NASA TECHNICAL REPORT

NASA TR R-325

C. 1

NASA TR R-325

A CRITICAL ANALYSIS OF THE GRAD APPROXIMATION FOR CLOSING OUT THE MAGNETOHYDRODYNAMIC EQUATIONS FOR PLASMAS

by Willard E. Meador

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1969

1. Report No. NASA TR R-325	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A CRITICAL ANALYSIS OF THE GRAD APPROXIMATION FOR CLOSING OUT THE MAGNETOHYDRODYNAMIC EQUATIONS FOR PLASMAS		5. Report Date December 1969
		6. Performing Organization Code
7. Author(s) Willard E. Meador		8. Performing Organization Report No. L-6523
9. Performing Organization Name and Address NASA Langley Research Center Hampton, Va. 23365		10. Work Unit No. 129-02-22-01-23
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Report
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A critical analysis is made of the utility of Grad's 13-moment distribution functions in closing out the generalized magnetohydrodynamic equations for plasmas. In particular, the inaccurate particle velocity dependence of the Grad approximation is shown to yield significant errors in the collisional transfers of such quantities as momentum and energy as functions of the moments (variables of the problem) considered. These errors directly affect the relations between moments which are derived by means of the aforementioned closing-out process; consequently, when the Grad functions are written so as to incorporate the moment relations, considerable doubt is cast upon the ability of such functions not only to describe the original 13 moments but also to predict accurately other plasma properties (e.g., the entropy) as well. Numerous comparisons with the exact first- and second-order perturbation solutions of the Boltzmann equation are made in order to substantiate these conclusions. A study also is made of the convergence of Grad-type functions as more moments are added, in consequence of which at least 16 moments are often necessary.

17. Key Words Suggested by Author(s)	18. Distribution Statement
Plasma physics Magnetohydrodynamic equations for plasmas Grad theory for plasmas Plasma dynamics	Unclassified – Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	38	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

CONTENTS

A CRITICAL ANALYSIS OF THE GRAD APPROXIMATION FOR CLOSING OUT THE MAGNETOHYDRODYNAMIC EQUATIONS FOR PLASMAS

By Willard E. Meador
Langley Research Center

SUMMARY

A critical analysis is made of the utility of Grad's 13-moment distribution functions in closing out the generalized magnetohydrodynamic equations for plasmas. In particular, the inaccurate particle velocity dependence of the Grad approximation is shown to yield significant errors in the collisional transfers of such quantities as momentum and energy as functions of the moments (variables of the problem) considered. These errors directly affect the relations between moments which are derived by means of the aforementioned closing-out process; consequently, when the Grad functions are written so as to incorporate the moment relations, considerable doubt is cast upon the ability of such functions not only to describe the original 13 moments but also to predict accurately other plasma properties (e.g., the entropy) as well. Numerous comparisons with the exact first- and second-order perturbation solutions of the Boltzmann equation are made in order to substantiate these conclusions. A study also is made of the convergence of Grad-type functions as more moments are added, in consequence of which at least 16 moments are often necessary.

INTRODUCTION

Macroscopic equations of change for such velocity moments of the Boltzmann kinetic equations as constituent number densities, mean velocities, temperatures, stresses, and heat fluxes in plasmas have been generated by many authors, but with widely varying accuracies with regard to the net effects of interparticle collisions. A particular example of the more rigorous approaches is the work of Everett (ref. 1), in which Grad's 13-moment velocity distribution functions are utilized for closing out the moment equations, that is, for expressing the collisional transfer terms as functions of the moments (variables) of the problem. When coupled with Maxwell's electromagnetic field equations, the resulting magnetohydrodynamic equations constitute a closed set.

The efficacy of such developments obviously depends on the accuracy of the distribution functions employed in the closing-out process. In particular, errors in the detailed dependence of the distribution functions upon particle velocities may lead to serious errors in the relations between the collisional transfer terms and the moments of the problem and also in the relations between moments which are deducible from the final closed set of macroscopic equations. Examples of moment relations include the specifications of heat fluxes and pressure tensor elements in terms of the electron diffusion velocity.

A prime purpose of the present research is to investigate the accuracy of the moment relations obtained by using the Grad 13-moment velocity distribution functions in the aforementioned closing-out process. This is accomplished by first assuming the distribution functions to be expressible as perturbation series, then employing the closed set of macroscopic equations to compute the heat flux, a certain higher moment, and the second-order contributions to the pressure tensor as functions of the electron diffusion velocity, and finally comparing the results with those obtained from exact first- and second-order perturbation solutions of the Boltzmann kinetic equations. The character of each perturbation series is such that the zeroth-order term corresponds to a local Maxwellian distribution function, whereas the first- and second-order terms correspond to corrections which are linear and quadratic, respectively, in the electron diffusion velocity or in its driving force.

Although this form of solution is not required by the Grad N-moment method, which is a more general approach that includes the perturbation expansion as a special case, there are two overriding advantages of the present procedure. They are, first, that such solutions are physically meaningful and, second, that exact solutions of the corresponding integro-differential perturbation equations are readily found if modifications to Meador's collision model (ref. 2) are substituted for the Boltzmann collision terms. Thus, all references herein to the order of a function or quantity indicate the highest power retained of the electron diffusion velocity or its driving force.

A study also is made of the convergence properties of Grad N-moment functions, whereupon it is observed that at least 16 moments are often necessary for the closing-out method to yield accurate moment relations and that the addition of higher first-order moments to the Grad functions is equivalent in a systematic fashion to the calculation of higher Sonine approximations to the Chapman-Enskog distribution functions (ref. 3). Several types of interparticle interaction potentials are considered, including those of the real (as opposed to Lorentz) fully ionized gas.

Another objective of the present research is to investigate the ability of Grad N-moment functions to predict accurately additional plasma properties (e.g., the entropy). This particular study involves the substitutions of the following two sets of moment relations into the original Grad functions: (1) the exact moment relations obtained from solutions of the Boltzmann kinetic equations and (2) those obtained from the closing-out process. The accuracies of the entropies thus calculated are found to be highly sensitive to the plasma conditions; more specifically, the errors are especially large when the electric field and the temperature gradient are so related as to yield zero heat flux with respect to the average electron motion.

Finally, the Grad 13-moment functions are modified through second order by first imposing entropy maximization, which is the approach of information theory, and then referring the electron particle velocity to the electron frame of reference, which is the method suggested by Everett (ref. 1). Although both modifications lead to improvements in the evaluation of the pressure tensor, the one proposed by Everett gives the best agreement with the exact calculations.

Unless otherwise noted, the following simplifying assumptions apply throughout the present report: time independence for all parameters, infinite mass for the heavy particles which are considered fixed scattering centers for the electrons, zero plasma flow velocity, zero applied magnetic field, and spatially homogeneous fluxes, temperatures, pressures, and all other quantities through the second perturbation order.

SYMBOLS

$A(\xi)$	ratio of impact parameter integrals
b	impact parameter
$\vec{B}$	magnetic field
$\hat{B}$	unit vector in direction of magnetic field
$\vec{c}_e$	electron particle velocity
$\vec{c}_j$	heavy-particle particle velocity
e	magnitude of electron charge
$\vec{E}$	electric field
f_e	electron distribution function
$f_e^{(o)}$	Maxwellian contribution to electron distribution function
f_j	heavy-particle distribution function

$g_1(\gamma), g_2(\gamma)$	functions defined in equations (81), (84), and (85)
i,j,k	indices
$\hat{i},\hat{j},\hat{k}$	unit vector in x-, y-, and z-direction, respectively
$\vec{j}$	electron current density
k	Boltzmann's constant
m_e	mass of electron
n_e	electron number density
n_j	number density of heavy particles
p_e	electron partial pressure
$P_{e,xy}$	xy-component of electron pressure tensor (subscript used indicates appropriate component)
$\overset{o}{\vec{P}}_e$	traceless electron pressure tensor
$\overset{o}{P}_{e,xy}$	xy-component of traceless electron pressure tensor referred to plasma motion (subscript used indicates appropriate component)
$\overset{o}{P}'_{e,xy}$	xy-component of traceless electron pressure tensor referred to electron diffusion velocity (subscript used indicates appropriate component)
$\vec{Q}_1$	heat flux vector, $\vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0$ (see table I)
$\vec{Q}_2$	vector, $\vec{\beta}_2 - 7\vec{\beta}_1 + \frac{35}{4}\vec{\beta}_0$ (see table II)
R_{ij}	integral defined by equation (16)
s	entropy density
$s^{(o)}$	equilibrium entropy density

$\vec{S}_j$ collisional transfer of reduced flux $\vec{\beta}_j$

t time

T_e electron temperature referred to plasma motion

T_e' electron temperature referred to electron diffusion velocity

$\overleftrightarrow{U}$ unit tensor

$\vec{v}_e$ electron diffusion velocity

x integration variable

x,y,z Cartesian coordinates

$\left(\frac{\partial f_e}{\partial t}\right)_c$ time rate of change of f_e due to collisions

$\left(\frac{\partial f_e}{\partial t}\right)_{c,\varphi_1}$ contribution to $\left(\frac{\partial f_e}{\partial t}\right)_c$ from first-order perturbation function

$\left(\frac{\partial f_e}{\partial t}\right)_{c,\varphi_2}$ contribution to $\left(\frac{\partial f_e}{\partial t}\right)_c$ from second-order perturbation function

α_1,α_2 constants

$\vec{\beta}_j$ velocity moment defined by equation (3); for example, $\vec{\beta}_0$ is reduced electron diffusion velocity and $\vec{\beta}_1$ is reduced heat flux

$\vec{\gamma}$ reduced electron particle velocity

δ variational operator

ϵ azimuthal angle for electron—heavy-particle collision

η heat flux parameter defined by equation (94)

θ polar angle in spherical coordinates

$\vec{\mu}_j$	Lagrange multiplier
ξ	effective interparticle interaction parameter
σ	electrical conductivity
τ_s	collision time for entropy production
τ_σ	collision time associated with electron diffusion
φ	azimuthal angle in spherical coordinates
φ_1	first-order perturbation function
φ_2	second-order perturbation function
φ_e	electron perturbation function
χ	scattering or deflection angle in electron—heavy-particle collision
ω	cyclotron frequency

Subscripts:

e	electrons
j	heavy particles
x,y,z	x-, y-, and z-components

Special notations:

$\langle\ \rangle$	average over velocity space
$\langle\ \rangle_0$	average over velocity space using only Maxwellian distribution
( )!!	factorial employing only odd numbers; for example, $(5)!! = 5 \cdot 3 \cdot 1 = 15$

Primed quantities in collision integrals denote conditions after collisions, as opposed to unprimed quantities which denote conditions before collisions.

MACROSCOPIC EQUATIONS OF CHANGE

A convenient initial step in the investigation of first-order moment relations is the derivation from the macroscopic equations of change of a recursion-type formulation of the collisional transfer terms. Such is the purpose of the present section, in which the fundamental working equations are deduced. Second-order moment relations are discussed in a subsequent section. As explained in the Introduction, the restriction to first order is imposed by neglecting squares and higher powers of the electron diffusion velocity (or its driving force).

The macroscopic equations of change for electrons are generated by taking velocity moments of the Boltzmann integro-differential equation (ref. 3)

$$\frac{\partial f_e}{\partial t} + \left(\frac{2kT_e}{m_e}\right)^{1/2} \vec{\gamma} \cdot \nabla f_e - \frac{e}{m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{E} \cdot \frac{\partial f_e}{\partial \vec{\gamma}} - \frac{e}{m_e}\left(\vec{\gamma} \times \vec{B}\right) \cdot \frac{\partial f_e}{\partial \vec{\gamma}} = \left(\frac{\partial f_e}{\partial t}\right)_c \tag{1}$$

in which $\left(\partial f_e/\partial t\right)_c$ represents the set of collision integrals and $\vec{\gamma}$ is the reduced electron particle velocity defined by

$$\vec{\gamma} = \left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{c}_e \tag{2}$$

There is one such equation of change for each of the moments regarded as variables of the problem.

As an example of this procedure, the equation of change for the moment

$$\vec{\beta}_j \equiv \left\langle \gamma^{2j}\vec{\gamma} \right\rangle = n_e^{-1} \int \gamma^{2j}\vec{\gamma} f_e \, d\vec{c}_e \tag{3}$$

is obtained from equation (1) by multiplying each member of that expression by $c_e^{2j}\vec{c}_e$, integrating over velocity space, and employing vector calculus and Gauss' divergence theorem in the usual manner (ref. 3). The result is

$$\frac{\partial}{\partial t}\left[n_e\left(\frac{2kT_e}{m_e}\right)^{(2j+1)/2} \vec{\beta}_j\right] + \nabla \cdot \left[n_e\left(\frac{2kT_e}{m_e}\right)^{j+1} \left\langle \gamma^{2j}\vec{\gamma}\vec{\gamma} \right\rangle\right] + \frac{en_e}{m_e}\left(\frac{2kT_e}{m_e}\right)^{j}\left(\left\langle \gamma^{2j} \right\rangle \vec{E} + 2j\vec{E} \cdot \left\langle \gamma^{2j-2}\vec{\gamma}\vec{\gamma} \right\rangle\right)$$
$$+ \frac{en_e}{m_e}\left(\frac{2kT_e}{m_e}\right)^{(2j+1)/2} \vec{\beta}_j \times \vec{B} = \left(\frac{2kT_e}{m_e}\right)^{(2j+1)/2} \vec{S}_j \tag{4}$$

where $\vec{S}_j$ refers to the collisional transfer of $\vec{\beta}_j$ and is expressed as

$$\vec{S}_j = \int \gamma^{2j}\vec{\gamma}\left(\frac{\partial f_e}{\partial t}\right)_c d\vec{c}_e \tag{5}$$

It is evident from equation (4) that if 0, 1, and 2 are the only values considered for the index j, the variables of the system (in an unreduced sense) will be comprised of the 28 moments corresponding to n_e, $\vec{\beta}_0$, $\vec{\beta}_1$, $\vec{\beta}_2$, and the symmetric tensors $\langle\vec{\gamma}\vec{\gamma}\rangle$, $\langle\gamma^2\vec{\gamma}\vec{\gamma}\rangle$, and $\langle\gamma^4\vec{\gamma}\vec{\gamma}\rangle$. Likewise, j selected as 0 and 1 involves the 19 moments corresponding to n_e, $\vec{\beta}_0$, $\vec{\beta}_1$, and the symmetric tensors $\langle\vec{\gamma}\vec{\gamma}\rangle$ and $\langle\gamma^2\vec{\gamma}\vec{\gamma}\rangle$, which set can be reduced to the familiar 13 moments of Grad (ref. 1) by using a distribution function to express $\langle\gamma^2\vec{\gamma}\vec{\gamma}\rangle$ in terms of the remaining variables. Such effort is part of the closing-out process, as is the elimination in a similar manner of the collisional transfer term $\vec{S}_j$ in equation (4). The number of moments ultimately retained as variables of the system is obviously a matter of some choice, there being no strict limitations on where the closing-out process must stop.

Because of the previous restrictions to spatial homogeneity and time independence, the first two terms on the left side of equation (4) are zero. In addition, since the electric current is assumed to generate the only magnetic field and the product $\vec{\beta}_j \times \vec{B}$ is therefore second order by reason of Maxwell's equations, the first-order form of equation (4) can be written

$$\begin{aligned}\vec{S}_j &= \frac{en_e}{m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\left(\langle\gamma^{2j}\rangle_0\vec{E} + 2j\vec{E}\cdot\langle\gamma^{2j-2}\vec{\gamma}\vec{\gamma}\rangle_0\right) \\ &= \frac{en_e}{3m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}(3 + 2j)\langle\gamma^{2j}\rangle_0\vec{E}\end{aligned} \tag{6}$$

The zero subscripts refer to velocity averages taken with respect to the zeroth-order Maxwellian distribution function given by

$$f_e^{(o)} = n_e\left(\frac{m_e}{2\pi kT_e}\right)^{3/2} e^{-\gamma^2} \tag{7}$$

Equation (6) is further reduced by utilizing

$$\langle\gamma^{2j}\rangle_0 = n_e^{-1}\int \gamma^{2j}f_e^{(o)}d\vec{c}_e = 2^{-j}(2j + 1)!! \tag{8}$$

to write the equation of motion

$$\vec{S}_0 = \frac{en_e}{m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\vec{E} \tag{9}$$

and the recursion relation

$$\vec{S}_{j+1} = \frac{1}{2}(5 + 2j)\vec{S}_j \tag{10}$$

Equation (10) depends only on the assumptions of first-order theory and spatial and time independence. This equation is the working equation for the computation of first-order moment relations, together with equation (5) and a formulation of $(\partial f_e/\partial t)_c$ yet to be given.

More specifically, if an electron distribution function of the Grad N-moment variety is employed in equations (5) and (10), all moments $\vec{\beta}_j$ can be expressed in terms of the reduced electron diffusion velocity $\vec{\beta}_0$. Hence, $\vec{\beta}_0$ is regarded as input data and thus is independent of the approximation to the distribution function. Equation (9) is of use only when the dependence of $\vec{\beta}_j$ upon the driving electric field is desired.

Grad N-Moment Distribution Functions

A convenient way of writing Grad's 13-moment electron distribution function for the purpose of closing out equations (5) and (10) is as follows (ref. 1):

$$f_e = f_e^{(o)}\left(1 + \varphi_e\right) \tag{11}$$

where

$$\varphi_e = 2\vec{\beta}_0 \cdot \vec{\gamma} + \frac{4}{5}\left(\gamma^2 - \frac{5}{2}\right)\left(\vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0\right) \cdot \vec{\gamma} + \frac{1}{p_e}\overset{o}{\overleftrightarrow{P}}_e : \vec{\gamma}\vec{\gamma} \tag{12}$$

and $\overset{o}{\overleftrightarrow{P}}_e$ is the traceless pressure tensor defined by

$$\overset{o}{\overleftrightarrow{P}}_e = 2p_e\left(\left\langle\vec{\gamma}\vec{\gamma}\right\rangle - \frac{1}{2}\overleftrightarrow{U}\right) = \frac{2p_e}{n_e}\int f_e^{(o)}\vec{\gamma}\vec{\gamma}\varphi_e \, d\vec{c}_e \tag{13}$$

Although equation (12) and its first-order form (see appendix A for an alternate derivation)

$$\varphi_e = 2\vec{\beta}_0 \cdot \vec{\gamma} + \frac{4}{5}\left(\gamma^2 - \frac{5}{2}\right)\left(\vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0\right) \cdot \vec{\gamma} \tag{14}$$

produce identities for the moments when the proper integrations are performed, this characteristic is not sufficient to guarantee accurate values of $\vec{S}_j$ when equations (11) and (14) are employed in equation (5). The integrals defining these moments are quite different from those defining the $\vec{S}_j$, so that errors in the detailed $\vec{\gamma}$-dependence of equation (14) will be reflected in the results from equation (5) and also in the moment relations subsequently derived from equation (10).

Before any distribution function can be employed in equation (5), however, the collisional derivative $(\partial f_e/\partial t)_c$ must be expressed in terms of φ_e. This is accomplished by using the following simplified first-order collision model developed by Meador (ref. 2):

$$\left(\frac{\partial f_e}{\partial t}\right)_c = -\frac{e^2 n_e R_{13}}{m_e \sigma R_{04}} \gamma^{1-(4/\xi)} f_e^{(o)} \varphi_e \tag{15}$$

in which R_{ij} is the integral

$$R_{ij} = \int_0^\infty x^{(4i/\xi)+j} e^{-x^2} dx \tag{16}$$

and ξ is an effective interaction parameter defined in such a way as to guarantee the correct relation between the heat flux and the electron diffusion velocity on the basis of the exact solution to the corresponding Boltzmann equation.

Reference 2 further shows that equation (15) is entirely adequate for describing a great variety of realistic plasmas, even though to the first Chapman-Enskog perturbation order the description is exact only for a Lorentz gas (i.e., negligible electron-electron interaction effects either in a slightly ionized gas or in a fully ionized gas having large ionic charges). The parameter ξ in the latter cases is the exponent in the electron—heavy-particle inverse-power interaction potential.

The substitution of equations (14) and (15) into equation (5) yields

$$\vec{S}_j(\text{13-moment}) = -\frac{e^2 n_e^2 R_{13} R_{-1,2j+5}}{2 m_e \sigma R_{04}^2 \xi}\left\{\left[(1 - 2j)\xi + 4\right]\vec{\beta}_0 + \frac{2}{5}\left[(1 + 2j)\xi - 4\right]\vec{\beta}_1\right\} \tag{17}$$

which relation combines with equation (10) and the identity

$$R_{-1,2j+7} = \xi^{-1}\left[(3 + j)\xi - 2\right]R_{-1,2j+5} \tag{18}$$

to give

$$\vec{\beta}_1(\text{13-moment}) = \frac{5\left[(11 + 6j)\xi^2 - 8(1 + j)\xi + 16\right]}{2\left[(13 + 6j)\xi^2 - 8(2 + j)\xi + 16\right]}\vec{\beta}_0 \tag{19}$$

In particular, when $j = 0$, the expression

$$\vec{\beta}_1(\text{13-moment}) = \frac{5\left(11\xi^2 - 8\xi + 16\right)}{2\left(13\xi^2 - 16\xi + 16\right)}\vec{\beta}_0 \tag{20}$$

represents the first-order culmination of the present method for closing out the macroscopic equations of change at the 13-moment level.

Likewise, for the other Grad functions of present interest, the substitution of

$$\varphi_e(\text{5-moment}) = 2\vec{\beta}_0 \cdot \vec{\gamma} \tag{21}$$

and equation (15) into equation (5) gives

$$\vec{S}_j(\text{5-moment}) = -\frac{e^2 n_e^2 R_{13} R_{-1,2j+5}}{m_e \sigma R_{04}^2}\vec{\beta}_0 \tag{22}$$

and the substitution of equation (A5) and equation (15) into equation (5) gives

$$\vec{S}_j(\text{16-moment}) = -\frac{e^2 n_e^2 R_{13} R_{-1,2j+5}}{2 m_e \sigma R_{04}^2 \xi^2}\left\{\frac{1}{4}\left[\left(3 - 8j + 4j^2\right)\xi^2 + 16(1 - j)\xi + 16\right]\vec{\beta}_0 + \frac{1}{5}\left[\left(3 + 4j - 4j^2\right)\xi^2 - 8(1 - 2j)\xi - 16\right]\vec{\beta}_1 - \frac{1}{35}\left[\left(1 - 4j^2\right)\xi^2 + 16j\xi - 16\right]\vec{\beta}_2\right\} \tag{23}$$

Accordingly, equations (10), (18), and (22) yield

$$\frac{5 + 2j}{2} = \frac{R_{-1,2j+7}}{R_{-1,2j+5}} \equiv \frac{(3 + j)\xi - 2}{\xi} \tag{24}$$

so that

$$\xi(\text{5-moment}) = 4 \tag{25}$$

In a similar manner, the following expression is deduced from equations (10), (18), and (23):

$$\vec{\beta}_2(\text{16-moment}) = 35\left[\left(23 + 56j + 20j^2\right)\xi^3 - 4\left(27 + 20j + 4j^2\right)\xi^2 + 16(5 + 4j)\xi - 64\right]^{-1}\left\{\frac{1}{4}\left[\left(21 - 32j - 20j^2\right)\xi^3 + 4\left(19 + 12j + 4j^2\right)\xi^2 - 16(1 + 4j)\xi + 64\right]\vec{\beta}_0 - \frac{1}{5}\left[\left(3 - 44j - 20j^2\right)\xi^3 + 4\left(19 + 16j + 4j^2\right)\xi^2 - 16(3 + 4j)\xi + 64\right]\vec{\beta}_1(\text{16-moment})\right\} \tag{26}$$

Equation (26) is further simplified by solving simultaneously the $j = 0$ and $j = 1$ expressions, which are

$$\vec{\beta}_2(\text{16-moment}) = \frac{7\left[5\left(21\xi^3 + 76\xi^2 - 16\xi + 64\right)\vec{\beta}_0 - 4\left(3\xi^3 + 76\xi^2 - 48\xi + 64\right)\vec{\beta}_1(\text{16-moment})\right]}{4\left(23\xi^3 - 108\xi^2 + 80\xi - 64\right)} \tag{27}$$

and

$$\vec{\beta}_2(\text{16-moment}) = \frac{7\left[4\left(61\xi^3 - 156\xi^2 + 112\xi - 64\right)\vec{\beta}_1(\text{16-moment}) - 5\left(31\xi^3 - 140\xi^2 + 80\xi - 64\right)\vec{\beta}_0\right]}{4\left(99\xi^3 - 204\xi^2 + 144\xi - 64\right)} \tag{28}$$

to obtain the first-order result

$$\vec{\beta}_1(\text{16-moment}) = \frac{5\left(349\xi^4 - 416\xi^3 + 672\xi^2 - 512\xi + 256\right)}{2\left(425\xi^4 - 816\xi^3 + 1120\xi^2 - 768\xi + 256\right)}\vec{\beta}_0 \tag{29}$$

Only one conclusion can be drawn from the differences between equations (20) and (29): In spite of the fact that the macroscopic variables $\vec{\beta}_0$ and $\vec{\beta}_1$ explicitly appear in each of equations (14) and (A5), the 13-moment and 16-moment distribution functions do not predict the same expressions for the collisional transfer terms $\vec{S}_j\left(\vec{\beta}_0,\vec{\beta}_1\right)$. Hence the accuracy of the closing-out procedure may be quite sensitive to the number of moments considered as variables in the original φ_e. The one exception occurs for Maxwellian molecules ($\xi = 4$), in which case all results agree with

$$\vec{\beta}_1(\text{5-moment}) = n_e^{-1} \int \gamma^2 \vec{\gamma} f_e^{(o)} \left(2\vec{\beta}_0 \cdot \vec{\gamma}\right) d\vec{c}_e = \frac{5}{2} \vec{\beta}_0 \tag{30}$$

and

$$\vec{\beta}_2(\text{5-moment}) = n_e^{-1} \int \gamma^4 \vec{\gamma} f_e^{(o)} \left(2\vec{\beta}_0 \cdot \vec{\gamma}\right) d\vec{c}_e = \frac{35}{4} \vec{\beta}_0 = 7\left(\vec{\beta}_1 - \frac{5}{4} \vec{\beta}_0\right) \tag{31}$$

because of the relation

$$\varphi_e(\text{13-moment},\xi=4) = \varphi_e(\text{16-moment},\xi=4) = \varphi_e(\text{5-moment}) \tag{32}$$

In analogy with equation (25), the combination of equation (20) with the $j = 1$ form

$$\vec{\beta}_1(\text{13-moment}) = \frac{5\left(17\xi^2 - 16\xi + 16\right)}{2\left(19\xi^2 - 24\xi + 16\right)} \vec{\beta}_0 \tag{33}$$

of equation (19) gives

$$\xi(\text{13-moment}) = 4 \text{ or } 4/3 \tag{34}$$

The bounds of the closing-out procedure are exceeded in both instances by the application of equation (10); in particular, the macroscopic equation of change for $\vec{\beta}_j$ with $j \geqq 1$ is used in deriving equation (25) and the macroscopic equation of change for $\vec{\beta}_2$ is used in deriving equation (33), whereas the accurate descriptions of the 5-moment and 13-moment distribution functions stop (unless $\xi = 4$) with $\vec{\beta}_0$ and $\vec{\beta}_1$, respectively. A definite upper limit on the index j is always imposed in equation (10) if the collisional transfer terms are computed from a Grad N-moment function.

Comparisons of Heat Fluxes

Reference 2 shows that the exact (or infinite moment) first-order distribution function corresponding to an applied electric field and the collision model of equation (15) can be written

$$\varphi_e(\infty\text{-moment}) = \frac{2R_{04}}{R_{13}} \gamma^{(4/\xi)-1} \vec{\beta}_0 \cdot \vec{\gamma} \tag{35}$$

Accordingly,

$$\vec{\beta}_1(\infty\text{-moment}) = 2\xi^{-1}(\xi + 1)\vec{\beta}_0 \tag{36}$$

and

$$\vec{\beta}_2(\infty\text{-moment}) = 2\xi^{-2}\left(3\xi^2 + 5\xi + 2\right)\vec{\beta}_0 \tag{37}$$

from equations (3) and (35).

The following summary of heat fluxes obtained from equations (30), (20), (29), and (36) can thus be used to assess the convergence properties of Grad-like distribution functions in closing out the macroscopic equations of change:

$$\vec{\beta}_1(5\text{-moment}) - \frac{5}{2}\vec{\beta}_0 = 0 \tag{38}$$

$$\vec{\beta}_1(13\text{-moment}) - \frac{5}{2}\vec{\beta}_0 = \frac{5\xi(4 - \xi)}{13\xi^2 - 16\xi + 16}\vec{\beta}_0 \tag{39}$$

$$\vec{\beta}_1(16\text{-moment}) - \frac{5}{2}\vec{\beta}_0 = \frac{10\xi(4 - \xi)\left(19\xi^2 - 24\xi + 16\right)}{425\xi^4 - 816\xi^3 + 1120\xi^2 - 768\xi + 256}\vec{\beta}_0 \tag{40}$$

and

$$\vec{\beta}_1(\infty\text{-moment}) - \frac{5}{2}\vec{\beta}_0 = \frac{4 - \xi}{2\xi}\vec{\beta}_0 \tag{41}$$

Numerical calculations based on equations (38) to (41) are given in table I for effective interparticle interaction potentials corresponding to a Lorentz fully ionized gas ($\xi = 1$), a real fully ionized gas including electron-electron collisions ($\xi = 1.6674$ from ref. 2), a Lorentz gas with $\xi = 2$, a Lorentz gas composed of Maxwellian molecules ($\xi = 4$), and a Lorentz gas of rigid spheres ($\xi = \infty$).

TABLE I.- NONDIMENSIONAL HEAT FLUX $\vec{Q}_1 = \vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0$

N	$\vec{Q}_1/\vec{\beta}_0$ for ξ of –				
	1	1.6674	2	4	∞
5	0.000	0.000	0.000	0.000	0.000
13	1.154	.764	.556	.000	-.385
16	1.521	.704	.507	.000	-.447
∞	1.500	.699	.500	.000	-.500

Although the errors in the heat flux at the 13-moment level are not especially serious in first-order calculations, they may well be large compared with the higher order corrections (to linear flux theory) discussed by Everett. In addition, the errors in $\vec{\beta}_1$ are fed back to the distribution functions themselves and will affect the strongest $\vec{\gamma}$-dependent terms therein; consequently, the errors may be compounded in other types of velocity integration (including the macroscopic equations of change for higher order quantities).

Another demonstration of the fact that the Grad function is less accurate in the present context than was commonly supposed is obtained from similar comparisons for $\vec{\beta}_2$. The expressions analogous to equations (38) to (41) are quite complicated, however, especially for the 16-moment velocity distribution; thus, it is more convenient in that case to compute the fluxes directly from equations (28) and (29). There results

$$\vec{\beta}_2(\text{16-moment},\xi=1) - 7\left[\vec{\beta}_1(\text{16-moment},\xi=1) - \frac{5}{4}\vec{\beta}_0\right] = 0.968\vec{\beta}_0 \tag{42}$$

$$\vec{\beta}_2(\text{16-moment},\xi=1.6674) - 7\left[\vec{\beta}_1(\text{16-moment},\xi=1.6674) - \frac{5}{4}\vec{\beta}_0\right] = -0.287\vec{\beta}_0 \tag{43}$$

$$\vec{\beta}_2(\text{16-moment},\xi=2) - 7\left[\vec{\beta}_1(\text{16-moment},\xi=2) - \frac{5}{4}\vec{\beta}_0\right] = -0.323\vec{\beta}_0 \tag{44}$$

$$\vec{\beta}_2(\text{16-moment},\xi=4) - 7\left[\vec{\beta}_1(\text{16-moment},\xi=4) - \frac{5}{4}\vec{\beta}_0\right] = 0 \tag{45}$$

and

$$\vec{\beta}_2(\text{16-moment},\xi=\infty) - 7\left[\vec{\beta}_1(\text{16-moment},\xi=\infty) - \frac{5}{4}\vec{\beta}_0\right] = 0.494\vec{\beta}_0 \tag{46}$$

The remaining examples are not as difficult to handle and the following equations may be derived from equations (3), (14), and (39) for the 13-moment function and from equations (31), (36), and (37) for the 5- and ∞-moment functions:

$$\vec{\beta}_2(\text{5-moment}) - 7\left[\vec{\beta}_1(\text{5-moment}) - \frac{5}{4}\vec{\beta}_0\right] = 0 \tag{47}$$

$$\vec{\beta}_2(\text{13-moment}) - 7\left[\vec{\beta}_1(\text{13-moment}) - \frac{5}{4}\vec{\beta}_0\right] = 0 \tag{48}$$

and

$$\vec{\beta}_2(\infty\text{-moment}) - 7\left[\vec{\beta}_1(\infty\text{-moment}) - \frac{5}{4}\vec{\beta}_0\right] = \frac{(\xi - 4)(3\xi - 4)}{4\xi^2}\vec{\beta}_0 \tag{49}$$

Numerical calculations based on equations (42) to (49) are given in table II, where it is seen that the 13-moment analysis is completely inadequate (except for Maxwellian molecules) and even the 16-moment analysis yields significant errors when compared with the exact answers.

TABLE II.- NONDIMENSIONAL FLUX $\vec{Q}_2 = \vec{\beta}_2 - 7\vec{\beta}_1 + \frac{35}{4}\vec{\beta}_0$

N	$\vec{Q}_2/\vec{\beta}_0$ for ξ of –				
	1	1.6674	2	4	∞
5	0.000	0.000	0.000	0.000	0.000
13	.000	.000	.000	.000	.000
16	.968	-.287	-.323	.000	.494
∞	.750	-.210	-.250	.000	.750

Entropy Calculations

Another important application of the present theory involves calculations of the entropy density difference (ref. 2)

$$s^{(o)} - s = \frac{k}{2}\int f_e^{(o)}\varphi_e^2 d\vec{c}_e \tag{50}$$

This and related quantities provide excellent criteria on the utility of the closing-out procedure for predicting plasma properties other than the basic moments appearing in the Grad-like functions.

It is further shown in reference 2 that if the collision time τ_s for entropy production is introduced according to the definition

$$\tau_s = \frac{\tau_\sigma\left(s^{(o)} - s\right)}{n_e k \beta_0^2} \tag{51}$$

where τ_σ is the familiar collision time

$$\tau_\sigma = \left(e^2 n_e\right)^{-1} m_e \sigma \tag{52}$$

the generalized Ohm's law can be written (through linear terms in $\omega\tau_s$) in a form completely analogous to the mean-free-path result. Hence,

$$\vec{j} = \sigma\left(\vec{E} + \omega\tau_s \hat{B} \times \vec{E}\right) \tag{53}$$

The computation of τ_s from equations (50) and (51) is thus equivalent to finding the Hall conductivity in the case of small magnetic fields.

The evaluation of equation (50) requires the following set of perturbation functions obtained from combinations of equations (21), (14), (A5), (35), and (38) to (46):

$$\varphi_e(\text{5-moment}) = 2\vec{\beta}_0 \cdot \vec{\gamma} \tag{54}$$

$$\varphi_e(\text{13-moment}) = \frac{4\left[\xi(4 - \xi)\gamma^2 + 9\xi^2 - 18\xi + 8\right]}{13\xi^2 - 16\xi + 16} \vec{\beta}_0 \cdot \vec{\gamma} \tag{55}$$

$$\varphi_e(\text{16-moment},\xi=1) = \frac{4}{35}\left(0.968\gamma^4 + 3.871\gamma^2 - 0.645\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{56}$$

$$\varphi_e(\text{16-moment},\xi=1.6674) = -\frac{4}{35}\left(0.287\gamma^4 - 6.937\gamma^2 - 2.669\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{57}$$

$$\varphi_e(\text{16-moment},\xi=2) = -\frac{4}{35}\left(0.323\gamma^4 - 5.807\gamma^2 - 5.807\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{58}$$

$$\varphi_e(\text{16-moment},\xi=4) = 2\vec{\beta}_0 \cdot \vec{\gamma} \tag{59}$$

$$\varphi_e(\text{16-moment},\xi=\infty) = \frac{4}{35}\left(0.494\gamma^4 - 6.588\gamma^2 + 29.65\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{60}$$

and

$$\varphi_e(\infty\text{-moment}) = \frac{2R_{04}}{R_{13}} \gamma^{(4/\xi)-1}\vec{\beta}_0 \cdot \vec{\gamma} \tag{61}$$

Numerical calculations based on equations (50), (51), and (54) to (61) appear in table III for a variety of effective interparticle interaction potentials and number N of moments in the Grad-like distribution. Except for the trivial case of $\xi = 4$, the 13-moment analysis yields an error in $\left(\tau_s/\tau_\sigma\right) - 1$ that ranges from 17 percent for $\xi = 1.6674$ to 67 percent for $\xi = \infty$.

TABLE III.- RATIO OF COLLISION TIMES τ_S AND τ_σ

N	τ_S/τ_σ for ξ of –				
	1	1.6674	2	4	∞
5	1.000	1.000	1.000	1.000	1.000
13	1.533	1.233	1.123	1.000	1.059
16	1.979	1.203	1.109	1.000	1.094
∞	1.933	1.199	1.104	1.000	1.178

The values of τ_S/τ_σ for $\xi = 1$ coincide exactly with the first, second, third, and infinite Sonine approximations (ref. 2), respectively, as found from solutions of the Boltzmann equation. That this is no accident follows from the observation that equation (55), for example, is identical for $\xi = 1$ with the second Sonine approximation derived in appendix C of reference 2. It is not immediately obvious why Grad-like functions reduce to Sonine approximations to the solution of the integro-differential kinetic equation when the higher moments $\vec{\beta}_j$ are related to $\vec{\beta}_0$ through the closing out of the macroscopic equations of change; however, a key factor may be the appearance of Sonine polynomials (in γ) in equation (A5). In any event, the fact that the addition of higher first-order moments to the Grad function in the present method corresponds in one-to-one fashion to higher Sonine approximations to the Chapman-Enskog distribution function is of considerable interest because it provides an alternate and equivalent method of solution.

One final illustration of this equivalence is given in table IV, where the ratio of τ_S to τ_σ computed in the present paper is compared for the real fully ionized gas ($\xi = 1.6674$) with the Sonine approximations (denoted by N') of reference 2. The small differences that do prevail at each level are caused by the use of the collision model of equation (15) in the one case and the more rigorous Chapman-Enskog treatment of electron-electron encounters in the other. Oddly enough, the much simpler approach yields slightly better values with respect to the exact ∞-moment numbers.

TABLE IV.- RATIO OF COLLISION TIMES τ_S AND τ_σ FOR A REAL FULLY IONIZED GAS ($\xi = 1.6674$)

N	τ_S/τ_σ (Grad-like)	N'	τ_S/τ_σ (Chapman-Enskog)
5	1.000	1	1.000
13	1.233	2	1.259
16	1.203	3	1.207
∞	1.199	∞	----

Other Applications of Grad-Like Functions

It has now been shown that Everett's method of utilizing the Grad 13-moment distribution function for closing out the macroscopic equations of change leads to substantial errors in the relations between $\vec{\beta}_0$ and the higher fluxes $\vec{\beta}_j$. These errors, in turn, are fed back to the distribution function itself, so that subsequent calculations of quantities such as the entropy are similarly affected. What is not clear, however, is whether the errors in the entropy are caused primarily by the approximate forms of the distribution function or by the use of incorrect moments therein. Even though such distinctions are improper in the strictest sense, the possibility that Grad-like functions employing exact $\vec{\beta}_j$-relations will yield improved entropy values should be investigated.

Two cases are considered for this purpose: (1) the spatially homogeneous Lorentz fully ionized gas ($\xi = 1$) already discussed; (2) a fully ionized Lorentz gas in which there is no heat flux relative to the average electron motion and which implies a temperature gradient in fixed relation to the electric field. Only in case (1) does the 13-moment function prove to be reliable.

Since equations (41) and (49) yield

$$\vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0 = \frac{3}{2}\vec{\beta}_0 \tag{62}$$

and

$$\vec{\beta}_2 - 7\left(\vec{\beta}_1 - \frac{5}{4}\vec{\beta}_0\right) = \frac{3}{4}\vec{\beta}_0 \tag{63}$$

for case (1), the Grad-like perturbation functions may be written as follows from equations (21), (14), (A5), and (35):

$$\varphi_e(\text{5-moment}) = 2\vec{\beta}_0 \cdot \vec{\gamma} \tag{64}$$

$$\varphi_e(\text{13-moment}) = \left(\frac{6}{5}\gamma^2 - 1\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{65}$$

$$\varphi_e(\text{16-moment}) = \left(\frac{3}{35}\gamma^4 + \frac{3}{5}\gamma^2 - \frac{1}{4}\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{66}$$

and

$$\varphi_e(\infty\text{-moment}) = \frac{\pi^{1/2}}{4}\gamma^3\vec{\beta}_0 \cdot \vec{\gamma} \tag{67}$$

Numerical calculations based on equations (50), (51), and (64) to (67) are presented in case (1) of table V, where it is evident that a major improvement is obtained over the corresponding 13-moment result in table III. What small error remains is thus attributable entirely to the form of the distribution function rather than to an incorrect relation between $\vec{\beta}_1$ and $\vec{\beta}_0$.

TABLE V.- RATIO OF COLLISION TIMES τ_s AND τ_σ FOR A LORENTZ FULLY IONIZED GAS

Conditions	N	τ_s/τ_σ
Case (1):	5	1.000
No gradients,	13	1.900
$\vec{\beta}_1 = 4\vec{\beta}_0$	16	1.932
	∞	1.933
Case (2):	5	1.000
Temperature gradient,	13	1.000
$\vec{\beta}_1 = 5\vec{\beta}_0/2$	16	1.804
	∞	1.865

Case (2) requires the solution of equation (1) with the temperature gradient included, which is available from reference 2 in the form

$$\varphi_e(\infty\text{-moment}) = -\frac{\tau_\sigma R_{04}}{R_{13}}\left(\frac{2kT_e}{m_e}\right)^{1/2}\gamma^{(4/\xi)-1}\left[\frac{e}{kT_e}\vec{E} + \left(\gamma^2 - \frac{5}{2}\right)\nabla \ln T_e\right]\cdot\vec{\gamma} \tag{68}$$

for zero pressure gradient. This expression can be used to evaluate $\vec{\beta}_1$ and $\vec{\beta}_0$ as

$$\vec{\beta}_1 = \pi^{-3/2}\int e^{-\gamma^2}\gamma^2\vec{\gamma}\varphi_e\, d\vec{\gamma} = -\frac{(\xi + 1)\tau_\sigma}{\xi^2 m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\left[2e\xi\vec{E} + k(\xi + 4)\nabla T_e\right] \tag{69}$$

and

$$\vec{\beta}_0 = \pi^{-3/2}\int e^{-\gamma^2}\vec{\gamma}\varphi_e\, d\vec{\gamma} = -\frac{\tau_\sigma}{2\xi m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\left[2e\xi\vec{E} + k(4 - \xi)\nabla T_e\right] \tag{70}$$

the combination of which with equation (68) and

$$\vec{\beta}_1 = \frac{5}{2}\vec{\beta}_0 \tag{71}$$

finally gives

$$\varphi_e(\infty\text{-moment}) = -\frac{\pi^{1/2}}{32}\gamma^3(3\gamma^2 - 20)\vec{\beta}_0 \cdot \vec{\gamma} \tag{72}$$

if $\xi = 1$.

Since the appropriate moments of equation (72) yield

$$\vec{\beta}_2 - 7\left(\vec{\beta}_1 - \frac{5}{4}\vec{\beta}_0\right) = -\frac{15}{4}\vec{\beta}_0 \tag{73}$$

the following set of perturbation functions are obtained when equations (71) and (73) are imposed on equations (21), (14), and (A5):

$$\varphi_e(5\text{-moment}) = \varphi_e(13\text{-moment}) = 2\vec{\beta}_0 \cdot \vec{\gamma} \tag{74}$$

and

$$\varphi_e(16\text{-moment}) = -\frac{3}{7}\left(\gamma^4 - 7\gamma^2 + \frac{49}{12}\right)\vec{\beta}_0 \cdot \vec{\gamma} \tag{75}$$

Numerical calculations based on equations (50), (51), (72), (74), and (75) are given in case (2) of table V, where it is seen that the 13-moment approximation to the ratio of τ_s to τ_σ is inadequate. The reason is obvious: Since φ_e(13-moment) is the same as φ_e(5-moment), and since f_e(5-moment) corresponds to a Maxwellian distribution referred to the electron frame of reference, the 13-moment function is unable in this case to describe the important frictional effects arising from electron-ion collisions. At least the 16-moment approximation is needed to safeguard against this occurrence.

Finally, one might consider the prediction of $\vec{\beta}_2$ by the 13-moment distribution function. Since the corresponding integration of equation (14) yields

$$\vec{\beta}_2 - 7\left(\vec{\beta}_1 - \frac{5}{4}\vec{\beta}_0\right) = 0 \tag{76}$$

regardless of the relation between $\vec{\beta}_1$ and $\vec{\beta}_0$, there can be no agreement at this level with equation (63) or (73). Serious difficulties are therefore unavoidable in some applications of the Grad function, even when the best values of the fluxes are employed.

SECOND-ORDER THEORY

Exact and Grad Solutions

As an example of the utility of the closing-out method for predicting higher order plasma properties, the second-order traceless pressure tensor so determined is compared in the present section with calculations involving a direct second-order solution of the Boltzmann equation. Errors in the results obtained by using the Grad 13-moment approximation are expected to arise from the fact that the pressure tensor term in equation (12) incompletely describes higher order contributions in much the same manner that the electron diffusion velocity and heat flux incompletely describe first-order effects.

Since an exact second-order solution is necessary in this investigation, the physical problem should be as tractable as possible and simpler than the one considered by Everett (ref. 1). The essential requirements are satisfied quite well by a hypothetical plasma in which an electric field is applied in the z-direction, the magnetic field is zero (in violation of Maxwell's equations), there are no spatial variations of any macroscopic quantity, and a Lorentz-like collision model is applicable.

The pertinent second-order perturbation equation can be written from equation (1) as follows:

$$\frac{\partial f_e}{\partial t} + \frac{\beta_0}{\tau_\sigma}\frac{\partial}{\partial \gamma_z}\left(f_e^{(o)}\varphi_1\right) = \left(\frac{\partial f_e}{\partial t}\right)_{c,\varphi_2} = -n_j f_e^{(o)}\gamma\left(\frac{2kT_e}{m_e}\right)^{1/2}\int\left(\varphi_2 - \varphi_2'\right)b\ db\ d\epsilon \qquad (77)$$

Primed symbols refer to quantities after a collision (as opposed to unprimed symbols for quantities before a collision) and φ_e is split into additive φ_1 and φ_2 parts, where φ_2 is second order and

$$\varphi_1 = \frac{2R_{04}}{R_{13}}\gamma^{(4/\xi)-1}\beta_0\gamma_z \qquad (78)$$

from equation (35). Also used in the derivation of equation (77) was Ohm's law and equation (52).

It is evident from the energy balance expression (ref. 3)

$$\frac{3}{2}\frac{\partial p_e}{\partial t} = \vec{E}\cdot\vec{j} = \frac{2p_e\beta_0^2}{\tau_\sigma} \qquad (79)$$

that the time derivative of the electron pressure is second order. Moreover, since the number density n_e is time independent by reason of the electron equation of continuity, there results

$$\frac{\partial f_e^{(o)}}{\partial t} = f_e^{(o)}\left(\gamma^2 - \frac{3}{2}\right)\frac{\partial \ln p_e}{\partial t} = \frac{4f_e^{(o)}\beta_0^2}{3\tau_\sigma}\left(\gamma^2 - \frac{3}{2}\right) \tag{80}$$

from equations (7) and (79).

A physically meaningful solution is obtained by substituting the trial function

$$\varphi_2 = \left(3\gamma_z^2 - \gamma^2\right)g_1(\gamma) + g_2(\gamma)t \tag{81}$$

into equation (77) and using equation (80), the relation

$$\frac{\partial}{\partial \gamma_z}\left(f_e^{(o)}\varphi_1\right) = \frac{2f_e^{(o)}R_{04}\beta_0}{R_{13}}\gamma^{(4/\xi)-3}\left[\gamma^2 - 2\left(\gamma^2 - \frac{4-\xi}{2\xi}\right)\gamma_z^2\right] \tag{82}$$

and the collision integral

$$\int\left(\gamma_z^2 - \gamma_z'^2\right)b\ db\ d\epsilon = \frac{A(\xi)R_{13}}{2n_j\tau_\sigma R_{04}}\left(\frac{m_e}{2kT_e}\right)^{1/2}\gamma^{-4/\xi}\left(3\gamma_z^2 - \gamma^2\right) \tag{83}$$

derived in appendix B. The resulting equations are

$$g_1(\gamma) = \frac{8R_{04}^2\beta_0^2}{9A(\xi)R_{13}^2}\gamma^{(8/\xi)-4}\left(\gamma^2 - \frac{4-\xi}{2\xi}\right) \tag{84}$$

and

$$g_2(\gamma) = \frac{4\beta_0^2}{3\tau_\sigma}\left[\frac{R_{04}}{R_{13}}\gamma^{(4/\xi)-1}\left(\gamma^2 - \frac{\xi+2}{\xi}\right) - \gamma^2 + \frac{3}{2}\right] \tag{85}$$

Since neither the electron number density

$$n_e \equiv \int f_e^{(o)}\ d\vec{c}_e + \int f_e^{(o)}\varphi_2\ d\vec{c}_e = n_e \tag{86}$$

nor the electron temperature

$$T_e \equiv \frac{m_e}{3n_e k}\left(\int c_e^2 f_e^{(o)} d\vec{c}_e + \int c_e^2 f_e^{(o)}\varphi_2\ d\vec{c}_e\right) = T_e \tag{87}$$

is affected by φ_2, the following combination of equations (78), (81), (84), and (85) is the correct perturbation function through second order:

$$\varphi_e = \frac{2R_{04}\beta_0}{R_{13}} \gamma^{(4/\xi)-1} \gamma_z + \frac{8R_{04}^2\beta_0^2}{9A(\xi)R_{13}^2} \gamma^{(8/\xi)-4}\left(\gamma^2 - \frac{4-\xi}{2\xi}\right)\left(3\gamma_z^2 - \gamma^2\right)$$

$$+ \frac{4\beta_0^2 t}{3\tau_\sigma}\left[\frac{R_{04}}{R_{13}} \gamma^{(4/\xi)-1}\left(\gamma^2 - \frac{\xi + 2}{\xi}\right) - \gamma^2 + \frac{3}{2}\right] \tag{88}$$

The traceless electron pressure tensor is determined by the substitution of equation (88) into equation (13), whereupon one obtains

$$\overset{o}{P}_{e,xy} = \overset{o}{P}_{e,xz} = \overset{o}{P}_{e,yz} = 0 \tag{89}$$

and

$$\overset{o}{P}_{e,zz} = -\overset{o}{P}_{e,xx} - \overset{o}{P}_{e,yy} = -2\overset{o}{P}_{e,xx} = \frac{64(\xi + 1)R_{04}R_{22}}{45\xi A(\xi)R_{13}^2} p_e\beta_0^2 \tag{90}$$

Equations (89) and (90) are the exact second-order tensor elements required for comparisons with the approximate Grad 13-moment predictions. It is essential, however, that the same physical problem be considered in all calculations; accordingly, the application and closing out of the macroscopic equation of change for $P_{e,zz}$ must correspond to the present plasma conditions, which are not the same as those of Everett.

The multiplication of each term of equation (1) by $m_e c_{e,z}^2$ and subsequent integration over velocity space yields the following $P_{e,zz}$ expression

$$\frac{\partial}{\partial t}\left(P_{e,zz}\right) + 2en_eEv_e = m_e \int c_{e,z}^2\left(\frac{\partial f_e}{\partial t}\right)_c d\vec{c}_e$$

$$= m_e \int c_{e,z}^2\left(f_e'f_j' - f_ef_j\right)\left|\vec{c}_e - \vec{c}_j\right| b\, db\, d\epsilon\, d\vec{c}_e\, d\vec{c}_j$$

$$= -n_j m_e\left(\frac{2kT_e}{m_e}\right)^{3/2} \int \gamma\left(\gamma_z^2 - \gamma_z'^2\right) f_e b\, db\, d\epsilon\, d\vec{c}_e \tag{91}$$

in which Chapman and Cowling's concepts (ref. 3) of inverse collisions and microscopic reversibility, together with several of the assumptions in the last paragraph of the Introduction, are employed in the successive simplifications of the collision integral.

Further simplifications of equation (91) are obtained by the use of Ohm's law and equations (79) and (83). There results

$$\beta_0^2 = \frac{9A(\xi)R_{13}}{64\pi R_{04}^2} \int e^{-\gamma^2} \gamma^{1-(4/\xi)} \left(3\gamma_z^2 - \gamma^2\right) \varphi_e \, d\vec{\gamma} \tag{92}$$

where the φ_e is that of equation (12) if the unmodified Grad 13-moment function is used for closing-out purposes. As mentioned previously, substantial errors may arise from the fact that the pressure tensor term in equation (12) incompletely describes second-order contributions in much the same manner as the electron diffusion velocity and heat flux incompletely describe first-order effects.

All such second-order corrections to equation (12) are available through straightforward extensions of the information theory outlined in references 4 and 5 and appendix A. More specifically, if the entropy is maximized subject to constraints on n_e, $\vec{\beta}_0$, $\vec{\beta}_1$, and $\overset{o}{\overline{\overline{P}}}_e$, which are interpreted as the observer's complete first-hand knowledge of the plasma, the following modified 13-moment perturbation function is obtained:

$$\varphi_e = 2\beta_0\gamma_z\left(\frac{2}{5}\eta\gamma^2 + 1 - \eta\right) + \frac{1}{2p_e}\overset{o}{P}_{e,zz}\left(3\gamma_z^2 - \gamma^2\right) + \frac{2\eta\beta_0^2\gamma_z^2}{25}\left[4\eta\gamma^4 + 20(1-\eta)\gamma^2 + 7(\eta - 10)\right] - \frac{4\eta(2\eta+5)\beta_0^2}{25}\left(\gamma^2 - \frac{5}{2}\right) \tag{93}$$

The parameter η introduced in this expression is defined by

$$\eta = \frac{\beta_1}{\beta_0} - \frac{5}{2} \tag{94}$$

A difficulty with equation (93) is apparent from the fact that the corresponding entropy is associated more with the observer's uncertainty about the plasma than with the disorder predicted by the exact solution to the Boltzmann equation. If the number of constraints employed in the entropy maximization is too small, the observer's uncertainty may be so unreasonably large that the two correction terms in equation (93) will substantially overcompensate the errors inherent in the unmodified Grad approximation.

The substitution of equation (93) into equation (92) finally yields

$$\overset{o}{P}_{e,zz}(\text{modified 13-moment}) = \left\{ \frac{40\xi R_{04}^2}{9(3\xi - 2)A(\xi)R_{13}R_{-1,5}} - \frac{4\eta}{75\xi^2}\left[\left(7\xi^2 - 32\xi + 16\right)\eta + 10\xi(\xi - 4)\right] \right\} p_e \beta_0^2 \tag{95}$$

As an examination of this development demonstrates, all that is necessary for the conversion of equation (95) to the unmodified Grad 13-moment result is the formal replacement of η with zero. Calculations of the latter type are presented in the second column of table VI for several electron—heavy-particle interaction potentials and may be compared with the exact results obtained from equation (90) and listed in the fourth column.

TABLE VI.- TRACELESS PRESSURE TENSOR COMPONENT

ξ	$\overset{o}{P}_{e,zz}/p_e\beta_0^2$		
	Unmodified 13-moment	Modified 13-moment	Exact
1	0.655	4.135	2.749
4	1.719	1.719	1.719
∞	1.964	2.137	2.513

Also given in table VI are the modified (by information theory) values of $\overset{o}{P}_{e,zz}$ as computed from equation (95) with

$$\eta = \frac{4 - \xi}{2\xi} \tag{96}$$

from equation (41). The previously mentioned possibility of overcorrections of errors inherent in the unmodified Grad distribution is especially apparent in the case of Coulomb interactions ($\xi = 1$), but evidently fails to occur when ξ is greater than 4. It is interesting that Maxwellian molecules ($\xi = 4$) again have the peculiar property whereby the corresponding exact distribution function of equation (88) is identical with that of equation (93) and entropy is maximized automatically.

Since a feature of the derivation of equation (93) from information theory is the fact that the square of the first-order function φ_1 contributes heavily to the resulting η-dependent terms and since the φ_1 of equation (12) rather than that of equation (78) was used in this regard in order to preserve the 13-moment characteristics of φ_e, it may be more consistent to substitute into equation (95) the following value of η obtained from the 13-moment equations (39) and (94) rather than the ∞-moment value in equation (96):

$$\eta = \frac{5\xi(4 - \xi)}{13\xi^2 - 16\xi + 16} \tag{97}$$

In particular,

$$\overset{o}{P}_{e,zz}\left(\text{modified 13-moment}, \xi=1, \eta=\frac{15}{13}\right) = 3.140 p_e \beta_0^2 \tag{98}$$

represents a substantial improvement over the corresponding value in table VI, but

$$\overset{o}{P}_{e,zz}\left(\text{modified 13-moment}, \xi=\infty, \eta=-\frac{5}{13}\right) = 2.114 p_e \beta_0^2 \tag{99}$$

does not.

These examples illustrate the difficulty in making general statements irrespective of the microscopic parameters. It does appear, however, that the type of consistency invoked in equations (97) to (99) is highly desirable over an important range of ξ and is not very damaging for the remaining values of ξ. In any event, it seems safe to assert that except for values of ξ in the neighborhood of 4, and especially for values of ξ somewhat less than 4, the unmodified 13-moment approximation is quite inadequate for closing-out purposes at the second-order level. This trend toward greater inaccuracies is expected to continue in the still higher orders, an example of which is the calculation of the third-order contribution to the electrical conductivity.

Reference Velocity Modification

It was shown in the preceding paragraphs that a consistent treatment of the modified (by information theory) 13-moment approximation leads to a substantial improvement over the unmodified approximation for an important range of the interaction parameter ξ. Another modification (ref. 1) consists of taking the electron particle velocity relative to the mean velocity of the electrons in the Grad approximation, instead of relative to the mean mass velocity of the plasma. Everett's argument for this procedure is as follows: Although the electrons still must be reasonably close to thermodynamic equilibrium among themselves because the dominant term is Maxwellian relative to the electron diffusion velocity, they need no longer be close to thermodynamic equilibrium with the heavy particles.

Such an argument seems inconclusive, however, because the description of the frictional distortion by electron—heavy-particle collisions of the Maxwellian distribution function for electrons does not appear to be generalized. Hence, the requirement of near thermodynamic equilibrium between species does not appear to be alleviated. The purpose of the present section is to investigate whether, in fact, Everett's modification is an improvement over that of information theory.

Everett's distribution function (ref. 1) through second-order terms can be written

$$f_e = n_e\left(\frac{m_e}{2\pi kT'_e}\right)^{3/2} \exp\left[-\frac{m_e}{2kT'_e}\left(\vec{c}_e - \vec{v}_e\right)^2\right]\left\{1 + \frac{4\eta\beta_0}{5}\left[\left(\vec{\gamma} - \vec{\beta}_0\right)^2 - \frac{5}{2}\right]\left(\gamma_z - \beta_0\right) + \frac{1}{2p_e}\overset{o}{P}{}'_{e,zz}\left(3\gamma_z^2 - \gamma^2\right)\right\} \tag{100}$$

the primes in which indicate quantities referred to the electron frame of reference. If the parameters α_1 and α_2 are introduced such that

$$T'_e = T_e\left(1 + \alpha_1\beta_0^2\right) \tag{101}$$

and

$$\overset{o}{P}{}'_{e,zz} = \overset{o}{P}_{e,zz} + \frac{4p_e\beta_0^2}{15}\alpha_2 \tag{102}$$

the subsequent expansion of equation (100) yields

$$f_e = n_e\left(\frac{m_e}{2\pi kT_e}\right)^{3/2} e^{-\gamma^2}\left(1 + \varphi_{e,Everett}\right) \tag{103}$$

and

$$\varphi_{e,Everett} = 2\beta_0\gamma_z\left(\frac{2}{5}\eta\gamma^2 + 1 - \eta\right) + \frac{1}{2p_e}\overset{o}{P}_{e,zz}\left(3\gamma_z^2 - \gamma^2\right) + \frac{2\alpha_2\beta_0^2}{15}\left(3\gamma_z^2 - \gamma^2\right) + \beta_0^2\left(2\gamma_z^2 + \alpha_1\gamma^2 - \frac{3}{2}\alpha_1 - 1\right) + \frac{4\eta\beta_0^2}{5}\left[\gamma_z^2\left(2\gamma^2 - 7\right) - \gamma^2 + \frac{5}{2}\right] \tag{104}$$

The requirements that the energy and $m_e c_{e,z}^2$ moments of equations (103) and (104) result in identities for T_e and $\overset{o}{P}_{e,zz}$, respectively, give

$$\alpha_1 = -\frac{2}{3} \tag{105}$$

and

$$\alpha_2 = -5 \tag{106}$$

so that equation (104) becomes

$$\varphi_{e,\mathrm{Everett}} = 2\beta_0\gamma_z\left(\frac{2}{5}\eta\gamma^2 + 1 - \eta\right) + \frac{1}{2p_e}\overset{o}{P}_{e,zz}\left(3\gamma_z^2 - \gamma^2\right)$$
$$+ \frac{4\eta\beta_0^2}{5}\left[\gamma_z^2\left(2\gamma^2 - 7\right) - \gamma^2 + \frac{5}{2}\right] \tag{107}$$

There is, indeed, a difference between $\varphi_{e,\mathrm{Everett}}$ and the unmodified 13-moment function of equation (12).

If equation (107) is substituted into equation (92) for the hypothetical plasma outlined previously, there results

$$\left(\overset{o}{P}_{e,zz}\right)_{\mathrm{Everett}} = \frac{8}{3}\left[\frac{5\xi R_{04}^2}{3(3\xi - 2)A(\xi)R_{13}R_{-1,5}} + \frac{(4 - \xi)\eta}{5\xi}\right]p_e\beta_0^2 \tag{108}$$

Accordingly, the values of $\overset{o}{P}_{e,zz}$ from equation (108) are $3.055p_e\beta_0^2$, $1.719p_e\beta_0^2$, and $2.231p_e\beta_0^2$ for $\xi = 1$, 4, and ∞, respectively, if η is computed from equation (96). Likewise, the values of $\overset{o}{P}_{e,zz}$ are $2.501p_e\beta_0^2$, $1.719p_e\beta_0^2$, and $2.169p_e\beta_0^2$ for $\xi = 1$, 4, and ∞, respectively, if η is computed from equation (97).

As seen in table VII, wherein all the calculations of $\overset{o}{P}_{e,zz}(\xi=1)$ are listed, Everett's function gives the best values yet of the traceless pressure tensor. It seems quite adequate for this purpose over the entire range of ξ and is definitely to be preferred over the distribution function derived from information theory.

Of some concern, however, is the fact that Everett's modifications in equation (107) are confined to the second perturbation order; consequently, his method fails to improve the inaccurate entropies and relations between first-order moments that were deduced

TABLE VII.- TRACELESS PRESSURE TENSOR COMPONENT

FOR COULOMB INTERACTIONS (ξ = 1)

$\overset{o}{P}_{e,zz}/p_e\beta_0^2$	Method
0.655	Unmodified Grad 13-moment.
4.135	13-moment with entropy maximization and exact heat flux.
3.140	13-moment with entropy maximization and approximate heat flux.
3.055	Everett 13-moment with exact heat flux.
2.501	Everett 13-moment with approximate heat flux.
2.749	Exact solution.

previously (tables I to V) by using the Grad distribution. Nor can it be said without additional calculations and comparisons that Everett's third-order results are valid.

CONCLUSIONS

The utility of Grad-like distribution functions has been investigated to the first and second perturbation orders with the following conclusions:

1. The 13 moments in Grad's velocity distribution function are deprived of their character as independent variables of the problem when the function is employed in the Everett manner to close out the macroscopic equations of change. The relations thus derived between the moments are not satisfied by the exact moments because of the inaccurate evaluations of the collisional transfer terms using the Grad approximation.

2. Although the errors in the Everett type of calculations of the heat flux relative to electron diffusion are not serious of themselves, the fact that the heat flux multiplies the strongest velocity polynomial in the 13-moment distribution function can lead to much larger errors in calculations of other quantities.

3. The substitution into the original Grad 13-moment distribution function of the incorrect heat flux obtained by the closing-out method results in poor predictions of both the entropy and the entropy production. This is especially pronounced if the important difference between the collision time for entropy production and the collision time associated with electron diffusion is considered. That such errors are caused primarily by incorrect moments, rather than by the approximate form of the 13-moment function, is demonstrated by the excellent value of the entropy found when the correct heat flux is substituted into the Grad distribution.

4. The Grad function is much more accurate in its prediction of entropy if, as originally intended, experimental values are substituted for the 13 moments. Even so, the

results are quite poor under special circumstances – namely, thermodynamic and field conditions for which the 13 moments are insufficient to describe the frictional effects arising from electron—heavy-particle collisions. In no case, however, does the 13-moment distribution function provide correct values of the moments higher than the heat flux.

5. A generalization of Everett's method to include Grad-like functions of arbitrary complexity shows a one-to-one correspondence between the number of moments considered and the number of Sonine polynomials employed in the first-order Chapman-Enskog solution. In particular, the 5-moment analysis corresponds to the first Sonine approximation and the 13-moment analysis corresponds to the second Sonine approximation. What benefit, if any, can be obtained by using the Everett method at these levels is probably associated with the mechanics of solution.

6. The unmodified 13-moment distribution function of Grad is unacceptable for closing-out purposes at the second-order level. Two modifications were considered with the following result: Everett's method of referring the Grad function to the electron frame of reference is more successfully used in predicting the traceless pressure tensor than is the distribution function obtained from information theory (entropy maximization). Neither technique improves the first-order computations.

7. Neither the Everett modification to the Grad function nor that of information theory depicts the same time dependence as the exact second-order solution derived in the present research. The interpretation is not altogether clear, but one must expect that second-order properties should depend on time in this problem through means more complex than the temperature. In any event, the present solution is a rather pronounced departure from that of Chapman and Enskog because the second-order perturbation function appears on the left side of the second-order perturbation equation.

8. Everett's calculations at the third and higher perturbation orders require further consideration before definite statements can be made regarding their validity.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 23, 1969.

APPENDIX A

DEVELOPMENTS FROM INFORMATION THEORY

It is shown in this appendix that the first-order Grad distribution function of equation (14) can be derived just as well from the concepts of information theory as from the original Grad expansion in terms of three-dimensional Hermite polynomials (ref. 1). A start in this direction was recently achieved by Stankiewicz (refs. 4 and 5), but he stopped short of the required number of moments.

The fundamental postulate of information theory states that the best distribution function that can be derived from limited experimental knowledge alone is the one which maximizes the entropy density

$$s - s^{(o)} = -k \int f_e \ln\left(f_e / f_e^{(o)}\right) d\vec{c}_e \approx -\frac{k}{2} \int f_e^{(o)} \varphi_e^2 \, d\vec{c}_e \tag{A1}$$

subject to the measured constraints. If the nonequilibrium contributions to the latter consist only of measurements of the $\vec{\beta}_j$ defined in equation (3), the application of the method of Lagrange multipliers $\vec{\mu}_j$ to $\delta s = 0$ and $\delta\vec{\beta}_j = 0$ yields the following variational expressions:

$$\int f_e^{(o)} \left(\varphi_e - \sum_j \gamma^{2j} \vec{\mu}_j \cdot \vec{\gamma}\right) \delta\varphi_e \, d\vec{c}_e = 0 \tag{A2}$$

and

$$\varphi_e = \sum_j \gamma^{2j} \vec{\mu}_j \cdot \vec{\gamma} \tag{A3}$$

The combination of equations (3) and (A3) yields

$$\vec{\beta}_k = \frac{1}{6} \sum_j 2^{-j-k} (2j + 2k + 3)!! \, \vec{\mu}_j \tag{A4}$$

the substitution of which into equation (A3) gives the 16-moment perturbation function

$$\varphi_e = 2\vec{\beta}_0 \cdot \vec{\gamma} + \frac{4}{5}\left(\gamma^2 - \frac{5}{2}\right)\left(\vec{\beta}_1 - \frac{5}{2}\vec{\beta}_0\right) \cdot \vec{\gamma} + \frac{4}{35}\left(\gamma^4 - 7\gamma^2 + \frac{35}{4}\right)\left(\vec{\beta}_2 - 7\vec{\beta}_1 + \frac{35}{4}\vec{\beta}_0\right) \cdot \vec{\gamma} \tag{A5}$$

if j assumes the values 0, 1, and 2. The description of this function as a 16-moment approximation refers to the addition of $\vec{\beta}_2$ to the original 13 moments of Grad.

Among other requirements of a similar nature, the condition

$$\vec{\beta}_2 = 7\left(\vec{\beta}_1 - \frac{5}{4}\vec{\beta}_0\right) \tag{A6}$$

must be satisfied before Grad's 13-moment function is consistent to first order with information theory and the full complement of $\vec{\beta}_j$ constraints.

APPENDIX B

COLLISION INTEGRALS

It is shown in this appendix that equation (83) follows directly from the collision dynamics when the electron—heavy-particle interaction potential is written as the inverse ξ-power of the separation distance.

The first step in such a development is the formulation of the reduced electron particle velocity after a collision $\vec{\gamma}'$ in terms of the reduced electron particle velocity before the collision $\vec{\gamma}$ and the corresponding scattering or deflection angle χ. This straightforward exercise in geometry yields

$$\vec{\gamma}' = \vec{\gamma} \cos \chi + \gamma \sin \chi \Big[\hat{i} (\sin \varphi \cos \epsilon + \cos \theta \cos \varphi \sin \epsilon) - \hat{j}(\cos \varphi \cos \epsilon - \cos \theta \sin \varphi \sin \epsilon) - \hat{k} \sin \theta \sin \epsilon\Big] \tag{B1}$$

where θ and φ are the polar and azimuthal angles, respectively, of $\vec{\gamma}$ in spherical coordinates and ϵ is the azimuthal angle of $\vec{\gamma}'$ with respect to the direction of $\vec{\gamma}$.

Accordingly,

$$\int \left(\vec{\gamma} - \vec{\gamma}'\right) b \, db \, d\epsilon = 2\pi\vec{\gamma} \int_0^\infty (1 - \cos \chi) b \, db \tag{B2}$$

The remaining integral involves, of course, the detailed collision dynamics, but that calculation is circumvented here by the following use (ref. 2) of equations (15) and (35):

$$\begin{aligned}\left(\frac{\partial f_e}{\partial t}\right)_{c,\varphi_1} &= -n_j f_e^{(o)} \gamma \left(\frac{2kT_e}{m_e}\right)^{1/2} \int \left(\varphi_1 - \varphi_1'\right) b \, db \, d\epsilon \\ &= -\frac{2n_j R_{04} f_e^{(o)}}{R_{13}} \left(\frac{2kT_e}{m_e}\right)^{1/2} \gamma^{4/\xi} \vec{\beta}_0 \cdot \int \left(\vec{\gamma} - \vec{\gamma}'\right) b \, db \, d\epsilon \\ &= -\frac{e^2 n_e R_{13}}{m_e \sigma R_{04}} \gamma^{1-(4/\xi)} f_e^{(o)} \varphi_1 = -\frac{2}{\tau_\sigma} f_e^{(o)} \vec{\beta}_0 \cdot \vec{\gamma}\end{aligned} \tag{B3}$$

and

$$\int \left(\vec{\gamma} - \vec{\gamma}'\right) b \, db \, d\epsilon = 2\pi\vec{\gamma} \int_0^\infty (1 - \cos \chi) b \, db = \frac{R_{13}}{n_j \tau_\sigma R_{04}} \left(\frac{m_e}{2kT_e}\right)^{1/2} \gamma^{-4/\xi} \vec{\gamma} \tag{B4}$$

where n_j is the number density of the j-type heavy particles.

Finally,

$$\int \left(\gamma_z^2 - \gamma_z'^2\right) b\ db\ d\epsilon = \int \left[\gamma_z^2\left(1 - \cos^2\chi\right) + 2\gamma^2 \sin\chi \cos\chi \sin\theta \cos\theta \sin\epsilon - \gamma^2 \sin^2\chi \sin^2\theta \sin^2\epsilon\right] b\ db\ d\epsilon$$

$$= \pi\left(3\gamma_z^2 - \gamma^2\right)\int_0^\infty \left(1 - \cos^2\chi\right) b\ db$$

$$= A(\xi)\pi\left(3\gamma_z^2 - \gamma^2\right)\int_0^\infty (1 - \cos\chi) b\ db$$

$$= \frac{A(\xi)R_{13}}{2n_j\tau_\sigma R_{04}}\left(\frac{m_e}{2kT_e}\right)^{1/2}\gamma^{-4/\xi}\left(3\gamma_z^2 - \gamma^2\right) \tag{B5}$$

from equations (B1) and (B4). The ratio $A(\xi)$ of the two impact parameter integrals has the values 2, 1.034, and 0.666, respectively, for $\xi = 1$, 4, and ∞. (See ref. 6, pp. 546-549.)

REFERENCES

1. Everett, Willis L.: Generalized Magnetohydrodynamic Equations for Nonequilibrium Plasma Systems. Rarefied Gas Dynamics, Vol. I, J. A. Laurmann, ed., Academic Press, 1963, pp. 1-25.

2. Meador, Willard E.: A Semiempirical Collision Model for Plasmas. NASA TR R-310, 1969.

3. Chapman, Sydney; and Cowling, T. G.: The Mathematical Theory of Non-Uniform Gases. Second ed., Cambridge Univ. Press, 1952. (Reprinted 1960.)

4. Stankiewicz, Norbert: Thermodynamic Study of Plasmas Using the Principle of Maximum Entropy. NASA TN D-4621. 1968.

5. Stankiewicz, N.: The Ellipsoidal Velocity Distribution Function. NASA TN D-5026, 1969.

6. Hirschfelder, Joseph O.; Curtiss, Charles F.; and Bird, R. Byron: Molecular Theory of Gases and Liquids. John Wiley & Sons, Inc., c.1954. (Reprinted with corrections 1964.)

NASA-Langley, 1969 —— 25 L-6523