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PERFORMANCE OF THE CROSS-PRODUCT STEERING LAW FOR THE TRANSEARTH INJECTION PHASE

NASA

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
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PERFORMANCE OF THE CROSS-PRODUCT STEERING LAW FOR THE TRANSEARTH INJECTION PHASE

By Thomas F. Gibson, Jr. and D. E. Steele

SUMMARY

Command module and service module (CSM) performance is evaluated using the proposed steering law $\left(\underline{v}_g \times \dot{\underline{v}}_g\right)$ for the transearth transfer between a circular lunar orbit and several V_∞ vectors. The study indicates that the performance of the steering law is virtually optimum when the engine ignition occurs at the right position in the parking orbit. The study shows that a very good position for engine ignition precedes by 3.50° the position in lunar orbit which minimizes the impulsive transearth velocity increment. Using this empirical ignition point, the penalty in characteristic velocity is less than 3 ft/sec. In addition, the study shows that a gain constant may be chosen such that the thrust vector remains nearly fixed inertially during the entire transearth burn with very little performance loss (< 5 ft/sec).

INTRODUCTION

The CSM is to be steered by the cross-product equations $\left(\underline{v}_g \times \dot{\underline{v}}_g\right)$ when transferring from the lunar parking orbit to the transearth trajectory. Cut-off occurs when the CSM has acquired a hyperbolic trajectory which in a two-body sense will ultimately attain a required velocity vector at infinity $\left(\underline{V}_\infty\right)$. Numerical studies have shown that aiming for a $\underline{V}_\infty$ is almost the same as aiming for reentry conditions at earth. This means that reentry errors are not highly dependent upon the thrust ignition point or the thrust cut-off position as long as the cut-off conditions satisfy the $\underline{V}_\infty$ vector. However, performance is dependent upon the ignition point. Unless engine ignition occurs at the right position, the performance will be impaired. This paper shows how to always obtain a "good" ignition point based upon analytic expressions and an empirical constant.

Massachusetts Institute of Technology (MIT) (ref. 1) has shown that the cross-product steering equations may be altered by a gain constant so that a near constant thrust direction results during the entire burn. This paper corroborates this for a considerable range of $\underline{V}_\infty$ vectors and also indicates that the performance penalty is small.

SYMBOLS

c	gain constant in $\underline{v}_g \times \dot{\underline{v}}_g$ equations
g_o	acceleration of earth gravity (32.2 ft/sec)
i_{sp}	specific impulse of CSM engine
$\underline{i}$	unit vector along X
$\underline{j}$	unit vector along Y
$\underline{k}$	unit vector along Z
L	angle $\underline{V}_\infty$ makes with orbit plane
m	CSM mass
$\underline{r}$	CSM position vector
r	magnitude of $\underline{r}$
$\underline{T}$	CSM thrust vector
T	magnitude of $\underline{T}$
t_b	time between ignition and cut-off
t_{togo}	time to cut-off
$\underline{V}_\infty$	velocity vector at infinity
V_∞	magnitude of $\underline{V}_\infty$

$\underline{v}$ CSM velocity vector

v magnitude of $\underline{v}$

v_c characteristic velocity associated with burn time t_b

$\underline{v}_r$ velocity vector required at position $\underline{r}$ to attain $\underline{V}_\infty$

v_r magnitude of $\underline{v}_r$

W_o CSM initial weight

X, Y, Z coordinate axes (defined in fig. 1)

$\Delta\underline{v}$ impulsive velocity increment vector

Δv magnitude of $\Delta\underline{v}$

$\Delta\theta$ largest angle that the thrust direction during burn makes with the initial thrust direction

μ gravitational parameter of moon

π pi

ω angle denoting thrust initiation position in orbit

ω_{opti} optimum position for engine ignition for impulsive transfer

ω_{optt} optimum position for engine ignition for $\underline{v}_g \times \dot{\underline{v}}_g$ thrust transfer

ω_p position for impulsive pericenter transfer

DISCUSSION

The CSM is considered to be in a circular orbit about the moon. The coordinate system used to define its state vector is shown in figure 1. This system has the origin at the moon's center and the +Z axis normal to the CSM orbit plane, and is the direction of $\underline{r} \times \underline{v}$. The +Y axis is parallel to $\left(\underline{r} \times \underline{v}\right) \times \underline{V}_\infty$ and the +X axis follows from the right hand rule. In this system $\underline{V}_\infty$ always is in the X-Z plane and is defined by its magnitude V_∞ and the angle L it makes with the orbit plane.

$$\underline{V}_\infty = V_\infty \left(\cos L\underline{i} + o\underline{j} + \sin L\underline{k}\right) \tag{1}$$

With the CSM orbit being circular, the position and velocity of the CSM is determined by the angle ω.

$$\underline{r} = r\left(\cos \omega\underline{i} + \sin \omega\underline{j} + o\underline{k}\right) \tag{2}$$

$$\underline{v} = v\left(-\sin \omega\underline{i} + \cos \omega\underline{j} + o\underline{k}\right) \tag{3}$$

where

$$v = \left(\frac{\mu}{r}\right)^{\frac{1}{2}}$$

Impulsive Transfer

The velocity required $\left(\underline{v}_r\right)$ at position $\underline{r}$ for a hyperbolic orbit which ultimately attains $\underline{V}_\infty$ is given by equation (4). The derivation of this equation may be found in reference 2.

$$\underline{v}_r = \frac{V_\infty}{2}\left[(A + 1)\underline{1}_\infty + (A - 1)\underline{1}_r\right] \tag{4}$$

where

$$\underline{1}_\infty = \frac{\underline{V}_\infty}{V_\infty} \qquad \underline{1}_r = \frac{\underline{r}}{r}$$

$$A = \left[1 + \frac{4\mu}{1 + \left(1 + \underline{1}_\infty \cdot \underline{1}_r\right)} \right]^{\frac{1}{2}} \tag{5}$$

The magnitude of $\underline{v}_r$ is given by equation (6).

$$v_r = \left(\frac{2\mu}{r} + V_\infty^{\,2} \right)^{\frac{1}{2}} \tag{6}$$

Note that v_r depends only on the magnitude of $\underline{r}$.

The impulsive velocity increment $\Delta\underline{v}$ to be added to $\underline{v}$ for the transfer is simply:

$$\Delta\underline{v} = \underline{v}_r - \underline{v} \tag{7}$$

When $\underline{V}_\infty$ and r are specified, $\Delta\underline{v}$ depends on ω. Since v_r and v are independent of ω, Δv is a function of the angle α between $\underline{v}$ and $\underline{v}_r$. Hence, the minimum value of Δv will occur at the value of ω which minimizes the angle α. The angle α can be made 0 when $\underline{V}_\infty$ lies in the CSM orbit plane $(L = 0)$. In this case, ω is chosen for a pericenter transfer and $\underline{v}_r$ is tangential. For this in-plane transfer the optimum point for transfer $\left(\omega_{opti}\right)$ is given by equation (8).

$$\omega_{opti} = \omega_p = \cos^{-1}\left(\frac{1}{1 + \dfrac{rV_\infty^{\,2}}{\mu}} \right) + \pi \tag{8}$$

where

$$\omega_p^{\,1} = \omega_p - \pi$$

In the more general case where $\underline{V}_\infty$ does not lie in the orbit plane, $(L \neq 0)$, α is always greater than 0. A closed form expressed for ω_{opti} has not been found for this case, hence ω_{opti} must be found numerically.

It can be shown simply that ω_{opti} always lies between π and $\frac{3}{2}\pi$. Moreover, when $|L| > \omega_{\rho}^{1}$, ω_{opti} will lie between $+\omega_p$ and $\frac{3}{2}\pi$. For this study ω_{opti} was obtained by computing five values of Δv from five values of ω in the interval containing ω_{opti}. A fourth order curve was fit to the data and ω_{opti} was selected on the basis of the curve fit. This method proved satisfactory.

$\underline{v}_g \times \dot{\underline{v}}_g$ Transfer

The CSM will steer out of lunar orbit by the $\underline{v}_g \times \dot{\underline{v}}_g$ scheme. This scheme is derived and discussed in references 1 and 2. A brief description is given here. The thrust vector $\underline{T}$ is pointed throughout the trajectory so that it satisfies equation (9).

$$\left(c\underline{b} - \underline{a}_T\right) \times \underline{v}_g = 0 \qquad (9)$$

where

$$\underline{b} = \frac{d\underline{v}_r}{dt} - \frac{\mu}{r^3}\underline{r}$$

$$\underline{a}_T = \frac{\underline{T}}{m}$$

$$\underline{v}_g = \underline{v}_r - \underline{v}$$

$$\underline{1}_g = \frac{\underline{v}_g}{v_g}$$

$$\frac{d\underline{v}_r}{dt} = \frac{V_\infty}{2r}(A - 1)\left(\underline{1}_r \cdot \underline{1}_g\right)\underline{1}_r + \frac{\mu}{rV_\infty^{\,2}\left(1 + \underline{1}_r \cdot \underline{1}_\infty\right)^2 A}\left[\left(\underline{1}_\infty + \underline{1}_r\right) \cdot \underline{v}_g\right]\left(\underline{1}_\infty + \underline{1}_r\right)$$

Equation (9) essentially says that the thrust vector is alined so that the difference between the total CSM acceleration (thrust and gravity) and the rate of change of the required velocity is parallel to the velocity error.

Equation (9) becomes undeterminable as $\underline{v}_g$ tends to 0, hence cut-off must be handled with care. The cut-off scheme used for the study has the time-to-go calculated at the beginning of each integration step by equation (10).

$$t_{togo} = \frac{v_g}{\left| \underline{a}_T + \frac{\mu}{r^3} \underline{r} \right|} \tag{10}$$

When time-to-go is less than 1.5 seconds, the thrust direction is frozen to the value used for previous integration step and one step of size time-to-go is taken. The cut-off error using the scheme has always been less than 0.1 ft/sec.

The $\underline{v}_g \times \dot{\underline{v}}_g$ scheme has a singularity when $\underline{r}$ is 180° away from $\underline{V}_\infty$. Fortunately, the optimum or near-optimum thrust phases do not include this singular point for the presently designed Apollo mission and CSM.

PERFORMANCE EVALUATION

The $\underline{v}_g \times \dot{\underline{v}}_g$ steering law will guide the CSM towards the desired $\underline{V}_\infty$ from any thrust initiation position in the lunar orbit. It will attain the proper cut-off conditions whenever it has enough fuel. Starting the thrusting phases at different positions in the orbit results in different fuel consumptions. For lunar missions, it is desirable to have engine ignition at a position which minimizes the fuel consumption. The performance of the $\underline{v}_g \times \dot{\underline{v}}_g$ steering law is evaluated by comparing its characteristic velocity with that velocity increment of the impulsive transfer. This is the same as comparing their fuel consumptions. The characteristic velocity v_c associated with this steering law, or any law, depends only upon the CSM parameters and the burn time $\left(t_b\right)$.

$$v_c = i_{sp}\, g_o \log \left[\frac{1}{1 - \frac{T\, t_b}{\left(W_o\, i_{sp}\right)}} \right] \tag{11}$$

Impulsive and $\underline{v}_g \times \dot{\underline{v}}_g$ transfers are made from various positions (different values of ω) in the lunar orbit. Parabolic shaped curves result when v_c and Δv are plotted against ω. The minimum value for the impulsive curve represents a characteristic value unattainable by any finite thrusting scheme. Therefore, the difference between the $\underline{v}_g \times \dot{\underline{v}}_g$ characteristic velocity and the impulsive minimum is indicative of its performance.

RESULTS

The transearth transfers are made from a circular orbit with a radius of 6 000 000 ft. This value for the radius represents a lower limit for the CSM orbit altitude, hence it was chosen because the gravity losses would be the largest. The transfers are made from the orbit to $\underline{V}_\infty$ magnitudes of 2000, 3000, and 4000 ft/sec. The $\underline{V}_\infty$ vectors make angles of 0°, 10°, and 20° with the orbit plane. The combination of these V_∞ angles and magnitudes span the set of $\underline{V}_\infty$ vectors for the Apollo mission. A V_∞ magnitude of 2000 ft/sec is slightly lower than the value needed for the minimum energy transearth trajectory (flight time $\sim$ 120 hr) while a V_∞ magnitude of 4000 ft/sec exceeds the value needed for the high energy return (flight time $\sim$ 60 hr). The plane change angle of 20° requires a velocity increment which exceeds the Δv budget for the CSM.

The impulsive and $\underline{v}_g \times \dot{\underline{v}}_g$ transfer velocity requirements are presented in figure 2 as function of engine ignition position (ω). The solid curves denote the $\underline{v}_g \times \dot{\underline{v}}_g$ performance when engine ignition occurs at ω, while the dotted curve denoted the impulsive transfer at ω. Figure 2(a) shows the performance for transfers to three vectors, which have a constant magnitude of 2000 ft/sec. The figure shows that both the impulsive and cross-product velocity requirements have similar parabolic shapes which have nearly the same minimum values. This verifies that the $\underline{v}_g \times \dot{\underline{v}}_g$ law is very near optimum. However, the minimum values occur at different ignition points. The curves connecting the minimum points for the two modes are separated by a constant value of 3° in the direction of ω. The figures also show that the velocity curves are "flat" in the region of the minimum. Also plotted on the graphs is the loci of pericenter (ω_p) impulsive transfers. Note that the position is not near the minimum position for large plane change. Figures 2(b) and 2(c) convey the same trends for V_∞ vectors with 3000 and 4000 ft/sec magnitude. However, the average difference between the loci of optimum ignition points increases to 3.5° for the 3000 ft/sec V_∞ and 4° for the 4000 ft/sec V_∞. Since the curves are flat over several degrees of ignition position, a very good ignition position may be obtained by initiating thrust 3.5° ahead of the optimum impulsive position. Using this ignition point, the performance penalty is less than 3 ft/sec.

All of the above results had the gain constant c in the $\underline{v}_g \times \dot{\underline{v}}_g$ equation set equal to 1.0. Figures 3 and 4 give results for variations in c between 0 and 1.0. Engine ignition occurs at the c = 1 optimum position $\left(\omega_{optt}\right)$. Time histories of the thrust direction from ignition to cut-off was computed for the transfers. The largest angle between the initial thrust direction and any subsequent thrust direction was recorded. This largest angle is denoted by $\Delta\theta$. Thus the thrust vector during the burn always lies within a cone of apex angle $\Delta\theta$, centered along the initial thrust direction. Figure 3 shows the variation of $\Delta\theta$ with c. The figure shows that $\Delta\theta$ can always be minimized to a small angle for small plane change angles. Even in the worst case where V_∞ equals 2000 ft/sec and the angle out of plane is 20°, $\Delta\theta$ may be kept to 4°. Figure 3 indicates that nearly constant thrust direction may be obtained for any of the $\underline{V}_\infty$'s. In all cases the value near 0.5 for c minimizes $\Delta\theta$.

The performance penalty for varying c is shown in figure 4. This figure shows that the performance penalty for using a c value which minimizes $\Delta\theta$ to be always less than 5 ft/sec.

CONCLUDING REMARKS

The MIT $\underline{v}_g \times \dot{\underline{v}}_g$ steering law is very nearly optimum for transfers to any Apollo V_∞ vectors when the ignition is in the right region. A gain constant can be found for any transfers which will minimize the thrust vector direction variation to less than 4° during the entire burn with practically no performance loss. The gain constant value will be near 0.5. A near optimum ignition position can be obtained by starting the engine 3.5° prior to the orbit position for optimum impulsive transfer.

REFERENCES

1. Copps, E. M. Jr.: Powered Flight Phases of CSM. SGA Memo No. 13-64 (Revision 1).

2. Battin, R. H.: Astronautical Guidance. McGraw-Hill, New York, 1964

TABLE I. - CSM CHARACTERISTICS AT ENGINE IGNITION

W_o	=	30 000 lbs
i_{sp}	=	313 sec
T	=	20 000 lbs

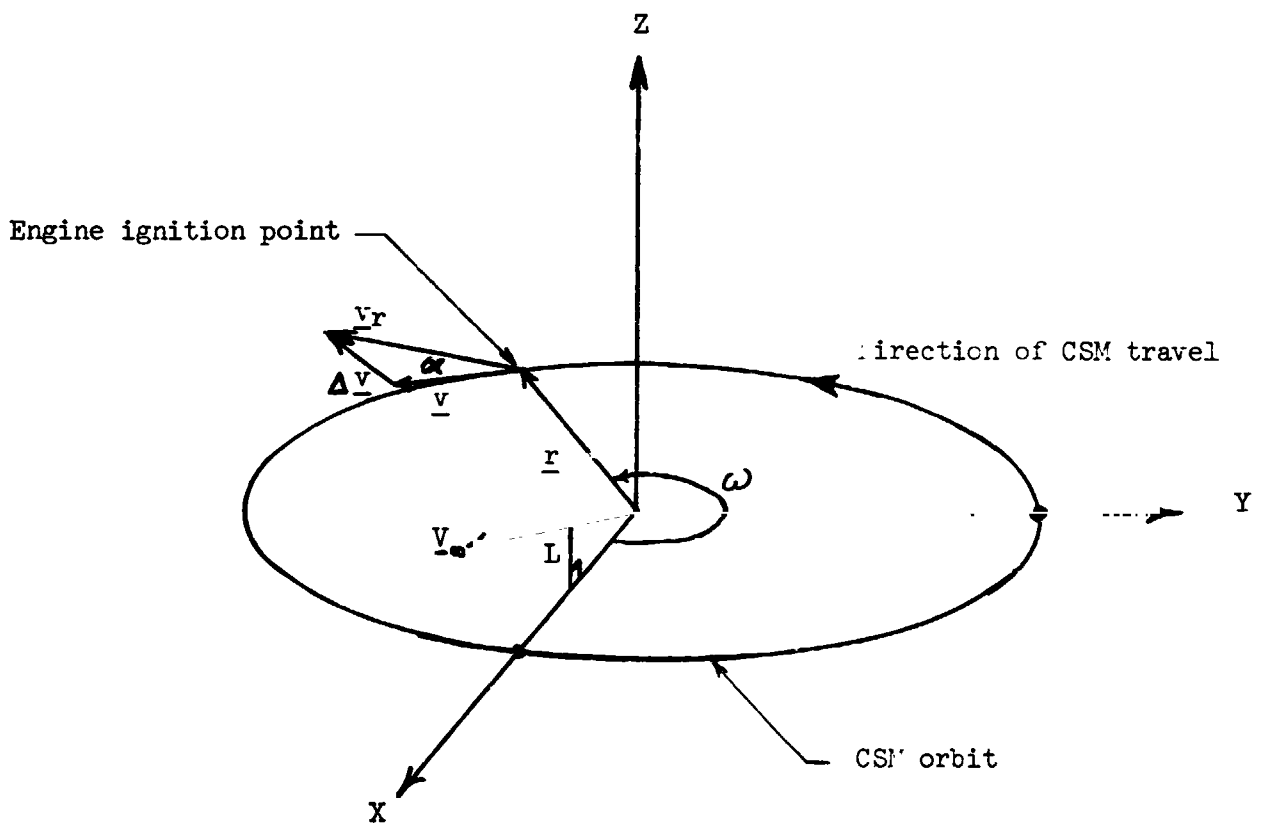
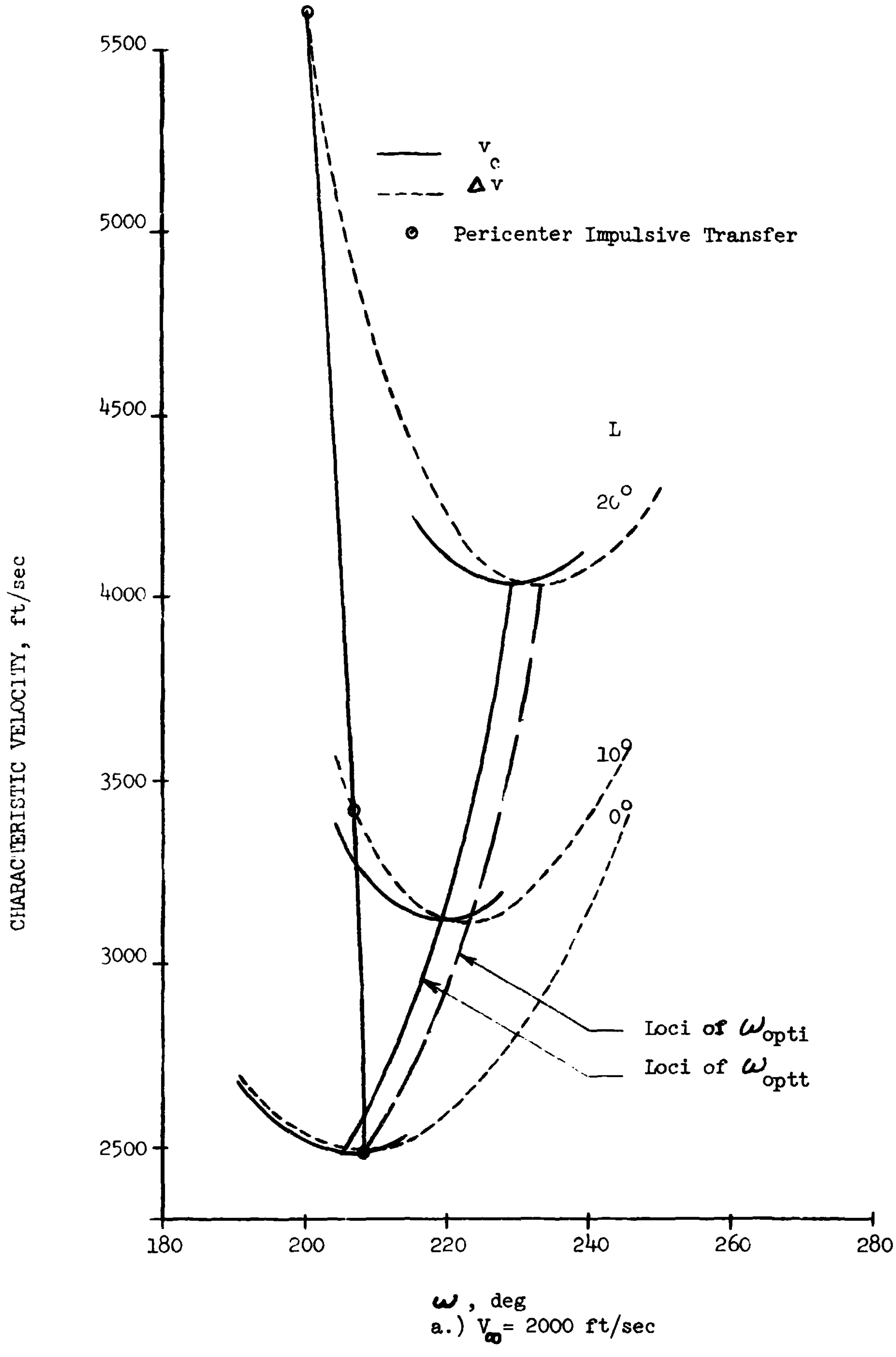
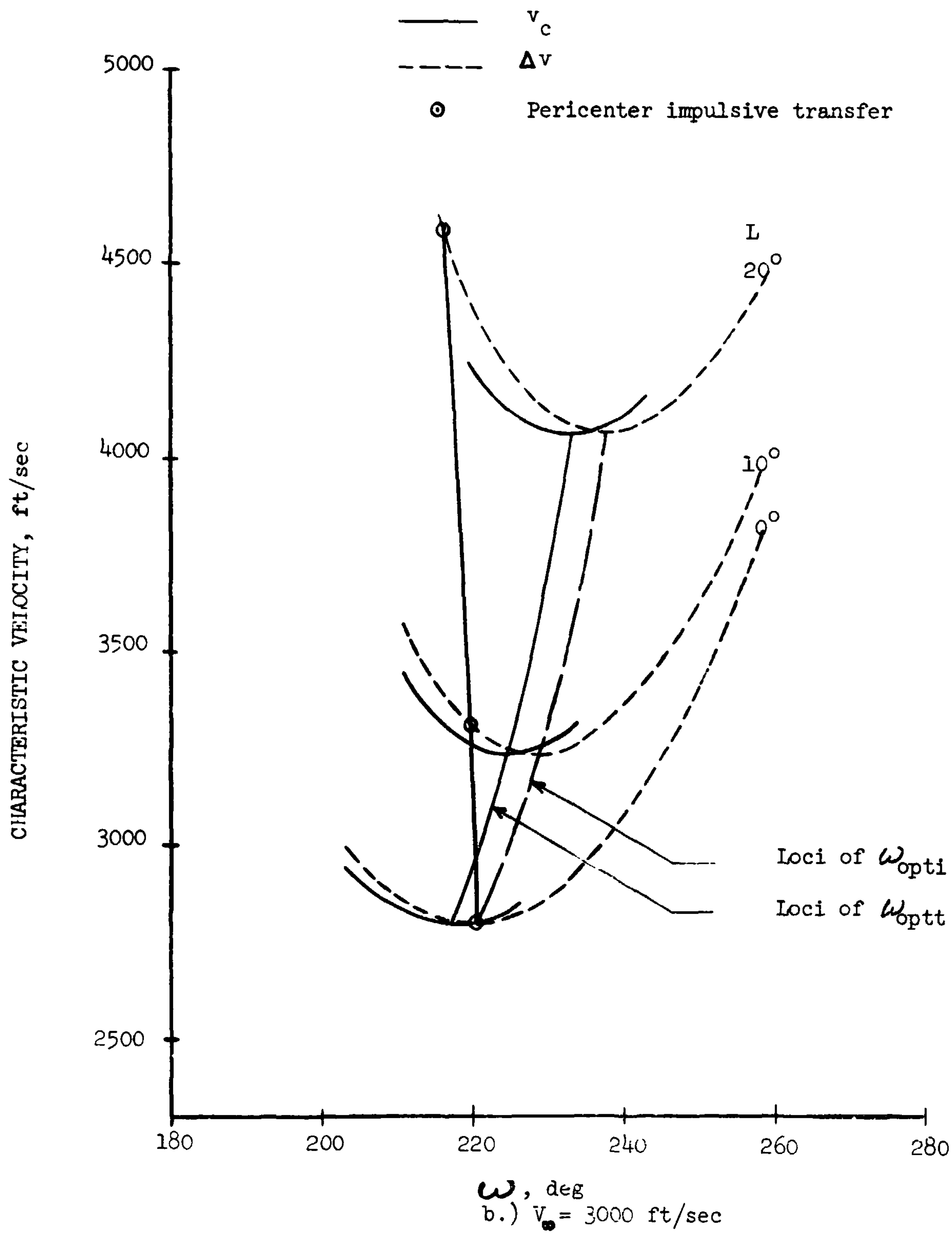


Figure 1.- CSM state vector at thrust ignition.



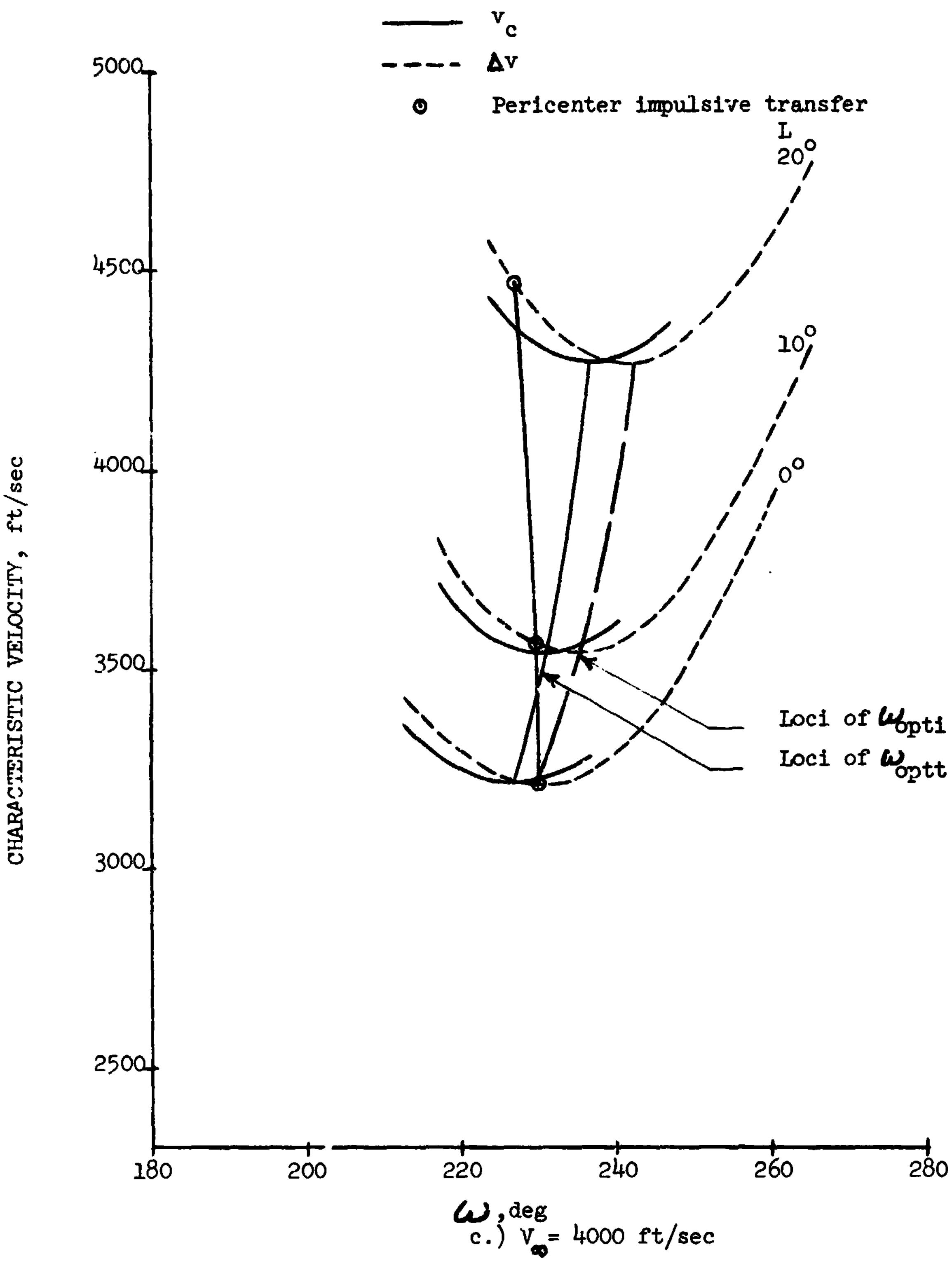
(a) V_∞ = 2000 ft/sec

Figure 2.- Characteristic velocity for impulsive and thrusting transearth transfers as a function of thrust initiation.



(b) V_∞ = 3000 ft/sec

Figure 2.- Continued



(c) V_∞ = 4000 ft/sec

Figure 2.- Concluded

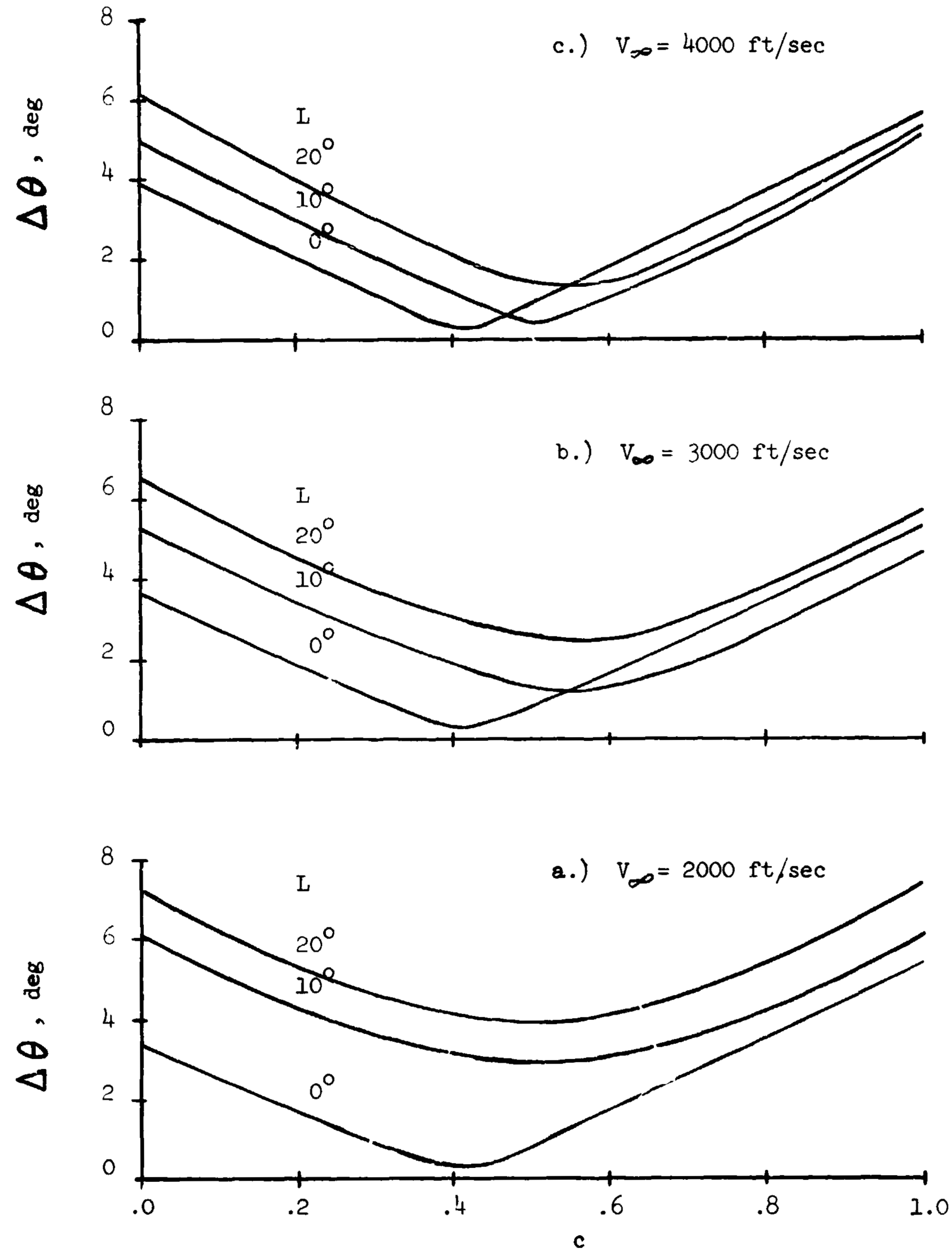


Figure 3.- Maximum variation of thrust direction from initial thrust direction during burn (Δθ) as a function of gain constant c.

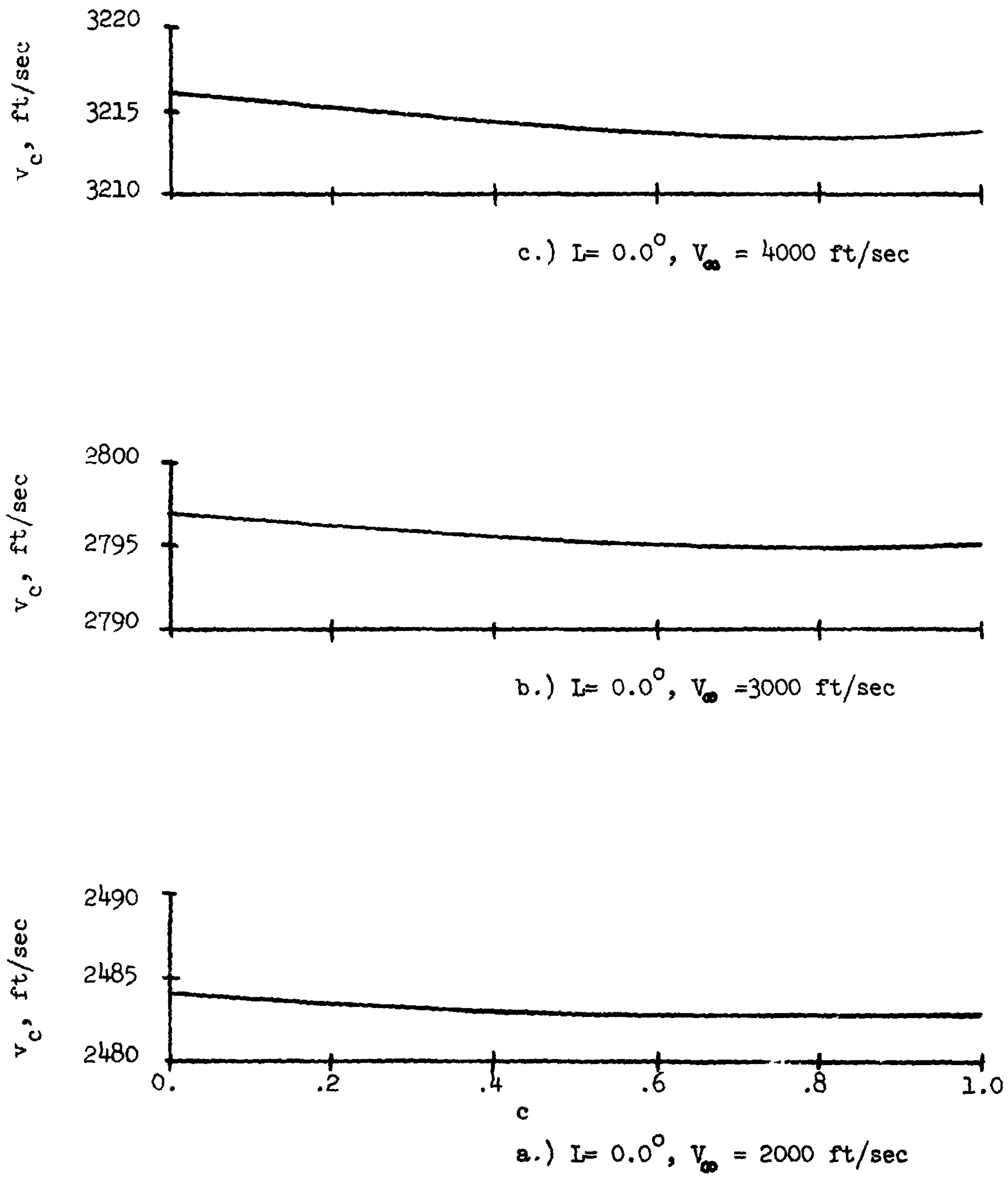
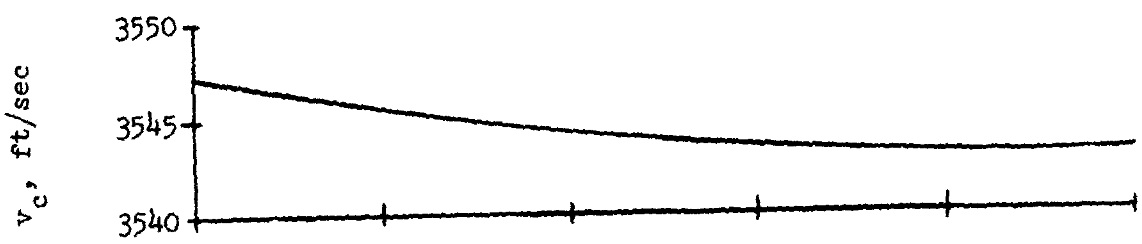
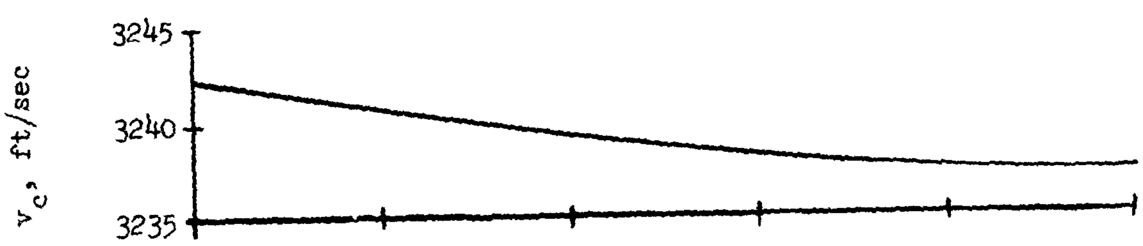


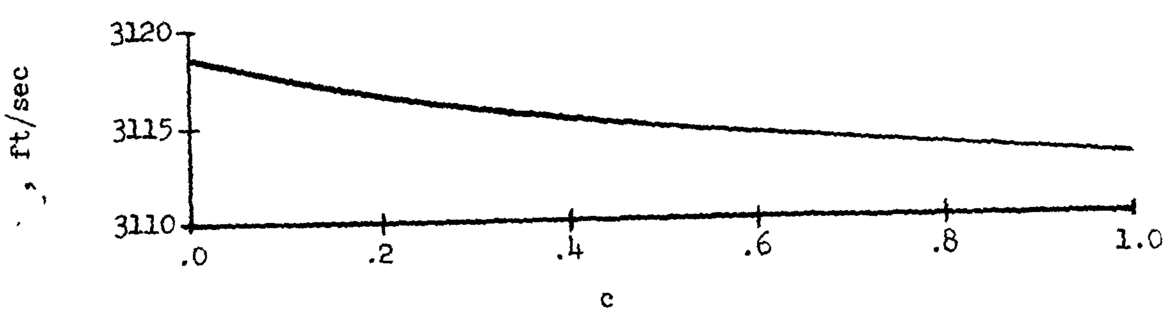
Figure 4.- Characteristic velocity variation versus gain constant c.



f.) L=10.°, V_∞ = 4000 ft/sec



e.) L= 10.°, V_∞ = 3000 ft/sec



d.) L= 10.°, V_∞ = 2000 ft/sec

Figure 4.- Continued

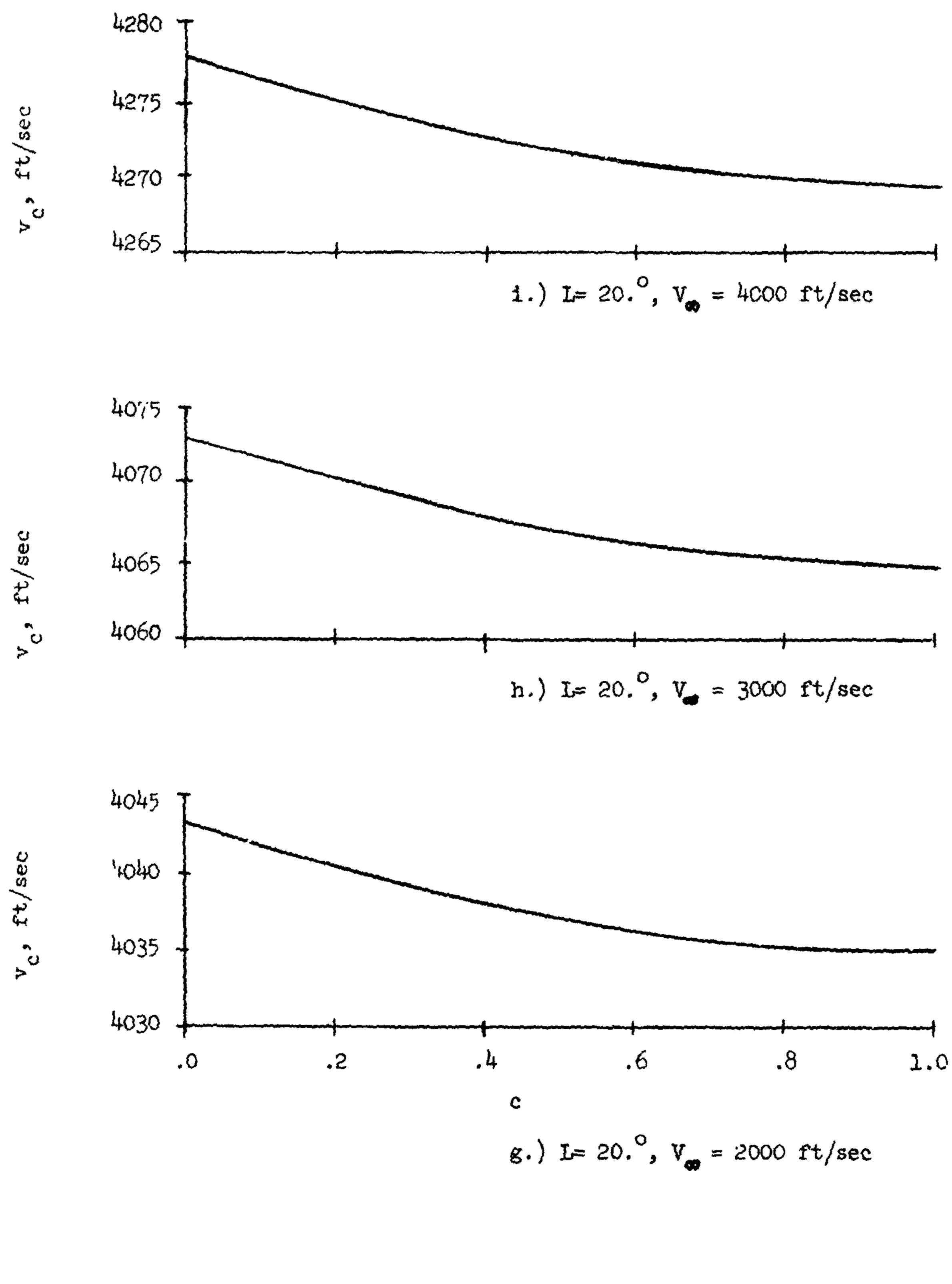


Figure 4.- Concluded