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ON THE CAPTURE PROBABILITIES OF RESONANCE ROTATION FOR MERCURY

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ABSTRACT

The computer solutions of Mercury's rotation near the 3:2 resonance spin rate disprove the theories of capture probabilities for resonance rotation in the solar system. The pitfalls of the averaging procedure in the theories of capture probabilities are pointed out. It is shown that the capture process of Mercury's rotation near the 3:2 resonance spin rate is not a probabilistic affair and that the values of capture probabilities for resonance rotation have been trapped into the pitfalls of the averaging methods.

ON THE CAPTURE PROBABILITIES OF RESONANCE ROTATION FOR MERCURY

INTRODUCTION

Liu and O'Keefe (Liu and O'Keefe, 1965) have developed a theory of rotation for the planet Mercury. They have shown that the rotation of Mercury is possibly locked with a period exactly two-thirds of the period of revolution and that the longest axis of the planet librates about the solar direction during every perihelion passage. According to the Liu-O'Keefe Theory, Liu (Liu, 1966) has calculated that the apparent circulational motion of Mercury at successive perihelia can be converted to a librational motion when Mercury rotates with any period within the range 58.65 ± 0.3 days for $(B-A)/C \cong 5 \times 10^{-5}$ if the tidal effect can be neglected.

On the other hand, Goldreich and Peale (Goldreich and Peale, 1966; 1968) have adopted the analogy with the motion of a pendulum to discuss the capture probabilities of resonance rotation in the solar system. They have argued that the resonance rotation of the planet Mercury cannot be captured at the 3:2 resonance spin rate with a constant tidal torque and that the probabilities of capture into the 3:2 resonance spin rate for several assumed forms of tidal torques might be one chance in five. Based on these arguments, Counselman and Shapiro (Counselman and Shapiro, 1969; 1970) have developed a theory of capture probabilities for Mercury which encompasses possible variations in orbital eccentricity as well as core-mantle coupling. According to the Counselman-Shapiro Theory the so-called probability of capture was reduced to about 0.02. Therefore, Shapiro has been entertaining the possibility that Mercury's spin rate may not be in the 3:2 resonance after all! (Shapiro, 1968)

The modern aesthetic values of the capture probability are quite offensive to common sense. Evidently, the motions of the planet Mercury near the 3:2 resonance spin rate (Liu and O'Keefe, 1965; Liu, 1966; Liu, 1970) cannot be covered with a single solution of a simple pendulum.

In this article, the computer solutions of the rotation of Mercury near the 3:2 resonance spin rate with constant tidal torques are presented. The pitfalls of the averaging procedure in the theory of capture probabilities are then pointed out. It is shown that the capture process of resonance rotation for Mercury is not a probabilistic affair. Therefore, the problem of capture probabilities does not exist.

EQUATION OF MOTION

The angle of rotation for Mercury has the form

$$\Phi = f + \varphi \tag{1}$$

where f is the true anomaly and φ is defined as the angular displacement of the longest axis of the planet from the solar direction. The rotation of Mercury is governed by (Liu and O'Keefe, 1965)

$$\frac{d}{dt}\left[C(t)\,\frac{d\Phi}{dt}\right] + \frac{3GMs}{r^3}\left[B(t)-A(t)\right]\cos\varphi\,\sin\varphi = -N \tag{2}$$

In equation (2), G is the gravitational constant, Ms is the mass of the Sun, r is the distance from the planet to the Sun, N is the torque due to solar tides and A(t), B(t) and C(t) are principal moments of inertia which are functions of the time t. If the independent variable is changed from the time t to the true anomaly f and A(t), B(t) and C(t) are assumed to be constants, then equation (2) becomes (Liu and O'Keefe, 1965; Goldreich, 1966; Goldreich and Peale, 1966)

$$\frac{d^2\varphi}{df^2} - \frac{2e \ \mathrm{Sin} f}{1+e \ \cos f}\left(\frac{d\varphi}{df} + 1\right) + \frac{3\cdot(B-A)}{2C(1+e \ \cos f)} \cdot \ \mathrm{Sin} \ 2\varphi = -q. \qquad (3)$$

With

$$q = \frac{N}{n^2 C} \cdot \frac{(1-e^2)^3}{(1+e \ \cos f)^4} \qquad (4)$$

where n is the orbital mean motion and e is the orbital eccentricity. The magnitude of q has been estimated in the order of $O(10^{-11})$ (Jeffreys, 1959). The value $(B-A)/C = 0.00005$ has been adopted by Liu and O'Keefe to investigate the resonance rotation (Liu and O'Keefe, 1965).

COMPUTER SOLUTIONS

By applying the Rung-Kutta integration algorithm we have solved equation (3) on the 360 computer. To transform the variable of function Φ from the true anomaly to the mean anomaly, the following elliptical equations have been used

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{E}{2} \qquad (5)$$

$$E - e \ \mathrm{Sin}E = nt = M \qquad (6)$$

where E and M are eccentric and mean anomaly respectively.

The computer program performs the necessary algorithm for successively updating expansion of equation (3). We chose $\varphi = 0$ when $t = 0$. The results of the value $\gamma = \Phi - \frac{3}{2} M$ at successive perihelia for different initial spin rates are presented in Table 1 for constant tidal torques $q = 10^{-11}$ and $q = 10^{-7}$.

From Table 1, we observe that the value γ librates about the solar direction at successive perihelia and will not decrease or increase indefinitely even if Mercury rotates with an initial spin rate which is smaller or larger than the resonance one. Thus we may draw the general conclusion that the planet can be trapped at a spin rate which is 3:2 commensurate with its orbital mean motion even if q is a constant tidal torque. We were astonished to find out that the resonance rotation of Mercury can be captured even if the assumed constant tidal torque is about 0.1 million times the conventional tides. The computer results of γ are recorded in Table 2. For this case, we have chosen initial spin rate 1.50965 n.

AVERAGING METHODS

Liu and O'Keefe (Liu and O'Keefe, 1965) have pointed out that a solution of equation (3) for small values of e and (B-A)/C at the 3:2 resonance spin rate can be found by the averaging methods. Let us choose the time τ reckoned from perihelion and related to the orbital period divided by 2π

$$\tau_{(f)} = 2 \tan^{-1} \left(\frac{1-e}{1+e}\right)^{\frac{1}{2}} \tan \frac{f}{2} - \frac{e(1-e^2)^{\frac{1}{2}} \operatorname{Sin} f}{1 + e \operatorname{Cos} f} \qquad (7)$$

The equation (3) takes the form

$$\frac{d^2\Phi}{d\tau^2} + \frac{3(B-A)}{2C} \cdot \frac{(1+e\ \cos f)^3}{(1-e^2)^3} \operatorname{Sin} 2\,(\Phi - f) = 0 \tag{8}$$

in which the tidal torque is neglected.

The solution of equation (8) may be sought in the form

$$\Phi = p\,\tau + \gamma \tag{9}$$

where p is a constant and γ is an unknown function. The resonances occur at $2p=m$ when m is an integer. We now substitute equation (9) in equation (8). Then for any integer m, the coefficients of equation (8) are τ-periodic with a period of 2π. To obtain a first order solution, the coefficients of equation (8) may be averaged over this period by holding γ fixed. Equation (8) reduces to

$$\frac{d^2\gamma}{d\tau^2} + \frac{3(B-A)}{2C}\, \Psi_{2p}\ (e)\ \operatorname{Sin} 2\gamma = 0 \tag{10}$$

where

$$\Psi_{2p}(e) = \frac{1}{\pi(1-e^2)^{3/2}} \int_0^{\pi} (1+e\ \operatorname{Cos} f)\ \operatorname{Cos}\left[m\ \tau_{(f)} - 2f\right] df \tag{11}$$

In equation (11), $\tau(f)$ is defined by equation (7). From equation (11), we have

$$\begin{aligned} m=1, &\qquad \Psi_1(e) = -\tfrac{1}{2}\,e + \ldots\ldots \\ m=2, &\qquad \Psi_2(e) = 1 - \tfrac{5}{2}\,e^2 + \ldots\ldots \\ m=3, &\qquad \Psi_3(e) = \tfrac{7}{2}\,e + \ldots\ldots \\ m=4, &\qquad \Psi_4(e) = \tfrac{17}{2}\,e^2 + \ldots\ldots \end{aligned} \tag{12}$$

$m \geq 5, \quad \Psi_m(e) = O(e^2).$

These results were obtained by Chernouško (Chernouško, 1963) and confirmed by Goldreich and Peale (Goldreich and Peale, 1966). Equation (10) or $\Psi_m(e)$ corresponds to equation (4) or $H(p,e)$ respectively in the paper by Goldreich and Peale. As pointed out by Goldreich and Peale and by Chernouško, in the procedure of derivation of equation (12) we have made two important assumptions. i.e.

(1) $2p=m$ is an integer.

(2) γ changes only slightly during each orbit (by holding γ fixed).

For the case of Mercury, $2p=3$, Equation (10) becomes

$$\frac{d^2\gamma}{d\tau^2} + \frac{21(B-A)}{4C}e \,\mathrm{Sin}2\gamma = 0 \tag{13}$$

Because the instantaneous spin rate varies at different positions along Mercury's orbit (Liu, 1970), the first assumption implies that the averaged spin rate over a period of revolution is exactly $(3/2)\,n$. As for the second assumption, γ changes slightly during each orbit only when the value $(B-A)/C$ is very small. Therefore, we are allowed to use equation (13) only when the averaged spin rate is exactly $(3/2)n$ and the value of $(B-A)/C$ is very small. In other words, if Mercury rotates with an averaged spin rate which is larger or smaller than $(3/2)n$, then $2p \neq 3$ and the value of γ will increase or decrease rapidly during each orbit. Consequently, the use of equation (13) to describe the motion of γ as a simple pendulum motion at such regimes of rotation is unjustified.

PITFALLS OF AVERAGING TECHNIQUES

The two constraints in the derivation of Equation (13) prohibit us from using equation (13) to describe the motion of γ when Mercury rotates with a spin rate in the regimes other than the resonance one. This is a fundamental mathematical principle in physics. Any mathematical exercises which violate this principle may be trapped into the pitfalls of the averaging methods. This fact can be seen from the following two examples by Goldreich and Peale (Goldreich and Peale, 1966; 1968)

As the planet's rotation slows, it will occasionally approach the 3:2 resonance spin rate. Goldreich and Peale have permitted themselves to use the following averaged equation of motion to describe the motion of γ <u>near</u> the 3:2 resonance spin rate

$$\ddot{\gamma} + \frac{3}{2} \cdot \frac{B-A}{C} n^2 H(p,e) \operatorname{Sin} 2\gamma = \langle T \rangle \qquad (14)$$

where

$$H(p,e) = \frac{7}{2} e + \ldots.$$

and $\langle T \rangle$ is a constant. The left hand sides of equation (13) and (14) are identical. By analogy with the behavior of a simple pendulum, Goldreich and Peale have argued that the resonance rotation of the planet cannot be captured near the resonance spin rate with a constant tidal torque. This argument is questionable because it may be trapped into the pitfalls of the average methods. The initial planetary spin which is faster than the resonant one corresponds to $p = (3/2) + \delta$. Since $\dot{\gamma} < 0$ corresponds to rotational rate less than the resonant one, we have $p = (3/2) - \delta$. For the cases of $p = (3/2) + \delta$ and $p = (3/2) - \delta$, the value of $2p$ is not an integer and the value of γ over an orbital period will increase or decrease rapidly. Therefore, the use of Cayley's table by Goldreich and Peale to obtain an averaged equation for investigation of the capture process is unjustified. As for evidence, the computer solutions of the resonance rotation near the resonance spin rate show that the argument made by Goldreich and Peale is completely wrong.

Furthermore, Goldreich and Peale have used the averaged equation to evaluate the capture probabilities for assumed tidal forms. According to their calculations, the capture probability is

$$P = \frac{2}{1 + \dfrac{\pi V}{2[3(B-A)\ H(p,e)/C]^{\frac{1}{2}}}} \tag{15}$$

where

$$V = p - \frac{(1 + \frac{15}{2} e^2 + \frac{45}{8} e^4 + \frac{5}{16} e^6)}{(1 + 3e^2 + \frac{3}{8} e^4)\ (1-e^2)^{3/2}}$$

For $p = 3/2$, $e = 0.206$, $(B-A)/C \cong 10^{-5}$, the capture probability is about only 0.02. It should be pointed out that this expression is illegal because it violated a basic mathematical principale in physics. As for further evidence, the computer solutions of Mercury's resonance rotation disparage the theories of capture probabilities by showing that the capture process of resonance rotation is not a probabilistic affair.

CONCLUSION

The computer solutions of Mercury's resonance rotation disprove the theories of capture probabilities. The capture process of Mercury's resonance rotation is not a probabilistic affair. We have shown that the values of capture probabilities have been buried in the pitfalls of the averaging methods and that the problem of capture probabilities for resonance rotation does not exist.

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TABLE 1. Values of $\gamma = \Phi - \frac{3}{2} M$ at successive perihelion for (B-A)/C $= 5 \times 10^{-5}$ and initial spin rates $\frac{d\Phi}{dt}\Big|_{t=0} = 1.4986\,9n$ and $\frac{d\Phi}{dt}\Big|_{t=0} = 1.50180n$ with assumed constant tidal torque $q = 10^{-11}$ and $q = 10^{-7}$.

f	γ rad.			
	$\frac{d\Phi}{dt}\Big\|_{t=0} = 1.49869$ n		$\frac{d\Phi}{dt}\Big\|_{t=0} = 1.50180$ n	
2π	$q = 10^{-11}$	$q = 10^{-7}$	$q = 10^{-11}$	$q = 10^{-7}$
1	-0.0084	-0.0084	0.0111	0.0111
2	-0.0167	-0.0167	0.0222	0.0222
3	-0.0251	-0.0251	0.0331	0.0331
4	-0.0333	-0.0333	0.0439	0.0439
5	-0.0413	-0.0414	0.0546	0.0546
6	-0.0493	-0.0494	0.0651	0.0650
7	-0.0570	-0.0571	0.0753	0.0752
8	-0.0645	-0.0646	0.0852	0.0851
9	-0.0717	-0.0719	0.0948	0.0947
10	-0.0787	-0.0789	0.1040	0.1038
11	-0.0853	-0.0856	0.1129	0.1126
12	-0.0917	-0.0920	0.1213	0.1210
13	-0.0977	-0.0981	0.1292	0.1288
14	-0.1033	-0.1037	0.1366	0.1362
15	-0.1085	-0.1090	0.1435	0.1431
16	-0.1133	-0.1138	0.1499	0.1494
17	-0.1177	-0.1183	0.1557	0.1551
18	-0.1216	-0.1222	0.1608	0.1602
19	-0.1251	-0.1258	0.1654	0.1647
20	-0.1280	-0.1288	0.1694	0.1686
21	-0.1305	-0.1313	0.1725	0.1717
22	-0.1325	-0.1334	0.1753	0.1744
23	-0.1339	-0.1349	0.1773	0.1763
24	-0.1349	-0.1359	0.1786	0.1776
25	-0.1353	-0.1364	0.1792	0.1781

TABLE 1. (Continued)

26	-0.1352	-0.1364	0.1792	0.1780
27	-0.1346	-0.1359	0.1784	0.1772
28	-0.1335	-0.1348	0.1770	0.1757
29	-0.1318	-0.1332	0.1749	0.1735
30	-0.1297	-0.1312	0.1722	0.1707
31	-0.1271	-0.1286	0.1688	0.1673
32	-0.1240	0.1255	0.1647	0.1631
33	-0.1204	-0.1220	0.1601	0.1584
34	-0.1163	-0.1180	0.1548	0.1530
35	-0.1118	-0.1136	0.1489	0.1471
36	-0.1068	-0.1087	0.1425	0.1406
37	-0.1015	-0.1034	0.1355	0.1336
38	-0.0958	-0.0977	0.1280	0.1260
39	-0.0896	-0.0916	0.1200	0.1179
40	-0.0832	-0.0852	0.1115	0.1094
41	-0.0764	-0.0785	0.1026	0.1005
42	-0.0693	-0.0714	0.0933	0.0912
43	-0.0620	-0.0641	0.0837	0.0815
44	-0.0544	-0.0566	0.0737	0.0715
45	-0.0466	-0.0488	0.0635	0.0612
46	-0.0387	-0.0409	0.0530	0.0507
47	-0.0305	-0.0328	0.0423	0.0400
48	-0.0222	-0.0245	0.0314	0.0291
49	-0.0140	-0.0163	0.0204	0.0181
50	-0.0056	-0.0079	0.0093	0.0070
51	0.0028	0.0005	-0.0018	-0.0041
52	0.0112	0.0089	-0.0128	-0.0151
53	0.0196	0.0173	-0.0239	-0.0262
54	0.0278	0.0255	-0.0349	-0.0372
55	0.0360	0.0338	-0.0457	-0.0480
56	0.0440	0.0418	-0.0563	-0.0586
57	0.0519	0.0497	-0.0668	-0.0690
58	0.0595	0.0573	-0.0769	-0.0791
59	0.0670	0.0648	-0.0868	-0.0890
60	0.0741	0.0720	-0.0963	-0.0984

TABLE 1. (Continued)

61	0.0810	0.0790	-0.1055	-0.1076
62	0.0876	0.0856	-0.1142	-0.1162
63	0.0938	0.0918	-0.1226	-0.1245
64	0.0997	0.0978	-0.1304	-0.1323
65	0.1051	0.1033	-0.1377	-0.1396
66	0.1102	0.1084	-0.1446	-0.1464
67	0.1149	0.1131	-0.1508	-0.1526
68	0.1191	0.1174	-0.1565	-0.1582
69	0.1228	0.1212	-0.1616	-0.1632
70	0.1261	0.1246	-0.1661	-0.1677
71	0.1289	0.1274	-0.1700	-0.1715
72	0.1312	0.1298	-0.1731	-0.1746
73	0.1330	0.1317	-0.1757	-0.1771
74	0.1343	0.1330	-0.1775	-0.1788
75	0.1351	0.1339	-0.1787	-0.1800
76	0.1353	0.1342	-0.1792	-0.1804
77	0.1351	0.1340	-0.1791	-0.1802
78	0.1342	0.1333	-0.1782	-0.1793
79	0.1330	0.1321	-0.1767	-0.1777
80	0.1312	0.1304	-0.1745	-0.1754
81	0.1289	0.1281	-0.1717	-0.1725
82	0.1261	0.1254	-0.1682	-0.1689
83	0.1228	0.1222	-0.1640	-0.1647
84	0.1190	0.1185	-0.1593	-0.1599
85	0.1148	0.1143	-0.1539	-0.1544
86	0.1102	0.1097	-0.1479	-0.1484
87	0.1051	0.1047	-0.1414	-0.1418
88	0.0996	0.0993	-0.1343	-0.1347
89	0.0937	0.0934	-0.1267	-0.1270
90	0.0875	0.0873	-0.1186	-0.1189

TABLE 1. (Continued).

91	0.0809	0.0807	-0.1101	-0.1103
92	0.0741	0.0739	-0.1011	-0.1013
93	0.0669	0.0668	-0.0918	-0.0920
94	0.0595	0.0594	-0.0821	-0.0822
95	0.0518	0.0517	-0.0721	-0.0722
96	0.0440	0.0439	-0.0618	-0.0619
97	0.0359	0.0359	-0.0513	-0.0513
98	0.0278	0.0278	-0.0405	-0.0405
99	0.0195	0.0195	-0.0296	-0.0296
100	0.0112	0.0112	-0.0186	-0.0186
101	0.0028	0.0028	-0.0076	-0.0076
102	-0.0056	-0.0056	0.0035	0.0035
103	-0.0140	-0.0140	0.0146	0.0146
104	-0.0224	-0.0224	0.0257	0.0257
105	-0.0306	-0.0306	0.0366	0.0366
106	-0.0387	-0.0387	0.0474	0.0474
107	-0.0467	-0.0467	0.0580	0.0579
108	-0.0545	-0.0546	0.0684	0.0683
109	-0.0620	-0.0622	0.0785	0.0784
110	-0.0693	-0.0695	0.0883	0.0882
111	-0.0765	-0.0767	0.0978	0.0976
112	-0.0832	-0.0835	0.1069	0.1067
113	-0.0896	-0.0900	0.1156	0.1153
114	-0.0958	-0.0962	0.1238	0.1235
115	-0.1015	-0.1019	0.1316	0.1312
116	-0.1069	-0.1073	0.1389	0.1385
117	-0.1118	-0.1123	0.1456	0.1451
118	-0.1163	-0.1169	0.1518	0.1513
119	-0.1204	-0.1210	0.1574	0.1568
120	-0.1240	-0.1247	0.1624	0.1617

TABLE 1. (Continued)

121	-0.1271	-0.1278	0.1667	0.1660
122	-0.1297	-0.1305	0.1705	0.1697
123	-0.1319	-0.1327	0.1736	0.1727
124	-0.1335	-0.1344	0.1760	0.1751
125	-0.1346	-0.1356	0.1778	0.1768
126	-0.1352	-0.1363	0.1789	0.1778
127	-0.1353	-0.1364	0.1793	0.1781
128	-0.1349	-0.1361	0.1790	0.1778
129	-0.1339	-0.1352	0.1780	0.1768
130	-0.1325	-0.1338	0.1764	0.1751
131	-0.1305	-0.1319	0.1741	0.1727
132	-0.1280	-0.1295	0.1712	0.1697
133	-0.1250	-0.1266	0.1676	0.1660
134	-0.2116	-0.1232	0.1633	0.1617
135	-0.1177	-0.1194	0.1584	0.1568
136	-0.1133	-0.1151	0.1530	0.1512
137	-0.1085	-0.1103	0.1469	0.1451
138	-0.1033	-0.1051	0.1403	0.1384
139	-0.0977	-0.0996	0.1331	0.1312
140	-0.0917	-0.0937	0.1255	0.1235
141	-0.0853	-0.0874	0.1173	0.1152
142	-0.0787	-0.0808	0.1087	0.1066
143	-0.0717	-0.0738	0.0997	0.0976
144	-0.0644	-0.0666	0.0903	0.0881
145	-0.0569	-0.0591	0.0805	0.0783
146	-0.0492	-0.0514	0.0705	0.0683
147	-0.0413	-0.0435	0.0601	0.0579
148	-0.0332	-0.0354	0.0496	0.0473
149	-0.0250	-0.0273	0.0388	0.0365
150	-0.0167	-0.0190	0.0279	0.0256

TABLE 1. (Continued)

151	-0.0083	-0.0106	0.0169	0.0146
152	0.0000	-0.0023	0.0058	0.0035
153	0.0085	0.0062	-0.0053	-0.0076
154	0.0168	0.0146	-0.0164	-0.0187
155	0.0251	0.0229	-0.0274	-0.0297
156	0.0333	0.0311	-0.0383	-0.0406
157	0.0414	0.0392	-0.0491	-0.0514
158	0.0493	0.0471	-0.0596	-0.0619
159	0.0570	0.0549	-0.0700	-0.0722
160	0.0645	0.0624	-0.0801	-0.0823

TABLE 2. Values of $\gamma = \Phi - \frac{3}{2} M$ at successive perihelion for $(B-A)/C = 5 \times 10^{-5}$ and initial spin rate $\frac{d\Phi}{dt}\Big|_{t=0} = 1.50965n$ with assumed constant tidal torque $q = 10^{-6}$. (0.1 million times the conventional tides).

$\frac{f}{2\pi}$	γ rad.	$\frac{f}{2\pi}$	γ rad.
1	0.060	21	0.998
2	0.120	22	1.026
3	0.180	23	1.052
4	0.239	24	1.077
5	0.297	25	1.100
6	0.353	26	1.121
7	0.409	27	1.140
8	0.463	28	1.158
9	0.516	29	1.175
10	0.566	30	1.190
11	0.615	31	1.204
12	0.663	32	1.216
13	0.708	33	1.227
14	0.751	34	1.237
15	0.792	35	1.246
16	0.831	36	1.253
17	0.869	37	1.259
18	0.904	38	1.264
19	0.937	39	1.268
20	0.969	40	1.271

TABLE 2 (Continued)

$\frac{f}{2\pi}$	γ rad.	$\frac{f}{2\pi}$	γ rad.
41	1.273	61	1.047
42	1.273	62	1.020
43	1.272	63	0.992
44	1.270	64	0.962
45	1.267	65	0.930
46	1.263	66	0.896
47	1.258	67	0.861
48	1.252	68	0.823
49	1.244	69	0.783
50	1.235	70	0.742
51	1.225	71	0.698
52	1.214	72	0.652
53	1.201	73	0.605
54	1.187	74	0.555
55	1.172	75	0.504
56	1.154	76	0.451
57	1.136	77	0.397
58	1.116	78	0.341
59	1.095	79	0.284
60	1.072	80	0.226

TABLE 2. (Continued)

$\frac{f}{2\pi}$	γ rad.	$\frac{f}{2\pi}$	γ rad.
81	0.167	101	-0,889
82	0.107	102	-0.926
83	0.047	103	-0.961
84	-0.014	104	-0.993
85	-0.074	105	-1.024
86	-0.134	106	-1.054
87	-0.194	107	-1.081
88	-0.253	108	-1.108
89	-0.311	109	-1.132
90	-0.368	110	-1.156
91	-0.424	111	-1.177
92	-0.478	112	-1.198
93	-0.531	113	-1.217
94	-0.582	114	-1.235
95	-0.632	115	-1.252
96	-0.680	116	-1.268
97	-0.725	117	-1.282
98	-0.769	118	-1.296
99	-0.811	119	-1.309
100	-0.851	120	-1.320

TABLE 2. (Continued)

$\frac{f}{2\pi}$	γ rad.	$\frac{f}{2\pi}$	γ rad.	$\frac{f}{2\pi}$	γ rad.
121	-1.331	141	-1.402	161	-1.201
122	-1.341	142	-1.399	162	-1.181
123	-1.350	143	-1.395	163	-1.160
124	-1.359	144	-1.391	164	-1.137
125	-1.367	145	-1.386	165	-1.112
126	-1.374	146	-1.381	166	-1.086
127	-1.380	147	-1.375	167	-1.059
128	-1.385	148	-1.368	168	-1.030
129	-1.390	149	-1.360	169	-0.999
130	-1.394	150	-1.352	170	-0.966
131	-1.398	151	-1.343	171	-0.932
132	-1.401	152	-1.333	172	-0.896
133	-1.403	153	-1.322	173	-0.859
134	-1.405	154	-1.311	174	-0.819
135	-1.406	155	-1.298	175	-0.777
136	-1.407	156	-1.285	176	-0.733
137	-1.407	157	-1.270	177	-0.688
138	-1.406	158	-1.254	178	-0.640
139	-1.405	159	-1.238	179	-0.592
140	-1.404	160	-1.220	180	-0.541

TABLE 2. (Continued)

f/2π	γ rad.	f/2π	γ rad.
181	-0.488	201	0.644
182	-0.434	202	0.690
183	-0.378	203	0.734
184	-0.322	204	0.776
185	-0.264	205	0.816
186	-0.205	206	0.854
187	-0.145	207	0.890
188	-0.085	208	0.924
189	-0.025	209	0.957
190	0.036	210	0.987
191	0.096	211	1.016
192	0.156	212	1.042
193	0.215	213	1.067
194	0.273	214	1.091
195	0.331	215	1.113
196	0.387	216	1.133
197	0.442	217	1.151
198	0.496	218	1.169
199	0.546	219	1.184
200	0.596	220	1.199