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THE CANONIZATION OF NICE VARIABLES

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The Canonization of Nice Variables

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Abstract

In a perturbed periodic classical motion the angle variable may be eliminated either by Kruskal's transformation to "nice variables" or — if the system is canonical — by the Poincaré-Von Zeipel method. For systems that possess a Hamiltonian, the present work (1) shows that Kruskal's transformation may be made canonical order by order, (2) derives a practical formula for achieving this result and (3) shows that the two methods are equivalent and may be matched order by order.

INTRODUCTION

A basic perturbation problem in celestial mechanics and in guiding center motion involves a set of n first-order differential equations which can be represented vectorially as

$$d\underline{x}/dt = \underline{F}(\underline{x}) \qquad (1)$$

and which has the following properties:

(1) $\underline{F}(\underline{x})$ depends on a small parameter $\varepsilon \ll 1$ and may be expanded in it:

$$\underline{F}(\underline{x}) = \underline{F}^{(0)}(\underline{x}) + \varepsilon \underline{F}^{(1)}(\underline{x}) + \ldots \qquad (2)$$

(2) In the limit $\varepsilon \to 0$ ("unperturbed case") the system may be solved and its solution is then periodic in time.

The problem is then to find an approximate solution valid for small but finite ε and useful for time intervals of the order of ε^{-1} periods.

Kruskal[1] devised a method for achieving this, which in many ways resembles the method of Bogoliubov and Zubarev and of Krylov and Bogoliubov[2]. The calculation in this case proceeds in two steps. First, a transformation to "intermediate variables" $\underline{y}(\underline{x})$ is performed, such that in the limit $\varepsilon \to 0$, y_n is an angle variable linear in time, while the remaining (n-1) components of $\underline{y}$ (which we shall collectively denote by $\tilde{\underline{y}}$) are constant. The equations according to which $\underline{y}$ evolves, derived from eq. (1), then have the form

$$d\underline{y}/dt = \sum_{k=0} \varepsilon^k \underline{g}^{(k)}(\underline{y}) \qquad (3)$$

In these equations, y_n appears in the $\underline{g}^{(k)}$ only as the angle argument of periodical functions and $\underline{g}^{(0)}$ has only one non-zero components, the last one[3]

$$\underline{g}^{(0)} = (0, 0, \ldots 0, 1) \tag{4}$$

The second step involves a near-identity transformation to "nice variables" $\underline{z}$

$$\underline{z} = \underline{y} + \sum_{k=1} \varepsilon^k \underline{\zeta}^{(k)}(\underline{y}) \tag{5}$$

such that in the new equations of evolution the transformed angle variable z_n no longer appears on the right-hand side

$$d\underline{z}/dt = \sum_{k=0} \varepsilon^k \underline{h}^{(k)}(\tilde{\underline{z}}) \tag{6}$$

where as before

$$\underline{z} = (\tilde{\underline{z}}, z_n)$$

and

$$\underline{h}^{(0)} = \underline{g}^{(0)} \tag{7}$$

A general recursion scheme for deriving $\underline{z}$ order by order has been described in an article by Stern[4] (henceforth referred to as I) and we shall follow the notation introduced there, which differs slightly from Kruskal's (a similar scheme for the related Krylov-Bogoliubov expansion has been given by Musen[5]). After equations (6) have been derived in this fashion, their first (n-1) components constitute an autonomous system for deriving the components of $\tilde{\underline{z}}$, which can then be independently solved. The problem is thus reduced to one with (n-1) variables.

If the system furthermore possesses a Hamiltonian, an additional variable may be eliminated by deriving a constant J of the motion which Kruskal terms the adiabatic invariant. It is defined as

$$J = \oint \underline{p} \cdot d\underline{q} = \int_0^1 \underline{p} \cdot (\partial \underline{q} / \partial z_n) \, dz_n \qquad (8)$$

with the integration performed using an arbitrary canonical set $(\underline{p}, \underline{q})$, over a group of points ("ring") all of which evolve according to (6) and possess the same $\underline{z}$ but with values of z_n that cover the full range for that variable (this property, if once established, is maintained throughout the ring's evolution in time).

If $\underline{z}$ itself forms a canonical set, with z_1 the momentum conjugate to z_n, then one may use this set in (8), leading to

$$J = \int_0^1 z_1 \, dz_n = z_1 \qquad (9)$$

which is a great simplification (if this is not the case, one is forced to retrace the transformations $\underline{x} \to \underline{y} \to \underline{z}$ until some canonical set, with which (8) can be evaluated, is reached). Kruskal did not derive nice canonical variables, but he showed them to be possible. Specifically, he proved that in any "nice" set, the following Poisson bracket relations are always satisfied

$$(\partial / \partial z_n) \left[z_i, z_j \right] = 0 \qquad (10)$$

$$\left[z_i , J \right] = \delta_{in} \qquad (11)$$

Following Nordheim and Fues[6], Kruskal then shows that J and z_n may be augmented by $(n-2)$ functions of the z_i to form a complete canonical set.

In what follows we shall assume that the intermediate variables y_i form a canonical set, in which case a near-identity transformation like (5) can, in principle, lead to nice canonical variables. This assumption is not unreasonable since, if a Hamiltonian for the system is known, such y_i can in general be derived by solving the unperturbed motion via the Hamilton-Jacobi method.

We then

(1) Show that the freedom allowed by Kruskal's method in the derivation of each order of eq. (5) is sufficient to assure the canonical character of the "nice variables" z_i .

(2) Derive a method for obtaining such z_i .

(3) Show that the result is equivalent to what is obtained by conventional perturbation methods based on the Hamilton-Jacobi equation.

THE POSSIBILITY OF STEP-BY-STEP DERIVATION

If $\underline{y}$ is canonical

$$\underline{y} = (\underline{p}, \underline{q})$$

then from (10)

$$(\partial/\partial z_n) \left\{ \left[z_i, z_j \right] - \left[y_i, y_j \right] \right\} = 0 \qquad (12)$$

If (5) is substituted here, the zero-order part cancels identically and the expression remaining inside the curly brackets separates into orders of ε and gives

$$(\partial/\partial z_n) \sum_{m=1} \varepsilon^m \left\{ [\zeta_i^{(m)}, y_j] - [\zeta_j^{(m)}, y_i] + \sum_{s=1}^{m-1} [\zeta_i^{(s)}, \zeta_j^{(m-s)}] \right\} = 0 \tag{13}$$

This condition is satisfied for <u>any</u> "nice" set $\underline{z}$. If $\underline{z}$ is not merely nice but also canonical, then the expressions in the curly brackets of both (12) and (13) are not only independent of z_n but actually vanish.

Let us now assume that the expansion (5) has already been derived and brought to canonical form, up to and including order (k-1). Then the first (k-1) orders of (13) do, in fact, vanish, leaving (after division by ε^k)

$$(\partial/\partial z_n) \left\{ [\zeta_i^{(k)}, y_j] - [\zeta_j^{(k)}, y_i] + \sum_{s=1}^{k-1} [\zeta_i^{(s)}, \zeta_j^{(k-s)}] \right\} + 0(\varepsilon) = 0 \tag{14}$$

As the next step, one may derive $\underline{\zeta}^{(k)}$ and thus extend the calculation of $\underline{z}$ one more order. The equation satisfied by $\underline{\zeta}^{(k)}$ is (I , eq. 15)

$$\partial \underline{\zeta}^{(k)}/\partial y_n - \underline{h}^{(k)}(\tilde{\underline{y}}) = \underline{\lambda}^{(k)} \tag{15}$$

where $\underline{\lambda}^{(k)}$ depends only on lower orders of $\underline{\zeta}^{(m)}$ and $\underline{h}^{(m)}$, assumed to be known at this stage. If $\langle \underline{\lambda}^{(k)} \rangle$ denotes the result of averaging over the angle variable y_n

$$\langle \underline{\lambda}^{(k)} \rangle = \int_0^1 \underline{\lambda}^{(k)}(\underline{y})\, dy_n \tag{16}$$

one gets (see I, eq. 19)

$$\underline{\zeta}^{(k)} = \int_0^{y_n} \left(\underline{\lambda}^{(k)} - \langle \underline{\lambda}^{(k)} \rangle \right) dy_n' + \underline{\mu}^{(k)}(\tilde{\underline{y}})$$

$$= \hat{\underline{\zeta}}^{(k)} + \underline{\mu}^{(k)}(\tilde{\underline{y}}) \qquad (17)$$

Here $\underline{\mu}^{(k)}$ is an arbitrary additive vector independent of y_n, allowed by the fact that the derivative of $\underline{\zeta}^{(k)}$ which enters (15) is unaffected by such an addition. The question now arises whether $\underline{\mu}^{(k)}$ may be selected so as to make the expansion (5) canonical to order k.

If this occurs, then the O(1) part of (14) must vanish. It helps here to introduce the concept of the conjugate vectors $\bar{\underline{y}}$ (I, eq. 4)

$$\bar{\underline{y}} = (\underline{q} , -\underline{p}) \qquad (18)$$

Then

$$\left[\zeta_i^{(m)}, y_j\right] - \left[\zeta_j^{(m)}, y_i\right] = \partial\zeta_i^{(m)}/\partial\bar{y}_j - \partial\zeta_j^{(m)}/\partial\bar{y}_i \qquad (19)$$

This has the form of a component of a curl dyadic in $\bar{\underline{y}}$ space, which one may call the conjugate curl, with components denoted by

$$(\bar{\nabla} \times \underline{\zeta}^{(m)})_{ij} \qquad (20)$$

Now it has been shown in an article by Stern[7] (henceforth referred to as II) that the general condition for an expansion (5) to be canonical is

$$\underline{\zeta}^{(m)} = \underline{f}^{(m)}(\underline{\zeta}) + \bar{\nabla}\chi^{(m)} \qquad (21)$$

$$(m = 1, 2, \dots)$$

where $\bar{\nabla}$ is a gradient operator in $\bar{\underline{y}}$ space, $\chi^{(m)}$ is an arbitrary scalar and $\underline{f}^{(m)}(\underline{\zeta})$ are vectors of a certain form, depending on orders of $\underline{\zeta}$ lower than the m-th and on their derivatives. Various choices of $\underline{f}^{(m)}$ are derived in II , all of which satisfy the identity (for canonical $\underline{\zeta}^{(s)}$)

$$(\bar{\nabla} \times \underline{f}^{(m)})_{ij} = - \sum_{s=1}^{m-1} [\zeta_i^{(s)}, \zeta_j^{(m-s)}] \tag{22}$$

Because the $\underline{\zeta}^{(s)}$ in this case are known to be canonical up to and including order (k-1), eq. (22) will hold for the lowest (k-1) values of m ; however, it will also hold for $m = k$, for even then the orders of $\underline{\zeta}$ appearing in the equation are all lower than the k-th. Substitution in (14) then yields

$$(\partial / \partial z_n) \left\{ \bar{\nabla} \times \left[\underline{\zeta}^{(k)} - \underline{f}^{(k)}(\underline{\zeta}) \right] \right\} + 0(\varepsilon) = 0 \tag{23}$$

Now let the transformation inverse to (5) be

$$y = \underline{z} + \sum_{s=1} \varepsilon^s \underline{\eta}^{(s)}(\underline{z}) \tag{24}$$

Given the expansion (5), the $\underline{\eta}^{(k)}$ may easily be derived (see appendix). Alternatively, they may be directly obtained from expanding the relation between (3) and (6), in a manner similar to what has been done for the expansion (5) in I . This is essentially the method of Krylov and Bogoliubov[2], as expanded by Musen[5] . Indeed, the elimination of the angle variable by Kruskal's method so resembles the Krylov-Bogoliubov approach that the two ought perhaps to be regarded as one method (which could be called the Krylov-Bogoliubov-Kruskal method; however, Kruskal's work proceeds past the elimination to the derivation of adiabatic invariants).

Since the inner part of (23) is a function of $\underline{y}$, one must transform

$$\partial/\partial z_n = \sum_i (\partial y_i/\partial z_n) \partial/\partial y_i$$

$$= \sum_i \left\{ \delta_{in} + \sum_s \varepsilon^s (\partial \eta_i^{(s)}/\partial z_n) \right\} \partial/\partial y_i \qquad (25)$$

or

$$\partial/\partial z_n = \partial/\partial y_n + 0(\varepsilon) \qquad (26)$$

Equation (23) thus becomes

$$\partial/\partial y_n \left\{ \bar{\nabla} \times \left[\underline{\zeta}^{(k)} - \underline{f}^{(k)}(\underline{\zeta}) \right] \right\} + 0(\varepsilon) = 0 \qquad (27)$$

One may now assume that all variables in (27) are expressed in terms of $\underline{y}$; in that case, each order in ε vanishes separately, including the zeroth. Because differential operators commute, this means

$$\bar{\nabla} \times \left\{ (\partial/\partial y_n) \left[\underline{\zeta}^{(k)} - \underline{f}^{(k)}(\underline{\zeta}) \right] \right\} = 0 \qquad (28)$$

This integrates to

$$(\partial/\partial y_n) \left[\underline{\zeta}^{(k)} - \underline{f}^{(k)}(\underline{\zeta}) \right] = \bar{\nabla}\Psi \qquad (29)$$

$$\underline{\zeta}^{(k)} - \underline{f}^{(k)}(\underline{\zeta}) = \bar{\nabla}\tau + \underline{u}(\tilde{\underline{z}}) \qquad (30)$$

where τ is the indefinite integral of Ψ and $\underline{u}$ is an additive function independent of z_n , allowed by the integration.

The above condition is satisfied by <u>any</u> $\underline{\zeta}^{(k)}$ belonging to a nice set of variables which is canonical to order (k-1). For instance, $\hat{\underline{\zeta}}^{(k)}$ defined in (17) will correspond to a certain τ and to a certain additive function, which may be denoted by $\hat{\underline{u}}$:

$$\hat{\underline{\zeta}}^{(k)} = \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla}\tau + \hat{\underline{u}}(\tilde{\underline{z}}) \qquad (31)$$

If one now chooses in (17)

$$\underline{\mu}^{(k)} = -\hat{\underline{u}}$$

then

$$\underline{\zeta}^{(k)} = \hat{\underline{\zeta}}^{(k)} - \hat{\underline{u}}$$

$$= \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla}\tau \qquad (32)$$

and by (21), $\underline{\zeta}^{(k)}$ satisfies the condition for canonical variables, making $\underline{z}$ canonical to order k . Thus the requirement can be met.

PRACTICAL CANONIZATION

In order to actually derive the "canonizing" $\underline{\mu}^{(k)}$, one must first investigate the amount of arbitrariness inherent in that vector. Let

$$\underline{\zeta}^{(k)} = \hat{\underline{\zeta}}^{(k)} + \underline{\mu}^{(k)}(\tilde{\underline{y}}) \qquad (33)$$

extend the canonical properties to $O(\varepsilon^k)$, i.e. let it satisfy

$$\hat{\underline{\zeta}}^{(k)} + \underline{\mu}^{(k)}(\tilde{\underline{y}}) = \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla}\chi^{(k)} \qquad (34)$$

Then if one replaces $\underline{\mu}^{(k)}$ with

$$\underline{\mu}'^{(k)} = \underline{\mu}^{(k)}(\tilde{\underline{y}}) + \bar{\nabla}\underline{\Phi}(\tilde{\underline{y}}) \qquad (35)$$

with Φ an arbitrary function independent of y_n , one finds

$$\hat{\underline{\zeta}}^{(k)} + \underline{\mu}'^{(k)}(\tilde{y}) = \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla}(\chi^{(k)} + \Phi) \qquad (36)$$

which still has the required form (21). The canonizing choice of $\underline{\mu}^{(k)}$ is thus arbitrary within the addition of a conjugate gradient of some scalar Φ which does not contain y_n .

In order to isolate $\underline{\mu}^{(k)}$ one operates on (34) with the averaging operator of (16); since $\underline{\mu}^{(k)}$ is independent of the angle variable, it equals its own average, giving

$$\begin{aligned}\underline{\mu}^{(k)}(\tilde{y}) &= \left\langle \underline{f}^{(k)}(\underline{\zeta}) - \hat{\underline{\zeta}}^{(k)} \right\rangle + \left\langle \bar{\nabla}\chi^{(k)} \right\rangle \\ &= \left\langle \underline{f}^{(k)}(\underline{\zeta}) - \hat{\underline{\zeta}}^{(k)} \right\rangle + \bar{\nabla}\left\langle \chi^{(k)} \right\rangle \qquad (37)\end{aligned}$$

The last term on the right is a conjugate gradient of some function of $\tilde{y}$, and it has already been established that such vectors, when added to $\underline{\mu}^{(k)}$, do not affect canonization. One may thus drop this part and obtain

$$\underline{\mu}^{(k)} = \left\langle \underline{f}^{(k)}(\underline{\zeta}) - \hat{\underline{\zeta}}^{(k)} \right\rangle \qquad (38)$$

To evaluate this, $\hat{\underline{\zeta}}^{(k)}$ must be derived from (17) while $\underline{f}^{(k)}$ may be obtained by methods given in II . The <u>most general</u> canonizing additive function is then

$$\underline{\mu}^{(k)} = \left\langle \underline{f}^{(k)}(\underline{\zeta}) - \hat{\underline{\zeta}}^{(k)} \right\rangle + \bar{\nabla}\Phi(\tilde{y}) \qquad (39)$$

with Φ arbitrary.

EQUIVALENCE TO CANONICAL PERTURBATION THEORY

A widely used method for solving perturbed periodic canonical systems is due to Poincaré and Von Zeipel[(5),(8)-(11)] and operates in the following manner. First, one expresses the Hamiltonian in terms of the solution

$$\underline{y} = (\underline{p}, \underline{q})$$

of the unperturbed Hamilton-Jacobi equation, with y_n an angle variable and y_1 the conjugate action variable. In the absence of "slowly varying" quantities the Hamiltonian then assumes the form

$$H = y_1 + \sum_{k=1} \varepsilon^k H^{(k)}(\underline{y}) \tag{40}$$

Next a near-identity transformation to new variables

$$\underline{z} = (\underline{P}, \underline{Q})$$

is sought, produced by the generating function

$$\sigma(\underline{P}, \underline{q}) = \sum P_i q_i + \sum_{k=1} \varepsilon^k \sigma^{(k)}(\underline{P}, \underline{q}) \tag{41}$$

and having the property that the new Hamiltonian $H^*(\underline{z})$ is independent of the transformed angle variable z_n, making its conjugate z_1 a constant of the motion. Since the transformation is a near-identity one, the lowest order of H^* has the same form as that of H, giving

$$H^*(\tilde{\underline{z}}) = z_1 + \sum_{k=1} \varepsilon^k H^{*(k)}(\tilde{\underline{z}}) \tag{42}$$

Methods then exist for deriving $\sigma^{(k)}$ and $H^{*(k)}$ order by order

Suppose now that two near-identity canonical transformations are given

$$\left.\begin{aligned} \underline{z} &= \underline{y} + \sum_{s=1} \varepsilon^s \underline{\zeta}^{(s)}(\underline{y}) \\ \underline{w} &= \underline{y} + \sum_{s=1} \varepsilon^s \underline{\psi}^{(s)}(\underline{y}) \end{aligned}\right\} \qquad (43)$$

either of which eliminates the angle variable from the Hamiltonian. Let furthermore

$$\underline{\zeta}^{(s)}(\underline{y}) = \underline{\psi}^{(s)}(\underline{y}) \qquad (44)$$

for

$$s = 1, 2, \ldots (k-1)$$

Then it will be shown that $\underline{\zeta}^{(k)}$ differs from $\underline{\psi}^{(k)}$ at most by a conjugate gradient of a function $\Phi(\tilde{\underline{y}})$ independent of y_n .

Clearly such a property would allow the two methods discussed in this work to be matched order by order. If $\underline{w}$, for instance, represents a solution of a given problem by the Poincaré-Von Zeipel method and $\underline{z}$ represents a solution by the Kruskal method, then $\underline{z}$ can be made equal to $\underline{w}$ to any desired order. To achieve this one must first choose $\underline{\mu}^{(k)}$ so as to make $\underline{z}$ canonical and then add to each order the appropriate $\Phi(\tilde{\underline{y}})$ which makes the corresponding $\underline{\zeta}^{(k)}$ and $\underline{\psi}^{(k)}$ equal to each other. Actually, one could also work in the opposite direction, since the Poincaré-Von Zeipel method also contains a certain amount of arbitrariness in each order, but we shall not consider this possibility here.

Let

$$H^*(\tilde{\underline{z}}) = \sum_{s=0} \varepsilon^s H^{*(s)}(\tilde{\underline{z}})$$

and

$$H^{**}(\tilde{\underline{w}}) = \sum_{s=0} \varepsilon^s H^{**(s)}(\tilde{\underline{w}})$$

be the two alternative forms of the Hamiltonian. Since the transformation is time-independent

$$H^*(\tilde{\underline{z}}) = H^{**}(\tilde{\underline{w}}) = H(\underline{y}) \tag{45}$$

Also, since it is a near-identity transformation, the $O(1)$ parts of the above equations must be equal, which leads to

$$\left.\begin{aligned} H^{*(0)}(\tilde{\underline{z}}) &= z_1 \\ H^{**(0)}(\tilde{\underline{w}}) &= w_1 \end{aligned}\right\} \tag{46}$$

We shall furthermore assume (and later justify)

$$H^{*(s)} = H^{**(s)} \tag{47}$$

$$(s = 1, 2, \ldots \; k\text{-}1)$$

Substituting the expansions in (45)

$$\sum \varepsilon^s H^{*(s)}(\,\breve{\underline{y}} + \sum \varepsilon^m \underline{\zeta}^{(m)}) = \sum \varepsilon^s H^{**(s)}(\,\breve{\underline{y}} + \sum \varepsilon^m \underline{Y}^{(m)}) \tag{48}$$

By means of expansion operators $T^{(m)}$ and $S^{(m)}$ (see I) this may be broken up into a series of equations, one for each order in ε . The equation for $O(\varepsilon^k)$ then is

$$\sum_{m=0}^{k} T^{(m)} {}_* H^{*(k-m)}(\tilde{\underline{y}}) = \sum_{m=0}^{k} S^{(m)} {}_* H^{**(k-m)}(\tilde{\underline{y}}) \qquad (49)$$

where $*$ marks "operates on" and where

$$T^{(0)} = S^{(0)} = 1$$

$$T^{(1)} = \underline{\zeta}^{(1)} \cdot \nabla$$

$$S^{(1)} = \underline{\Psi}^{(1)} \cdot \nabla$$

and in general

$$\left.\begin{aligned} T^{(m)} &= \underline{\zeta}^{(m)} \cdot \nabla + N^{(m)}(\underline{\zeta}) \\ S^{(m)} &= \underline{\Psi}^{(m)} \cdot \nabla + N^{(m)}(\underline{\Psi}) \end{aligned}\right\} \qquad (50)$$

with $N^{(m)}$ an operator depending only on orders of its argument that are lower than the m-th. Substitution and use of (46) give

$$H^{*(k)} + \underline{\zeta}^{(k)} \cdot \nabla y_1 + \left\{ N^{(k)}(\underline{\zeta}) * y_1 + \sum_{m=1}^{k-1} T^{(m)} {}_* H^{*(k-m)} \right\} =$$

$$= H^{**(k)} + \underline{\Psi}^{(k)} \cdot \nabla y_1 + \left\{ N^{(k)}(\underline{\Psi}) * y_1 + \sum_{m=1}^{k-1} S^{(m)} {}_* H^{**(k-m)} \right\} \qquad (51)$$

The two expressions in curly brackets depend only on lower orders; they are therefore equal to each other and may be dropped, leaving

$$H^{*(k)}(\tilde{\underline{y}}) - H^{**(k)}(\tilde{\underline{y}}) = (\underline{\Psi}^{(k)} - \underline{\zeta}^{(k)}) \cdot \nabla y_1 \qquad (52)$$

Now by (24), scalars $\chi^{(k)}$ and $\tau^{(k)}$ must exist such that

$$\underline{\zeta}^{(k)} = \underline{f}^{(k)}(\underline{\zeta}) + \bar{\nabla}\chi^{(k)}$$
$$\underline{\Psi}^{(k)} = \underline{f}^{(k)}(\underline{\Psi}) + \bar{\nabla}\tau^{(k)} \tag{53}$$

Since lower orders of the two expansions are equal, the two $\underline{f}^{(k)}$ vectors are equal too, leaving

$$\underline{\zeta}^{(k)} - \underline{\Psi}^{(k)} = \bar{\nabla}(\chi^{(k)} - \tau^{(k)}) = \bar{\nabla}\Phi \tag{54}$$

To prove our assertion we must show that Φ is independent of y_n. Substituting in (52)

$$H^{*(k)}(\tilde{\underline{y}}) - H^{**(k)}(\tilde{\underline{y}}) = \partial\bar{\Phi}/\partial\bar{y}_1 = \partial\Phi/\partial y_n \tag{55}$$

Now Φ is allowed to depend on y_n only in a periodic manner, from which it follows that $\partial\Phi/\partial y_n$ also depends periodically on y_n. However, the left-hand side is independent of that variable, so that Φ must be independent of y_n. This proves the main assertion. Incidentally, one also finds

$$H^{*(k)} = H^{**(k)} \tag{56}$$

which justifies equation (47).

APPENDIX : THE INVERSE TRANSFORMATION

Adding (5) and (24) and cancelling zeroth order terms gives

$$\sum{}' \varepsilon^k \underline{\eta}^{(k)}(\underline{z}) = -\sum{}' \varepsilon^k \underline{\zeta}^{(k)}(\underline{y})$$

$$= -\sum{}' \varepsilon^k \underline{\zeta}^{(k)}\left(\underline{z} + \sum \varepsilon^m \underline{\eta}^{(m)}(\underline{z})\right) \tag{A-1}$$

If ∇ is in z-space and $*$ denotes operation

$$\underline{\zeta}^{(k)}\left(\underline{z} + \sum \varepsilon^m \underline{\eta}^{(m)}(\underline{z})\right) = \exp\left(\sum_{m=1} \varepsilon^m \underline{\eta}^{(m)}(\underline{z}) \cdot \nabla\right) * \underline{\zeta}^{(k)}(\underline{z})$$

$$= \sum_{m=0} \varepsilon^m S^{(m)} * \underline{\zeta}^{(k)}(\underline{z}) \tag{A-2}$$

where

$$S^{(0)} = 1 \qquad S^{(1)} = \underline{\eta}^{(1)} \cdot \nabla$$

$$S^{(2)} = \underline{\eta}^{(2)} \cdot \nabla + \tfrac{1}{2}\, \underline{\eta}^{(1)} \underline{\eta}^{(1)} : \nabla\nabla \tag{A-3}$$

and so forth. If this is substituted in (A-1), orders of ε may be individually equated since everything is now expressed in $\underline{z}$. Separating the $m = 0$ term from the rest then brings the $O(\varepsilon^k)$ relation to the form

$$\underline{\eta}^{(k)} = -\underline{\zeta}^{(k)} - \sum_{m=1}^{k-1} S^{(m)} * \underline{\zeta}^{(k-m)} \tag{A-4}$$

If all lower orders of $\underline{\zeta}^{(m)}$ are known, those of $\underline{\eta}^{(m)}$ may be derived and used for constructing the $S^{(m)}$.

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