structures. The volume of a Michell structure is a function of the displacements of the points of application of the applied forces; hence for the dipolar and spherical coordinate systems, the generalized displacements are obtained for the Michell strain pattern. Some interesting loading systems result when the two coordinate systems are used in developing Michell optimum structures. The dipolar system gives rise to a structure which could conceivably be employed for an entry body that had a displaced center of gravity with respect to the geometric center. Another application of this system is a structure which distributes a concentrated load over a wide bearing area, thereby reducing the possibility of large deformations of the material. An application of the spherical system gives rise to a spherical prestressed structure, under a suitable force system.

INTRODUCTION

In aeronautics structures are needed that can withstand a given load system and that have a minimum volume of material. For a given load system this objective is more easily achieved if the probability of failure is allowed to be small instead of zero (ref. 1). However, the development and application of the theory of optimum structures, which is the subject of this paper, is strictly a problem in structural mechanics; hence, in this case, the idealized version of failure (of probability zero) is adopted.

For research in the field of structural optimization, the structural design is usually represented by design parameters of the appropriate shape and size, the merit function is expressed in terms of these parameters and minimized with respect to them, within limits of the constraints, by using nonlinear programming techniques (see ref. 2). On the other hand, a theory based on displacement or strain criterion could be used to find the structure with the absolute minimum volume for the given load system. The form of the structure could be developed as a consequence. Such an approach gives the absolute minimum volume structure for the given load system. Michell's paper (ref. 3) inspired research along these lines. For a given system of self-equilibrating forces, Michell has laid down a deformation criterion which if satisfied by a framework will give a structure of a minimum volume, that is, of a minimum amount of material. Cox (ref. 4) drew the attention of engineers to Michell's results and, by contributing to the application of these results, championed the cause of optimum structural design. Hemp (ref. 5) has incorporated the Michell criterion of strain into a two-dimensional theory of optimum frameworks and has developed some special forms. Further, Michell's theorem has been applied to develop optimum frameworks for some useful practical load systems (see refs. 6, 7, 8, and 9).

The present paper is presented in three independent parts. In the first part a theorem is presented on the superposition of Michell structures. This theorem is then used to develop structures for two force systems. In the second part, solutions are presented to the governing equations developed by Hemp for the form of the minimum volume framework. The resulting structure is then developed, that is, the coordinate curves along which the members lie are laid out. It is then shown how certain arbitrary functions of the solution can be adjusted to match a given external force system. In the third part, two systems of well-known orthogonal coordinate curves, namely, the dipolar and the spherical coordinate systems, are selected for the layout of members of the Michell optimum structures. The corresponding strain-displacement equations incorporating the Michell strain criterion are solved to give the displacements in terms of corresponding curvilinear coordinates. Some force systems are illustrated as applications of the two coordinate systems in the development of Michell optimum structures.

SYMBOLS

A,B	unit arc lengths along α and β members
ds_1,ds_2	arc lengths along α and β members
e	magnitude of strain in any member of a Michell optimum structure
$e_{\alpha\alpha},e_{\beta\beta},e_{\gamma\gamma}$	normal strains in the curvilinear system (α,β,γ)
$e_{\alpha\beta},e_{\beta\gamma},e_{\gamma\alpha}$	shear strains in the curvilinear system (α,β,γ)
$\overline{F}_i$	forces acting at the points of application $\overline{r}_i$
T_1,T_2	forces per unit length parallel to the α and β members
t_1,t_2	thickness of the α,β members
(u,v)	displacements corresponding to the curvilinear system (α,β)
$(u_\alpha,u_\beta,u_\gamma)$	displacements corresponding to the curvilinear system (α,β,γ)
V^*	volume of a Michell optimum structure
α,β,γ	curvilinear coordinates
σ_c	allowable stress in compression
σ_t	allowable stress in tension
ψ_1,ψ_2	angles made by the α and β curves with the x axis

MICHELL CRITERION OF STRAIN

The Michell criterion of strain for a minimum volume framework to equilibrate a given force system states that the members of the structure must all be strained by the same amount (e), the sign depending on the sign of the axial stress carried by the member.

The volume V^* of the minimum volume framework to carry a system of forces $\overline{F}_i$ with allowable stresses σ_c and σ_t in compression and tension is given by (see ref. 3)

$$V^* = \frac{\sigma_t + \sigma_c}{2\sigma_t\sigma_c} \sum_{i=1}^{n} \bar{F}_i \cdot \left(\frac{\bar{v}_i}{e}\right) - \frac{\sigma_t - \sigma_c}{2\sigma_t\sigma_c} \sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i \qquad (1)$$

where $\bar{r}_i$ denotes the points of applications of the forces $\bar{F}_i$ and $\bar{v}_i$ denote the displacements of these points.

This deformation prescribed by Michell imposes certain restrictions upon the layout of the members of a Michell structure. At a node of this framework, the members follow the principal directions of strain. If members carrying loads of the same sign meet at a node, there is no restriction on their layout since for this case the Michell deformation is a pure dilatation or contraction and the principal axes of strain are indeterminate. If at a node, members carrying loads of opposite sign meet, these members must be at right angles. The members of this class of frames form curves of orthogonal systems.

SUPERPOSITION OF MICHELL STRUCTURES

In this section the indeterminacy of Michell structures is discussed. Then the utility of superposing Michell structures is presented. A theorem is presented on the superposition of Michell structures which relates the volume of the superposed structures for the corresponding force systems to the volume of the structure for the resultant of the force systems. The usefulness of the superposition theorem is demonstrated by means of two examples.

Indeterminacy of Michell Structures

In the development of a Michell structure to equilibrate a given force system, the geometry of the layout of the members is also unknown in addition to the forces in the members. The problem of determining the sizes of the members, for an optimum member layout, to equilibrate a given force system involves the solution of the following equations: (i) equilibrium equations obtained by considering the equation of an element $d\alpha$ by $d\beta$, where α, β are the curvilinear coordinates; (ii) equations of compatibility of strain in the curvilinear coordinate system; and (iii) equilibrium conditions along the boundary. Once the members' sizes are known, the forces in the members are obtained by specifying the allowable stress in the members.

It is seen that, for Michell structures, the areas of the members and hence the forces in the members for an allowable stress are obtained from the deformation pattern or kinematics. For a statically determinate structure, the determination of member forces for a given external force system requires the solution of only the equilibrium equations. A Michell structure is not statically determinate since, in addition to equilibrium equations, the deformation equations are needed to determine the forces in the members.

Superposition

The strain pattern prescribed by Michell for a minimum volume framework is such that solutions satisfying the Michell criterion of strain exist only for a few special cases. It often happens that it is not possible to find an optimum framework to equilibrate a force system. However, if the force system were resolved into component force systems, optimum frameworks could be determined for the component force systems. Suppose a composite structure is obtained by the superposition of the constituent Michell structures while continuity of displacements between the superposed structures is insured. To determine whether this composite structure can be used as the optimum structure for the given loading in the absence of a unique solution for that load system, a theorem is proposed which states that the volume of the composite structure is the same as the volume of the unique structure for the given load system. This theorem justifies the use of the composite structure as the optimum structure for the load system considered. The theorem will now be enunciated, proved, and applied to two force systems.

<u>Theorem</u>: Let $\overline{F}_{1i}$, i = 1, 2, 3, denote a self-equilibrating system of forces acting at points $\overline{r}_i$, i = 1, 2, 3, and let V_1 denote the volume of the Michell structure S_1 which equilibrates the force system $\overline{F}_{1i}$. Let $\overline{F}_{2j}$, j = 1, 2, 4, denote another self-equilibrating force system acting at points r_j, j = 1, 2, 4, and V_2 denote the volume of the Michell structure S_2 which equilibrates the force system $\overline{F}_{2j}$. Let $\overline{F}_{3k}$, k = 1, 2, 3, 4, represent the resultant of the force systems $\overline{F}_{1i}$ and $\overline{F}_{2j}$ acting at points $\overline{r}_k$, k = 1, 2, 3, 4, and let V_3 denote the volume of the Michell structure S_3 which equilibrates the force system $\overline{F}_{3k}$. If a structure be obtained by the superposition of the two Michell structures S_1 and S_2 such that there is a compatibility of strain between the two superposed structures (i.e., there is a continuity of displacements at nodes common to both the structures), then the volume of this composite structure will be $V_1 + V_2$. It is proposed, then, that $V_3 = V_1 + V_2$.

<u>Proof</u>: Consider a domain D containing the points $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$, $\overline{r}_4$. In this domain an orthogonal coordinate system of curves (or systems of curves, provided there is continuity of displacement along the lines of junction) links points $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$, $\overline{r}_4$. The Michell strain pattern is now imposed on this coordinate system so that the strains along the orthogonal directions are $\pm e$ and the shear strain is zero. Let the displacements at the points $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$, $\overline{r}_4$ be $\overline{v}_1$, $\overline{v}_2$, $\overline{v}_3$, $\overline{v}_4$, respectively.

Now at the points $\overline{r}_i$, i = 1, 2, 3, the system $\overline{F}_{1i}$ is made to act. The corresponding Michell structure S_1, to equilibrate the system $\overline{F}_{1i}$, will consist only of those members of the coordinate system that transmit forces; the remaining members of the coordinate system can be said to have zero area and, hence, do not form part of the structure S_1. Then the volume V_1 of S_1 is (from Eq. (1)),

$$V_1 = \frac{\sigma_t + \sigma_c}{2\sigma_t\sigma_c} \sum_{i=1,2,3} \bar{F}_{1i} \cdot \left(\frac{\bar{v}_i}{e}\right) - \frac{\sigma_t - \sigma_c}{2\sigma_t\sigma_c} \sum_{i=1,2,3} \bar{F}_{1i} \cdot \bar{r}_i \qquad (2)$$

Next, at points $\bar{r}_j$, j = 1, 2, 4, the system $\bar{F}_{2j}$ is made to act. The corresponding Michell structure S_2, to equilibrate the system $\bar{F}_{2j}$, will again consist only of those members that transmit forces. The volume V_2 of the structure S_2 is given by

$$V_2 = \frac{\sigma_t + \sigma_c}{2\sigma_t\sigma_c} \sum_{j=1,2,4} \bar{F}_{2j} \cdot \left(\frac{\bar{v}_j}{e}\right) - \frac{\sigma_t - \sigma_c}{2\sigma_t\sigma_c} \sum_{j=1,2,4} \bar{F}_{2j} \cdot \bar{r}_j \qquad (3)$$

The system F_{3k}, k = 1, 2, 3, 4, is now made to act at the points $\bar{r}_k$. The volume V_3 of the corresponding Michell structure S_3, to equilibrate the system $\bar{F}_{3k}$, is given by

$$V_3 = \frac{\sigma_t + \sigma_c}{2\sigma_t\sigma_c} \sum_{k=1,2,3,4} \bar{F}_{3k} \cdot \left(\frac{\bar{v}_k}{e}\right) - \frac{\sigma_t - \sigma_c}{2\sigma_t\sigma_c} \sum_{k=1,2,3,4} \bar{F}_{3k} \cdot \bar{r}_k \qquad (4)$$

Now since $\bar{F}_{3k}$ is the resultant of systems $\bar{F}_{1i}$ and $\bar{F}_{2j}$, it follows that

$$\bar{F}_{3k} = \bar{F}_{1i} + \bar{F}_{2j} \qquad (5)$$

that is,

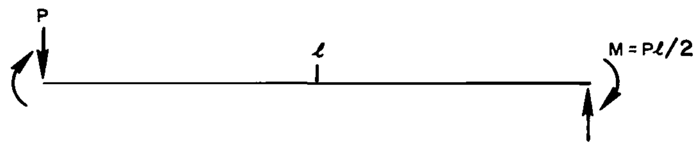
$$\left.\begin{aligned} \bar{F}_{31} &= \bar{F}_{11} + \bar{F}_{21} \\ \bar{F}_{32} &= \bar{F}_{12} + \bar{F}_{22} \\ \bar{F}_{33} &= \bar{F}_{13} \\ \bar{F}_{34} &= \bar{F}_{24} \end{aligned}\right\} \qquad (6)$$

Structures S_1 and S_2 have common points $\bar{r}_1$ and $\bar{r}_2$ at which they have equal displacements $\bar{v}_1$ and $\bar{v}_2$. Hence the structures S_1 and S_2 can be superposed. The resulting composite structure will have the volume $(V_1 + V_2)$. From equations (2), (3), (4), and (5), it is seen that the volume V_3 of the Michell structure S_3 which equilibrates the resultant of the force systems $\bar{F}_{1i}$ and $\bar{F}_{2j}$ is equal to the volume $(V_1 + V_2)$ of the composite structure obtained by superposing the two Michell structures S_1 and S_2 corresponding to the load systems $\bar{F}_{1i}$ and $\bar{F}_{2j}$; that is,

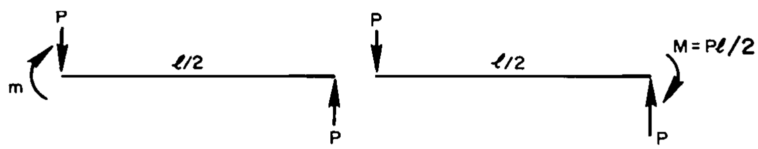
$$V_3 = V_1 + V_2 \qquad (7)$$

Application of the Superposition Theorem

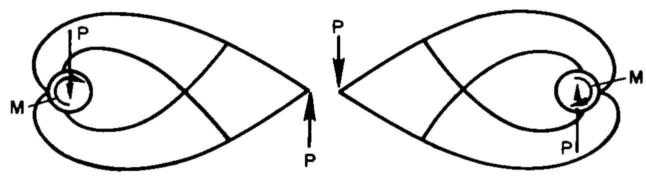
The theorem is now used to develop optimum structures for the load systems of figures 1(a) and 2(a). The load system of figure 1(a), representing a pure shear beam-type loading, is resolved into two constitutive systems, as shown in figure 1(b). Each system lends itself to a solution, in the sense that there is a definite Michell structure to equilibrate each system. These structures are shown independently in figure 1(c) and are combined as shown in figure 1(d) to give the optimum structure for the load system under consideration.



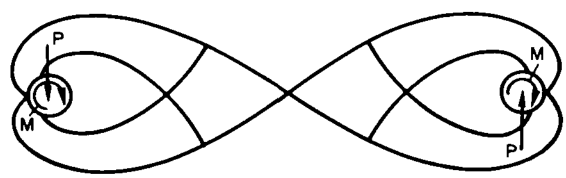
(a) Force system 1 (pure shear beam-type loading).



(b) Component systems for above force system.



(c) Michell structures for component systems.



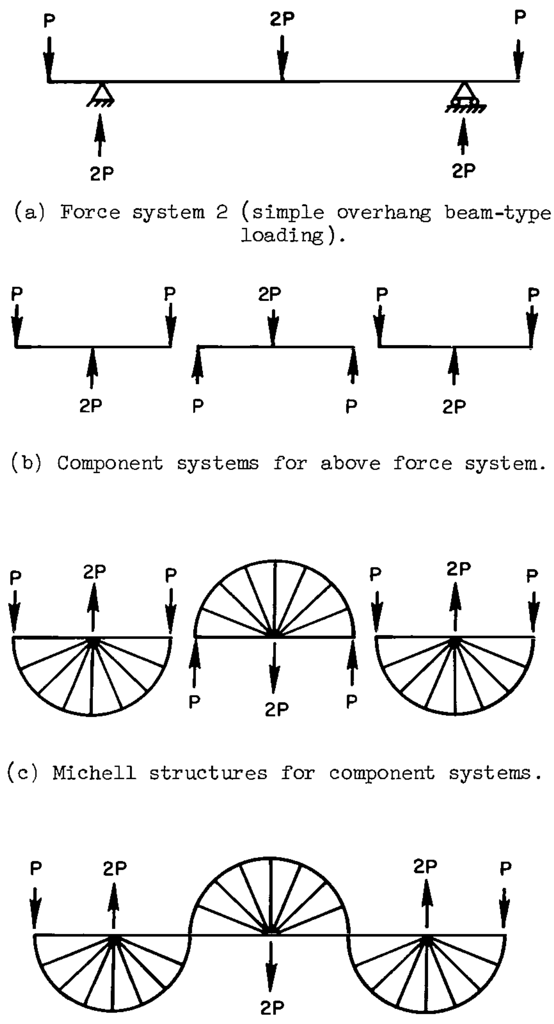
(d) Composite structure (optimum structure for the above force systems).

Figure 1.- Michell structure for a pure shear beam-type loading.

The volume of each individual unit is (see ref. 9)

$$V_c = \pi P \frac{l}{2}\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_t}\right)$$

$$= \frac{\pi M}{2}\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_t}\right) \tag{8}$$



(a) Force system 2 (simple overhang beam-type loading).

(b) Component systems for above force system.

(c) Michell structures for component systems.

(d) Composite structure (optimum structure for above force system).

Figure 2.- Michell structure for an overhang beam-type loading.

The volume of the Michell structure (fig. 1(d)) to carry the load system of figure 1(a) is, then,

$$V^* = 2V_c$$

$$= \pi M\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_t}\right) \tag{9}$$

The load system of figure 2(a) is resolved as shown in figure 2(b). The Michell structure for each component load system is shown in figure 2(c). The units are combined as shown in figure 2(d). The volume of each unit is (see ref. 9)

$$V_c = \frac{\pi}{6} Pl\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_t}\right) \tag{10}$$

The volume of the Michell structure (fig. 2(d)) to carry the load system of figure 2(a) is

$$V^* = 3V_c$$

$$= \frac{\pi}{2} Pl \left(\frac{1}{\sigma_c} + \frac{1}{\sigma_t}\right) \tag{11}$$

THE DEVELOPMENT OF A SYSTEM OF COORDINATE CURVES FOR MEMBER LAYOUT OF A MICHELL OPTIMUM STRUCTURE

The governing equations for the form of the orthogonal coordinate curves and the equations of equilibrium of a curvilinear element are presented in this section. These equations have been derived by Hemp (ref. 5); however, their derivation will be briefly represented to maintain continuity of thought. A solution is then obtained to Hemp's governing equations which dictate the form of the orthogonal coordinate system that satisfies the condition of compatibility for the Michell criterion of strain in two dimensions. This solution gives the layout of coordinate curves which constitute the lines of principal strain along which the members of the corresponding Michell structure lie. These curves are then enclosed within a suitable boundary by matching, at the boundary, the forces in the members of the structure and the external forces acting on the structure (one gets the nature of external forces on the boundary of the structure). Indirectly, then, one obtains the force system which the developed Michell structure (the members lie along the coordinate curves and are represented by the solution obtained from Hemp's governing equations for the form of Michell structures) equilibrates. Inversely, one can say that the solution (i.e., a Michell structure) has been obtained for this force system.

Equations Governing the Form of the Coordinate Curves

The outline of the development of Hemp's governing equations for the form of the Michell optimum structure is now presented. (For details see section 3 of ref. 5.) Let $x = x(\alpha,\beta)$, $y = y(a,\beta)$ be a set of orthogonal coordinate curves representing lines of principal strain. The parametric pair (α,β) represents curvilinear coordinates in the plane of the rectangular Cartesian coordinate system. The functions $x(\alpha,\beta)$ and $y(a,\beta)$ are continuous and have first and second derivatives with respect to α and β. Along α-coordinate curves, α varies and β is constant; along β-coordinate curves, β varies and α is constant. Positive directions along both of these curves are those along which α and β are increasing. The arc lengths ds_1 and ds_2 along α and β curves are given by

$$ds_1 = A(\alpha,\beta)d\alpha, \qquad ds_2 = B(\alpha,\beta)d\beta \tag{12}$$

where

$$\left.\begin{aligned} A &= \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2\right]^{1/2} \\ &\text{and} \\ B &= \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right]^{1/2} \end{aligned}\right\} \tag{13}$$

Let ψ_1 and ψ_2 represent the angles made by the positive tangents to α and β curves, respectively, with the x axis such that

$$\bar{\omega} = \psi_2 - \psi_1 \tag{14}$$

It follows that

$$\left.\begin{aligned} \cos\psi_1 &= \frac{dx}{ds_1} = \frac{1}{A}\frac{\partial x}{\partial \alpha}, &\quad \cos\psi_2 &= \frac{dx}{ds_2} = \frac{1}{B}\frac{\partial x}{\partial \beta} \\ \sin\psi_1 &= \frac{dy}{ds_1} = \frac{1}{A}\frac{\partial y}{\partial \alpha}, &\quad \sin\psi_2 &= \frac{dy}{ds_2} = \frac{1}{B}\frac{\partial y}{\partial \beta} \end{aligned}\right\} \tag{15}$$

Now

$$\begin{aligned} \cos\bar{\omega} &= \cos(\psi_2 - \psi_1) \\ &= \frac{1}{AB}\left(\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta}\right) \end{aligned} \tag{16}$$

From equations (13) and (16), the derivatives $\partial\psi_1/\partial\alpha$ and $\partial\psi_2/\partial\beta$ are given by[1]

$$\left.\begin{aligned} \frac{\partial\psi_1}{\partial\alpha} &= -\frac{1}{B\sin\bar{\omega}}\left[\frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha}(B\cos\bar{\omega})\right] \\ \frac{\partial\psi_2}{\partial\beta} &= \frac{1}{A\sin\bar{\omega}}\left[\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(A\cos\bar{\omega})\right] \end{aligned}\right\} \tag{17}$$

Also, from equations (14) and (17) the following relation can be shown to exist between A, B, and $\bar{\omega}$:

$$\frac{\partial}{\partial\alpha}\left\{\frac{1}{A\sin\bar{\omega}}\left[\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(A\cos\bar{\omega})\right] - \frac{\partial\bar{\omega}}{\partial\beta}\right\} + \frac{\partial}{\partial\beta}\left\{\frac{1}{B\sin\bar{\omega}}\left[\frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha}(B\cos\bar{\omega})\right]\right\} = 0 \tag{18}$$

For $\bar{\omega} = \pi/2$, this relation becomes

[1]For details refer to appendix A of reference 5.

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \tag{19}$$

In the curvilinear coordinate system, the deformation is characterized by the strains $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$. The expressions for the strains are given by (ref. 5, p. 59)

$$\left.\begin{aligned} e_{\alpha\alpha} &= \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{v}{AB}\frac{\partial A}{\partial\beta} \\ e_{\beta\beta} &= \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{u}{AB}\frac{\partial B}{\partial\alpha} \\ e_{\alpha\beta} &= \frac{1}{2}\left[\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right)\right] \end{aligned}\right\} \tag{20}$$

where u and v are the displacements along α and β curves, respectively.

In the deformed state, the arc lengths ds_1 and ds_2 are increased by $1 + e_{\alpha\alpha}$ and $1 + e_{\beta\beta}$, respectively, and the angle $\overline{\omega}$ is deformed from $\pi/2$ to $(\pi/2) - a\alpha\beta$. Replace A, B, and $\overline{\omega}$ by $A(1 + e_{\alpha\alpha})$, $B(1 + e_{\beta\beta})$, and $(\pi/2) - 2\alpha\beta$, respectively, in equation (18); if the resulting equation is developed correctly to the first order of strain, the equation of compatibility of strain is obtained as follows:

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial e_{\beta\beta}}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial e_{\alpha\alpha}}{\partial\beta}\right) - 2\frac{\partial^2 e_{\alpha\beta}}{\partial\alpha\,\partial\beta} - \frac{\partial}{\partial\alpha}\left[\frac{1}{A}\frac{\partial B}{\partial\alpha}(e_{\alpha\alpha} - e_{\beta\beta})\right]$$

$$+ \frac{\partial}{\partial\beta}\left[\frac{1}{B}\frac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta})\right] - 2\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial A}{\partial\beta}e_{\alpha\beta}\right) - 2\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial B}{\partial\alpha}e_{\alpha\beta}\right) = 0 \tag{21}$$

Now the Michell criterion of strain, for members carrying strains of opposite sign, is defined by

$$e_{\alpha\alpha} = e\ , \qquad e_{\beta\beta} = -e\ , \qquad e_{\alpha\beta} = 0 \tag{22}$$

For this strain condition, the equation of compatibility of strain becomes

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) - \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \tag{23}$$

It follows from equations (19) and (23) that

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = 0 \ , \qquad \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \tag{24}$$

Thus,

$$\left.\begin{aligned} \frac{1}{B}\frac{\partial A}{\partial\beta} &= -F_1'(\alpha) \\ \frac{1}{A}\frac{\partial B}{\partial\alpha} &= F_2'(\beta) \end{aligned}\right\} \tag{25}$$

where $F_1(\alpha)$, $F_2(\beta)$ are arbitrary functions and $F_1'(\alpha)$, $F_2'(\beta)$ are their derivatives. The following transformation is now introduced:

$$\alpha = \varphi_1(\overline{\alpha}) \ , \qquad \beta = \varphi_2(\overline{\beta}) \tag{26}$$

where φ_1, φ_2, φ_1', and φ_2' are continuous functions. Further, φ_1 and φ_2 are chosen so that

$$\varphi_1(\overline{\alpha}) = F_1^{-1}(\pm\overline{\alpha}) \ , \qquad \varphi_2(\overline{\beta}) = F_2^{-1}(\pm\overline{\beta}) \tag{27}$$

where F_1^{-1}, F_2^{-1} are the inverse functions of F_1, F_2 and the upper or lower sign is taken accordingly as $F_1'F_2'$ is positive or negative. The transform of equations (25) can now be written in terms of α and β as follows:

$$\frac{1}{B}\frac{\partial A}{\partial\beta} = -1 \ , \qquad \frac{1}{A}\frac{\partial B}{\partial\alpha} = 1 \tag{28}$$

These are the governing equations for the arc lengths A and B of the system defined by equations (12).

Now, the angles ψ_1 and ψ_2, defining the directions of α and β curves, are obtained from equations (14) with $\overline{\omega} = \pi/2$, (17) and (28) as follows:

$$\psi_1 = \alpha + \beta \ , \qquad \psi_2 = \frac{\pi}{2} + \alpha + \beta \tag{29}$$

In order to select a set of orthogonal coordinate curves (12) that satisfy the strain conditions (22), the corresponding rates of changes of arc lengths A and B have to satisfy equations (28). When A, B have been found, equations (12), along with equations (29), can be employed to determine the equations of the lines of principal strains.

Equations of Equilibrium

The equations derived by Hemp (appendix B, ref. 5) for the equilibrium of a curvilinear element $d\alpha$ by $d\beta$ are now presented. Let t_1 and t_2 denote the thicknesses of the α and β members. Also, let T_1, T_2 be the forces per unit length parallel to the α and β curves and S be the shear per unit length. From the equilibrium of a curvilinear element $d\alpha$ by $d\beta$, the following equations are obtained:

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}(BT_1) + \frac{\partial}{\alpha\beta}(AS) + \frac{\partial A}{\partial\beta}S - \frac{\partial B}{\partial\alpha}T_2 = 0 \\ \frac{\partial}{\partial\alpha}(BS) + \frac{\partial}{\partial\beta}(AT_2) - \frac{\partial A}{\partial\beta}T_1 + \frac{\partial B}{\partial\alpha}S = 0 \end{aligned}\right\} \quad (30)$$

For Michell criterion, the shear stress equals zero. Further, from equations (22) it follows that

$$T_1 = \sigma_t t_1 \ , \qquad T_2 = \sigma_c t_2 \quad (31)$$

and hence the equations of equilibrium can be written as follows:

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}(B\sigma_t t_1) - \frac{\partial B}{\partial\alpha}(\sigma_c t_2) = 0 \\ \frac{\partial}{\partial\beta}(A\sigma_c t_2) - \frac{\partial A}{\partial\beta}(\sigma_t t_1) = 0 \end{aligned}\right\} \quad (32)$$

Determination of a System of Coordinate Curves and the Development of the Corresponding Michell Optimum Framework

A solution is now sought to the governing equations (28). Hemp has obtained a few solutions in the form of certain functions for the arc lengths A and B and has presented some layouts (i.e., coordinate curves) for the members of the Michell optimum structures. Herein, solutions to the governing equations are obtained in terms of different functions for A and B, which then give us a different layout (i.e., a set of coordinate curves) for the members of a Michell structure. This layout, then, gives Michell structures which can equilibrate new force systems; thus, solutions (Michell structures) are obtained for these new systems. To determine a system of coordinate curves that satisfy the Michell criterion of strain (22), equations (28) have to be solved. The following functions for the unit arc lengths A and B satisfy equations (28):

$$\left.\begin{aligned} A = 4\cos(\alpha + \beta) \\ B = 4\sin(\alpha + \beta) \end{aligned}\right\} \quad (33)$$

Equations (15), (29), and (33) then give

$$x = \int \cos\psi_1 \, ds_1 = 4\int \cos^2(\alpha + \beta)d\alpha$$

$$= \sin 2(\alpha + \beta) + 2\alpha + C_1(\beta)$$

where $C_1(\beta)$ is an arbitrary function of β Also,

$$x = \int \cos \psi_2 \, ds_2 = -4\int \sin^2(\alpha + \beta) d\beta$$

$$= -2\beta + \sin 2(\alpha + \beta) + C_2(\alpha)$$

where $C_2(\alpha)$ is an arbitrary function of α. On identifying the two expressions for x, it is seen that $C_1(\beta)$ is indeed -2β and $C_2(\alpha)$ is 2α. Hence, the expression for x is given by

$$x = 2(\alpha - \beta) + \sin 2(\alpha + \beta) \tag{34}$$

Similarly,

$$y = -\cos 2(\alpha + \beta) \tag{35}$$

Now, eliminating α and β, the Cartesian forms of the α and β curves (representing lines of principal strain) for $-1 \leq y \leq 1$ are, respectively,

$$\left.\begin{aligned} x &= -4\beta + \cos^{-1}(-y) + \sqrt{1 - y^2}\,, \quad && \alpha \text{ curves} \\ x &= 4\alpha - \cos^{-1}(-y) + \sqrt{1 - y^2}\,, \quad && \beta \text{ curves} \end{aligned}\right\} \tag{36}$$

The above equations represent the coordinate curves for the layout of the Michell structure. The curves (illustrated in fig. 3) are enclosed in a rectangular domain $-1 \leq y \leq 1$, $-\pi \leq x \leq \pi$, and can be used to solve plane stress problems for rectangular strips loaded with suitable self-equilibrating forces. The α and β curves are in tension and compression, respectively. These tension and compression members are indicated in figure 3. From equations (15) the angles ψ_1 and ψ_2 made by the α and β curves with the x axis are given by

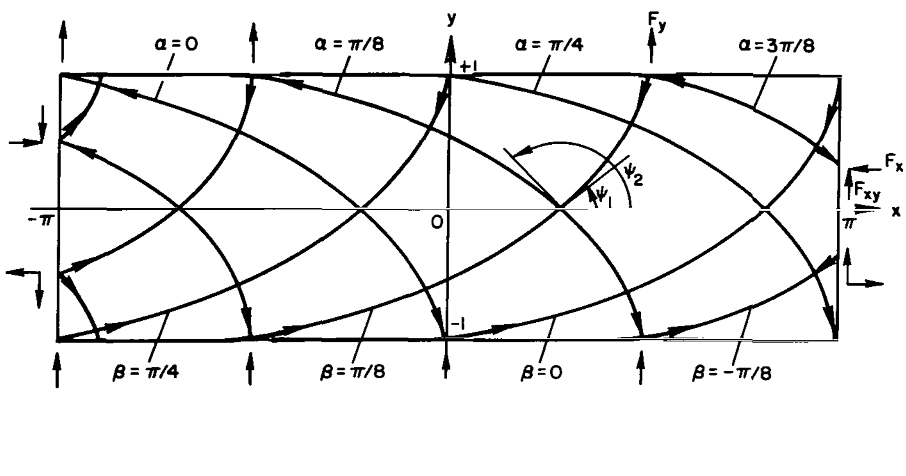


Figure 3.- Coordinate curves forming a Michell structure within a rectangular domain.

$$\left.\begin{aligned} \cos \psi_1 &= \sqrt{\frac{1 - y}{2}} & \sin \psi_1 &= \sqrt{\frac{1 + y}{2}} \\ \cos \psi_2 &= -\sqrt{\frac{1 + y}{2}}, & \sin \psi_2 &= \sqrt{\frac{1 - y}{2}} \end{aligned}\right\} \tag{37}$$

The areas of the members (i.e., thicknesses t_1 and t_2 of the α and β members) of the structure are now determined from the equilibrium of a curvilinear element of the structure (eqs. (32)) and from the force equilibrium at the boundary. If $\sigma_t = \sigma = -\sigma_c$ in the equilibrium equations (32) and equation (34) is used, the equilibrium equations can be written:

$$\left.\begin{aligned} \left[(t_1 + t_2) + 2(1 + y)\frac{\partial t_1}{\partial y}\right]\sqrt{\frac{1 - y}{2}} &= 0 \\ \left[(t_1 + t_2) - 2(1 - y)\frac{\partial t_2}{\partial y}\right]\sqrt{\frac{1 + y}{2}} &= 0 \end{aligned}\right\} \tag{38}$$

The solution to the above equations is given by

$$\left.\begin{aligned} t_1 &= f(x)\sqrt{\frac{1 - y}{1 + y}} + g(x) \\ t_2 &= f(x)\sqrt{\frac{1 + y}{1 - y}} - g(x) \end{aligned}\right\} \tag{39}$$

where $f(x)$ and $g(x)$ are arbitrary functions of x. It is seen that $y = 1$ presents a singularity. The functions $f(x)$ and $g(x)$ are determined from the total force equilibrium on the boundary.

The variation of the forces in the structure for $|\sigma_c| = |\sigma_t| = \sigma$ is given as: (i) The tangential force per unit length (T_{xy}) is proportional to

$$t_1 \cos \psi_1 + t_2 \sin \psi_1 = t_1\sqrt{\frac{1 - y}{2}} + t_2\sqrt{\frac{1 + y}{2}} \tag{40}$$

where t_1 and t_2 are given by equations (39); (ii) the normal force per unit length parallel to the x axis (T_x) is proportional to

$$t_1 \cos \psi_1 - t_2 \sin \psi_1 = t_1\sqrt{\frac{1 - y}{2}} - t_2\sqrt{\frac{1 + y}{2}} \tag{41}$$

(iii) the normal force per unit length parallel to the y axis (T_y) is proportional to

$$t_1 \sin \psi_1 - t_2 \cos \psi_1 = t_1\sqrt{\frac{1 + y}{2}} - t_2\sqrt{\frac{1 - y}{2}} \tag{42}$$

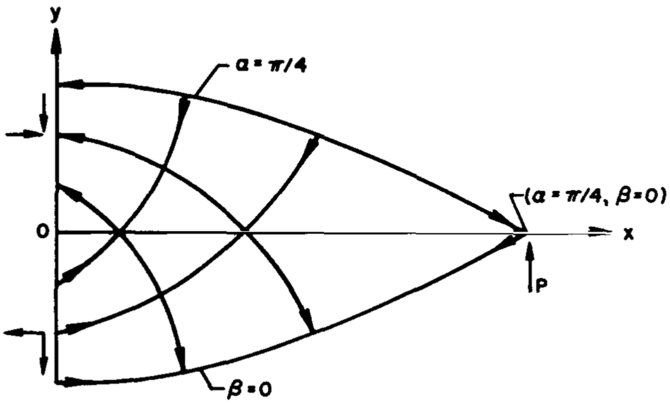


Figure 4.- Coordinate curves forming a Michell structure within a domain enclosed by α, β curves and a y ordinate.

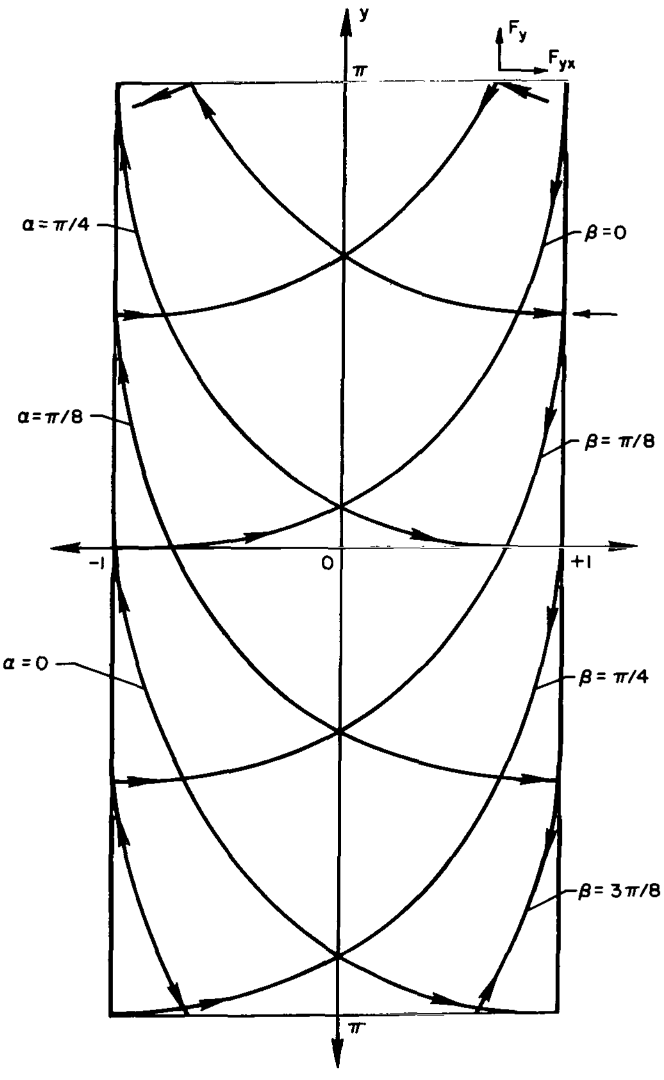


Figure 5.- Coordinate curves forming a Michell structure within a rectangular domain.

Although the external forces carried by the structure should be self-equilibrating, their distribution is governed by the distribution of T_x, T_y, T_{xy} (eqs. (40), (41), (42)).

The above development shows how the system of coordinate curves under consideration develops the Michell optimum structure for a rectangular domain under a system of external forces. These coordinate curves could also be enclosed between an α curve (say $\beta = 0$), a β curve (say $\alpha = \pi/4$), and a y ordinate (say $x = 0$). When thus bounded, as shown in figure 4, a concentrated force P at $\alpha = \pi/4$, $\beta = 0$ (i.e., at $x = 1 + (\pi/2)$, $y = 0$) can be equilibrated by means of normal and tangential forces distributed along the y ordinate $x = 0$.

Alternatively, the following solutions to equations (29) can be had:

$$\left.\begin{aligned} A &= 4\sin(\alpha + \beta) \\ B &= -4\cos(\alpha + \beta) \end{aligned}\right\} \qquad (43)$$

The equations for these curves in a rectangular Cartesian coordinate system are given by

$$\left.\begin{aligned} y &= 4\alpha - \cos^{-1}(-x) - \sqrt{1 - x^2} \\ y &= 4\beta + \cos^{-1}(-x) - \sqrt{1 - x^2} \end{aligned}\right\} \qquad (43a)$$

The corresponding curves are illustrated in figure 5. In nature, this system of curves is the same as the one considered earlier (given by eqs. (33)) except for a 90° clockwise rotation about the origin.

CONSIDERATION OF TWO SPECIFIC ORTHOGONAL COORDINATE SYSTEMS FOR THE MEMBER LAYOUT OF MICHELL OPTIMUM STRUCTURES

Two specific orthogonal coordinate systems - the dipolar and the spherical - are selected for the member layout of Michell optimum structures for each system, and the strain-displacement equations (20) are solved for the displacements, with the strains $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\alpha\beta}$ corresponding to the Michell strain criterion. The system of curves is enclosed within some regular boundaries and a self-equilibrating system of forces is made to act along the boundary. A Michell optimum structure then obtains for the selected force system. The volume of this structure can now be obtained from equation (1) since the displacements of the points of application of the forces are determined. Examples of some force systems are presented as applications of the coordinate systems for member layouts of the Michell optimum frameworks.

The Dipolar Coordinate System

The dipolar coordinate system, figure 6, is defined by the following transformation:

$$x = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}, \qquad y = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta} \tag{44}$$

where (α,β) represent curvilinear coordinates in the plane of the rectangular coordinate system and a is a constant parameter. The corresponding unit arc lengths A and B are determined from equations (13) and are obtained as

$$A = B = \frac{a}{\cosh \alpha - \cos \beta} \tag{45}$$

For the above values of A and B, the equation of compatibility of strains, equation (21), can only be satisfied if $(e_{\alpha\alpha} - e_{\beta\beta})$ is zero. It follows that this system only admits of Michell strains of the same sign. The corresponding Michell criterion for strains is taken as

$$e_{\alpha\alpha} = e_{\beta\beta} = \pm e, \qquad e_{\alpha\beta} = 0 \tag{46}$$

Since this system can only admit Michell strains of the same sign, the resulting layout of the members will give an all-tension or an all-compression structure. The coordinate curves, along which the members will lie, dictate the natural boundaries and the force system that can be equilibrated. Inversely then, a Michell structure is obtained for that force system. The displacements of the system (obtained by solving the strain-displacement equations for the strain system of eq. (46)) give the displacements of the points of application of the forces from which the weight of the

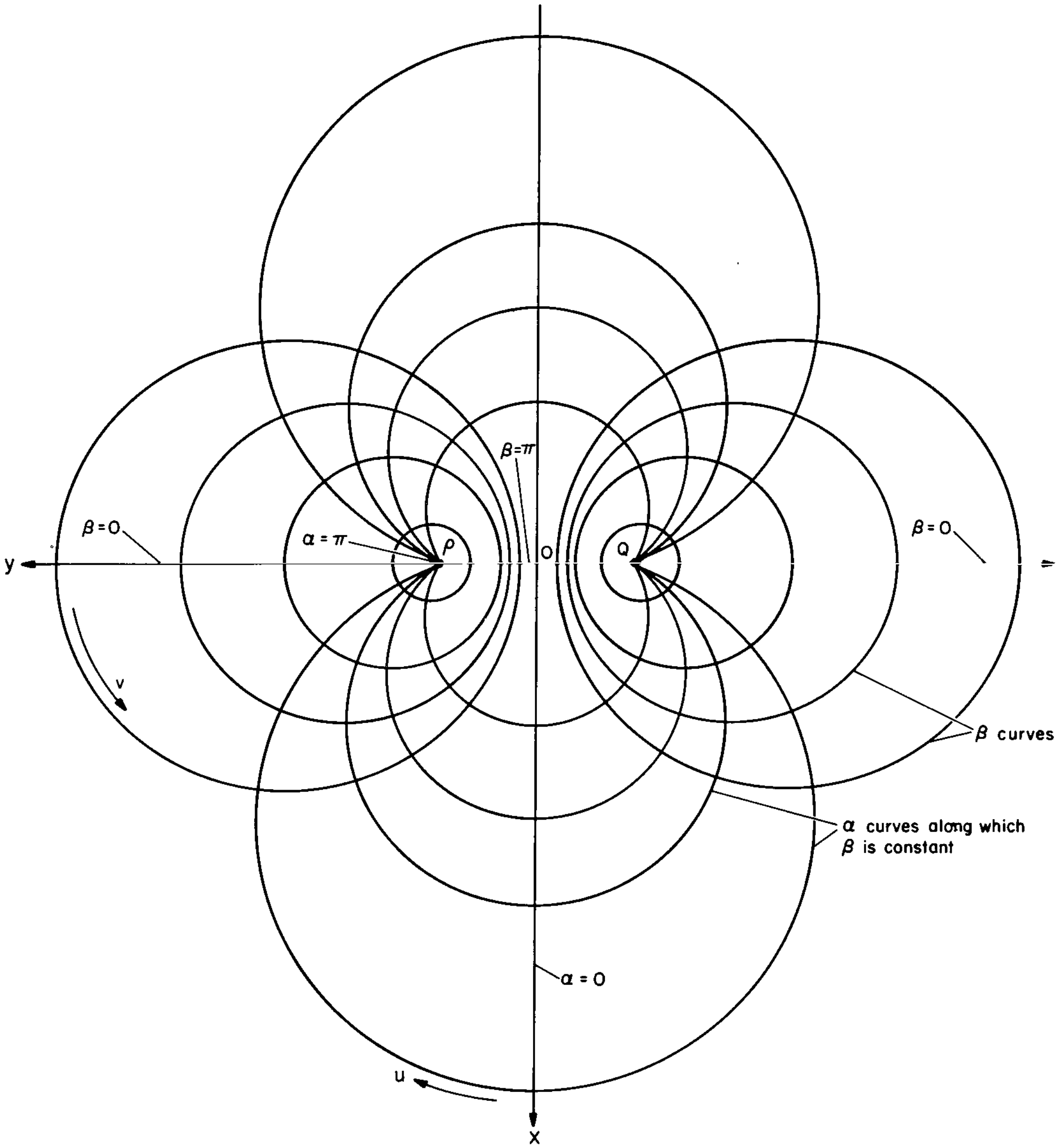


Figure 6.- The dipolar coordinate system.

structure is determined. Hence, the strain-displacement equations for the system will now be solved for the strain pattern of equations (46).

The expressions for strains in this coordinate system are obtained by substituting the expressions for A and B from equation (45) into equations (20). Then, by employing the Michell criterion defined by equations (46), the differential equations for the displacements u and v along the α and β curves, respectively, are obtained as

$$\left.\begin{aligned} \frac{\cosh\alpha - \cos\beta}{a}\frac{\partial u}{\partial \alpha} - \frac{\sin\beta}{a} v &= \pm e \\ \frac{\cosh\alpha - \cos\beta}{a}\frac{\partial v}{\partial \beta} - \frac{\sinh\alpha}{a} u &= \pm e \\ \frac{\cosh\alpha - \cos\beta}{a}\frac{\partial v}{\partial \alpha} + v\frac{\sinh\alpha}{a} + \frac{\cosh\alpha - \cos\beta}{a}\frac{\partial u}{\partial \beta} + u\frac{\sin\beta}{a} &= 0 \end{aligned}\right\} \tag{47}$$

When $\psi = \cosh\alpha - \cos\beta$, the above system of equations is rewritten as

$$\psi\frac{\partial u}{\partial \alpha} - \frac{\partial \psi}{\partial \beta} v = \pm ae \tag{48a}$$

$$\psi\frac{\partial v}{\partial \beta} - \frac{\partial \psi}{\partial \alpha} u = \pm ae \tag{48b}$$

$$\frac{\partial}{\partial \alpha}(\psi v) + \frac{\partial}{\partial \beta}(\psi u) = 0 \tag{48c}$$

From equations (48a) and (48b),

$$\frac{\partial}{\partial \alpha}(\psi u) = \frac{\partial}{\partial \beta}(\psi v) \tag{49}$$

Now equations (48c) and (49) give

$$\frac{\partial^2}{\partial \alpha^2}(\psi u) + \frac{\partial^2}{\partial \beta^2}(\psi u) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial \alpha^2}(\psi v) + \frac{\partial^2}{\partial \beta^2}(\psi v) = 0 \tag{50}$$

Multiplying equation (48a) by ψ, differentiating with respect to α, eliminating $(\partial/\partial\alpha)(\psi v)$ by means of equation (48c), and employing the first of equations (50) yields a second-order differential equation that contains derivatives of u with respect to β only:

$$\psi\frac{\partial^2}{\partial \beta^2}(\psi u) - \frac{\partial \psi}{\partial \beta}\frac{\partial}{\partial \beta}(\psi u) + \left(\frac{\partial^2 \psi}{\partial \alpha^2}\right)(\psi u) = \mp ae\frac{\partial \psi}{\partial \alpha} \tag{51}$$

Similarly, from equations (48b), (48c), and the second of equations (50),

$$\psi \frac{\partial^2}{\partial \alpha^2} (\psi v) - \frac{\partial \psi}{\partial \alpha} \frac{\partial}{\partial \alpha} (\psi v) + \left(\frac{\partial^2 \psi}{\partial \beta^2}\right) (\psi v) = \mp ae \frac{\partial \psi}{\partial \beta} \tag{52}$$

When the general solutions to equations (51) and (52) are obtained, the resulting expressions for the displacements u and v must be of the form

$$\left.\begin{aligned} u(\alpha,\beta) &= \frac{f_1(\alpha)\sin\beta + f_2(\alpha)(\cos\beta - \operatorname{sech}\alpha) \mp ae \tanh\alpha}{\psi(\alpha,\beta)} \\ v(\alpha,\beta) &= \frac{f_3(\beta)\sinh\alpha + f_4(\beta)(\cosh\alpha - \sec\beta) \mp ae \tan\beta}{\psi(\alpha,\beta)} \end{aligned}\right\} \tag{53}$$

for arbitrary functions $f_1(\alpha)$, $f_2(\alpha)$, $f_3(\beta)$, and $f_4(\beta)$. Equations (48a-c) now determine the form of the arbitrary functions and yield finally:

$$\left.\begin{aligned} u(\alpha,\beta) &= \frac{(k \sinh\alpha + l\cosh\alpha)\sin\beta + (\mp ae \sinh\alpha + n\cosh\alpha)\cos\beta - n}{\psi(\alpha,\beta)} \\ v(\alpha,\beta) &= \frac{(n\sin\beta - l\cos\beta)\sinh\alpha + (\mp ae\sin\beta - k\cos\beta)\cosh\alpha + k}{\psi(\alpha,\beta)} \end{aligned}\right\} \tag{54}$$

where k, l, and n are arbitrary constants.

Equations (54) form the general solutions to the system of equations (48a-c) or (47). A general solution has been obtained for the displacements of the dipolar coordinate system for the Michell strain condition defined by equations (46). If the system of curves is enclosed within suitable boundaries, a system of forces can be equilibrated. Then the constants in solutions (54) can be determined from the physical conditions of the problem. This will now be demonstrated with respect to two applications of the dipolar system in the development of Michell optimum structures.

Application 1.- For the first application, a region bounded by two nonconcentric circles $\alpha = \alpha_1$ and $\alpha = \alpha_2$ is taken (see fig. 7). The loading on the resulting structure is taken to consist of uniform external and internal pressure p_1 and p_2 acting on the β boundaries $\alpha = \alpha_1$, $\alpha = \alpha_2$, respectively, as shown in the figure. Both sets of members are in compression. The general solutions for displacements for the pertinent strain system of $e_{\alpha\alpha} = -e = e_{\beta\beta}$, $e_{\alpha\beta} = 0$, are given by

$$
\left.
\begin{aligned}
u(\alpha,\beta) &= \frac{ae \sinh\alpha \cos\beta}{\psi} + \frac{k \sinh\alpha \sin\beta}{\psi} + \frac{l \cosh\alpha \sin\beta}{\psi} \\
&\quad + n\,\frac{\cosh\alpha \cos\beta - 1}{\psi} \\
v(\alpha,\beta) &= \frac{ae \sin\beta \cosh\alpha}{\psi} + \frac{k(1 - \cos\beta \cosh\alpha)}{\psi} - \frac{l \cos\beta \sinh\alpha}{\psi} \\
&\quad + n\,\frac{\sinh\alpha \sin\beta}{\psi}
\end{aligned}
\right\} \qquad (55)
$$

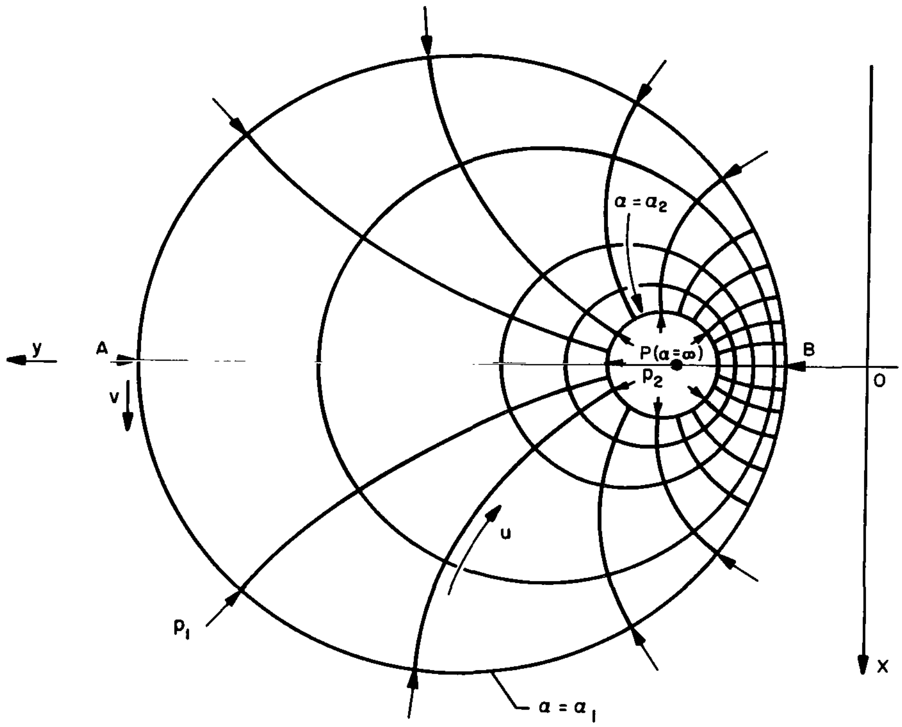


Figure 7.- Application 1 of the dipolar coordinate system: two nonconcentric circles under external and internal normal pressure - a candidate for entry body configuration.

The boundary conditions are as follows:

(i) At $P(\alpha = \infty)$ for any β direction, $u = 0$.

(ii) Along AB (i.e., for $\beta = 0$ or π and any α) due to the symmetry of line AB, the displacement $v = 0$. At $(\alpha, \beta = \pi)$, the condition $v = 0$ gives (from eqs. (55))

$$k = \frac{-l \sinh \alpha}{1 + \cosh \alpha} \tag{56}$$

At $(\alpha, \beta = 0)$, condition $v = 0$ gives

$$k = \frac{l \sinh \alpha}{1 - \cosh \alpha} \tag{57}$$

However, since k and l are constants, it follows, from equations (56) and (57), that

$$k = l = 0 \tag{58}$$

At $(\alpha = \infty, \beta)$, the condition $u = 0$ gives

$$n = -ae \tag{59}$$

From equations (55) through (59) it follows that the displacements corresponding to the structure and its loading are given by

$$\left.\begin{aligned} u(\alpha,\beta) &= \frac{ae}{\cosh \alpha - \cos \beta} (\sinh \alpha \cos \beta - \cosh \alpha \cos \beta + 1) \\ v(\alpha,\beta) &= \frac{ae}{\cosh \alpha - \cos \beta} (\cosh \alpha \sin \beta - \sinh \alpha \sin \beta) \end{aligned}\right\} \tag{60}$$

Once the displacements are known, the solution to the problem is complete, for its volume can be obtained by means of equation (1). The volume is given by (for $|\sigma_c| = |\sigma_t| = \sigma$)

$$V^* = \frac{1}{\sigma} \int \left[\left(p_1 \, ds_2 \, \frac{u}{e}\right)_{\alpha=\alpha_1} + \left(p_2 \, ds_2 \, \frac{u}{e}\right)_{\alpha=\alpha_2}\right]$$

$$= \frac{2}{\sigma} \int_{\beta=0}^{\beta=\pi} \left[\left(Bp_1 \, \frac{u}{e}\right)_{\alpha=\alpha_1} + \left(Bp_2 \, \frac{u}{e}\right)_{\alpha=\alpha_2}\right] d\beta \tag{61}$$

where B, obtained by using equations (13), is given by $B = (a/\cosh \alpha - \cos \beta)$ and u is given by equation (60).

Since the application of the dipolar coordinate system in obtaining the Michell structure for the loading illustrated in figure 7 has been presented, it would be interesting to consider some practical use of that load system. It is conceivable that such a configuration could be used for an entry body. When thus employed, the inner circle could represent the payload compartment which is displaced with respect to the geometric center of the body; the external and internal pressures, then, would represent the aerodynamic forces and the inertia forces, respectively. The α members form the structural members to carry the forces due to the aerodynamic loading. Also, at impact the α members could buckle and thus act as energy absorbers.

Application 2 (see fig. 8).- Herein the Michell structure is bounded by α curves $\beta = \beta_o$ and $\beta = \pi$ (or the y axis). The loading consists of a distributed pressure (p_1) loading along PQ ($\beta = \pi$) and reacted by normal

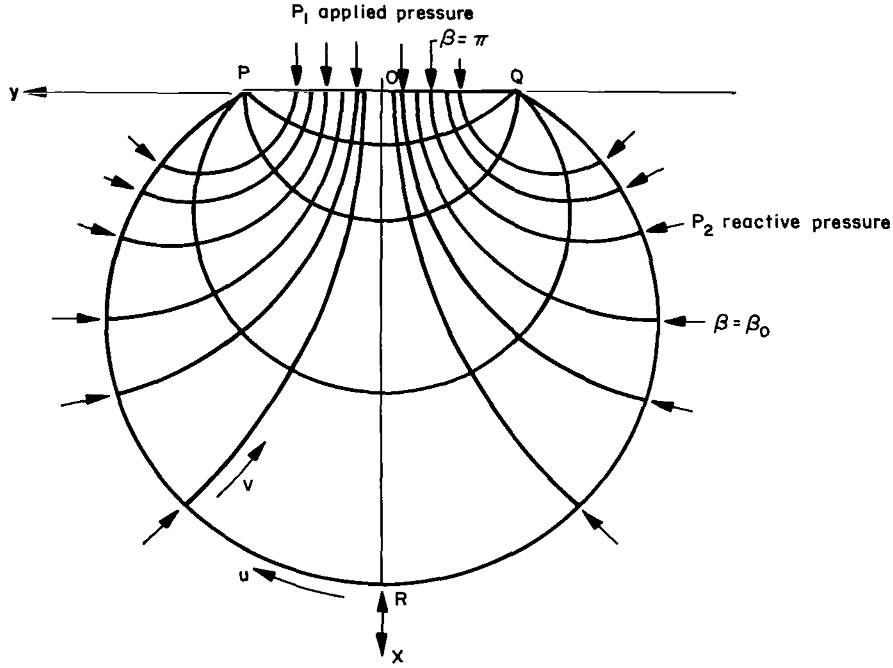


Figure 8.- Application 2 of the dipolar coordinate system: transmission of applied pressure from one surface to another of greater surface area.

pressure (p_2) along the surface $\beta = \beta_o$. All the members, that is, the α and the β sets, are in compression. The Michell strain criterion is again $e_{\alpha\alpha} = -e = e_{\beta\beta}$, $e_{\alpha\beta} = 0$ and the corresponding general expressions for the displacements u and v are given by equations (55). The boundary conditions are as follows:

(i) Along $\alpha = 0$, for all β (i.e., along the x axis) $u = 0$ due to symmetry.

(ii) Along $\beta = \pi$ (i.e., the y axis), the displacement v is zero.

These boundary conditions give

$$k = l = n = 0 \tag{62}$$

Thus, the displacements are given by

$$u(\alpha,\beta) = \frac{ae \sinh \alpha \cos \beta}{\psi}$$

$$v(\alpha,\beta) = \frac{ae \sin \beta \cosh \alpha}{\psi} \qquad (63)$$

The volume of the structure is now given by (for $|\sigma_c| = |\sigma| = \sigma$)

$$V^* = \frac{1}{\sigma} \int \left[p_1 \, ds_1 \, \frac{v}{e}\Big|_{\beta=\pi} + p_2 \, ds_1 \, \frac{v}{e}\Big|_{\beta=\beta_o} \right]$$

$$= \frac{2}{\sigma} \int_{\alpha=o}^{\alpha=\infty} A\left[\left(p_1 \frac{v}{e}\right)_{\beta=\pi} + \left(p_2 \frac{v}{e}\right)_{\beta=\beta_o}\right] d\alpha \qquad (64)$$

where $A = (a/\cosh \alpha - \cos \beta)$ and v is obtained from equations (63).

It is conceivable that this form of configuration could be used as a bearing structure for a load acting on a weak material or soil (i.e., one having a low bearing value). The shape of the outer boundary ($\beta = \beta_o$) along which the material or soil forces act is obtained from the geometry of the dipolar system for the corresponding applied pressure distribution along $\beta = \pi$. Along this boundary ($\beta = \beta_o$), no shearing forces are developed; consequently the possibility of slip, and hence large settlement is reduced. The structure then efficiently distributes this load from the bearing area PQ over a larger area (along the surface of the $\beta = \beta_o$ curve) and hence reduces the tendency of large deformations in the material.

The Spherical Coordinate System for Member Layout

The spherical coordinate system given by the transformation

$$\left.\begin{aligned} x &= \alpha \sin \beta \cos \gamma \\ y &= \alpha \sin \beta \sin \gamma \\ z &= \alpha \cos \beta \end{aligned}\right\} \qquad (65)$$

where the curvilinear coordinates α, β, γ are illustrated in figure 9, is proposed for the member layout of a Michell optimum framework. Expressions for displacements are obtained for a Michell strain criterion; they represent the general solution of the strain-displacement equation for a Michell strain criterion in a three-dimensional spherical coordinate system. Any self-equilibrating force system can now be made to act upon a domain enclosing a spherical system of curves, the proper boundary conditions on the displacements could be imposed, and the volume of the structure could then be determined.

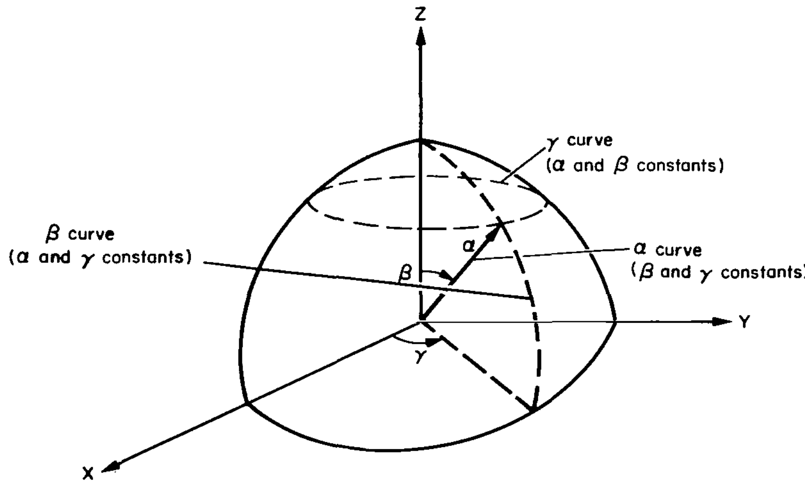


Figure 9.- Spherical coordinate system.

The parameters of the coordinate system are given by

$$h_1 = \left\{ \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2 \right]^{1/2} \right\}^{-1} \tag{66}$$

with h_2 and h_3 similarly given by differentiation with respect to β and γ, respectively. For the system, represented by equations (65), the parameters are obtained as follows:

$$h_1 = 1 , \qquad h_2 = \frac{1}{\alpha} , \qquad h_3 = \frac{1}{\alpha \sin \beta} \tag{67}$$

The corresponding strain-displacement equations are given by (see p. 54, ref. 11)

$$\left.\begin{aligned}
e_{\alpha\alpha} &= h_1 \frac{\partial u_\alpha}{\partial \alpha} + h_1 h_2 u_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) + h_1 h_3 u_\gamma \frac{\partial}{\partial \gamma}\left(\frac{1}{h_1}\right) = \frac{\partial u_\alpha}{\partial \alpha} \\
e_{\beta\beta} &= h_2 \frac{\partial u_\beta}{\partial \beta} + h_2 h_3 u_\gamma \frac{\partial}{\partial \gamma}\left(\frac{1}{h_2}\right) + h_1 h_2 u_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) = \frac{1}{\alpha}\frac{\partial u_\beta}{\partial \beta} + \frac{1}{\alpha} u_\alpha \\
e_{\gamma\gamma} &= h_3 \frac{\partial u_\gamma}{\partial \gamma} + h_1 h_3 u_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{h_3}\right) + h_2 h_3 u_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{h_3}\right) = \frac{1}{\alpha \sin \beta}\frac{\partial u_\gamma}{\partial \gamma} + \frac{1}{\alpha} u_\alpha \\
&\quad + \frac{\cos \beta}{\alpha \sin \beta} u_\beta \\
e_{\beta\gamma} &= \frac{h_2}{h_3}\frac{\partial}{\alpha\beta}(h_3 u_\gamma) + \frac{h_3}{h_2}\frac{\partial}{\alpha\gamma}(h_2 u_\beta) = -\frac{\cos \beta}{\alpha \sin \beta} u_\gamma + \frac{1}{\alpha}\frac{\partial u_\gamma}{\partial \beta} + \frac{1}{\alpha \sin \beta}\frac{\partial u_\beta}{\partial \gamma} \\
e_{\gamma\alpha} &= \frac{h_3}{h_1}\frac{\partial}{\alpha\gamma}(h_1 u_\alpha) + \frac{h_1}{h_3}\frac{\partial}{\partial \alpha}(h_3 u_\gamma) = \frac{1}{\alpha \sin \beta}\frac{\partial u_\alpha}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \alpha} - \frac{1}{\alpha} u_\gamma \\
e_{\alpha\beta} &= \frac{h_1}{h_2}\frac{\partial}{\partial \alpha}(h_2 u_\beta) + \frac{h_2}{h_1}\frac{\partial}{\partial \beta}(h_1 u_\alpha) = \frac{\partial u_\beta}{\partial \alpha} - \frac{1}{\alpha} u_\beta + \frac{1}{\alpha}\frac{\partial u_\alpha}{\partial \beta}
\end{aligned}\right\} \quad (68)$$

where u_α, u_β, u_γ are the displacements along the α, β, and γ directions, respectively.

For a Michell strain criterion it is necessary that

$$e_{\alpha\alpha} = \pm e , \quad e_{\beta\beta} = \pm e , \quad e_{\gamma\gamma} = \pm e , \quad e_{\alpha\beta} = 0 = e_{\beta\gamma} = e_{\gamma\alpha} \qquad (69)$$

The equations (68) are now solved for the following general strain system:

$$e_{\alpha\alpha} , \quad e_{\beta\beta} , \quad e_{\gamma\gamma} , \quad e_{\alpha\beta} = 0 = e_{\beta\gamma} = e_{\gamma\alpha} \qquad (70)$$

It will, however, be shown that equations (68) are only solvable for certain relations between $e_{\alpha\alpha}$, $e_{\beta\beta}$, and $e_{\gamma\gamma}$. These relations will then eliminate some of the eight cases shown in equations (69) and will be the conditions of compatibility of strain. For the strain system of equation (70), the strain-displacement equations (68) are as follows:

$$\frac{\partial u_\alpha}{\partial \alpha} = e_{\alpha\alpha} \qquad (71a)$$

$$\frac{1}{\alpha}\frac{\partial u_\beta}{\partial \beta} + \frac{u_\alpha}{\alpha} = e_{\beta\beta} \tag{71b}$$

$$\frac{1}{\alpha \sin \beta}\frac{\partial u_\gamma}{\partial \gamma} + \frac{1}{\alpha} u_\alpha + \frac{\cot \beta}{\alpha} u_\beta = e_{\gamma\gamma} \tag{71c}$$

$$\frac{-\cot \beta}{\alpha} u_\gamma + \frac{1}{\alpha}\frac{\partial u_\gamma}{\partial \beta} + \frac{1}{\alpha \sin \beta}\frac{\partial u_\beta}{\partial \gamma} = e_{\beta\gamma} = 0 \tag{71d}$$

$$\frac{1}{\alpha \sin \beta}\frac{\partial u_\alpha}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \alpha} - \frac{u_\gamma}{\alpha} = e_{\gamma\alpha} = 0 \tag{71e}$$

$$\frac{\partial u_\beta}{\partial \alpha} - \frac{1}{\alpha} u_\beta + \frac{1}{\alpha}\frac{\partial u_\alpha}{\partial \beta} = e_{\alpha\beta} = 0 \tag{71f}$$

Equations (71a), (71b), and (71f) contain only displacements u_α and u_β and yield the general integrals

$$u_\alpha(\alpha,\beta,\gamma) = e_{\alpha\alpha}\alpha + C_1(\gamma)\cos \beta - C_2'(\gamma)\sin \beta \tag{72}$$

$$u_\beta(\alpha,\beta,\gamma) = (e_{\beta\beta} - e_{\alpha\alpha})\alpha\beta - C_1(\gamma)\sin \beta - C_2'(\gamma)\cos \beta - \alpha l'(\gamma) \tag{73}$$

where $C_1(\gamma)$, $C_2(\gamma)$, and $l(\gamma)$ are arbitrary functions of γ, and the primes indicate differentiation with respect to γ. When the derivatives of u_γ have been eliminated between equations (71d) and (71e), it is seen that the function $C_1(\gamma)$ is, in fact, a constant. Equation (71c) can now be integrated to yield

$$u_\gamma(\alpha,\beta,\gamma) = [(e_{\gamma\gamma} - e_{\alpha\alpha})\sin \beta - (e_{\alpha\alpha} - e_{\beta\beta})\beta \cos \beta]\alpha\gamma$$

$$+ C_2(\gamma) + \alpha l(\gamma)\cos \beta + \alpha\lambda(\alpha,\beta) \tag{74}$$

with $\lambda(\alpha,\beta)$ arbitrary. Now equation (71e) shows that

$$C_2(\gamma) = D_1 \sin \gamma + D_2 \cos \gamma + D_3 \tag{75}$$

and

$$\lambda(\alpha,\beta) = \alpha k(\beta) - D_3 \tag{76}$$

where D_1, D_2, and D_3 are constants, and k is an arbitrary function of β. Finally, equation (71d) shows that

$$l(\gamma) = D_4 \sin \gamma + D_5 \cos \gamma + D_6 \tag{77}$$

where D_4, D_5, D_6 are arbitrary constants. Moreover, equation (71d) can be satisfied if and only if

$$e_{\beta\beta} - e_{\alpha\alpha} = 0 \tag{78}$$

which is the compatibility condition for the differential system (71). With the arbitrary constants renamed, the general solution representing the displacements is written:

$$u_\alpha(\alpha,\beta,\gamma) = e_{\alpha\alpha}\alpha - A \cos \beta + (B \cos \gamma + C \sin \gamma)\sin \beta \tag{79a}$$

$$u_\beta(a,\beta,\gamma) = A \sin \beta + (C \cos \beta + E\alpha)\sin \gamma + (B \cos \beta - D\alpha)\cos \gamma \tag{79b}$$

$$u_\gamma(\alpha,\beta,\gamma) = \alpha\gamma(e_{\gamma\gamma} - e_{\alpha\alpha})\sin \beta + F\alpha \sin \beta + (D\alpha \cos \beta - B)\sin \gamma + (E\alpha \cos \beta + C)\cos \gamma \tag{79c}$$

From equation (78) it is seen that $e_{\alpha\alpha}$ and $e_{\beta\beta}$ have to be equal in magnitude as well as in sign eliminating four of the eight cases of equation (69). The four possible cases are as follows:

$e_{\alpha\alpha}$	$e_{\beta\beta}$	$e_{\gamma\gamma}$
+e	+e	+e
+e	+e	−e
−e	−e	+e
−e	−e	−e

(80)

The term $e_{\gamma\gamma} - e_{\alpha\alpha}$ in equation (79c) provides for handling any of the above four cases.

The formal solution of the spherical coordinate system is now complete; for once the displacements are known, and the volume can be obtained for a set of self-equilibrating external forces.

The solution given by equations (79) for considering applications to some load systems shows that when $e_{\gamma\gamma} = e_{\alpha\alpha}$, equations (79) can represent the solutions for a uniform radially loaded spherical structure. When $e_{\gamma\gamma} = -e_{\alpha\alpha}$, u_γ (see eq. (79c)) is a multivalued expression that leads to an interesting application. Consider a structure with a spherical boundary surface. Let cuts be made along two longitudinal planes, separated by a small angle $d\gamma$, up to the axis of the sphere. If the portion of the sphere enclosed between these two planes is removed and the cut surfaces are rejoined, a structure with initial stresses is obtained. Thus, when $e_{\gamma\gamma} = -e_{\alpha\alpha}$, equations (79) can represent the solution for a structure with a spherical bounding surface with initial locked stresses in it (i.e., a prestressed spherical structure) and under a suitable force system.

CONCLUDING REMARKS

The Michell strain criterion has been successfully applied to develop minimum volume structures for a variety of domains to equilibrate some interesting force systems of significance to the aeronautics and space industry.

The indeterminacy of Michell structures was discussed in the first section. The utility of the superposition theorem for Michell structures was presented, and the theorem was proved and applied to the two load systems of figures 1 and 2. The superposition theorem now makes it possible to solve a broader spectrum of force systems which by themselves do not have a unique solution (i.e., do not lend themselves to a unique optimum structure) but, on resolution into two or more systems, do lend themselves to separate solutions which can then be superposed.

In the second section the governing equations, developed by Hemp, for the coordinate curves of the member layout of Michell structures were presented and two solutions were obtained. These solutions give rise to the coordinate systems shown in figures 3 and 5 for the member layouts of Michell structures. These coordinate systems are then applied to offer solutions to a variety of force systems acting on a plane rectangular domain as well as on a curvilinear domain as illustrated in figure 4.

In the third section, two well-known orthogonal coordinate systems, namely the dipolar and the spherical systems, were considered for member layout of Michell structures. General solutions were obtained for displacements to satisfy Michell strain criterion in the two coordinate systems. Two interesting structures arise as a result of the application of the dipolar system; one of them suggests interesting possibilities for an entry body configuration having a displaced center of gravity with respect to the geometric center, which could be helpful in orienting the body during descent. The other structure enables a concentrated load to be distributed over a wide bearing area, thereby reducing the possibility of large deformations of the bearing material.

For the spherical coordinate system, the general solution of principal strain-displacement equations has been derived. Applications to physical problems could further the utility of this solution. It is shown that the spherical coordinate system admits solution to the Michell strain criterion for the four cases listed in equation (80). Some interesting applications of the solution are discussed.

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REFERENCES

1. Ghista, Dhanjoo N.: Structural Optimization With Probability of Failure Constraint. NASA TN D-3777, 1966.

2. Ghista, Dhanjoo N.: Weight Optimization by Structural Synthesis. J. Sci. Eng. Res., vol. 10, pt. 2, 1966.

3. Michell, A. G. M.: The Limits of Economy of Material in Frame-Structures. Philosophical Magazine (London), Ser. 6, vol. 8, no. 47, Nov. 1904, pp. 589-597.

4. Cox, H. L.: The Theory of Design. British A. R. C., no. 19791, Jan. 13, 1958.

5. Hemp, W. S.: Theory of Structural Design. Rep. 115, College of Aeronautics, Cranfield, Great Britain, Aug. 1958.

6. Chan, A. S. L.: The Design of Michell Optimum Structures. Rep. 142, College of Aeronautics, Cranfield, Great Britain, Dec. 1960.

7. Chan, H. S. Y.: Optimum Michell Frameworks for Three Parallel Forces. Rep. 167, College of Aeronautics, Cranfield, Great Britain, Aug. 1963.

8. Chan, H. S. Y.: Tabulation of Some Layouts and Virtual Displacement Fields in the Theory of Michell Optimum Structures. Rep. 161, College of Aeronautics, Cranfield, Great Britain, Feb. 1964.

9. Ghista, Dhanjoo N.: Optimum Frameworks Under Single Load System. J. Structural Division, ASCE Proc., vol. 92, no. ST 5, Oct. 1966, pp. 261-286.

10. Love, A. E.: A Treatise on the Mathematical Theory of Elasticity. Fourth ed., Dover Pub.

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