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INTERPLANETARY TRAJECTORY ENCKE METHOD FORTRAN PROGRAM MANUAL FOR THE I.B.M. SYSTEM/360

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INTERPLANETARY TRAJECTORY ENCKE METHOD
FORTRAN PROGRAM MANUAL
FOR THE I. B. M. SYSTEM/360

September 1970

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

ITEM PROGRAM MANUAL

FORTRAN IV VERSION

by

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A program of this nature must, of necessity, take a period of several years for its development; it is thus impossible to mention the names of all those who have contributed to its growth. It was originally conceived by S. Pines and H. Wolf at Republic Aviation Corporation under contract to NASA (NASA-109) beginning in 1959. This version is issued under contract to the Theoretical Mechanics Branch, Laboratory for Space Physics of the Goddard Space Flight Center. Major contributors have been C. Bergren, C. Hipkins, M. Wachman, L. Lefton, F. Shaffer, F. Whitlock and N. Levine.

Numerous additions and improvements have been made to the current version including reprogramming for the I.B.M. Operating System 360 Computers, and development is a continuing effort. This edition covers the conversion of the original machine language program to Fortran IV.

The program has been and is available for general use to interested organizations.** The authors express their appreciation to Mrs. Beatrice Boccucci: for her assistance in the final preparation of this manual.

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TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF FIGURES

I. INTRODUCTION

This report describes a general purpose Interplanetary Trajectory Encke Method (ITEM) Program, programmed in the FORTRAN IV language. The method employed is designed to give the maximum available accuracy without incurring prohibitive penalties in machine time. On the basis of research described in Reference 4, the Encke method was selected as best satisfying these requirements. However, the classical Encke method was modified to eliminate some of its objectionable features. This modified Encke method is described in Appendix A.

The perturbations included in this program are the gravitational attractions of the Earth, Moon, Sun, Mercury and the outer planets. The outer planets are considered as point masses. Additionally, the effects of the zonal and tesseral harmonics of the Earth, as well as aerodynamic drag, small corrective thrusts, and radiation pressure including the shadow effect of the Earth, are considered. The input may be prepared in any one of several common systems and a great variety of output options are available.

Additional options that are currently under development are explained in Section XIII.

II. NOTATION

Upper case - vectors; Hats - unit vectors; Lower case - magnitudes

Description	Symbol	Units
Cartesian coordinatesof vehicle with respect to reference body	$x\ y\ z$	km
Velocity components in Cartesian Coordinates	$\dot{x}\ \dot{y}\ \dot{z}$	km/sec
Time	t	hrs.
Longitude measured from Greenwich, + East (used in Section Iv and Appendix H)	θ	degrees
Longitude of vernal equinox	θ_o	degrees
Speed	v	km/sec
Geodetic altitude*	h	km
Geodetic latitude	φ	degrees
Geodetic flight path angle	γ	degrees
Geodetic flight path azimuth	A	degrees
Acceleration parameter (defined in Appendix E)	u	
Right ascension	RA	degrees
Astronomical units	AU	
Earth radii	ER	
Earth mass	m_e	
Semi-major axis	a	ER

*Note: The following 3 symbols with primes denote the corresponding geocentric quantities. Geocentric in this report refers to a spherical earth, i.e., $e^2 = 0$. In this case $\varphi' = \delta =$ declination.

Description	Symbol
Vehicle position vector	R
Distance to vehicle	r
Perturbation displacement vector	ΔR
Perturbation displacement vector components	ξ, η, ζ
Perturbation acceleration	F
Coordinate functions and their time derivatives	$f, g, \dot{f}, \dot{g}$
Mass parameter	μ
Earth's eccentricity as used in Appendices H, I, L, S	e
Mean motion	n
Unit vectors for classical two-body orbit solution	$\hat{P}, \hat{Q}$
Eccentric anomaly as used in Appendix T	E
Elevation angle as used in Appendix I	E
$R_O \cdot \dot{R}_O$	d_O
Inclination of orbital plane	i
Right ascension of the ascending node	Ω
Vector from the body to vehicle	R_{VT}
Greenwich Hour Angle	G.H.A.
Argument of perigee	ω
Parameters which account for polar oblateness of the earth, defined in Appendix H	c, s
Right ascension of the station meridian	RA_S
Range measured from observation station	ρ
Direction cosines measured in a topocentric coordinate system	λ, μ
Declination	δ

SUBSCRIPTS

	Symbol
Vehicle	v
ith perturbing body	i
Quantity obtained from Keplerian solution of two-body problem	k
Reference body as used in Appendix B	c
Station	s
$R_A - R_B$	R_{AB}
Value at rectification time	o
Corresponds to x, y, z components respectively	n = 1, 2, 3
Value at perigee	p

III. GENERAL PROCEDURE FOR USING PROGRAMS

Initial conditions, terminal conditions and print frequency, as well as other parameters controlling the flow of the program, are read as input. The computation of the trajectory then proceeds until one of the terminal conditions (e.g. maximum time) has been reached or an error is encountered. At this time the program prints the reason for its termination and proceeds to the next case. When an end of file is encountered on the input tape, control is transferred to the monitor.

IV. INITIAL CONDITIONS

The initial conditions necessary for the specification of a trajectory are:

1. Initial position of the vehicle relative to the reference body.

2. Initial velocity of the vehicle relative to the reference body.

3. Initial time of launch referenced to a base time.

For specification of the intial conditions, the reference systems and units shown below may be used.

A. <u>Cartesian Coordinates</u>

The coordinate system is defined as follows:

1. The origin is at the center of the reference body.

2. The x-axis is in the direction of the mean equinox of December 31.0 of the year of launch.

3. The xy plane is the mean equatorial plane of the Earth.

   Initial position is given by the x, y, z coordinates of the vehicle. Initial velocity is given by the $\dot{x}$, $\dot{y}$, $\dot{z}$ components of the vehicle. Initial time of launch from base time(1) (t) is also given. If the program is used in its standard form, the units(2) to be used for the above are:

   x, y, z - kilometers

   x, y, z - kilometers/second

   t - month, day, hours, minutes, and seconds from base time

   The year of launch must also be given.

---

(1) The base time is 0.0^h UT December 31 of the year previous to the year of launch.

(2) Scale factors are used to convert the input units to the units used internally (ER or AU a[illegible] A[illegible]y other set of units may be used by changing thes[illegible] sc[illegible]s with the appropriate ID card as described in Sect[illegible]

B Geodetic Polar Coordinates

Initial position of the vehicle is given by:

1. Geodetic latitude (φ)

2. Longitude[3] (θ), measured from Greenwich

3. Geodetic altitude (h)

4. Longitude of vernal equinox[3] at initial time (θ_0). This quantity may be computed by the program or may be loaded

Initial velocity of the vehicle is given by:

1. Speed (v) with respect to the center of the Earth.

2. Flight path azimuth (A) measured clockwise from north in a plane normal to the geodetic altitude.

3. Flight path angle (γ) measured from a plane normal to the geodetic altitude.

Initial time of launch from base time[1] (t) must also be given.

The following units must be used with the above quantities:

1. φ, θ, and θ_0 - degrees; h - kilometers

2. A and γ - degrees; v - kilometers/second

3. t - month, day, hours, minutes and seconds

C Geocentric Polar Coordinates

Ordinarily an input given in polar coordinates will be interpreted as described in the preceding Paragraph B. However, if NOPT(1) = 1[4], the program will interpret latitude as declination, height as distance above a spherical Earth of equatorial radius, and flight path angle and azimuth with reference to a plane normal to the radius vector.

- -

(3) If the right ascension (RA) at initial time is known, it may be used in place of longitude (θ). The longitude of the vernal equinox (θ_0) is then set to zero.

(4) See INPUT Section.

D. Osculating Element Input

The osculating elements to be input are:

Argument of perigee

Longitude of ascending node

Inclination

Semi-major axis (in Earth radii)

Eccentricity

Time of perigee, mean anomaly, eccentricity anomaly, or true anomaly (only one)

The program converts the above to Cartesian coordinates and then continues normally. (See Section IX, ID = 10.)

E. Comments

1. The program computes in Cartesian coordinates. Units used internally in the computation are:

   (a) Position: Earth Radii (ER) or Astronomical Units (AU).

   (b) Velocity: ER/HOUR or AU/HOUR

   (c) Time : Hours

   (Earth Radii units are used in the Earth or Moon Reference. Astronomical Units are used in the Sun, Mercury, Venus or outer planet reference).

2. The user is restricted to Cartesian coordinates when launching from any body other than the Earth as directed in preceding versions.

3. Equations for converting the initial conditions from Polar to Cartesian coordinates are shown in Appendix H.

V. TERMINATING CONDITIONS

The set of conditions which will terminate a trajectory may be summarized as:

1. Maximum time of flight - hours.
2. Maximum distance from any possible reference body considered in the solution. Last value in R-vector of integration and print block "ARRAY (1,n,J)."
3. Minimum distance from any possible reference body considered in the solution. First value in R-vector of integration and print block ARRAY (1,n,1)*

Any of these conditions will terminate a trajectory. Loading a large number into any of the maxima and a zero into any of the minima will make the corresponding conditions inoperative. A proper choice of these numbers will permit complete computation of the desired trajectory, avoid extensive unnecessary computation and guard against faulty input.

---

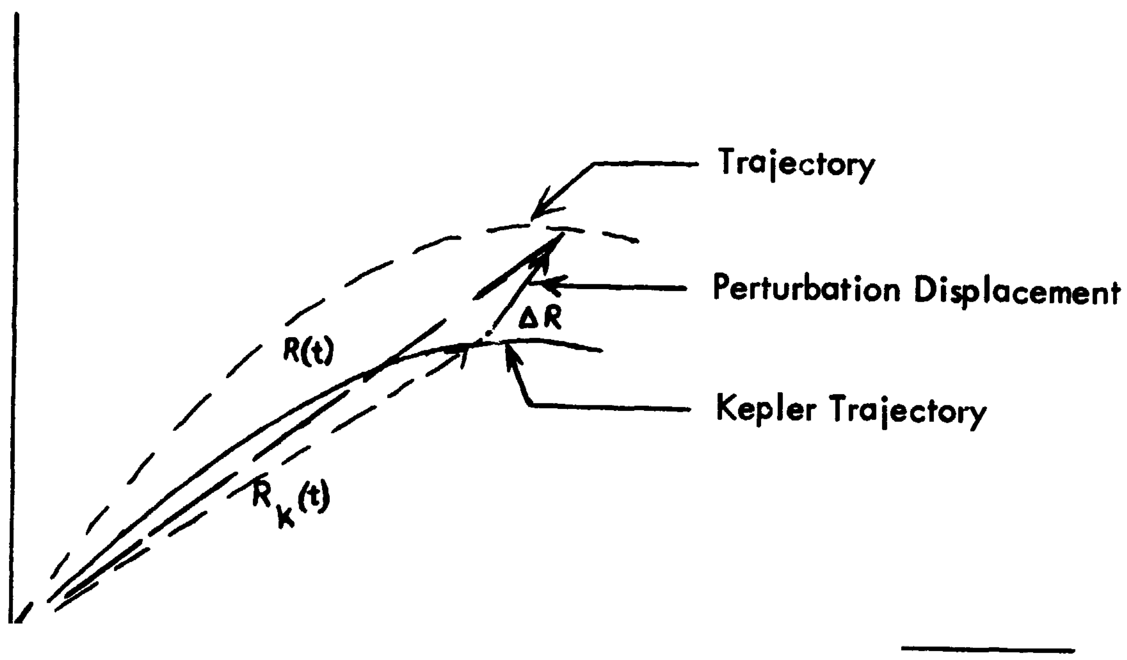
* n designates reference body.

n = 1 Earth	n = 7 Saturn
n = 2 Moon	n = 8 Uranus
n = 3 Sun	n = 9 Neptune
n = 4 Venus	n = 10 Pluto
n = 5 Mars	n = 12 Mercury
n = 6 Jupiter	

VI. PERMISSIBLE PERTURBATIONS

The trajectory computation consists of two parts, the exact solution to the two-body problem and integrated additions to this solution for the effect of perturbations. The successful control of round-off errors in the modified Encke method depends on preventing the accumulated round-off error in the integrated perturbation displacement from affecting the computed position. This is achieved by always keeping the perturbation displacement small and rectifying whenever the perturbation exceeds specified limits. The constants mentioned below are used in determining the allowable limits as ratios of the perturbation position and velocity to the two-body position and velocity, respectively.

This ratio for the position vector is shown in the following sketch.



Encke Method

The recommended values for these ratios are as follows:

Position Ratio

POSRCS $\left(\frac{\Delta r}{r}\right)^2 \leq .0001$

Velocity Ratio

VELRCS $\left(\frac{\Delta \dot{r}}{\dot{r}}\right)^2 \leq .0001$

and these are incorporated into the program. Modifications may be made by altering the data subroutine, or by reading them in under ID = 12, or by using Subroutine Modif.

VII. RADAR INFORMATION PROGRAM

The program may be used to simulate radar data if desired. A maximum of 30 stations can be processed at one time. The following information is required for each station considered:

1. Station Name - for identification purposes

2. Position of Radar Station

   a. Longitude (θ) of the station from Greenwich - degrees, minutes and seconds* - positive eastward.

   b. Geodetic latitude (φ) of the station - degrees, minutes and seconds* - positive north.

   c. Altitude (h) of station above sea level - feet.

The simulated radar information consists of azimuth, elevation, topocentric right ascension and declination, slant range, and range rate. It is printed at every normal print time for which the elevation angle is positive. Refraction is not considered.

This section is coded as a subroutine and may be called at any time.

---

* Alternatively, these quantities may be given in degrees and decimals. Zero's must be loaded into the positions reserved for minutes and seconds.

The fractional parts will not appear in the printout reproducing the station coordinates. They will, however, be included in the computation.

VIII. SUBROUTINE MODIF

Modifications to program constants, which normally remain unchanged during the running of a number of cases, may be made by using the appropriate common in conjunction with a compilation of a subroutine called MODIF. This subroutine may include data statements, FORTRAN G or H level statements, and/or read statements. The use of read statements is suggested to facilitate stacking of cases. Any modification included in this subroutine will be operative for all succeeding cases, unless it is revoked.

Modifications required more frequently may be accomplished through the use of ID(I)'s, as described in the INPUT section (Section IX).

A.

Radiation Pressure may be included by loading a coefficient into

RACOE __________**

The number to be loaded is:

$$\frac{KC_r A}{m}$$

C_r is the radiation pressure in dynes/cm^2 at a distance of 1 AU from the Sun.

$$C_r = 4.6 \times 10^{-5} \frac{\text{dynes}}{\text{cm}^2} \text{ (estimated value)}$$

\- -

** Real number

A area in cm^2

m mass in grams

K scaler $3600^2(23455.)^2/6378.165 \times 10^5 = .11178 \times 10^8$ seconds to hours, ER to AU, cm to ER

The radiation pressure will only be active if sunlight impinges on the vehicle. For correct results the radiation pressure should therefore, be run only in conjunction with the optional shadow computation as described in Appendix O.

If, however, the expected trajectory may be safely assumed to be entirely out of the Earth's shadow, shadow testing may be avoided with a consequent saving in machine time. In this case, the following modification cards should be included in MODIF:

```
SHDN    =  1.D0**
SHDN1   =  1.D0**
NOPT(17) =  0
```

B. Aerodynamic Drag

If inclusion of the aerodynamic drag is desired, the drag parameter 1/2 C_D A/M may be initialized by ID = 23, as described in (Section IX) or loaded into subroutine MODIF by means of the following card:

COEFL = ______**

The units of C_D A/M are the area in cm^2 and the mass in grams. A layered atmosphere rotating with the Earth is assumed. The density is obtained by a linear interpolation of the density-altitude table. The above may be incorporated into the Block DATA subroutine.

** Real number(s)

C. Atmospheric Tables

Atmospheric tables for the drag computation are stored in core. They correspond to COSPAR International Reference Atmosphere of 1961, contained in the Report of the Preparatory Group for an International Reference Atmosphere, accepted at the COSPAR meeting in Florence, April 1961. The units for the air density are grams/cm^3 and the height is given in ER from the center of the Earth. If it is desired to change this atmosphere, the following procedure should be followed:

NTAR = _______*	The number to be entered is N - the number of entries in the density table.
RTBL(I) = _______**	I = 1, 2, - - -, N - the values of r for which densities are given, in ascending order (a maximum of 50).
RHOT(I) = _______**	I = 1, 2, - - -, N - the values for the air density in grams/cm^3 in respective order corresponding to the preceding r table.

If other units are used for the density table, the drag parameter described in Part B of this section must be read in with like units and the constant (-6378.165D5)** normally in DRSC has to be changed accordingly. The negative sign directs the drag force opposite to the velocity. This constant converts the drag from the units used for A, M and ρ to ER/hour^2.

- -

* Integer

** Real number

D. Printout

The program provides a special printout near the Earth, Moon, Sun, or T-Planet reference. This prinout occurs at every integration step and is useful for observing the behavior of these relevant quantities during ascent and re-entry. This feature is triggered by the following modification cards:

SERE(I) = __________** I = 1, 2, ... 12

For printout near	Earth use index	1	in	ER units.
	Moon	2		ER
	Sun	3		AU
	Venus	4		AU
	Mars	5		AU
	Jupiter	6		AU
	Saturn	7		AU
	Uranus	8		AU
	Neptune	9		AU
	Pluto	10		AU
	Mercury	12		AU

The numbers used above are the radial distances within which the special printout is to be effective. The units are earth radii for the Earth and Moon references and astronomical units for the remaining planet references. A zero in any of the SERE(I)'s suppresses this feature.

** Real number(s)

E. <u>Ephemeris Time</u>

The planetary coordinates are interpolated using ephemeris time.

$$ET = UT + \Delta T E$$

An approximate value of Δ T E (35 seconds) is used. To change this quantity, the following card, giving Δ T E in hours, is inserted:

ETMUT = ________** The value for Δ T E - hours.

To restore original quantity:

ETMUT = .009888888889** Δ T E is 35 seconds.

- -

** Real number(s)

F. Inclusion or Exclusion of Perturbations

Ordinarily included are the graviational attractions of the Moon, Sun, Mercury, and the outer planets. The gravitational field of the earth (2nd, 3rd and 4th zonal harmonics) are included if NOPT(50) = 1.* A maximum of 15 zonal coefficients may be used. To exclude any or all of these perturbations, the following modifications should be included in subroutine MODIF:

	MOON	MEI(2) = 0.DO
	SUN	MEI(3) = 0.DO
	VENUS	MEI(4) = 0.DO
	MARS	MEI(5) = 0.DO
	JUPITER	MEI(6) = 0.DO
	SATURN	MEI(7) = 0.DO
	URANUS	MEI(8) = 0.DO
	NEPTUNE	MEI(9) = 0.DO
	PLUTO	MEI(10) = 0.DO
	MERCURY	MEI(12) = 0.DO
	ZONAL and/or TESSERAL HARMONICS**	
Zonals	EFJ (2,3,4,........ 15) = 0.DO	
Tesserals	ECG(I,J) = 0.DO ESG(I,J) = 0.DO	$\}$ I and J = 1 to 15

---

* Described in the INPUT Section (Section IX)

** Described in Appendix G

IX. INPUT

Input to the program is read in as follows:

Each set of input is preceded by an ID card which contains an integer number terminating in column 10. This card may also contain Hollerith information starting in column 11.

ID = 1 Permits one card of Hollerith information - usually used for case identification.

ID = 2 Permits one card containing a set of 72 fixed point 1's or 0's. Each flag (1 or 0) corresponds to the same numbered subroutine. A zero is used for normal operation and a one is used to print diagnostic information in the proper subroutine. A blank card after ID = 2 will be necessary if the system does not zero out core before load time and normal operation is desired. In the program, these flags are referred to as NC(I).

ID = 3 Performs similarly to ID = 2. It allows a card of 1's and 0's to be read into NOPT(I) (I = 1 to 72). These flags permit the incorporation of various options into the program. Following are the currently available options:

NOPT(1) = 1 indicates polar geocentric coordinates

= 2 indicates geodetic coordinates when polar load is used.

NOPT (2-13) are used for print options.

= 0 indicates no print

= 1 indicates print

NOPT(2) is associated with XR print

NOPT(3) is associated with XRDT print

NOPT(4) is associated with XVE print

NOPT(5) is associated with XVM print

NOPT(6) is associated with XME print

NOPT(7)	is associated with XVS print
NOPT(8)	is associated with XVVN print
NOPT(9)	is associated with XVMR print
NOPT(10)	is associated witn XVJP print
NOPT(11)	is associated with XI print
NOPT(12)	is associated with XIDT print
NOPT(13)	is associated with D2XI print
NOPT(14) = 1	prints data statement parameters
NOPT(15) = 1	deletes regular print after rectification
NOPT(16) = 1	deletes print in rectification
NOPT(17) = 1	activates shadow computations
NOPT(20) = 2	activates predictor only integrator*
NOPT(20) = 3	activates predictor-corrector integrator*
NOPT(25) = 1	prints spin axis angles. This option may be used in conjunction with NC(47) to allow the user to input a desired spin vector.
NOPT(33) = 2	rotates positions and velocities from equatorial to ecliptic coordinates for printing
NOPT(40-44) = 0	indicates no print
NOPT(40 and 41) = 1	is associated with XVP1 print
NOPT(40 and 42)=1	is associated with XVP2
NOPT(40 and 43)=1	is associated with XVP3
NOPT(40 and 44)=1	is associated with XVP4
NOPT(40 and 45)=1	is associated with XVP5
NOPT(40 and 46)=1	is associated with XVP6
NOPT(50) = 0	No zonal or Tesseral harmonics are used.
NOPT(50) = 1	Includes the gravitational field of the Earth (2nd, 3rd, and 4th zonal harmonics)
NOPT(50) = 2	Includes zonal and tesseral harmonics

---

* Described in Section XIV Methods of Integration

NOPT(51) = 1	activates restart feature (See ID=9).
NOPT(65) = 1	activates trajectory search (ID=18 must be included).
NOPT(68) = 1	activates residual computations with radar stations. (ID=21 must be included).
NOPT(68) = 2	activates residual computations without radar stations. (ID=21 must be included).
NOPT(69) = 1	activates solar engine logic for small corrective thrusts. This option can be made available upon request.
NOPT(70) = 1	activates element roation. This option is used in conjunction with a variable called IELD. IELD should be set in subroutine MODIF as follows:
IELD=2	Permits ecliptic elements for input, resulting in equatorial osculating elements output.
IELD=3	Permits equatorial elements for input, resulting in ecliptic element output.
NOPT(70) = 2	Permits osculating element input, Brouwer mean
NOPT(70) = 3	element output. This option can be made available upon request.
NOPT(71) = 1	activates Beta Integrator*
NOPT(72) = 1	permits element load with the period (HRS) substituted for the semi-major axis A(ER).

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* Described in Section XIV Methods of Integration.

ID = 4 Used to read in start time of flight in year, month, day, hours, minutes, and seconds: starting reference body; and target reference using the following format (5I5, F5.2, 2I5). The reference bodies are numbered as follows:

1 = Earth	4 = Venus	7 = Saturn	10 = Pluto
2 = Moon	5 = Mars	8 = Uranus	12 = Mercury
3 = Sun	6 = Jupiter	9 = Neptune	

ID = 5 Used for polar load and reads in θ, φ, h, v, A, γ, and θ_o. The format used is (3D20.0). The program expects all angles in degrees, altitude in kilometers measured from the surface of the Earth, and the velocity in kilometers per second. If θ_o is read in as 1000.0D0, the program will compute the proper θ_o.

ID = 6 Used for Cartesian input. x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$ are read in with format (3D20.0). The program expects these coordinates to be equatorial in kilometers and kilometers per second, with the starting reference body as center.

ID = 7 This option generates initial conditions for a trajectory which is designed to get a spacecraft to the target within a specified number of days, without thrust. The input is: the Julian date of start time, the flight time in days, option number, and the radius of the Earth's sphere of influence. The format used is (3D20.0).

Option number 1 starts in Sun Reference.

Option number 2 and 3 are not activated, however, they can be made available upon request. (See Section XIII).

When Option = 1 is used, input for ID=7 must be followed by 4 cards

Card 1 = 0, 0, 0.51373647D-6* Format (I5,D12.6,D18.8)

Card 2 = Blank

Card 3 = Blank

Card 4 = Blank

ID = 8 This ID permits one to read in a vector of special print times. The first card after the ID card contains the number of such print times from 1 to 50, format (I10). The following card or cards contain the times, format (3D20.0). If this ID is used in conjunction with NOPT(69) (solar engine option), these times are used forstarting and stopping the solar engine. Odd numbers start the engine, the even numbers shut the engine off.

ID = 9 This ID reads in the following: PRSP, RACOE, TIMEL, STI, VI, CCNT, ENPLAN, and TR using format (3D20.0).

PRSP	A non-zero value will suppress normal print times until the specified time in hours has been reached.
RACOE	A non-zero value will activate radiation pressure computations.
TIMEL	Maximum time of flight in hours.
STI	Not activated
VI	Not activated
CCNT	Triggers nodal crossing print. (See ID = 14 also). If CCNT and ICCNT are set to non-zero, ICCNT will take precedent.
ENPLAN	A non-zero activates the number of perturbing bodies to be used in the calculations. This must be an integer 1 or 12. If 1 is used the ephemeris data tape is neigher required nor used.

---

* Gravitational constant for the Sun

TR A non-zero value, in conjunction with NOPT(51), will activate the restart feature. TR should be set to the desired restart time in hours.

ID = 10 This ID permits one to load the initial conditions as osculating elements of an ellipse. The following are read in: SOMEG, LOMEG, INC, A, ECC, ELOAD, ELTRIG with format (3D20.0).

SOMEG is the argument of perigee

LOMEG is the longitude of the ascending node

A is the semi-major axis

ECC is the eccentricity

ELOAD depends on ELTRIG

If	ELTRIG = 1	ELOAD = time of perigee
	ELTRIG = 2	ELOAD = mean anomaly
	ELTRIG = 3	ELOAD = eccentric anomaly
	ELTRIG = 4	ELOAD = true anomaly

ID = 11 Permits one to alter the integration and print intervals of the various reference bodies. The number of cards to be read is a function of ENPLAN. (See Sample Input Data in Section IX). The data expected are read in with format (3D20.0), and terminated by any integer greater than or equal to 13 - (FORMAT I10) as follows:

Card 1 contains eight distances from Earth in ER

2 contains seven integration intervals in hrs.

3 contains seven print intervals in hrs.

4 contains eight distances from the Moon in ER

5 contains seven integration intervals in hrs.

6 contains seven print intervals in hrs.

7 contains eight distances from the Sun in AU

8 contains seven integration intervals in hrs.

9 contains seven print intervals in hrs.

10	contains eight distances from Venus in AU
11	contains seven integration intervals in hrs.
12	contains seven print intervals in hrs.
13	contains eight distances from Mars in AU
14	contains seven integration intervals in hrs.
15	contains seven print intervals in hrs.
16	contains eight distances from Jupiter in AU
17	contains seven integration intervals in hrs.
18	contains seven print intervals in hrs.
LAST	contains an integer .GE. 13 (FORMAT I10).

ID = 12 Permits one to make changes in the program's built-in data or to read in other-than-normal inputs. This can be done by using a subroutine called MODIF which must contain the proper block common.

ID = 13 Allows one to change input and output scale factors, using format (3D20.0). The card following the ID card contains TSCL, REKM, and XMDKM.

TSCL is the time scale factor and sits in the program as 3600. It is used to change seconds to hours and hours to seconds

REKM sits in the program as 6378.165, the number of kilometers in one ER.

XMDKM sits as 14.9599×10^7 and is the number of kilometers in one AU.

ID = 14 sets triggers for apogee, perigee, and nodal crossing prints The following card reads in ICANT, ICPNT, and ICCNT, using format (3I10)

ICANT = n prints every n^{th} apogee

ICPNT = n prints every n^{th} perigee

ICCNT = n prints every n^{th} nodal crossing

ID = 15 Not activated

ID = 16 Allows radar station data to be read in. The card following the ID card contains the number of stations to be read in with format (I10). The next two cards contain the name and coordinates of the first station with format (A46/7D10.0). This last format is repeated until all stations have been accounted for.

ID = 17 is used for reading in solar engine information with cor-corrective thrusts. This option is not activated*.

ID = 18 is used for the iterator. The card following the ID card is read in with the format (10I5)

IPS	=	Number of the dependent parameter(s).
ITMAX	=	Maximum number of iterations.
NSL	=	Number of the independent parameter(s).
IPOFL(IPS)	=	Identification number(s) of the quantities to be achieved.
IVAR(NSL)	=	Input quantities to be varied in the units and sequence of the polar load, i.e., $\theta,\varphi,h,v,A,\gamma$.

The remining cards are read in with the format (7D10.0)

YEPS(IPS) = The tolerance to which convergence is desired. if the solution converges to within the special tolerance, the iteration will stop.

---

* See Section XIII

XEPS(NSL) = Epsilon values of the independent va iable.

XVAR(NSL) = Epsilons to be used for generating secant partials.

YCON(IPS) = The desired values of the dependent variables.

A maximum of any six dependent variables may be selected. The identification number(s) of the quantities to be achieved are as follows:

1. B · T Miss parameters
2. B · R Miss parameters
3. Earth, Lunar or T-planet inclination.*
4. Earth, Lunar or T-planet ascending node.*
5. Earth, Lunar or T-planet argument of perigee.*
6. Earth, Lunar or T-planet pericynthion radius.*

ID = 20 Starts the program.

ID = 21 Used in conjunction with NOPT(68) for reading in nominal and variational triggers with (out) radar simulation. If NOPT(68) = 1, the card following the ID card is read with format (4I10).

NOMTRI 1 = Nominal trajectory, 0 = variational

NOQAN = The number of radar stations

IOSCTR 0 = Radar residuals, 1 = osculating element residuals.

The user may find this option convenient for checking out numerical derivatives. If NOPT(68) = 2, the card following the ID card is read with format (I10).

NOMTRI 1 = Nominal trajectory, 0 = variational

- -

* See Appendix R.

ID = 23 Used to initialize aerodynamic drag computation

The card following the ID card is read in with format (3D20.0).

COEFL* $1/2\ C_D\ A/M$.

DRSC*

DNTAR* number of entries in the density table.

ID = 4 must precede ID = 5. Except for this condition and ID = 20 which must be read in last, the ID's may be read in randomly.

---

* See Section VIII-B

```
C                         SAMPLE LIST OF SUBROUTINE MODIF
C
C
      SUBROUTINE MODIF
      IMPLICIT REAL*8 (A—H, O—Z)
      COMMON/HENRY/H1(480),MEI(12),H2(226)
      REAL*8 MEI
C**** *************************************************************
      MEI (4) = 0.0D0
      MEI (6) = 0.0D0
      MEI (9) = 0.0D0
      RETURN
      END


      SUBROUTINE MODIF
      IMPLICIT REAL*8 (A—H, O—Z)
      COMMON/GLOBE/FJG(15),CG(15,15),SG(15,15), EF      ECG(15,15),
     XFSG(15, 15), EJG(4), ES22, EC22, ES31, EC31, ES33, EC33, N1Z, N2S, N3T, NOQ
C**** *************************************************************
      EFJ(2) = 1082.30D-6
      EFJ(3) = -2.30D-6
      EFJ(4) = -1.80D-6
      RETURN
      END
```

SAMPLE INPUT DATA

```
//GO.DATA5 DD *
          1                   TEST CASE FOR I.B.M. 360 VERSION
                                 NOPT(50) = 1 FOR OBLATENESS
          2
000000000000000000000000000000000000000000000000000000000000000000000000
          3
211111111111110000000000000000000000000000000000001000000000000000000000
          6           CARTESIAN LOAD.
 6095.8926D0         604.36567D0          2406.3589D0
 1.101985D0          9.8589445D0          -4.4549856D0
          4
 1965     5    30    12     70.0      1    4
         14
          0          0          0
          9
 0.0D0               0.0D0                10.0D0
 0.0D0               0.0D0                0.0D0
 1.0D0               0.0D0
         11     SET VECTORS FOR DESIRED PLANETS
          1          SET EARTH VECT
 1.0D0               4.0D0                8.0D0
 125.0D0

 0.0625D0            0.0625D0             0.5D0


 1.D0                2.D0                 6.D0


         13          SET K TO TERMINATE READ.
         16          RADAR STATIONS
          2
 ROSMAN
     277.0 7.0       4.2       35.        12.          .7         2874.56
 CANBERA
 148.      57.       20.9      -35.       37.          52.7       3766.588
         20     STARTS THE PROGRAM.
/*
```

```
//GO.DATA5 DD *
            1    ITERATOR DATA FOR MARS TRAJ. 1973
                        NOPT(65) = 1 FOR SEARCH.
            2
000000000000000000000000000000000000000000000000000000000000000000000000
            3     POLAR GEOC
111100101000011100000000000000000000000000000000010000000000000001000000
            4    START TIME
 1973       8    1    13    4   1.0     1     5
            5     POLAR LOAD     (LONG,LAT,ALT,VEL,AZ,FPA,LVE)
 112.30999OD0         -25.700D0               262.12960D0
 11.6050000D0         77.7800000D0            6.6500000000
 1000.0D0
            9
 8000.0D0             0.0D0                   8000.0D0
 0.0D0                0.0D0                   0.0D0
 12.0D0
           11    SET VECTORS FOR DESIRED PLANETS
            1          SET EARTH VECTOR
 1.0D0                4.0D0                   8.0D0
 125.0D0

 0.0625D0             0.125D0                 0.5D0


 12.0D0               12.0D0                  48.0D0


            3          SET SUN VECTOR
 0.0D0                1.30979948D0            10.D0


 12.D0                12.D0


 24.0D0               24.0D0


            5       SET MARS VECTOR
 .000040776D0         .00016D0                .00032D0
 .005D0

 0.0625D0             0.125D0                 0.5D0


 1.0D0                12.0D0                  12.0D0


           13          SET K TO TERMINATE READ.
           14
            0          0             1
           18      ITERATOR
     2     10     2    1    2        1    4
 100.0    -100.0     .00001        .000001 -.0002      -.000001      0.0
 0.0       1.        1.            1.       1.          1.           1.
 1.        1.
           20     START
/*
```

X. OUTPUT

A. Program Outputs

The following information is printed as the output of the program.

1. Title
2. Case number and any identifying titles.
3. Launch time - year, month, day, hour, min, sec.
4. Input - in the same units as they were entered into the program.
5. List of parameters used in run.
6. At each rectification the following data are printed:

__(b)__ RECTIFICATION PRINT __(a)__ REFERENCE
PERT OVER UNPERT = __(c)__ TIME = __(d)__ DELTA T = __(c)__

(a) Reference body

(b) and (c) indicate the reason for rectification

(c) If (c) = 0, rectification may be due either to switch of reference body or to change of integration interval.

If (c) $\neq$ 0, then the position, velocity, perturbations or the incremental eccentric anomaly have exceeded the permissible limits and (b) indicates which has been exceeded (see Section VI). These indications are given as:

PO - Position
VL - Velocity
TH - Incremental eccentric anomaly

(d) Time of rectification

(e) Integration interval

TIME IN YEAR, MONTH, DAY, HOUR, MIN, SEC, ____ ____ ____ ____ ____ ____		(a)
T = ________ HOURS FROM EPOCH		(b)
XR ______ YR ______ ZR ______ RR ______		(c)
XRDT ______ YRDT ______ ZRDT ______ RRDT ______		(d)
RIGHT ASCENSION (DEG) = ______ DECL = ______		(e)
TH SUBSAT POINT	LONG = ______	(f)
	LAT = ______	(g)
	HT = ______	(h)
	GHA = ______	(i)
GEOCENTRIC AZIMUTH	= ______	(j)
ELEVATION	= ______	(k)
GEODETIC AZIMUTH	= ______	(ℓ)
ELEVATION	= ______	(m)

(a) Year, Month, day, hour, second from time of launch

(b) Print time in hours from time of launch

(c) Position coordinates and magnitude of radius vector with respect to the reference body - kilometers/second

(d) Velocity components and magnitude of velocity vector with respect to the reference body - kilometers/second.

(e) Right ascention and declination in Earth reference system - degrees

(f) Longitude or sub-satellite point - degrees

(g) Latitude (geodetic) - degrees

(h) Geodetic height above the Earth's surface - kilometers

(i) Greenwich hour angle - degrees

(j) Geocentric flight path azimuth - degrees

(k) Geocentric flight path angle - degrees

(ℓ) Geodetic flight path azimuth - degrees

(m) Geodetic flight path angle - degrees

MOON SUBSAT POINT	LONG	=	________	(a)
	LAT	=	________	(b)
	AZIM	=	________	(c)
	ELEV	=	________	(d)
OSCULATING ELEMENTS AT TIME	T	=	________	
TRUE ANOMALY		=	________	(e)
SEM MAJ AXIS		=	________	(f)
ECCENT		=	________	(g)
PERICENT		=	________	(h)
APOCENT		=	________	(i)
INCLINATION		=	________	(j)

(a) Moon longitude - angle between the projection of the vector from the Moon to the vehicle onto the Moon's orbital plane and the Moon-Earth vector (Moon reference only) - degrees

(b) Moon latitude - angle between the radius vector connecting the Moon and the vehicle and its projection onto the orbital plane of the Moon about the Earth (Moon reference only) - degrees

(c) Selenocentric flight path azimuth - degrees

(d) Selenocentric flight path angle - degrees

(e) True anomaly - degrees

(f) Semi-major axis of trajectory - ER   + = ellipse; - = hyperbola

(g) Eccentricity of trajectory**

(h) Closest distance to the reference body (not necessarily the Earth)** - kilometers

(i) Farthest distance from the reference body (not necessarily the Earth)**(meaningful only for elliptic orbits) - kilometers

(j) Inclination of the orbital plane defined as the angle between the positive polar axis and the angular momentum vector** - degrees

---

** These are the osculating values and hence only constitute an estimate of the quantities described.

ARG PERIC = ____________ (a)

PERIOD = ____________ (b)

MEAN MOT = ____________ (c)

R A ASC NODE = ____________ (d)

M ANOMALY = ____________ (e)

E ANOMALY = ____________ (f)

T PERIC = ____________ (g)

UNIT PERICENTER POSITION VECTOR = ____________ (h)

UNIT ANGULAR MOMENTUM VECTOR = ____________ (i)

(a) Argument of pericenter - angle measured from the ascending node to the pericenter vector**- degrees.

Set to zero for circular orbits and poorly determined for near-circular orbits.

(b) Period**- hours

(c) Mean motion** - radians/hour

(d) Right ascension of the ascending node measured from the vernal equinox eastward along the equator**- degrees

(e) Mean anomaly** - degrees

(f) Eccentric anomaly** - degrees

(g) Time of nearest pericenter** - hours

(h) Components of the unit vector directed from reference toward pericenter**

(i) Components of the unit angular momentum vector

** Osculating values

B. Optional Outputs

1.	XVE = _____	YVE = _____	ZVE = _____	RVE = _____	(a)
	XVM = _____	YVM = _____	ZVM = _____	RVM = _____	(b)
	XME = _____	YME = _____	ZME = _____	RME = _____	(c)
	XVS = _____	YVS = _____	ZVS = _____	RVS = _____	(d)
	XVVN = _____	YVVN = _____	ZVVN = _____	RVVN = _____	(e)
	XVMR = _____	YVMR = _____	ZVMR = _____	RVMR = _____	(f)
	XVJP = _____	YVJP = _____	ZVJP = _____	RVJP = _____	(g)
	XVP1 = _____	YVP1 = _____	ZVP1 = _____	RVP1 = _____	(h)
	XVP2 = _____	YVP2 = _____	ZVP2 = _____	RVP2 = _____	(i)
	XVP3 = _____	YVP3 = _____	ZVP3 = _____	RVP3 = _____	(j)
	XVP4 = _____	YVP4 = _____	ZVP4 = _____	RVP4 = _____	(k)
	XVP5 = _____	YVP5 = _____	ZVP5 = _____	RVP5 = _____	(ℓ)
	XVP6 = _____	XVP6 = _____	ZVP6 = _____	RVP6 = _____	(m)
	XI = _____	ETA = _____	ZETA = _____	PERT = _____	(n)
	XIDT = _____	ETADT = _____	ZETADT = _____	VPERT = _____	(o)
	D2XI = _____	D2ETA = _____	D2ZETA = _____	APERT = _____	(p)

The above optional output appears between XRDT and RIGHT ASCENSION in the standard output. For instructions on how to obtain, see Section IX, ID = 3.

(a) Coordinates of vehicle with respect to the Earth - kilometers
(b) Coordinates of vehicle with respect to the Moon - kilometers
(c) Coordinates of the Moon with respect to the Earth - kilometers
(d) Coordinates of vehicle with respect to the Sun - kilometers
(e) Coordinates of vehicle with respect to Venus - kilometers
(f) Coordinates of vehicle with respect to Mars - kilometers
(g) Coordinates of vehicle with respect to Jupiter - kilometers

(h) Coordinatesof vehicle with respect to Saturn - kilometers

(i) Coordinates of vehicle with respect to Uranus - kilometers

(j) Coordinates of vehicle with respect to Neptune - kilometers

(k) Coordinates of vehicle with respect to Pluto - kilometers

(ℓ) Coordinates of vehicle with respect to E-M Barycenter - kilometers

(m) Coordinates of vehicle with respect to Mercury - kilometers

(n) Perturbation vector and magnitude of the perturbations with respect to the reference body - kilometers

(o) Perturbation velocity vector and magnitude - kilometers/second

(p) Perturbation acceleration vector and magnitude - kilometers/second^2

2. Shadow Print*

PASSAGE FROM	SHADOW PENUMBRA SUN PENUMBRA	TO	PENUMBRA SHADOW PENUMBRA SUN

AT _(a)_ TIME IN	SHADOW PENUMBRA SUN PENUMBRA	_(b)_ ACCUMULATED TIME _(c)_	

The above optional output appears before TIME IN YEAR, MONTH, DAY, HOUR, MIN, SEC in the standard output. It is controlled through the INPUT subroutine [NOPT(17)] (see Section IX, ID = 3).

(a) Time at which vehicle traverses denoted shadow boundary - hours

(b) Total time the vehicle spends in denoted shadow region during current traverse - hours

(c) Total accumulated time spend in denoted shadow region since launch - hours

---

* Appendix N

3. Radar Output*

STATION	(a)
AZIMUTH	(b)
ELEVATION	(b)
TOPOC. R A	(c)
TOPOC. DECL.	(c)
SLT RNG	(d)
RANGE	(e)

This output appears at the tail end of a normal printout. An ID card in the INPUT subroutine will control this segment of the program (see Section IX, ID = 16).

(a) Station name (identification) for each station

(b) Azimuth and elevation with respect to each station - degrees

(c) Topocentric right ascension and declination with respect to each station - degrees

(d) The slant range to each station - kilometers

(e) Rate of change of slant range for each station - kilometers/second

If the elevation is negative (the vehicle is below the horizon), this print is suppressed for the station in question.

---

* Appendix I.

4. <u>Reentry Output</u>

REENTRY PRINT	TIME	INERTIAL SPEED (kilometers/second)

Right ascension, declination, Earth subsatellite points and flight path azimuth and angle as given above.

The above optional output appears between GEOD ELEV and MOON SUBSAT POINT in the standard output.

5. <u>Trajectory Search Output</u>

The output consists of the normal ITEM output for a nominal trajectory and the same trajectory output for each variation requested for each iteration. The output format used only for the trajectory search follows:

VARIATION IN INITIAL CONDITIONS	(a)	(b)	(c)	(d)	(e)	(f)	(g)

(a) Change in latitude - degrees
(b) Change in longitude - degrees
(c) Change in altitude - kilometers
(d) Change in velocity - kilometers/second
(e) Change in azimuth - degrees
(f) Change in flight path angle - degrees
(g) Change in initial time - hours

5. Trajectory Search Output (cont.)

QUANTITY CODE		(a)	
DESIRED VALUES OF ABOVE QUANTITIES		(b)	
REQUIRED ACCURACY		(c)	
MATRIX OF PARTIAL DERIVATIVES		(d)	
RESIDUALS AND HCANGES IN INITIAL CONDITIONS	(e)	(f)	(g)

(a) Code indicating quantities to be searched for.

(b) Desired values of above quantities - degrees, kilometers, seconds.

(c) Tolerances allowed on above values - degrees, kilometers, seconds.

(d) Matrix with the dependent variables arranged by row. The independent by column.

(e) Residuals (desired-nominal) of quantities designated by the quantity code.

(f) Change required in initial conditions.

(g) Normalized changes in initial quantities in order of the variations.

The option associated with trajectory search routines is initiated by an ID card in the INPUT subroutine (see Section IX, ID = 18)

6. <u>Impact Parameter Output</u>*

SHAT	(a)	
RHAT	(b)	
THAT	(c)	
B	(d)	
B·T, B·R	(e)	(f)

(a) Unit vector in the direction of the incoming asymptote.

(b) Unit vector normal to THAT and the asymptote in a right-hand sense.

(c) Unit vector parallel to the ecliptic (or Moon orbital plane) and perpendicular to the incoming asymptote.

(d) Vector from the body to the vehicle as it crosses the impact plane.

(e) The dot product of R_{VT} at the crossing and THAT.

(f) The dot product of R_{VT} at the crossing and RHAT.

---

* Appendix U

7. Apogee, Perigee, Nodal Crossing Print

Apogee and perigee print times are computed more accurately than formerly, and the nodal crossings are found by iteration. The trigger for apogee is ICANT; for perigee ICPNT; for nodal crossing ICCNT. The apogee and perigee print times are found by fitting a parabola through three neighboring points and determining the minimum or maximum respectively. The crossing time is found iteratively by

$$t_{CN} = t_{CN-1} - \frac{Z_{N-1}}{\dot{Z}_{N-1}}$$

t_{CO} is the time for which Z changed in sign.

XI INTERNAL PROCEDURES

A. Units

The units used internally are Earth Radii and Earth Radii/hour in the Earth and Moon references, and Astronomical Units and Astronomical Units/hour in the Sun, Mercury, Venus thru Pluto Reference systems.

B. Ephemeris Tape

The relative positions of the solar system bodies are obtained from a tape generated by the Jet Propulsion Laboratory. A separate program prepares a binary tape referred to the mean equinox of MID-FILE*, containing 16 days perrecord, in a form compatible with the main program. The Ephemeris Subroutine searches the tape and reads in the proper file and record, keeping 32 days of tables in core storage at a time.

The first record on each file* consists of the year, number of records and number of files* in fixed decimal form. Each of the successive records contains the following information:

Word 1: Initial time of record in hours from base time. (0.0^hUT December 31 of year previous to launch).

---

* Pseudo file. Tape is prepared in overlapping two-year groups.

Equatorial coordinates of Mercury in two-day intervals follow (9 x values, 9 y values, 9 z values). Then 27 consecutive five-word blocks containing the equatorial coordinates, in four-day intervals of

XVNE	YVNE	ZVNE	Venus with respect to the Sun
XSE	YSE	ZSE	Sun with respect to the Earth
XAS	YAS	ZAS	Mars with respect to the Sun
XJS	YJS	ZJS	Jupiter with respect to the Sun
XSAS	YSAS	ZSAS	Saturn with respect to the Sun
XUS	YUS	ZUS	Uranus with respect to the Sun
XNS	YNS	ZNS	Neptune with respect to the Sun
XPS	YPS	ZPS	Pluto with respect to the Sun
XBS	YBS	ZBS	Earth-Moon barycenter with respect to the Sun

are followed by three 32-word blocks containing the equatorial coordinates of the

XME	YME	ZME	Moon with respect to Earth

The Moon coordinates are stored in half-day intervals (0.0^h, $12^h.0$ UT) with distance measured in ER. All other tables are in AU.

The equatorial coordinates of the planets and of the Moon are followed by their velocities, in exactly the same order. Moon velocities are in ER/day. All other velocities are in AU/day.

At present, an ephemeris tape is available for 1965 - 1969, and 1968 - 1982, written in 5 and 15 two year groups respectively, each of which overlaps one year.

C. Ephemeris in Core

The astronomical tables are stored in core in 96-hour intervals for the Sun and the planets, and 12-hour intervals for the Moon. There are always 32-days of tables available, arranged in such a way that the value of time for which the interpolation takes place is not near either end of the table.

In location TABLE(1), the time of the first entry from the initial time is stored. In TABLE(2) to TABLE(10) there are 9 x coordinates of the Sun with respect to the Earth. The following chart indicates the storage locations of the remaining astronomical data to be saved.

TABLE(11)	to	TABLE(19)	y coordinates of the Sun with respect to the Earth
TABLE(20)	to	TABLE(28)	z coordinates of the Sun with respect to the Earth
TABLE(29)	to	TABLE(55)	x, y, z coordinates of Jupiter with respect to the Sun
TABLE(56)	to	TABLE(82)	x, y, z coordinates of Mars with respect to the Sun
TABLE(83)	to	TABLE(109)	x, y, z coordinates of Venus with respect to the Sun
TABLE(110)	to	TABLE(136)	x, y, z coordinates of Saturn with respect to the Sun
TABLE(137)	to	TABLE(163)	x, y, z coordinates of Uranus with respect to the Sun
TABLE(164)	to	TABLE(190)	x, y, z coordinates of Neptune with respect to the Sun
TABLE(191)	to	TABLE(217)	x, y, z coordinates of Pluto with respect to the Sun
TABLE(218)	to	TABLE(244)	x, y, z coordinates of the Earth-Moon Barycenter with respect to the Sun
TABLE(245)	to	TABLE(296)	x, y, z coordinates of Mercury with respect to the Sun
TABLE(299)	to	TABLE(363)	x coordinates of the Moon with respect to the Earth

TABLE(364)	to	TABLE(428)	y coordinates of the Moon with respect to the Earth
TABLE(429)	to	TABLE(493)	z coordinates of the Moon with respect to the Earth

These are followed by the velocities:

TABLE(494)	to	TABLE(520)	$\dot{x}$, $\dot{y}$, $\dot{z}$ coordinates of the Sun with respect to the Earth
TABLE(521)	to	TABLE(547)	$\dot{x}$, $\dot{y}$, $\dot{z}$ coordinates of Jupiter with respect to the Sun
TABLE(548)	to	TABLE(601)	$\dot{x}$, $\dot{y}$, $\dot{z}$ coordinates of Venus with respect to the Sun
TABLE(791)	to	TABLE(985)	$\dot{x}$, $\dot{y}$, $\dot{z}$ coordinates of the Moon with respect to the Earth

D. <u>Perturbation Program</u>

The perturbation program solves three differential equations for XI, ETA, ZETA. The differential equation for XI, with the various terms replaced by the storages containing them, is representative of all three equations and is given below:

$$
\begin{array}{lll}
\mathrm{D2XI} = & - \mathrm{GME} & [\mathrm{VCOR}(1)/\mathrm{VCOR}(4) - \mathrm{COMP}(1)/\mathrm{COMP}(4)] \\
& - \mathrm{GMVN} & [\mathrm{VCOR}(19)/\mathrm{VCOR}(22) - \mathrm{COMP}(19)/\mathrm{COMP}(22)] \\
& - \mathrm{GMS} & [\mathrm{VCOR}(13)/\mathrm{VCOR}(16) - \mathrm{COMP}(13)/\mathrm{COMP}(16)] \\
& - \mathrm{GMMR} & [\mathrm{VCOR}(25)/\mathrm{VCOR}(28) - \mathrm{COMP}(25)/\mathrm{COMP}(28)] \\
& - \mathrm{GMJP} & [\mathrm{VCOR}(31)/\mathrm{VCOR}(34) - \mathrm{COMP}(31)/\mathrm{COMP}(34)] \\
& - \mathrm{GMM} & [\mathrm{VCOR}(7)/\mathrm{VCOR}(10) - \mathrm{COMP}(7)/\mathrm{COMP}(10)] \\
& + \text{OTHER PERTURBATIONS} &
\end{array}
$$

where, for example, in the first term GME = K^2 is the mass of the Earth, and VCOR(4) is the length cubed of the vector [VCOR(1), VCOR(2), VCOR(3)]. Similarly, in the other terms the denominator is the length cubed of the vector containing the corresponding numerator. In the case where the two terms within each of the brackets are nearly equal, they are computed by the special method described in Appendix E to avoid loss of accuracy.

The contents of the COMP storage at any time, t, depends upon the reference origin at that time.

CONTENTS OF COMP STORAGE

Earth Ref.	Moon Ref.	Sun Ref.	Venus Ref.	Mars Ref.	Jupiter Ref.	COMP(I)
XVEO	XME	XSE	XVNE	XMRE	XJPE	(1), (2), (3) = x, y, z (4), (5), (6) = R^3,R,R^2
XEM	XVMO	XSM	XVMM	XMRM	XJPM	(7), (8), (9)
XES	XMS	XVSO	XVNS	XMRS	XJPS	(13), (14), (15)
XEVN	XMVN	XSVN	XVVNO	XMRVN	XJPVN	(19), (20), (21)
XEMR	XMMR	XSMR	XVNMR	XVMRO	XJPMR	(25), (26), (27)
XEJP	XMJP	XSJP	XVNKP	XMRJP	XVJPO	(31), (32), (33)

Here XVE refers to the x component of the vehicle with respect to the Earth, with corresponding definitions for the other quantities. An additional subscript of 0 denotes quantity derived from the two-body problem.

CONTENS OF VCOR STORAGE

All Refs.	VCOR(I)
XVE	(1), (2), (3), (4), (5), (6) = x, y, z, R^3, R, R^2
XVM	(7), (8), (9)
XVS	(13), (14), (15)
XVVN	(19), (20), (21)
XVMR	(25), (26), (27)
XVJP	(31), (32), (33)

XII. REFERENCES

1. Anon.; "Goddard Minimum Variance Orbit Determination Program", Report No. X-640-62-191, NASA Goddard Space Flight Center, Special Projects Branch, Theoret 1 Division, October 18, 1962.

2. Whitney, C. A.; "The Structure of the High Atmosphere," Smithsonian Institution, Astrophysical Observatory, Special Report No. 25, April 20, 1959.

3. Minzner, R. A. and Ripley, W. S.; "The ARDC Model Atmosphere, 1956," AFCRCTW-56-204, Astia Doc. 110233, December 1956.

4. Pines, S., Payne, M. and Wolf, H.; "Comparison of Special Perturbation Methods in Celstial Mechanics, ARL TR 60-281, August 1960.

5. Morrison, J. and Pines, S.; "The Reduction from Geocentric to Geodetic Coordinates," A.J. 66, No. 1, February 1961.

6. Campbell, J., Moore, W. E. and Wolf, H.; "MINMAX - A General Purpose Adaptive Iterator for Nonlinear Problems," Analytical Mechanics Associates, Inc. Report on Contract NAS 9-2527, June 30, 1964.

7. Pines, S., Lefton, L., Levine, N., Whitlock, F.; "A Variable Order Interpolation Scheme for Integrating Second Order Differential Equations," Report on Contract NAS 5-11738, July 1969.

8. Eades, Jr., James B.; "Elements of Orbital Transfer," Virginia Polytechnic Institute, Special Report No. 3, Volume 58, March 1965.

9. Battin, R. H. (1964); "Astronautical Guidance," McGraw-Hill, New York.

10. Kallmann-Bijl, H., Boyd, R. F., Lagow, H., Poloskov, S. M., Priester, W. (1961): "Cospar International Reference Atmosphere".

11. Pines, S., Austin, G.; "Gravitational Acceleration of a Point Mass Due to a Rotating Nonspherical Body," Analytical Mechanics Associates, Inc., Report on Contract NAS 9-7805, May 1969.

XIII. Available Upon Request

Numerous additions and improvements are under development to the current OS/360 version. These additions can be made available for general use to interested organizations.** A brief description of some of the additions are as follows:

1. ltiple Vehicles.

   It is now possible to integrate N trajectories simultaneously (N = 1 to 30). The user has the option of using the two-body solution for all trajectories or separate two-bodies for each trajectory.

2. Lambert's Problem.

   The program has the capability of generating its own initial conditions when one is interested in a specific interplanetary trajectory. This option requires a starting Julian date, a desired flight time, and a target planet. Within this option, there is a further option which computes the initial conditions on the sphere of influence of the Earth or on a parking orbit inside the Earth's sphere of influence.

3. J. P. L. Ephemeris.

   It is now possible to read the J.P.L. Ephemeris directly rather than by the method described in Section XI-1. This capability is obtained by adding a module of subroutines that would permit the trajectories to be integrated with respect to the mean equinox of 1950.

4. Small Corrective Thrust.

5. Trajectory Search

   It is planned to automate the iteration scheme to go from

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two-body, to patched conic, to full trajectory, and to increase the number of variables to be adjusted, in optimal fashion.

Section XIV Methods of Integration

The integration scheme employed by the Interplanetary Trajectory Encke Method program is a sixth order backward difference scheme, initiated by a Runge-Kutta scheme. The routine used is a Newton-Gregory integration scheme for general second order difference equations. (See Appendix C.).

The program has an option (NOPT(71) $\neq$ 0) for integrating Beta instead of time. (See Appendix M.). Computation is much faster in this mode, however, the user is cautioned to choose delta beta with care.

The program has an option (NOPT(20) = 2 or 3) for integrating second order differential equations by means of interpolating a table of second order derivatives. The size of the table and the option of using predictor-corrector, or predictor only, are inputs to the program. (See Appendix J).

MATHEMATICAL APPENDIX

A. INTRODUCTION

The problem of orbit determination over long time periods requires a precise technique for integrating the equations of motion. Reference 4 contains an analysis of an integration procedure that yields the minimum loss of information due to the accumulation of numerical round-off errors. The Encke perturbation method has been shown to require minimum machine computation time for a minimum loss of numerical accuracy. The orbit prediction scheme presented herein uses a modified form of the Encke method with the initial position and velocity vectors replacing the conventional P and Q vectors of the Encke scheme. By avoiding reference to the position of perigee, it is possible to avoid numerical ambiguities arising from near-circular orbits and orbits of low inclination.

B. EQUATIONS OF MOTION

In a Newtonian system, the equations of motion of a particle in the gravitational field of n attracting bodies and subject to other perturbing accelerations such as thrust, drag, oblateness, radiation pressure, etc. are given by

$$\ddot{R}_v = - \sum_{i=1}^{n} \mu_i \frac{R_{vi}}{r_{vi}^3} + \sum_j F_j \tag{B.1}$$

These equations are put into observable form by referring them to a reference body c. The equations of motion of the reference body are

$$\ddot{R}_c = - \sum_{\substack{i=1 \\ i \neq c}}^{n} \mu_i \frac{R_{ci}}{r_{ci}^3} \tag{B.2}$$

Subtraction of Equation (B.2) from Equation (B.1) results in the equations of motion of the vehicle with respect to the reference body c.

$$\ddot{R}_{vc} = - (\mu_v + \mu_c) \frac{R_{vc}}{r_{vc}^3} - \sum_{\substack{i=1 \\ i \neq c}}^{n} \mu_i \left[\frac{R_{vi}}{r_{vi}^3} - \frac{R_{ci}}{r_{ci}^3} \right] + \sum_j F_j \tag{B.3}$$

C. INTEGRATION PROCEDURE

If Equation (B.3) is integrated directly by some numerical scheme, there results, after a number of step-by-step integrations, an accumulation of error which leads to inaccurate results. To avoid this loss in precision, it is convenient to write Equation (B.3) in the form

$$\ddot{R}_{vc} = \ddot{R}_k + \Delta\ddot{R} \tag{C.1}$$

The velocity and displacement vectors can be written as

$$\dot{R}_{vc} = \dot{R}_k + \Delta\dot{R} \tag{C.2}$$

$$R_{vc} = R_k + \Delta R \tag{C.3}$$

The reference body (the one in whose sphere of influence the vehicle travels) is chosen so as to minimize the perturbations.

In this method $\ddot{R}_k$ is taken as

$$\ddot{R}_k = -(\mu_v + \mu_c)\frac{R_k}{r_k^{\,3}} \tag{C.4}$$

and

$$\ddot{\xi} = -(\mu_v + \mu_c)\left[\frac{R_{vc}}{r_{vc}^{\,3}} - \frac{R_k}{r_k^{\,3}}\right] - \sum_{\substack{i=1\\ i\neq c}}^{n}\mu_i\left[\frac{R_{vi}}{r_{vi}^{\,3}} - \frac{R_{ci}}{r_{ci}^{\,3}}\right] + \sum_j F_j \tag{C.5}$$

Equations (C.4) constitute the equations of motion of the Kepler problem and are solved as described in Appendix D.

Equations (C.5) are integrated numerically. The integration scheme employed by the ITEM program is a sixth order backward difference scheme, initiated by a Runge-Kutta scheme. The routine used is a Newton-Gregory integration scheme for second order difference equations written by S. Pines and J. Mohan of Analytical Mechanics Associates, Inc.

As derived in Appendix D, the solution of the Kepler problem may be represented by the vectors R_O, $\dot{R}_O$, the scalar a and the rectification time t_O.

The rectification process consists of moving R_{vc}, $\dot{R}_{vc}$ into the locations R_O and $\dot{R}_O$, t into t_O and the computation of a and n.

For computational convenience, the coefficients appearing in Equations (D.2) are also computed during rectification.

D. SOLUTION OF THE KEPLER TWO-BODY PROBLEM

The unified formulation of the two-body problem is used for both elliptic and hyperbolic cases.

$$\beta = \sqrt{|a|} \cdot \theta$$

$$\alpha = \beta^2 \left(\frac{1}{a}\right)$$

$$F_1(\alpha) = \frac{1}{6} - \frac{\alpha}{120} + \frac{\alpha^2}{5040} - - - - = \sum_{i=0}^{\infty} \frac{(-\alpha)^i}{(2i + 3)!}$$

$$F_2(\alpha) = \frac{1}{2} - \frac{\alpha}{24} + \frac{\alpha^2}{720} - - - - = \sum_{i=0}^{\infty} \frac{(-\alpha)^i}{(2i + 2)!} \qquad \text{(D.1)}$$

$$F_3(\alpha) = 1 - \alpha F_1$$

$$F_4(\alpha) = 1 - \alpha F_2$$

and

$$f = 1 - \frac{1}{r_o} \beta^2 F_2$$

$$g = \frac{r_o}{\sqrt{\mu}} \beta F_3 \quad \frac{d_o \beta^2 F_2}{\mu} \qquad \text{(D.2)}$$

$$\dot{f} = - \frac{\sqrt{\mu}}{r_o} \beta F_3$$

$$\dot{g} = 1 - \frac{1}{r} \beta^2 F_2$$

where

$$d_o = R_o \cdot R_o$$

$$r = \beta^2 F_2 + r_o F_4 + \frac{d_o}{\sqrt{\mu}} \beta F_3$$

$$R = f R_o + g \dot{R}_o$$

$$\dot{R} = \dot{f} R_o + \dot{g} \dot{R}_o$$

$$a = \left(\frac{2}{r_o} - \frac{V_o^2}{\mu} \right)^{-1}$$

$$V_o^2 = \dot{R}_o \cdot \dot{R}_o$$

α is determined from the modified Kepler equation

$$\sqrt{\mu} \quad \Delta t = \beta^3 F_1 + r_o \beta F_3 + \frac{d_o}{\sqrt{\mu}} \beta^2 F_2 \qquad (D.3)$$

See Figure 1 for the two-body orbit which results from the solution of Equation (C.4) with the initial conditions:

$$\begin{aligned} R_k(t_o) &= R_{vc}(t_o) = R_o \\ \dot{R}_k(t_o) &= \dot{R}_{vc}(t_o) = \dot{R}_o \end{aligned} \qquad (D.4)$$

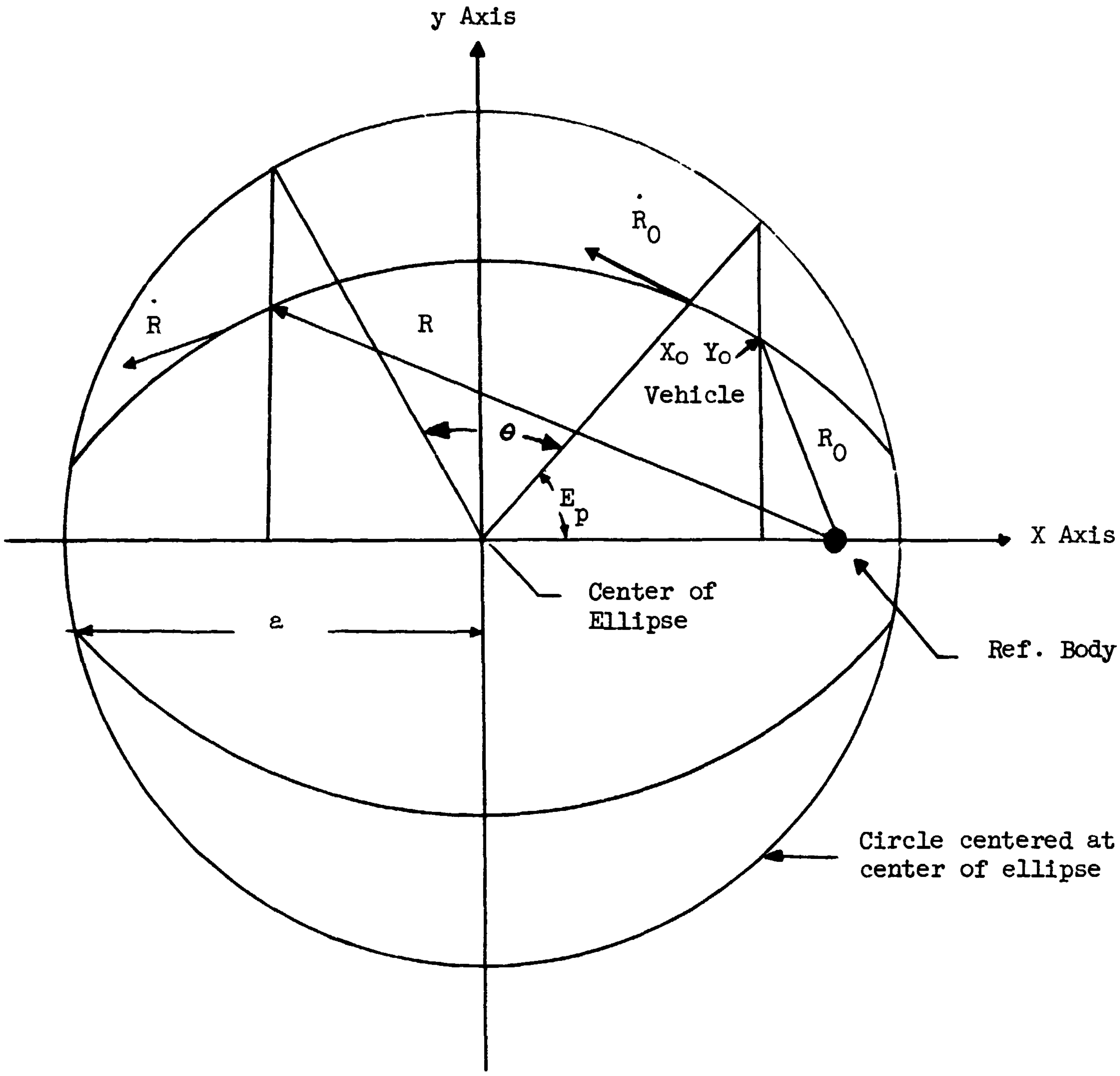


Figure 1. Geometry of the Elliptic Two-Body Orbit

E. COMPUTATION OF PERTURBATION TERMS

The terms accounting for the Encke term and the planetary perturbations appearing on the right hand side f Equation (C.5) involve numerous terms of the form $\frac{R}{r^3} - \frac{R_o}{r_o^3}$ where R and R_o may differ only by small amounts. For the Encke term, for instance $R - R_o = \xi$ which is small, and for the planetary perturbations, the difference is R_{vc} which also often is small.

A computation scheme, which avoids loss of precision due to the subtraction of nearly equal terms and which also is correct when R_{vc} is not small, is employed. This scheme is described below: Find

$$\frac{R}{r^3} - \frac{R_o}{r_o^3}$$

$$u = \frac{2}{r_o^2} \left(R_o + \frac{1}{2} \Delta R\right) \cdot \Delta R \tag{E.1}$$

$$\frac{R}{r^3} - \frac{R_o}{r_o^3} = \frac{\Delta R}{r_o^3} + \frac{R(u^3 + 3u^2 + 3u)}{\left(1 + \frac{r^3}{r_o^3}\right)}$$

F. CONCLUSIONS

The method presented yields accurate trajectories using relatively little computer time. Summarizing some of the important features:

1. All significant solar system bodies may be included without undue complications.

2. Since the perturbations only are integrated, the allowable integration interval is fairly large over most of the path. Even in the vicinity of Earth or another planet a relatively large interval (compared to other schemes) may be used without limiting the stability and accuracy of the solutions.

3. The perturbations are kept small in two ways. First, the two-body orbit is rectified whenever the perturbations exceed a specified maximum value compared to the corresponding unperturbed values. This limits error build-up with respect to particular reference body. Second, the reference body of the two-body problem is changed from Earth, to Sun, to planet accordingly, as that reference body would contribute the largest perturbing force otherwise.

4. This method will handle circular orbits, zero inclination, etc. The problem is defined in terms of parameters which have real physical significance (namely, the position and velocity vectors) which are directly relatable to measurable quantities.

G. OBLATENESS TERMS

A subroutine called GOBL was modifed to obtain, inertial Cartesian coordinates, the gravitational perturbation acceleration due to a rotating nonspherical body whose mass coefficients are given in terms of the zonal and tesseral harmonics. The method described herein avoids the classical singularity, which occurs for polar passage, when using spherical coordinates to describe the gravitational potential. This method also minimizes the numerical error incurred on double application of coordinate transformation from inertial to body-fixed and back again in the case of a rotating nonspherical gravitational body.

DERIVATION OF THE EQUATIONS

The potential at a point R, in the coordinate system fixed in the body, is given by

$$\varphi = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \left[J_n P_n(u) - \sum_{m=1}^{n} P_{n,m}(u)(C_{n,m} \cos m\lambda + S_{n,m} \sin m\lambda)\right]\right\} \quad (1)$$

where

$$u = \frac{z}{r}$$

$$\tan \lambda = \frac{y}{x}$$

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n} (u^2 - 1)^n \quad (2)$$

$$P_{n,m}(u) = (1 - u^2)^{m/2} \frac{d^m}{du^m} P_n(u)$$

The accelerating force vector, in the body-fixed Cartesian coordinates, is given by

$$F = \frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x_i} + \frac{\partial\varphi}{\partial\lambda}\frac{\partial\lambda}{\partial x_i} \quad (3)$$

where

$$\frac{\partial r}{\partial x_i} = \frac{R}{r} = \hat{R}$$

$$\frac{\partial u}{\partial x_i} = \frac{1}{r}\hat{k} - \frac{u}{r}\hat{R} \quad (4)$$

$$\frac{\partial\lambda}{\partial x_i} - \frac{1}{r(1-u^2)}\hat{k} \times \hat{R}$$

$$k = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Combining the scalar coeff..ients of the vectors $\hat{R}$, $\hat{k}$, and $\hat{k} \times \hat{R}$, we have

$$F = \left[-\frac{\mu}{r^2} + \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \left\{ J_n [u P_n' + (n+1) P_n] \right\} - \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \left\{ \sum_{m=1}^{n} [u P_{n,m}' + (n+1) P_{n,m}] [C_{n,m} \cos m\lambda + S_{n,m} \sin m\lambda] \right\} \right] \hat{R}$$

$$- \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \left[J_n P_n' - \sum_{m=1}^{n} P_{n,m}' (C_{n,m} \cos m\lambda + S_{n,m} \sin m\lambda) \right] \hat{k} \tag{5}$$

$$+ \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \left[\sum_{m=1}^{n} \frac{m}{(1-u^2)} P_{n,m} (S_{n,m} \cos m\lambda - C_{n,m} \sin m\lambda) \right] \hat{k} \times \hat{R}$$

Examination of Eq. (5) indicates that, for a polar passage with $u = 1$, a singularity occurs in the coefficient of the $\hat{k} \times \hat{R}$ term and in the derivatives of the associated Legendre polynomials $P_{n,m}(u)$. In order to remove this singularity and to avoid numerical inaccuracy in trajectories close to $u = 1$, it is convenient to change the coordinates from the spherical r, u, λ system to the four-parameter system defined below.

Let the body-fixed Cartesian vector be

$$R = r \begin{bmatrix} s \\ t \\ u \end{bmatrix} \tag{6}$$

where

$$s = \frac{x}{r}$$

$$t = \frac{y}{r} \tag{6a}$$

$$u = \frac{z}{r}$$

In this coordinate system, the potential, φ, is given by

$$\varphi = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n}\left\{J_n A_{n,o}(u) - \sum_{m=1}^{n} A_{n,m}(u)\left[C_{n,m}R_m(s,t) + S_{n,m}I(s,t)\right]\right\}\right] \tag{7}$$

where

$$A_{n,m} = \frac{1}{2^n n!}\frac{d^{n+m}}{du^{n+m}}(u^2 - 1)^n$$

$$R_m(s,t) = \text{Real Part of } (s + i\,t)^m \tag{7a}$$

$$I_m(s,t) = \text{Imaginary Part of } (s + i\,t)^m$$

$$i = \sqrt{-1}$$

In this system, the acceleration force vector in body-fixed Cartesian coordinates is given by

$$F = \frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial\varphi}{\partial s}\frac{\partial s}{\partial x_i} + \frac{\partial\varphi}{\partial t}\frac{\partial t}{\partial x_i} + \frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x_i} \tag{8}$$

where

$$\frac{\partial r}{\partial x_i} = \frac{R}{r} = \hat{R}$$

$$\frac{\partial s}{\partial x_i} = \frac{1}{r}\,\hat{i} - \frac{s}{r}\,\hat{R}$$

$$\frac{\partial t}{\partial x_i} = \frac{1}{r}\,\hat{j} - \frac{t}{r}\,\hat{R} \qquad (8a)$$

$$\frac{\partial u}{\partial x_i} = \frac{1}{r}\,\hat{k} - \frac{u}{r}\,\hat{R}$$

$$\hat{i} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad ; \quad \hat{j} = \begin{bmatrix}0\\ \\0\end{bmatrix} \quad ; \quad \hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Combining the scalar coefficients of the vectors $\hat{R}$, $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$F = \left(\frac{\partial\varphi}{\partial r} - \frac{s}{r}\frac{\partial\varphi}{\partial s} - \frac{t}{r}\frac{\partial\varphi}{\partial t} - \frac{u}{r}\frac{\partial\varphi}{\partial u}\right)\hat{R} + \frac{1}{r}\frac{\partial\varphi}{\partial s}\,\hat{i} + \frac{1}{r}\frac{\partial\varphi}{\partial t}\,\hat{j} + \frac{1}{r}\frac{\partial\varphi}{\partial u}\,\hat{k} \qquad (9)$$

or

$$F = \alpha_r\hat{R} + \alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k} \qquad (9a)$$

For the perturbation coefficient of R, we have

$$\sum_{n=2}^{\infty}\frac{\mu\, a^n}{r^{n+2}}\Big\{J_n\big[u\,A_{n,1}+(n+1)A_{n,o}\big] - \sum_{m=1}^{n}\big[u\,A_{n,m+1}+(n+1)A_{n,m}\big](R_m C_{n,m}+I_m S_{n,m})$$

$$-\sum_{m=1}^{n} m\,A_{n,m}\big[(s\,R_{m-1}-t\,I_{m-1})C_{n,m} + (s\,I_{m-1}+t\,R_{m-1})S_{n,m}\big]\Big\} \qquad (10)$$

However

$$s R_{m-1} - t I_{m-1} = R_m$$
$$s I_{m-1} + t R_{m-1} = I_m \tag{11}$$

Therefore, the two inner summations over the m index can be combined to read

$$-\sum_{m=1}^{n} \left[j A_{n,m+1} + (n+m+1)A_{n,m} \right] (R_m C_{n,m} + I_m S_{n,m}) \tag{10a}$$

Furthermore, if we let

$$J_n = - C_{n,o}$$

the expression for the coefficient of $\hat{R}$ may be written as

$$\alpha_r = -\sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \sum_{m=0}^{n} \left[u A_{n,m+1} + (n+m+1)A_{n,m} \right] (R_m C_{n,m} + I_m S_{n,m}) \tag{10b}$$

A recursion equation for $A_{n,m}$ may be derived, which yields

$$A_{n+1,m+1} = u A_{n,m+1} + (n+m+1)A_{n,m} \tag{12}$$

The final compact expression for the coefficient of $\hat{R}$ is given by

$$\alpha_r = -\sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \sum_{m=0}^{n} A_{n+1,m+1}(u) \left[R_m(s,t)C_{n,m} + I_m(s,t)S_{n,m} \right] \tag{13}$$

Turning to the coefficient of $\hat{k}$, we have

$$\alpha_3 = - \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \left[J_n A_{n,1} - \sum_{m=1}^{n} A_{n,m+1} (R_m C_{n,m} + I_m S_{n,m}) \right] \tag{14}$$

Using the convention that $J_n = -C_{n,o}$, Eq. (13) becomes

$$\alpha_3 = \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \sum_{m=0}^{n} A_{n,m+1} \left[R_m(s,t) C_{n,m} + I_m(s,t) S_{n,m} \right] \tag{14a}$$

The coefficients of $\hat{i}$ and $\hat{j}$, respectively, follow without modification.

$$\alpha_1 = \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \sum_{m=1}^{n} n\, A_{n,m}(u) \left[R_{m-1}(s,t) C_{n,m} + I_{m-1}(s,t) S_{n,m} \right] \tag{15}$$

and

$$\alpha_2 = \sum_{n=2}^{\infty} \frac{\mu a^n}{r^{n+2}} \sum_{n=1}^{n} m\, A_{n,m}(u) \left[R_{m-1}(s,t) S_{n,m} - I_{m-1}(s,t) C_{n,m} \right] \tag{16}$$

RECURSION EQUATIONS FOR THE MODIFIED LEGENDRE POLYNOMIALS AND THEIR DERIVATIVES

We seek a recursion equation for the modified Legendre polynomial, $A_{n,m}$. We have

$$A_{n,m}(u) = \frac{d^m}{du^m} P_n(u) \tag{17}$$

In any standard reference on Legendre polynomials, we may obtain the two recursion equations,

$$(n+1)P_n + u\, P_n' = P_{n+1}' \tag{18}$$

and

$$(2n+1)P_n + P_{n-1}' = P_{n+1}' \tag{19}$$

In terms of $A_{n,m}(u)$, Equations (18) and (19) become

$$(n+1)\ A_{n,0} + u\ A_{n,1} = A_{n+1,1} \qquad (18a$$

and

$$(2n+1)\ A_{n,0} + A_{n-1,1} = A_{n+1,1} \qquad (19a$$

Combining Eqs. (18a) and (19a) and eliminating $A_{n,0}$, we solve for $A_{n+1,1}(u)$ as follows:

$$A_{n+1,1} = P_{n+1}' = u\ (2+\frac{1}{n})P_n' - (1+\frac{1}{n})P_{n-1}' \tag{20}$$

or

$$A_{n+1,1} = u(2 + \frac{1}{n})A_{n,1} - (1 + \frac{1}{n})A_{n-1,1} \qquad (20a)$$

Successive differentiation of Eq. (20a) yields

$$A_{n,m} = (m-1)\left(2 + \frac{1}{n-1}\right)A_{n-1,m-1} + u\left(2 + \frac{1}{n-1}\right)A_{n-1,m} - \left(1 + \frac{1}{n-1}\right)A_{n-2,m} \qquad (21)$$

This is the required recursion equation. Furthermore, we have

$$A_{n,m} = 0, \qquad m > n$$

$$A_{n,n} = 1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot (2n-1) \qquad (22)$$

$$A_{n,n-1} = u\, A_{n,n}$$

Starting with Eq. (22), we may generate each row of $A_{n,m}$, for fixed n, by retaining the two previous rows of $A_{n,m}$ and through application of Eq. (21) for m = n-2, n-3, , , , to m=1. Thus, only three rows of $A_{n,m}$ need be retained for a given α.

To obtain the result given in Eq. (12), we need only differentiate Eq. (18a) and obtain

$$A_{n+1,m+1} = u\, A_{n,m+1} + (n+m+1)A_{n,m} \qquad (12)$$

An alternate recursion formula for $A_{n,m}$, which is more stable than Eq. (21), is

$$A_{n,m} = \frac{1}{n - m}\left(u\, A_{n,m+1} - A_{n-1,m+1}\right) \qquad (21a)$$

RECURSION EQUATIONS FOR R_m AND I_m, AND THE ACCELERATION EQUATIONS FOR A ROTATING BODY

For the zero power of s + it, we have

$$(s + it)^{o} = 1 \tag{23}$$

Thus

$$\begin{aligned} R_0 &= 1 \\ I_0 &= 0 \end{aligned} \tag{23a}$$

From Eq. (11), we obtain the recursion equations for R_m and I_m. Given s and t, we have

$$\begin{aligned} R_m(s,t) &= s\ R_{m-1} - t\ I_{m-1} \\ I_m(s,t) &= S\ I_{m-1} + t\ R_{m-1} \end{aligned} \tag{11}$$

We may define

$$\begin{aligned} D_{n,m} &= R_m(s,t)E_{n,m} + I_m(s,t)S_{n,m} \\ E_{n,m} &= R_{m-1}(s,t)E_{n,m} + I_{m-1}(s,t)S_{n,m} \\ F_{n,m} &= R_{m-1}(s,t)S_{n,m} - I_{m-1}(s,t)C_{n,m} \end{aligned} \tag{24}$$

The final desired form of the equations for α_r, α_1, α_2, and α_3 are given by

$$\alpha_r = -\sum_{n=2}^{\infty} \frac{\mu\, a^n}{r^{n+2}} \sum_{m=0}^{n} A_{n+1,m+1}(u)\; D_{n,m}$$

$$\alpha_1 = \sum_{n=2}^{\infty} \frac{\mu\, a^n}{r^{n+2}} \sum_{m=0}^{n} mA_{n,m}(u)\; E_{n,m}$$

$$\alpha_2 = \sum_{n=2}^{\infty} \frac{\mu\, a^n}{r^{n+2}} \sum_{m=1}^{n} m\, A_{n,m}(u)\; F_{n,m} \tag{25}$$

$$\alpha_3 = \sum_{n=2}^{\infty} \frac{\mu\, a^n}{r^{n+2}} \sum_{m=1}^{n} A_{n,m+1}(u)\; D_{n,m}$$

In the body-fixed system, we have, for the acceleration,

$$F = \alpha_r\, \hat{R} + \alpha_1\, \hat{i} + \alpha_2\, \hat{j} + \alpha_3\, \hat{k} \tag{9a}$$

Let the rotation matrix N(3x3) represent the transformation of a rotating body from inertial to body-fixed coordinates.

$$\hat{R} = N\, \hat{R}_{inert} \tag{26}$$

Then the inertial acceleration is

$$F_{inert} = N^T\, F \tag{27}$$

Or, in the inertial system, we have

$$F_{inert} = \alpha_r\, \hat{R}_{inert} + \alpha_1\, \hat{N} + \alpha_2\, \hat{N}_2 + \alpha_3\, \hat{N}_3 \tag{28}$$

where

$$\hat{N}_1 = \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \end{bmatrix} \tag{29}$$

$$N_3 = \begin{bmatrix} n_{31} \\ n_{32} \\ n_{33} \end{bmatrix}$$

H. TRANSFORMATION EQUATIONS FROM GEODETIC POLAR COORDINATES TO CARTESIAN COORDINATES*

The geodetic polar coordinates in the program are referred to an ellipsoid of revolution. The equation of a cross section is given by

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{H.1}$$

where

$$b^2 = a^2(1 - e^2)$$

The slope of the normal, along which h is measured is given by

$$\tan \varphi = -\frac{1}{\frac{dz}{dx}} = \frac{a^2 z}{b^2 x} \quad \text{(See Figure 2)} \tag{H.2}$$

and

$$\tan \varphi' = \frac{z}{x} = \frac{b^2}{b^2} \tan \varphi = (1 - e^2) \tan \varphi$$

Eliminating x between equations (H.1) and (H.2) and solving for z results in:

$$z = \frac{a(1 - e^2) \sin \varphi}{(1 - e^2 \sin^2 \varphi)}$$

*For geocentric (i.e. $e^2 = 0$) polar coordinates, c = s = 1. In this case the latitude input is interpreted as declination.

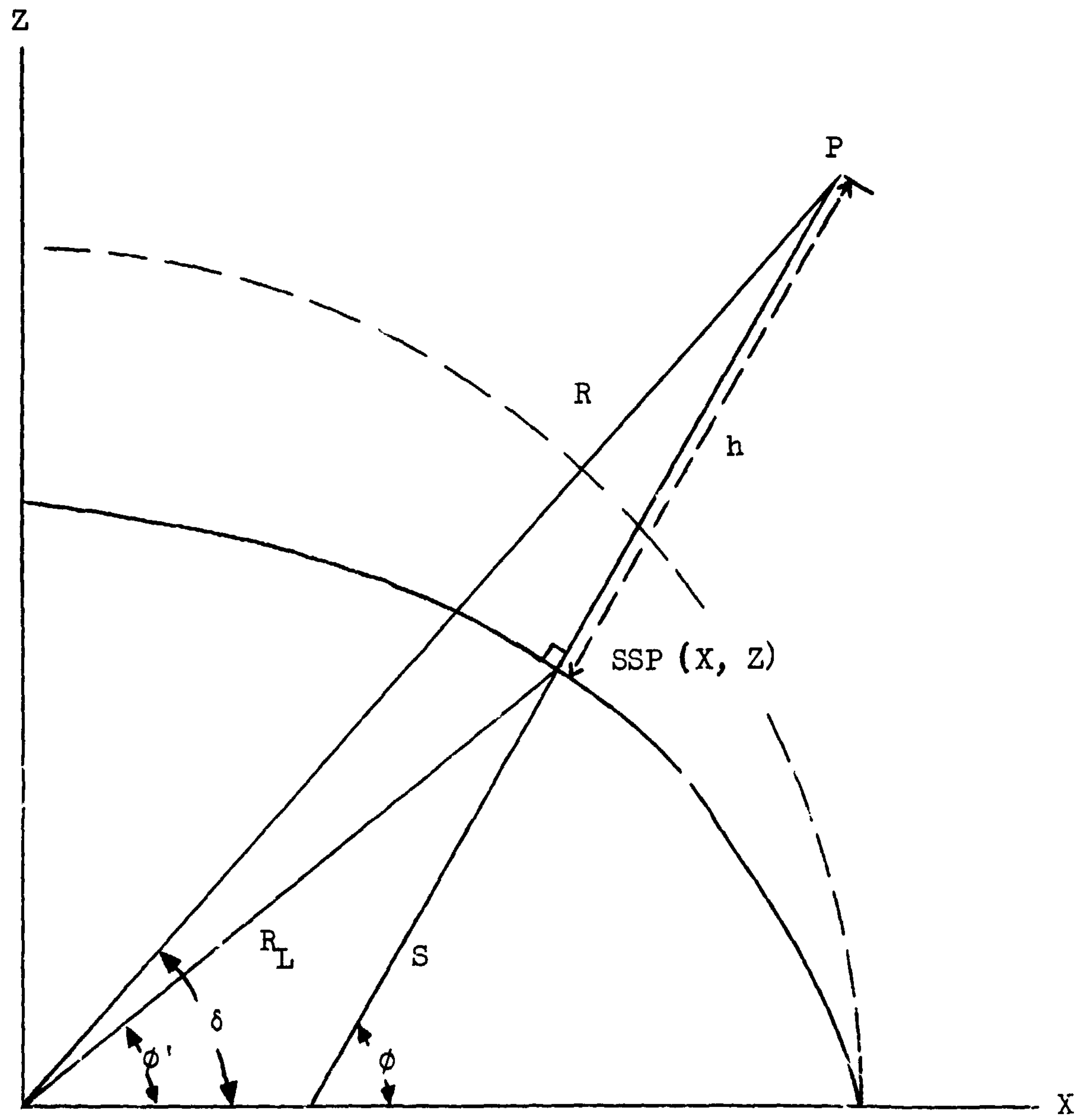


Figure 2. Relation Between Declination, Geocentric and Geodetic Latitudes

and from equation (H.2) then

$$x = \frac{a \cos \varphi}{(1 - e^2 \sin^2 \varphi)^{\frac{1}{2}}}$$

In units of a, R and R are then given by equation (H.3)

$$
\begin{aligned}
e &= (1 - e^2 \sin^2 \varphi)^{-\frac{1}{2}} \\
s &= (1 - e^2)\, c \\
x &= (c + h) \cos \varphi \cos (\theta - \theta_o) \\
y &= (c + h) \cos \varphi \sin (\theta - \theta_o) \\
z &= (s + h) \sin \varphi \\
\dot{x} &= v \Big\{ (\sin \gamma \cos \varphi - \cos \gamma \cos A \sin \varphi) \cos (\theta - \theta_o) \\
&\qquad - \cos \gamma \sin A \sin (\theta - \theta_o) \Big\} \\
\dot{y} &= v \Big\{ (\sin \gamma \cos \varphi - \cos \gamma \cos A \sin \varphi) \sin (\theta - \theta_o) \\
&\qquad + \cos \gamma \sin A \cos (\theta - \theta_o) \Big\} \\
\dot{z} &= v \Big\{ \sin \gamma \sin \varphi + \cos \gamma \cos A \cos \varphi \Big\}
\end{aligned}
\qquad (H.3)
$$

These equations include the effect of the rotation of the earth. The longitude of the vernal equinox (θ_o) at launch time is computed by the program from Newcomb's formula.

I. TRANSFORMATION EQUATIONS FOR RADAR SIMULATION

The program computes sight angles (in an azimuth-elevation system), slant range and range rate data for up to 30 radar stations. The vehicle coordinates are transformed from a system of geocentric cartesian coordinates (xyz), the x-axis in the direction of the vernal equinox and the x-y plane in the equatorial plane of the earth to the required topocentric azimuth elevation system. This is accomplished by a series of coordinate transformations as follows:

1. A rotation of the coordinate system about the z-axis through an angle RA_x so that x y plane is in the meridian plane of the station.

$$
\begin{aligned}
x' &= x \cos RA_s + y \sin RA_s \\
y' &= -x \sin RA_s + y \cos RA_s \\
z' &= z
\end{aligned}
\tag{I.1}
$$

The velocity transformation must take the rotational velocity of the new coordinate system into account.

$$
\begin{aligned}
\dot{x}' &= y' \omega_e + \dot{x} \cos RA_s + \dot{y} \sin RA_s \\
\dot{y}' &= -x' \omega_e - \dot{x} \sin RA_s + \dot{y} \cos RA_s \\
\dot{z}' &= \dot{z}
\end{aligned}
$$

where x', y', z' are the rotated coordinates and RA_s is the right ascension of the station and ω_e is the sidereal rate of the earth's rotation. The G.H.A. necessary to obtain RA_s from the station longitude is computed by the program.

2. A translation of the origin of the coordinate system from the center of the earth to the station in question

$$x'' = x' - (c + h) \cos \varphi$$

$$y'' = y'$$

$$z'' = z' - (s + h) \sin \varphi \qquad (I.2)$$

where

$$c = (1 - e^2 \sin^2 \varphi)^{-\frac{1}{2}}$$

$$s = (1 - e^2)\ c$$

$$\dot{x}'' = \dot{x}'; \qquad \dot{y}'' = \dot{y}'; \qquad \dot{z}'' = \dot{z}'$$

where x'', y'', z'' are the translated coordinates. φ is the geodetic latitude and h the height above sea level of the station in question.

3. A rotation of $(90 - \varphi)$ about the y'' axis to place the (x'',z'') plane into the horizon plane

$$x''' = x'' \sin \varphi + z'' \cos \varphi$$

$$y''' = y''$$

$$z''' = - x'' \cos \varphi + z'' \sin \varphi$$

$$\dot{x}''' = \dot{x}'' \sin \varphi + \dot{z}'' \cos \varphi \qquad (I.3)$$

$$\dot{y}''' = \dot{y}''$$

$$\dot{z}''' = - \dot{x} \cos \varphi + \dot{z}'' \sin \varphi$$

Now x''', y''', z''' are the coordinates of the vehicle in a topocentric azimuth elevation system, with z''' axis pointing to zenith and the x''' pointing south along the meridian. Range, range rate, azimuth and elevation are then given by

$$\rho = (x'''^2 + y'''^2 + z'''^2)^{\frac{1}{2}} = \text{Slant range} \tag{I.4}$$

$$\dot{\rho} = \frac{x'''\,\dot{x}''' + y'''\,\dot{y}''' + z'''\,\dot{z}'''}{\rho} \tag{I.5}$$

$$E = \tan^{-1} \frac{z'''}{(x'''^2 + y'''^2)^{\frac{1}{2}}} = \text{Elevation} \tag{I.6}$$

$$A' = \tan^{-1} \frac{y'''}{x'''}$$

$$A = \left\{ \begin{array}{ll} \pi - A' & A' < \pi \\ 3\pi - A' & A' > \pi \end{array} \right\} \tag{I.7}$$

J. A VARIABLE ORDER INTERPOLATION SCHEME FOR INTEGRATING SECOND ORDER DIFFERENTIAL EQUATIONS

A subroutine called AMAINT was programmed to integrate second order differential equations by means of interpolating a table of second order derivatives. The size of the table and the option of using predictor-corrector, or predictor only, are inputs to the program. This subroutine operates in a fixed step-size mode only.

The program uses a self-starting scheme instead of the usual Runge-Kutta starter to build a table of second derivatives. This scheme employs the following technique; a first guess for the tables is made by stepping up the independent variable and calling the derivative routine; now, using this table and the predictor only formulas, calculate every point on the table in succession. When either a) all of the first derivatives, or b) all of the second derivatives, from one iteration to the next agree to 15 digits, we consider the scheme converged and we now have a starting table. Normally this converges in 8 iterations. For a twelfth-order integrator, our starter routine calls the derivative routine 89 times; the Runge-Kutta starter would require 176 calls. Whenever a step size other than the normal is required, the program can take that step using the stored table of second derivatives, rather than using Runge-Kutta again as is usually done.

Derivation of Equations Used

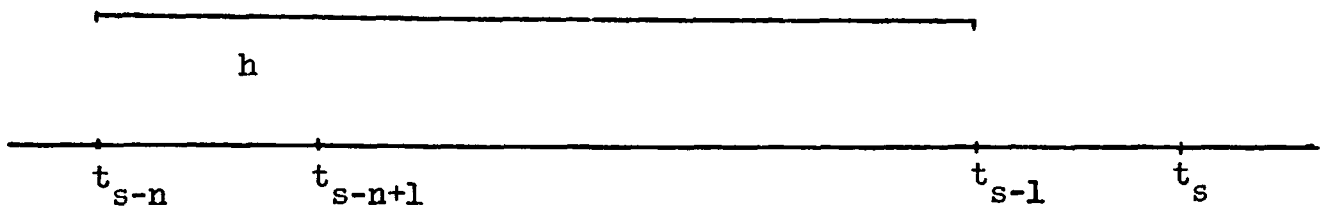
The function to be integrated is:

$$\ddot{y}(t) = f[\dot{y}(t), y(t), t] \tag{1.0}$$

We assume that we have n function values

$$f_{s-n}, ---, f_0, ---, f_{s-1}$$

over some constant increment h of t



where s is chosen to be at the midpoint or close to it. We choose s in such a manner that our coefficients will be integers for a large n. The maximum n depends on the word size of the computer being used.

The function may be extrapolated to t_s using an nth order Lagrange interpolation formula:

$$\ddot{y}(t) = \sum_{j=s-n}^{s-1} \frac{\prod\limits_{\substack{i = s-n \\ i \neq j}}^{s-1} (t - t_i)}{\prod\limits_{\substack{i = s-n \\ i \neq j}}^{s-1} (t_j - t_i)} f_j \tag{1.1}$$

This results in a polynomial in t

$$\ddot{y}(t) = \sum_{j=1}^{n} \left(\sum_{i=n-1}^{0} S_{j,i}\, t^{i} \right) f_j \; ; \qquad S_{j,n-1} = 1$$

Integrating we get:

$$\dot{y}(t) = \sum_{j=1}^{n} \left[\left(\sum_{i=n-1}^{0} \frac{S_{j,i} t^{i+1}}{i+1} \right) f_j + c_j \right]$$

$$y(t) = \sum_{j=1}^{n} \left[\left(\sum_{i=n-1}^{0} \frac{S_{j,i} t^{i+2}}{(i+1)(i+2)} \right) f_j + c_j\, t \right] + d$$

$$y(t_s) = \sum_{j=1}^{n} \left[\sum_{i=n-1} \frac{S_{j,i}(t_s^{\,i+2} - t_{s-1}^{i+2})}{(i+1)\,(i+2)} - (t_s - t_{s-1} \right.$$

$$\left. \sum_{i=n-1}^{0} \frac{S_{j,i} t_{s-1}^{i+1}}{(i+1)} \right] f_j + y(t_{s-1})(t_s - t_{s-1}) + y(t_{s-1}) \qquad (1.2)$$

$$\dot{y}(t_s) = \sum_{j=1}^{n} \left[\sum_{i=n-1} \frac{S_{j,i}(t_s^{i+1} - t_{s-1}^{i+1})}{(i+1)} \right] f_j + y(t_{s-1}) \qquad (1.3)$$

Consider each interval as 1 since they are equally spaced. Now $t_s = h \cdot s$. $S_{j,i}$ is an nxn matrix $S \cdot D_{nxn}$ where S_{nxn} is formed in the following manner: The first row is all one's; the second is (-) the sum of the t_i's of Equation (1.1) in the numerators of the varying j's; the next is the sum of the products of the t_i's taken two at a time; etc. Thus we have a matrix of the coefficients of the polynomials formed by the numerators of (1.1) sitting column-wise. The D matrix is a diagonal matrix whose elements are the denominators of (1.1).

We now form matrices A and B where

$$A_{nxn} = a_{ij} = \frac{(s-n-1+i+e_k)^{n+1-j} - (s-n-1+i)^{n+1-j}}{n+1-j}$$

and

$$B_{nxn} = b_{ij} = \frac{(s-n-1+i+e_k)^{n+2-j} - (s-n-1+i)^{n+2-j} - (n+2-j)e_k(s-n-1+i)^{n+1-j}}{(n+1-j)(n+2-j)}$$

Now

$$ASD = \alpha_{ij} \qquad BSD = \beta_{ij}$$

Equation (1.2) becomes

$$y(t_s) = (y(t_{s-1}) + h_1\, \dot{y}(t_{s-1}) + h^2 \sum_{j=1}^{n} \beta_{n,j} f_j \tag{2.1}$$

and Equation (1.3) becomes

$$y(t_s) = y(t_{s-1}) + h \sum_{j=1}^{n} \alpha_{n,j} f_j \tag{2.2}$$

where $e_k = h_1/h$, $h_1 \neq h$ when $t_s - t_{s-1} \neq h$. These are the predictor equations.

If we desire a corrector formula, we generate

$$\alpha^c_{n,j} \quad \text{and} \quad \beta^c_{n,j} \qquad j = 1, n+1$$

We use Eqs. (2.1) and (2.2) to find $y(t_s)$ and $\dot{y}(t_s)$. Then using Eq.

(1.0), we find $\ddot{y}(t_s)$. Now we have a table of n+1 f_j's and we can find our corrected functions and first derivatives by:

$$y_c(t_s) = y(t_{s-1}) + h\,\dot{y}(t_{s-1}) + h^2 \sum_{j=1}^{n+1} \beta^c_{n,j}\, f_j$$

$$\dot{y}_c(t_s) = y(t_{s-1}) + h \sum_{j=1}^{n+1} \alpha^c_{n,j}\, f_j$$

When $h_1 \neq h$, the corrector formulas are not valid. Therefore, when an odd integration step is required, predictor only may be used.

The α_{ij} and β_{ij} matrices are used to generate the initial table of f_j's at each restart by means of an iterative method. In generating the coefficients α_{ij}, β_{ij}, $\alpha^c_{n,j}$, $\beta^c_{n,j}$ on a computer, we attempt to keep then integers by limiting n to conform to the word size of the computer being used. We also use a least common denominator technique to perform one, instead of n, divisions. Thus we see that the word size of the computer controls the order of the integrator to be used. When our constants do not sit as integers in the machine, the resulting round-off causes biased errors.

A study was made using this routine on an IBM 360 Model 91 computer to integrate the sine and cosine functions. The Univac 1108 computer was also used to runa a small number of cases. Orders from T to 12, and integration intervals from $\pi/16$ to $\pi/526$, predictor only and precitor-corrector, were used with the following results:

n = Number of points (not differences)

Predictor-Corrector	Δt	Sin π
n = 6	$\pi/32$	$.68 \times 10^{-9}$
	$\pi/64$	$.27 \times 10^{-11}$
	$\pi/128$	$.12 \times 10^{-13}$
	$\pi/256$	$.28 \times 10^{-14}$
	$\pi/512$	$.53 \times 10^{-14}$
n = 7	$\pi/32$	$.75 \times 10^{-10}$
	$\pi/64$	$.40 \times 10^{-12}$
	$\pi/128$	$.12 \times 10^{-13}$
	$\pi/256$	$.29 \times 10^{-14}$
	$\pi/512$	$.54 \times 10^{-14}$
n = 8	$\pi/32$	$.58 \times 10^{-11}$
	$\pi/64$	$.49 \times 10^{-14}$
	$\pi/128$	$.16 \times 10^{-14}$
	$\pi/256$	$.26 \times 10^{-14}$
	$\pi/512$	$.49 \times 10^{-14}$
n = 9	$\pi/32$	$.12 \times 10^{-12}$
	$\pi/64$	$.18 \times 10^{-14}$
	$\pi/128$	$.17 \times 10^{-14}$
	$\pi/256$	$.30 \times 10^{-14}$
n = 10	$\pi/32$	$.52 \times 10^{-13}$
	$\pi/64$	$.11 \times 10^{-14}$
	$\pi/128$	$.19 \times 10^{-14}$
	$\pi/256$	$.31 \times 10^{-14}$

Predictor-Corrector	Δt	Sin π
n = 11	$\pi/16$	$.12 \times 10^{-11}$
	$\pi/32$	$.84 \times 10^{-15}$
	$\pi/64$	$.11 \times 10^{-14}$
	$\pi/128$	$.18 \times 10^{-14}$
n = 12	$\pi/16$	$.34 \times 10^{-11}$
	$\pi/32$	$.81 \times 10^{-13}$
	$\pi/64$	$.74 \times 10^{-13}$
	$\pi/128$	$.71 \times 10^{-13}$
Predictor Only		
n = 6	$\pi/32$	$.42 \times 10^{-6}$
	$\pi/64$	$.69 \times 10^{-8}$
	$\pi/128$	$.11 \times 10^{-9}$
	$\pi/256$	$.17 \times 10^{-11}$
	$\pi/512$	$.21 \times 10^{-13}$
n = 7	$\pi/32$	$.84 \times 10^{-8}$
	$\pi/64$	$.43 \times 10^{-10}$
	$\pi/128$	$.19 \times 10^{-12}$
	$\pi/256$	$.18 \times 10^{-14}$
	$\pi/512$	$.45 \times 10^{-14}$
n = 8	$\pi/32$	$.37 \times 10^{-10}$
	$\pi/64$	$.15 \times 10^{-10}$
	$\pi/128$	$.62 \times 10^{-13}$
	$\pi/256$	$.31 \times 10^{-14}$
	$\pi/512$	$.51 \times 10^{-14}$

Predictor Only	Δt	Sin π
n = 9	$\pi/32$	$.82 \times 10^{-10}$
	$\pi/64$	$.11 \times 10^{-12}$
	$\pi/128$	$.14 \times 10^{-14}$
	$\pi/256$	$.25 \times 10^{-14}$
	$\pi/512$	$.46 \times 10^{-14}$
n = 10	$\pi/16$	$.16 \times 10^{-7}$
	$\pi/32$	$.31 \times 10^{-10}$
	$\pi/64$	$.34 \times 10^{-13}$
	$\pi/128$	$.15 \times 10^{-14}$
n = 11	$\pi/16$	$.48 \times 10^{-9}$
	$\pi/32$	$.73 \times 10^{-12}$
	$\pi/64$	$.66 \times 10^{-16}$
	$\pi/128$	$.39 \times 10^{-15}$
n = 12	$\pi/16$	$.33 \times 10^{-9}$
	π [illegible]	$.23 \times 10^{-12}$
	$\pi/64$	$.40 \times 10^{-13}$
	$\pi/128$	$.41 \times 10^{-13}$

Univac 1108 Only

Predictor-Corrector	Δt	Sin π
n = 12	$\pi/16$	$.34 \times 10^{-11}$
	$\pi/32$	$.17 \times 10^{-15}$
	$\pi/64$	$.27 \times 10^{-15}$
	$\pi/128$	$.29 \times 10^{-15}$

From the above we see that, as we increase n, we can also increase the Δt up to a point. The optimum appears at n = 11 for the IBM 360 Model 91. On the Univac 1108, the optimum was at n = 12. When using the predictor-corrector method, we can maintain the same accuracy as predictor only, using twice the integration interval at the optimum n. Therefore, we conclude that using the predictor-corrector judiciously (i.e., with the proper integration interval for the function to be integrated) may prove to be better than predictor only, with a very slight cost in computer time.

Our optimum Δt for the sine function was $\pi/64$ for predictor only and $\pi/32$ for predictor-corrector. The preceding data are true for the IBM Model 91 computer and for the function tested. We believe that, with a computer having a larger word size capacity, we would find a larger n to be optimal since it would be possible to generate integers for the integration coefficients.

A change in the hardware of the IBM 360 Model 91 was made which increased precision in multiplication. Using this machine, we then ran trajectories similar to those discussed in the Fehlberg paper.[1] Instead of using a rotating coordinate system, we used an inertial coordinate system and integrated in an Encke mode, using β as an independent variable instead of time. We converted Fehlberg's initial conditions ($x_0 = 1.2$, $\dot{x}_0 = 0$, $y_0 = 0$, $\dot{y}_0 = 1.049357509830320$, $\mu_m = 1/82.45$) to our system and they became

$$x_0 = 1.212128562765311$$
$$\dot{x}_0 = 0.$$
$$y_0 = 0.$$
$$\dot{y}_0 = .162771052934991$$
$$\mu_m = 1/82.45$$
$$\mu_e = .987871447234689$$
$$x_{o_m} = 1.$$
$$\dot{x}_{o_m} = 0$$
$$y_{o_m} = 0.$$
$$\dot{y}_{o_m} = 1$$

(The o_m subscript denotes the Moon's initial conditions.)

The starting point of our trajectories was at apogee behind the Moon, on the line of centers of the Earth and Moon. Our integrations traced a figure eight around the Moon and terminated at the second apogee. We ran six different trajectories, using 3 different $\Delta\beta$'s with 11th and 12th order predictor-corrector schemes for each. The results were as follows:

---

1 "New One-Step Integration Methods of High-Order Accuracy Applied to Some Problems of Celestial Mechanics", Erwin Fehlberg, NASA George C. Marshall Space Flight Center, Huntsville, Ala.

$\Delta\beta$	No. of Steps	Order of Integration	Starting Apogee	2nd Apogee
.0078125	1258	12	1.2128562765311	1.2128562765114
.0078125	1258	11	1.2128562765311	1.2128562764704
.00390625	2515	12	1.2128562765311	1.2128562765340
.00390625	2515	11	1.2128562765311	1.2128562765350
.001953125	5028	12	1.2128562765311	1.2128562765353
.001953125	5028	11	1.2128562765311	1.2128562765368

For this particular problem, $\Delta\beta$ = .00390625 with the 12th order predictor-corrector scheme appears to be the optimum trajectory.

Use of Routine:

In order to utilize this subroutine, a programmer must supply the following common cards:

```
COMMON/AM1/DELT, T, DTI, XI(30), XID(30), D2XI(30), IFST(30).
COMMON/AMI/M(14), IS, N1, N2, N21, NEQN.
```

The common names may not be changed.

The variable names may be changed providing the notation for integer or real numbers is maintained.

M(14) is a dummy block of storage used internally by the routine.

IS is the reference point for generating the integration coefficients.

N1 is the order of integration.

N2 = N1+1 is used for the corrector.

N21 = 0 is used for the predictor only.

N21 = 1 is used for the predictor-corrector.

NEQN is the number of equations to be integrated.

DELT is the normal integration interval.

T is the independent variable.

DTI is the current integration interval, oddball or normal.

XI(I) is the i^{th} function value.

XID(I) is the i^{th} first derivative.

D2XI(I) is the i^{th} second derivative.

IFST(I) = 0 means the i^{th} equation is second order.

IFST(I) = 1 means the i^{th} equation is of first order.

For COMMON AMI, the user must define N1, N21, and NEQN in his program.

For COMMON AM1, the user also must define DELT, DTI, XI(I), XID(I), D2XI(I), IFST(I) (I = 1, NEQN) at T = T_0.

The user must also supply a subroutine called DERIV, with the proper COMMON included and which computes D2XI(I) as a function of XI(I), XID(I) and T. If first order equations are desired, the first derivative should be stored in D2XI(I), and D2XI(I) should be a function of XI(I) and T.

The following calls to AMAINT are used:

Call AMAINT (0) - initializes the integrator and calls DERIV.

Call AMAINT (3) - generates integration table. DELT may be changed before this call and the user must make sure that the proper Call DERIV has been made.

Call AMAINT (1) - takes one normal integration step and calls DERIV.

Call AMAINT (2) - requires that DTI be defined prior to the call, takes an integration step of size DTI, and calls the derivative routine.

K. DRAG COMPUTATION

The drag acceleration is computed, assuming a spherically symmetric atmosphere rotating with the earth. Thus:

$$D = -1/2 \, \rho \, A \, |V_{eff}| V_{eff}$$

$$\Delta \ddot{R}_D = \frac{D}{m} \qquad (K.1)$$

where

$$V_{eff} = \dot{R} - \omega \times R$$

ω is the sideral rotation rate vector of the Earth.

L. COMPUTATION OF SUBSATELLITE POINT

The geodetic coordinates of the subsatellite point are computed by the following method:

The geocentric latitude (declination) is obtained from

$$\sin \varphi' = \frac{z}{r} \qquad \text{(L.1)}$$

This latitude is then corrected to geodetic latitude by the formula

$$\varphi = \varphi' + a_2 \sin 2\varphi' + a_4 \sin 4\varphi' + a_6 \sin 6\varphi' + a_8 \sin 8\varphi' \qquad \text{(L.2)}$$

where

$$a_2 = \frac{1}{1024r}\left\{512e^2 + 128e^4 + 60e^6 + 35e^8\right\}$$

$$+ \frac{1}{32r^2}\left\{e^6 + e^8\right\} - \frac{3}{256r^3}\left\{4e^6 + 3e^8\right\}$$

$$a_4 = -\frac{1}{1024r}\left\{64e^4 + 48e^6 + 35e^8\right\}$$

$$+ \frac{1}{16r^2}\left\{4e^4 + 2e^6 + e^8\right\} + \frac{15e^8}{256r^3} - \frac{e^8}{16r^4}$$

$$a_6 = \frac{3}{1024r}\left\{4e^6 + 5e^8\right\} - \frac{3}{32r^2}\left\{e^6 + e^8\right\}$$

$$+ \frac{35}{768r^3}\left\{4e^6 + 3e^8\right\} \qquad \text{(L.3)}$$

$$a_8 = \frac{e^8}{2048}\left\{-\frac{5}{r} + \frac{64}{r^2} - \frac{252}{r^3} + \frac{320}{r^4}\right\}$$

e = the eccentricity of the earth

r = the distance from earth's center

See Reference 5.

The geodetic height is then given by

$$h = r \cos (\varphi - \varphi') - \sqrt{1-e^2\sin^2\varphi} \qquad (L.4)$$

The longitude is obtained by subtracting the sidereal time of Greenwich from the right ascension given by

$$\tan RA = \frac{y}{x} \qquad (L.5)$$

M. CHANGE OF INDEPENDENT VARIABLE - BETA MODE

According to the standard Encke method, we introduce a differential equation

$$\ddot{\rho} = -\mu \frac{\rho}{|\rho|^3} \tag{Y.1}$$

In the construction of the closed-form solution for (Y.1), a parameter β arises. It is related to t by Kepler's equation,

$$t = t_0 + \frac{f(\beta)}{\sqrt{\mu}} \tag{Y.2}$$

where f is a transcendental function of β and is obtained by summing several power series.

If t is taken as the independent variable, Equation (Y.2) has to be solved for β by an iterative method, requiring numerous time-consuming evaluations of the function f for each integration step. Using β as the independent variable, however, only requires a single evaluation.

It remains, of course, to see what becomes of equation (Y.1) and

$$\ddot{\xi} = \ddot{x} - \ddot{\rho}$$

$$= -\mu \left(\frac{x}{|x|^3} - \frac{\rho}{|\rho|^3} \right) + F \tag{Y.3}$$

If β is the independent variable. We have, from Kepler's equation, that

$$\frac{dt}{d\beta} = \frac{|\rho|}{\sqrt{\mu}} \tag{Y.4}$$

at any point along the solution of (Y.1). Thus

$$\rho = \rho' \frac{\sqrt{\mu}}{|\rho|} \text{ and } \rho' = \rho \frac{|\rho|}{\sqrt{\mu}}$$

at any point along the solution of (Y.1) and the initial conditions become

$$\rho(\beta_0) = x_0 \quad \text{and} \quad \rho'(\beta_0) = \frac{\dot{x}_0 |x_0|}{\sqrt{\mu}}$$

when

$$\beta_0 = \beta(t_0) = 0$$

Now the solution for (Y.1), ρ and ρ', can be written in closed form for any β. As auxiliary quantities in this solution, we have $|\rho|$ and

$D = \frac{\rho \cdot \rho'}{|\rho|}$. They are computed as functions of β before ρ and ρ' are known; that is, with accuracy at least as good as that of ρ and ρ'. Not only are they needed and easy to compute, but they also have the interesting property that

$$\frac{dt}{d\beta} = \frac{|\rho|}{\sqrt{\mu}}$$

and (Y.5)

$$\frac{d^2t}{d\beta^2} = \frac{D}{\sqrt{\mu}}$$

Thus Equation (Y.1) is solved more economically in terms of β than in terms of t.

Now we turn to Equation (Y.3). To treat it, we want to express ξ'' in terms of $\ddot{\xi}$. From (Y.5) we have that

$$\xi' = \dot{\xi}\,\frac{dt}{d\beta} = \dot{\xi}\,\frac{|\rho|}{\sqrt{\mu}} \tag{Y.6}$$

Differentiating with respect to β,

$$\xi'' = \frac{d\dot{\xi}}{d\beta}\,\frac{|\rho|}{\sqrt{\mu}} + \dot{\xi}\,\frac{d}{d\beta}\left(\frac{|\rho|}{\sqrt{\mu}}\right)$$

$$= \ddot{\xi}\left(\frac{|\rho|}{\sqrt{\mu}}\right)^2 + \xi'\,\frac{\sqrt{\mu}}{|\rho|}\,\frac{D}{\sqrt{\mu}}$$

$$= -\,|\rho|^2\left(\frac{x}{|x|^3} - \frac{\rho}{|\rho|^3}\right) + \frac{|\rho|^2}{\mu}\,F + \xi'\,\frac{D}{|\rho|} \tag{Y.7}$$

Thus (Y.7) is the equation to be integrated numerically, instead of (Y.3). The coefficients $\frac{|\rho|^2}{\mu}$ and $\frac{D}{|\rho|}$ can be calculated with much more accuracy than the factors involving ξ, since they depend only on the two-body solution. For analysis of error propagation, we write (Y.7) as

$$\xi'' = -\,\frac{1}{|\rho|}\left[(\rho + \xi)\,\frac{|\rho|^3}{|\rho + \xi|^3} - \rho\right] + \frac{|\rho|^2}{\mu}\,F + \xi'\,\frac{D}{|\rho|} \tag{Y.8}$$

The mechanics of the procedure, then, are easy to enumerate. The initial conditions are x_0 and $\dot{x}_0$. Let

$$\rho(t_0) = x_0$$
$$\rho'(t_0) = \frac{\dot{x}_0|x_0|}{\sqrt{\mu}} \qquad (Y.9)$$

Using these initial conditions, evaluate t, $\frac{|\rho|^2}{\mu}$, $\frac{D}{|\rho|}$, ρ, ρ' for each value of β to be considered.

Let $\xi_0 = \xi_0' = 0$. Using these initial conditions, integrate Equation (Y.7) to get $\xi(\beta)$ and $\xi'(\beta)$. Note that the first two terms on the right-hand side of Equation (Y.7) are functions of x and possibly x'. These are obtained by

$$x(\beta) = \rho(\beta) + \xi(\beta)$$
$$x'(\beta) = \rho'(\beta) + \xi'(\beta) \qquad (Y.10)$$

If at any point $\dot{x}$ is required, it can be found from

$$\dot{x}[t(\beta)] = x'(\beta) \frac{\sqrt{\mu}}{|\rho(\beta)|} \qquad (Y.11)$$

Depending on the recitification control logic, there will be places where the solution to Equation (Y.1) must be started over. At this point, the values t, x, $\dot{x}$ become the new t_0, x_0, and $\dot{x}_0$, while β, ξ, and ξ' are reset to zero.

...arison of Modes

a) It is immediately apparent that eliminating the necessity of iteratively solving Equation (Y.2) will substantially increase the speed of computation.

b) An important advantage arises further from eliminating the sometimes ponderous logic which supplies initial guesses for the iterative process and guarantees convergence of the solution.

c) A third advantage of the β-method is not quite so apparent, but no less important. It is well known that the size of the integration time step can be increased as the distance from the center of attraction increases. This change of the integration interval requires a cumbersome restart procedure. An examination of Equation (Y.5) shows that equal intervals of β correspond to tim[illegible]ervals of increasing lengths as the distance increases. The time interval thus automatically expands and contracts correctly without outside intervention.

d) Geometric stopping and printing conditions can [illegible]ally be conveniently expressed in terms of β, whereas they often require iterative determinations of the time. This advantage, however, is slight.

e) If state vectors are required at fixed times, an iteration is necessary to find the corresponding value of β. In this case the β-method is no better, and no worse, than the standard methods. If such vectors are required at frequent, closely spaced, time points (as in orbit determination, for instance), the advantage of the β-method is marginal.

N. SHADOW LOGIC

A coordinate system is set up in the plane defined by the centers of the light-emitting source, the shadowing body, and the probe. Both bodies are assumed to be spherical, and hence all testing can be carried out in this plane. The diagram in Figure 3 shows this plane.

The coordinates are defined by unit vectors i and j:

$$\underline{i} = \frac{\underline{R}_{c\ell}}{r_{c\ell}}; \qquad \underline{i} \cdot \underline{i} = 1; \qquad \underline{i} \cdot \underline{j} = 0 \tag{N.1}$$

$$\ell \underline{j} = - d\underline{i} + \underline{R}_{vc}; \qquad \underline{j} \cdot \underline{j} = 1 \tag{N.2}$$

where

$$d = \underline{R}_{vc} \cdot \underline{i}$$

Vehicle coordinates in this system are given by:

$$x_v = \underline{R}_{vc} \cdot \underline{i} = d \tag{N.3}$$

$$y_v = \underline{R}_{vc} \cdot \underline{j} = [-d^2 + r_{vc}^{\ 2}]^{1/2} \tag{N.4}$$

$$z_v = \underline{R}_{vc} \cdot \underline{K} = 0 \tag{N.5}$$

1. Shadow Parameters

a) The tips of the umbra and penumbra cones are:

$$d_u = \frac{r_{c\ell}}{\frac{r_\ell}{r_c} - 1} , \qquad d_p = -\frac{r_{c\ell}}{\frac{r_\ell}{r_c} + 1}$$

b) The slopes of the bounding lines are:

$$\sin \alpha_u = \cos \theta_u = \frac{r_c}{d_u}$$

$$-\cos \alpha_u = \sin \theta_u = \left[1 - \left(\frac{r_c}{d_u}\right)^2\right]^{\frac{1}{2}}$$

$$\tan \alpha_u = \frac{\sin \alpha_u}{\cos \alpha_u}$$

$$\tan \theta_u = \frac{\sin \theta_u}{\cos \theta_u}$$

$$\cos \theta_p = \sin \alpha_p = \frac{r_c}{|d_p|}$$

$$\sin \theta_p = \cos \alpha_p = \left[1 - \left(\frac{r_c}{d_p}\right)^2\right]^{\frac{1}{2}}$$

$$\tan \alpha_p = \frac{\sin \alpha_p}{\cos \alpha_p}$$

$$\tan \theta_p = \frac{\sin \theta_p}{\cos \theta_p}$$

c) Refraction Correction: (UMBRA)

$$\alpha'_u = \alpha_u - \epsilon , \qquad \theta'_u = \theta_u - \epsilon$$

$$\sin \alpha'_u = \sin \alpha_u \cos \epsilon - \cos \alpha_u \sin \epsilon$$

$$\tan \alpha'_u = \frac{\tan \alpha_u - \tan \epsilon}{1 + \tan \alpha_u \tan \epsilon}$$

$$\tan \theta'_u = \frac{\tan \theta_u - \tan \epsilon}{1 + \tan \theta_u \tan \epsilon}$$

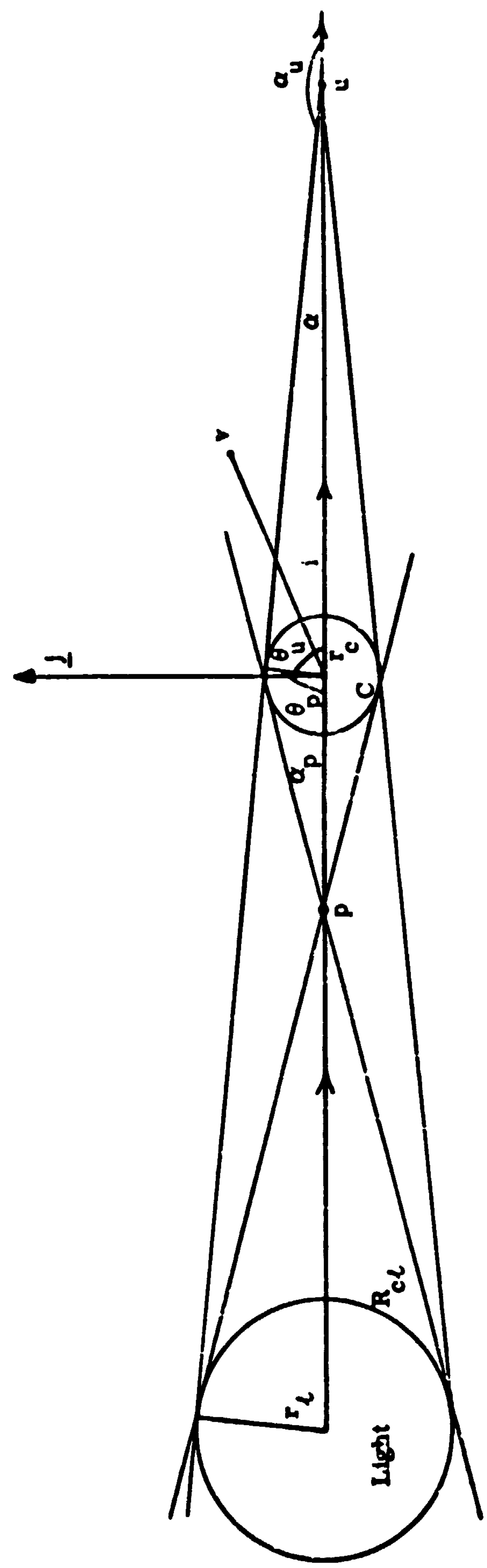


Figure 3.

$$d'_u = \frac{r_c}{\sin \alpha'_u}$$

d) Refraction Correction: (PENUMBRA)

Both $\epsilon < \alpha_p$ $\epsilon > \alpha_p$; $\alpha'_p = \alpha_p - \epsilon$

$$\sin \alpha'_p = |\sin \alpha_p \cos \epsilon - \cos \alpha_p \sin \epsilon|$$

$$\tan \alpha'_p = \frac{\tan \alpha_p - \tan \epsilon}{1 + \tan \alpha_p \tan \epsilon}$$

$$\tan \theta'_p = \frac{\tan \theta_p - \tan \epsilon}{1 + \tan \theta_p \tan \epsilon}$$

$$d'_p = - \text{sign} (\tan \alpha'_p) \frac{r_c}{\sin \alpha'_p}$$

The equations of the bounding lines are given below.

2. <u>The Testing Procedure</u>

θ_p Line

$$Q_1 = \begin{cases} \dfrac{|y_v|}{\tan \theta'_p} - x_v \geq 0 & \text{Sunlight} \\[2ex] \dfrac{|y_v|}{\tan \theta'_p} - x_v < 0 & \text{Go to next test} \end{cases}$$

$$Q_2 = \begin{cases} |y_v| - (x_v - d_p') \tan \alpha_p' > 0 & \text{Sunlight} \\ |y_v| - (x_v - d_p') \tan \alpha_p' = 0 & \text{Sunlight penumbra boundary} \\ < 0 & \text{Go to next test} \end{cases}$$

If $R_\ell = 0$, exit here.

$$Q_3 = \frac{|y_v|}{\tan \theta_u'} - x_v \quad \begin{cases} \geq 0 & \text{Penumbra} \\ < 0 & \text{Go to next test} \end{cases}$$

$$Q_4 = |y_v| - (x_v - d_u') \tan \alpha_u' \quad \begin{cases} > 0 & \text{Penumbra} \\ = 0 & \text{Shadow penumbra boundary} \\ < 0 & \text{Shadow} \end{cases}$$

Q_2 and Q_4 are stored and saved. The crossing times are found by linearly interpolating for 0-values of Q_2 and Q_4 respectively, to guarantee that crossing from one region into another always occurs across these boundaries.

O. SOLAR RADIATION PRESSURE

The radiation pressure subroutine computes the force of solar radiation on the spacecraft if an appropriate pressure coefficient is used. The calculation relies on the shadow routine to set a trigger to multiply the pressure coefficient by 1.0, 0.5, or 0.0 for full sunlight, penumbra or umbra respectively. Therefore, the shadow subroutine must be used in conjunction with the radiation pressure routine for most cases. If the spacecraft is known to be continually in sunlight See Section VIII-2 to avoid elaborate shadow testing.

$$P_{RP} = \frac{C_R A R_{VS}}{m\, r^3_{VS}} \tag{O.1}$$

(See Section VIII-A-2 for definition of symbols.)

This radiation pressure subroutine has been found to be inexact for satellites of large area to mass ratio since it only controls the pressure to the nearest integration step. For such spacecraft (e.g. balloons) several degrees error in true anomaly may result after 100 days unless the integration is carried exactly to the boundaries. A modification to achieve this increased precision is available and will be included in future versions of the program.

P. ECLIPTIC COORDINATES

The ecliptic coordinates are an approximate set obtained by a simple rotation of the equatorial coordinates about the x-axis through a fixed angle $i = 23^{\circ}26'35''$ which is approximately the true obliquity for Jan 0.0, 1970. More exact coordinates may be obtained by changing SNE (unit normal to the ecliptic) as desired.

Q. MOON ROTATING AND FIXED COORDINATE SYSTEM

Geocentric coordinates of the vehicle based on the earth-moon plane are generated from the geocentric equatorial radius vector to the vehicle, R_{VE}, the geocentric unit vector in the direction of the moon R_{ME}, and the vector in the direction of the moon's velocity, $\dot{R}_{ME}$.

Coordinates in the rotating system, XROT etc., are found by using the current values of these vectors at each time step in the relations

$$XROT = R_{VE} \cdot \hat{R}_{ME}$$

$$YROT = R_{VE} \cdot \left(\hat{H}_{ME} \times \hat{R}_{ME}\right)$$

$$ZROY = R_{VE} \cdot \hat{H}_{ME} \qquad (Q.1)$$

where

$$H_{ME} = \frac{\hat{R}_{ME} \times \dot{R}_{ME}}{|\hat{R}_{ME} \times \dot{R}_{ME}|}$$

For the fixed axis system XINJ, etc., the initial vectors $\hat{R}_{ME}(t_o)$ and $\dot{R}_{ME}(t_o)$ at the time of injection are used with the current value of R_{VE}.

R. TRAJECTORY SEARCH

The program provides a search routine to obtain selected trajectories. The search is based on linear theory and varies the polar load input quantities (independent variables) to search for desired dependent variables. A maximum of any six dependent variables may be selected. However, at this writing, the user is restricted to two dependent variable. Modifications to incorporate the remaining five are available and will be included in future versions of the program. The quantities at present are $B \cdot \hat{T}$ and $B \cdot \hat{R}$.

The components of the impact parameter vector ($B \cdot \hat{T}_{IMP}$, $B \cdot \hat{R}_{IMP}$) are referred to the ecliptic plane for T-planet trajectories and the Moon's orbital plane for lunar trajectories. The number of independent variables must equal the number of dependent variables for this routine to function.

This version of the search routine is time consuming if the initial conditions are poorly approximated. Before using this routine, two things should be done.

1. A first guess of the initial conditions of the nominal trajectory should be obtained from a patched conic or a similar search program.

2. The number of variables should be kept to a minimum. It is planned to automate the iteration scheme to go from two-body, to patched conic, to full trajectory, and to increase the number of variables to be adjusted, in optimal fashion. Even in its present form, however, it is extremely useful.

The iterator uses a modified version of the MIN-MAX Principle (Reference 6).

A_{ij}	is the matrix of partials
Δx_i	is the vector of changes in the independent variables
λ_{ii}	is a diagonal matrix of weights
y_i	is a vector of residuals

The system to be solved is

$$(A_{ji}A_{ij} + \lambda_{ii})\Delta x_i = A_{ji}y_i$$

$$A_{ji}A_{ij} + \lambda_{ii} = B_{ij}$$

$$A_{ji}y_i = z_i$$

Procedure

The system

$$B_{ij}\Delta x_i = z_i$$

is solved. If the value of Δx_i is greater than SIZER, some arbitrary amount, set

$$B_{ij} = B_{ij} + \lambda_{ii}$$

and solve the system again. Repeat these operations until Δx_i is less than or equal to SIZER. Now, run a new nominal trajectory with the new independent variable

$$x_i = x_i + \Delta x_i$$

a) If the new residuals y_i are greater than the previous ones, set

$$B_{ij} = B_{ij} + \lambda_{ii}$$

Solve the system again and continue solving until the new residuals are less than the old. Now the system is ready for a new iteration.

b) If the new residuals y_i are less than the old, set

$$B_{ij} = B_{ij} - \lambda_{ii}$$

Solve the system again and continue solving until the new residuals are greater or equal to the old.

The iteration continues until either the maximum number of iterations (input) is exceeded or the residuals are less than or equal to an input tolerance.

S. EQUATIONS FOR FLIGHT PATH AZIMUTH AND FLIGHT PATH ANGLE

A subroutine computes the flight path azimuth and flight path angle with the following equations:

1. Flight path angle

$$\gamma = \sin^{-1}\left[\frac{\dot{R}}{V} \cdot \hat{N}\right] \tag{S.1}$$

$\hat{N}$ is the vertical unit vector. In the geodetic system $\hat{N}$ is given by

$$\hat{N} = [\cos \varphi \cos (\theta - \theta_0), \cos \varphi \sin (\theta - \theta_0, \sin \varphi]$$

In the geocentric system φ is replaced by φ'. Alternatively, in the latter system

$$\hat{N} = \frac{R}{r}$$

2. Flight path azimuth

$$A = \sin^{-1}\left[\frac{1}{\cos \gamma}\left\{\frac{\dot{y}}{v} \cos (\theta-\theta_0) - \frac{\dot{x}}{v} \sin (\theta-\theta_0)\right\}\right]$$

$$(S.2)$$

$$A = \cos^{-1}\left[\frac{1}{\cos \gamma \cos \varphi}\left\{\frac{\dot{z}}{v} - \sin \gamma \sin \varphi\right\}\right]$$

Both formulas are used to determine the proper quadrant of A. To obtain the geocentric output, $e^2 = 0$, φ is replaced by declination $\delta = \varphi'$.

T. OSCULATING ELEMENTS

The osculating elements are obtained from the following equations:

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} \tag{T.1}$$

$$n = \mu^{1/2}|a|^{-3/2} \tag{T.2}$$

$$\left.\begin{array}{l} e \cos E \\ e \cosh E \end{array}\right\} = \left(1 - \frac{r}{a}\right) \tag{T.3}$$

$$\left.\begin{array}{l} e \sin E \\ e \sinh E \end{array}\right\} = \frac{d}{\sqrt{|\mu a|}} \tag{T.4}$$

$$M = \left\{\begin{array}{l} E - e \sin E \\ e \sinh E - E \end{array}\right. \tag{T.5}$$

$$t_p = t - \frac{M}{n} \tag{T.6}$$

The angles Ω, ω, i are obtained from the vectors H and $\hat{P}$, where

$$H = R \times \dot{R} \tag{T.7}$$

$$eP = \left(\frac{1}{r} - \frac{1}{a}\right) R - \frac{d}{\mu} \dot{R} \tag{T.8}$$

In terms of these vectors.

$$\cos i = \frac{H_z}{h} \text{ in the first or fourth quadrant} \tag{T.9}$$

$$\sin \Omega = \frac{H_x}{h \sin i}$$

$$\cos \Omega = \frac{-H_y}{h \sin i} \tag{T.10}$$

$$\cos \omega = P_z \cos \Omega + P_y \sin \Omega$$

$$\sin \omega = \frac{P_z}{\sin i} . \tag{T.11}$$

U. IMPACT PARAMETERS

The "impact parameters" are coordinates in the "impact" plane. This plane passes through the body (planet or the moon) and is normal to the incoming asymptote. The direction cosines of the asymptote are given by equations (U.1, U.2) in terms of unit vectors $\hat{P}$ (Appendix T) and

$$\hat{Q} = \frac{H}{h} \times \hat{P} \tag{U.1}$$

$$\hat{S} = \frac{1}{e}\left[\hat{P} + \sqrt{(e^2 - 1)}\ \hat{Q}\right] \tag{U.2}$$

In the plane defined by S as its normal, two unit vectors $\hat{T}_{IMP}$ and $\hat{R}_{IMP}$ are defined. $\hat{T}_{IMP}$ is parallel to the ecliptic plane for T-planet impacts, and to the moon's orbital plane for moon impacts. Explicitly

$$\hat{T}_{IMP} = \frac{\hat{N} \times \hat{S}}{|\hat{N} \times \hat{S}|} \tag{U.3}$$

where $\hat{N}$ is the unit normal to the ecliptic plane, or the moon's orbital plane. $\hat{R}_{IMP}$ is normal to both $\hat{S}$ and $\hat{T}_{IMP}$. B_{IMP} is the vector from the body to the vehicle as it crosses the impact plane. The data computed are the dot products

$$B_{IMP} \cdot \hat{T}_{IMP}$$

and

$$B_{IMP} \cdot \hat{R}_{IMP}$$

V. MOON'S ORBITAL PLANE INPUT AND OUTPUT

A polar coordinate system is available for input and output which uses as its reference plane the moon's orbital plane and the vector from the moon to earth as unit vector. Polar coordinates in this system are defined analogous to geocentric polar coordinates. The cartesian coordinates in this system are computed by equations (H.3) with

$$c = s = r_B$$

and

$$\theta_0 = 0$$

Here r_B is the radius of the body of departure (earth or moon).

These coordinates are then transformed to equatorial coordinates by a matrix C computed as follows:

$$\hat{i} = \frac{R_{EM}}{r_{EM}}$$

$$\hat{k} = \frac{R_{EM} \times \dot{R}_{EM}}{|R_{EM} \times \dot{R}_{EM}|} \qquad \text{(V.1)}$$

$$\hat{j} = \hat{k} \times \hat{i}$$

The transformation matrix C is then given by

$$C = \begin{bmatrix} i_x & j_x & k_x \\ i_y & j_y & k_y \\ i_s & j_s & k_s \end{bmatrix} \qquad \text{(V.2)}$$

and

$$\begin{aligned} R &= CR_{MOP} \\ \dot{R} &= C\dot{R}_{MOP} \end{aligned} \qquad (V.3)$$

The matrix C is unitary, and $C^{-1} = C^*$, permitting easy inversion of equations (V.2).

W. EQUATIONS FOR TRANSLUNAR PLANE INPUT

The translunar plane input is designed to permit easy visualization of the geometric relationships between initial conditions for circumlunar trajectories and the motion of the moon.

The initial conditions are given in a coordinate system referred to the translunar plane. This system has its x axis along the ascending node of the vehicle with respect to the moon's orbital plane, its y axis in the translunar plane at right angles to the ascending node, in the direction of motion. In this coordinate system, initial position and velocity vectors are given by

$$x_{TL} = (r_B + h)\cos \Psi$$

$$y_{TL} = (r_B + h)\sin \Psi \qquad \text{(W.1)}$$

$$z_{TL} = 0$$

Here r_B is the radius of the body of departure (earth or moon).

$$\dot{x}_{TL} = v \sin (\gamma - \Psi)$$

$$\dot{y}_{TL} = v \cos (\gamma - \Psi) \qquad \text{(W.2)}$$

$$\dot{z}_{TL} = 0$$

The translunar plane is positioned by giving its inclination i_{TL} with respect to the moon's orbital plane and the lunar lead angle O, the angle between the moon's position at injection and the descending node. The vectors R_{TL} and $\dot{R}_{TL}$ may then be transformed into the equatorial system by the following series of rotations:

1. A rotation $-i_{TL}$ about the x_{TL} axis will rotate the translunar plane into the moon's orbital plane.

2. A rotation of $\pi - (\lambda_M + \varphi)$ about the new z-axis will refer the moon's orbital plane coordinate system to the ascending node of the moon's orbital plane (with respect to the equator) as x-axis.

Here λ_M stands for the argument of latitude of the moon. These rotations are performed by multiplying R_{TL} and $\dot{R}_{TL}$ by the matrix:

$$A = \begin{bmatrix} -\cos(\lambda_M + \varphi) & \sin(\lambda_M + \varphi) & -\sin(\lambda_M + \varphi)\sin i_{TL} \\ -\sin(\lambda_M + \varphi) & -\cos(\lambda_M + \varphi)\cos i_{TL} & \cos(\lambda_M + \varphi)\sin i_{TL} \\ 0 & \sin i_{TL} & \cos i_{TL} \end{bmatrix} \quad (W.3)$$

3. The moon's orbital plane (MOP) is rotated about its node through an angle $-i_M$ (the inclination of the MOP)

4. The ascending node is brought into coincidence with the vernal equinox by a rotation $-\Omega_M$. These two rotations are embodied in the matrix

$$B = \begin{bmatrix} \cos\Omega_M & -\sin\Omega_M \cos i_M & \sin\Omega_M \sin i_M \\ \sin\Omega_M & \cos\Omega_M \cos i_M & -\cos\Omega_M \sin i_M \\ 0 & \sin i_M & \cos i_M \end{bmatrix} \quad (W.4)$$

and thus:

$$R = (BA)\, R_{TL}$$
$$\dot{R} = (BA)\, \dot{R}_{TL} \quad (W.5)$$