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ELECTROMAGNETIC SCATTERING BY CYLINDERS AN INTRODUCTION

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An introduction to the scattering of electromagnetic waves by cylinders is given. The theory of electromagnetism is reviewed, and the theory of light scattering is developed. The general features of scattering by cylinders are presented by use of the Rayleigh-Gans theory.

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DEFINITION OF SYMBOLS

Symbol	Definition
$\underset{\sim}{A}$	Vector potential
a	Radius of cylinder
$\underset{\sim}{B}$	Magnetic induction
c	Velocity of light
$\underset{\sim}{D}$	Electric displacement
$\underset{\sim}{E}$	Electric field
$\underset{\sim}{H}$	Magnetic field
$\underset{\sim}{J}$	Current density
$\underset{\sim}{k}$	Propagation constant $(2\pi\hat{n}/\lambda)$
L	Energy flux density at a surface
ℓ	Length of cylinder
$\hat{\ell}, \hat{m}, \hat{n}, \hat{r}, \hat{\epsilon}$	Unit vectors
$\underset{\sim}{M}$	Magnetization
m	Refractive index
$\underset{\sim}{P}$	Polarization vector
$\underset{\sim}{p}$	Dipole moment
$\underset{\sim}{S}$	Poynting vector
$\underset{=}{S}$	Scattering matrix
S_i	Scattering amplitude functions
v	Velocity

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
α	Polarizability
α	Tilt angle
β	Bisectrix angle
ϵ	Permittivity or dielectric constant
ϵ_0	Permittivity of free space
θ	Scattering angle
λ	Wavelength
μ	Permeability
μ_0	Permeability of free space
ρ	Charge density
σ	Electric conductivity
$\frac{d\sigma}{d\Omega}$	Differential scattering cross section
ϕ	Scalar electric potential
ϕ	Azimuth angle between the plane of incident beam and z-axis and the scattering plane
ω	Angular frequency

TECHNICAL MEMORANDUM X-64545

ELECTROMAGNETIC SCATTERING BY CYLINDERS – AN INTRODUCTION

SUMMARY

This report, the first of a series of papers, presents an introduction to the scattering of electromagnetic waves by cylinders. The basic scattering features particular to cylinders are discussed from the Rayleigh-Gans theory.

In the first part of the report a review of electromagnetism and a derivation of dipole radiation are presented. In the second part, the basic scattering theory is given. From the equations for dipole radiation, the theory of scattering by cylinders according to the Rayleigh-Gans theory is developed.

A computer program to calculate the scattering intensity functions is presented.

INTRODUCTION

This report is the first of a series of papers describing the scattering of electromagnetic waves by cylinders. The applications are numerous, e.g., in studying cylindrical plasma, polymers, paints, rodlets, and platelets; in the fields of astronomy, chemistry, meteorology, and physics. Cylindrical scattering differs from spherical scattering in several important ways. There is a loss in the degree of symmetry introducing anisotropic processes which can generally be characterized by a shape factor. There are off-diagonal components in the scattering matrix which arise from the mixing of the electric field components.

The general methods to be discussed in these reports are (1) Rayleigh, (2) Rayleigh-Gans, (3) infinite cylinders, (4) finite cylinders by neglecting end effects, (5) geometric scattering, (6) Watson transformations, (7) Wiener-Hopf technique, (8) perturbation techniques, (9) symmetry techniques, (10) S-matrix theory, and (11) diffraction theory.

ELECTROMAGNETIC THEORY

Maxwell Equations

Light, electromagnetic radiation, is described by Maxwell's equations. All theoretical optics problems, including scattering problems, can be solved formally by using Maxwell's equations.

The state of excitation (force) which is established in space and detected by the presence of an electric charge is said to constitute an electromagnetic field. The electromagnetic field is a vector field, and associated with it are the two vectors $\underset{\sim}{E}$ and $\underset{\sim}{B}$, called the electric field vector and the magnetic induction vector, respectively. The spatial and temporal derivatives of these two vectors are defined by Maxwell's simultaneous partial differential equations [1]:

$$\nabla \cdot \underset{\sim}{D} = \rho \tag{1}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \tag{2}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \tag{3}$$

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J} + \frac{\partial \underset{\sim}{D}}{\partial t} \tag{4}$$

where ρ is the electric charge density. The vectors $\underset{\sim}{D}$, $\underset{\sim}{H}$, and $\underset{\sim}{J}$, are the electric displacement, the magnetic vector, and the electric current density, respectively. For macroscopic media, the dynamical response of the aggregates of the atoms is summarized in the constitutive relations for isotropic, permeable, conducting dielectrics [2]:

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E} \tag{5}$$

$$\underset{\sim}{J} = \sigma \underset{\sim}{E} \tag{6}$$

$$\underset{\sim}{H} = \frac{\underset{\sim}{B}}{\mu} \tag{7}$$

where ϵ is the electric permittivity, σ is the specific electrical conductivity, and μ is the magnetic permeability. In this work, ϵ and μ are assumed to be scalars.

Maxwell's equations are the results of particular experimental laws: equation (1) from Coulomb's law, equation (4) from Ampere's law, equation (3) from Faraday's law, and equation (2) from the observed nonexistence of magnetic monopoles.

Table 1 gives the units of the electromagnetic quantities.

TABLE 1. ELECTROMAGNETIC QUANTITIES IN SI UNITS

Symbol	Quantity	Units[a]
$\underset{\sim}{E}$	Electric Field	N/C
$\underset{\sim}{H}$	Magnetic Field	C/m·sec
$\underset{\sim}{D}$	Electric Displacement	C/m^2
$\underset{\sim}{B}$	Magnetic Induction	N·sec/C·m
$\underset{\sim}{S}$	Poynting's Vector	$N{\cdot}m/sec{\cdot}m^2$
c	Speed of Light	m/sec
ϵ	Electric Permittivity	$C^2/N{\cdot}m^2$
μ	Magnetic Permeability	$N{\cdot}sec^2/C^2$

a. N = newtons, C = coulombs, and m = meters.

Wave Equations

Maxwell's equations predict the existence of electromagnetic waves propagating with the velocity of light, which lead to the electromagnetic theory of light. The wave equations are obtained by vector operations [3] on Maxwell's equations. The equations are usually solved in applications using the appropriate boundary conditions for a particular problem. Assuming the isotropic case (the constitutive equations),

$$\underset{\sim}{H} = \frac{\underset{\sim}{B}}{\mu}$$

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E}, \text{ and } \underset{\sim}{J} = \sigma \underset{\sim}{E}$$

then in a region with no free charges, Maxwell's equations are:

$$\nabla \cdot \epsilon \underset{\sim}{E} = 0 \tag{8}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \tag{9}$$

$$\nabla \times \underset{\sim}{E} = - \frac{\partial \underset{\sim}{B}}{\partial t} \tag{10}$$

$$\nabla \times \frac{\underset{\sim}{B}}{\mu} = \sigma \underset{\sim}{E} + \epsilon \frac{\partial \underset{\sim}{E}}{\partial t} \tag{11}$$

Taking the curl of equation (10), we have

$$\nabla \times (\nabla \times \underset{\sim}{E}) = - \frac{\partial}{\partial t} (\nabla \times \underset{\sim}{B}) \tag{12}$$

On substituting equation (11) for the curl $\underset{\sim}{B}$, we obtain

$$\nabla \times (\nabla \times \underset{\sim}{E}) = - \frac{\partial}{\partial t} \left(\mu \sigma \underset{\sim}{E} + \mu \epsilon \frac{\partial \underset{\sim}{E}}{\partial t} \right) \tag{13}$$

Now, for any vector it is true that in Cartesian coordinates

$$\nabla \times (\nabla \times \underset{\sim}{E}) = \nabla (\nabla \cdot \underset{\sim}{E}) - \nabla^2 \underset{\sim}{E}$$

but $\nabla \cdot \underset{\sim}{E} = 0$ in the charge free region. Therefore, equation (13) becomes

$$\boxed{\nabla^2 \underset{\sim}{E} - \mu \sigma \frac{\partial \underset{\sim}{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = 0} \quad . \tag{14}$$

Similarly, take the curl of equation (11):

$$\nabla \times (\nabla \times \underset{\sim}{B}) = \sigma \mu \nabla \times \underset{\sim}{E} + \mu \epsilon \frac{\partial}{\partial t} (\nabla \times \underset{\sim}{E}) \tag{15}$$

Then, upon substituting from equation (10):

$$\nabla \times (\nabla \times \underset{\sim}{B}) = -\sigma \mu \frac{\partial \underset{\sim}{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \underset{\sim}{B}}{\partial t^2}$$

or

$$\boxed{\nabla^2 \underset{\sim}{B} - \sigma\mu \frac{\partial \underset{\sim}{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \underset{\sim}{B}}{\partial t^2} = 0} \quad . \tag{16}$$

Equations (14) and (16) are the inhomogeneous vector wave equations for $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

In a nonconducting medium, $\sigma = 0$ and the second term in equations (14) and (16) vanishes, leaving a propagation equation for waves with a velocity $v = 1/\sqrt{\mu\epsilon}$.

The boundary conditions at an interface between two media are [4]:

1. The normal component of $\underset{\sim}{B}$ is continuous.

$$(\underset{\sim}{B}_2 - \underset{\sim}{B}_1) \cdot \hat{n} = 0$$

2. There is a discontinuity in the normal component $\underset{\sim}{D}$ equivalent to K, the surface charge density,

$$(\underset{\sim}{D}_2 - \underset{\sim}{D}_1) \cdot \hat{n} = K$$

3. The tangential component of $\underset{\sim}{E}$ is continuous,

$$(\underset{\sim}{E}_2 - \underset{\sim}{E}_1) \times \hat{n} = 0$$

4. There is a discontinuity in the tangential component of $\underset{\sim}{H}$ equal to $\underset{\sim}{L}$, the surface current density,

$$(\underset{\sim}{H}_2 - \underset{\sim}{H}_1) \times \hat{n} = \underset{\sim}{L}$$

Plane Waves

We now consider the homogeneous vector wave equations ($\sigma = 0$) in which the field depends only on one coordinate r and on time. Such waves are said to be "plane":

$$\nabla^2 \underset{\sim}{E} - \mu\epsilon \frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = 0$$

$$\nabla^2 \underset{\sim}{B} - \mu\epsilon \frac{\partial^2 \underset{\sim}{B}}{\partial t^2} = 0$$

Assume

$$\boxed{\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E}_0 f(\underset{\sim}{r}, t)}$$

where $f(\underset{\sim}{r}, t)$ satisfies the scalar wave equation:

$$\nabla^2 f(\underset{\sim}{r}, t) - \mu\epsilon \ddot{f}(\underset{\sim}{r}, t) = 0 \quad . \tag{17}$$

Now let

$$f(\underset{\sim}{r}, t) = f(\hat{n} \cdot \underset{\sim}{r} - vt) \quad ;$$

then

$$\nabla f(\underset{\sim}{r}, t) = f'(\hat{n} \cdot \underset{\sim}{r} - vt)\, \hat{n} \cdot \nabla \underset{\sim}{r}$$

but $\nabla \underset{\sim}{r}$ is the idemfactor [3]. Thus,

$$\hat{n} \cdot \nabla \underset{\sim}{r} = \hat{n}$$

and

$$\nabla f(\underset{\sim}{r}, t) = \hat{n} f'(\hat{n} \cdot \underset{\sim}{r} - vt)$$

$$\nabla^2 f(\underset{\sim}{r}, t) = \nabla \cdot \nabla f = \nabla \cdot \hat{n} f' = \hat{n} \cdot \nabla f' = \hat{n} \cdot \hat{n} f'' = f''$$

Also

$$\ddot{f} = v^2 f'' \tag{18}$$

From (17) and (18) we have

$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{19}$$

Hence, the solution for $\underset{\sim}{E}$ and $\underset{\sim}{B}$ is

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \underset{\sim}{E}_0 f(\hat{n} \cdot \underset{\sim}{r} - vt)$$

$$\underset{\sim}{B}(\underset{\sim}{r}, t) = \underset{\sim}{B}_0 g(\hat{n} \cdot \underset{\sim}{r} - vt)$$

The wave equations do not relate f, g, $\underset{\sim}{E}$, and $\underset{\sim}{B}$, but Maxwell's equations do.

From Equation (8)

$$\nabla \cdot \underset{\sim}{E} = 0$$

and substituting

$$\underset{\sim}{E} = \underset{\sim}{E}_0 \, f(\hat{n} \cdot \underset{\sim}{r} - vt) \quad ,$$

we have

$$\nabla \cdot (\underset{\sim}{E}_0 \, f) = \underset{\sim}{E}_0 \cdot \nabla f = \underset{\sim}{E}_0 \cdot \hat{n} \, f' = 0$$

Therefore,

$$\boxed{\underset{\sim}{E}_0 \cdot \hat{n} = 0} \quad , \tag{20}$$

which states that $\underset{\sim}{E}$ is transverse to the direction of propagation. Similarly, for $\underset{\sim}{B}$

$$\boxed{\underset{\sim}{B}_0 \cdot \hat{n} = 0} \tag{21}$$

From equation (10), we have:

$$-\dot{\underset{\sim}{B}} = \nabla \times \underset{\sim}{E}$$

$$-\dot{\underset{\sim}{B}} = \nabla \times (\underset{\sim}{E}_0 \, f)$$

$$-\dot{\underset{\sim}{B}} = \nabla f \times \underset{\sim}{E}_0 = \hat{n} \times \underset{\sim}{E}_0 \, f'$$

From equations (18), (19) and

$$\underset{\sim}{B} = \underset{\sim}{B}_0 \, g(\hat{n} \cdot \underset{\sim}{r} - vt) \quad ,$$

we have

$$-\dot{\underset{\sim}{B}} = -\underset{\sim}{B}_0 \dot{g} = -\underset{\sim}{B}_0 \frac{g'}{\sqrt{\mu\epsilon}}$$

hence,

$$\frac{-\underset{\sim}{B}_0 g'}{\sqrt{\mu\epsilon}} = \hat{n} \times \underset{\sim}{E}_0 f'$$

$$\frac{1}{\sqrt{\mu\epsilon}} \underset{\sim}{B}_0 dg = \hat{n} \times \underset{\sim}{E}_0 df$$

$$\boxed{\underset{\sim}{B} = \sqrt{\mu\epsilon}\ \hat{n} \times \underset{\sim}{E}} \quad . \tag{22}$$

$\underset{\sim}{E}$ and $\underset{\sim}{B}$ are mutually orthogonal and are perpendicular to the direction of propagation $\hat{n}$.

Propagation Constant

For a plane wave propagating along the positive z-axis, the component of $\underset{\sim}{E}$ along the x-axis can be written as:

$$E_x = A\, e^{i(kz - \omega t)} \tag{23}$$

where only the real part of the exponential is to be taken as the physical quantity. Substituting equation (23) into equation (14) yields

$$-k^2 A\, e^{i(\omega t - kz)} + \mu\sigma \left(i\omega A\, e^{i(\omega t - kz)}\right) + \mu\epsilon\, \omega^2 A\, e^{i(\omega t - kz)} = 0$$

Therefore, the propagation constant, k, is related to the constitutive constants of the medium

$$\boxed{k^2 = \mu \epsilon \omega^2 + i \mu \sigma \omega} \quad , \tag{24}$$

and may be represented as [4]

$$k = \alpha_1 + i\beta_1$$

where

$$\alpha_1 = \omega \left\{ \frac{\mu\epsilon}{2} \left[\left(1 + \frac{\sigma^2}{\epsilon^2 \omega^2} \right)^{1/2} + 1 \right] \right\}^{1/2}$$

and

$$\beta_1 = \omega \left\{ \frac{\mu\epsilon}{2} \left[\left(1 + \frac{\sigma^2}{\epsilon^2 \omega^2} \right)^{1/2} - 1 \right] \right\}^{1/2}$$

Thus, for a conducting or dissipative medium $(\sigma > 0)$, the propagation constant is complex, providing a damping part $e^{-\beta_1 z}$ to the wave which corresponds to absorption.

Polarization [2]

The plane wave

$$\underset{\sim}{E} = \hat{\epsilon}_1 E_0 f(\hat{n} \cdot \underset{\sim}{r} - vt)$$

is a wave with its electric field vector always in the direction $\hat{\epsilon}_1$. Such a wave is said to be linearly polarized in the direction of $\hat{\epsilon}_1$. To describe a general state of polarization, two linearly polarized independent waves are needed. Consider harmonic plane waves

$$f(r, t) = e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)}$$

where

$$\frac{\omega}{k} = v \quad , \qquad \text{and} \qquad \underset{\sim}{k} = k\hat{r}$$

Then the general state of polarization can be described with the following two waves:

$$\underset{\sim}{E}_1 = \hat{\epsilon}_1 E_1 e^{i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t}$$

$$\underset{\sim}{E}_2 = \hat{\epsilon}_2 E_2 e^{i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t}$$

with

$$\underset{\sim}{B}_j = \sqrt{\mu\epsilon}\ \hat{n} \times \underset{\sim}{E}_j \qquad j = 1, 2$$

and $\hat{\epsilon}_1$ perpendicular to $\hat{\epsilon}_2$. The amplitudes E_1 and E_2 are complex numbers to allow the possibility of a phase difference between the waves. The general harmonic wave solution is a linear combination of $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_2$,

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\ e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)} \tag{25}$$

If E_1 and E_2 have the same phase, equation (25) represents a linearly polarized wave, with its polarization vector making an angle $\theta = \tan^{-1} (E_2/E_1)$ with $\hat{\epsilon}_1$ and having a magnitude $E = \sqrt{E_1^2 + E_2^2}$.

If E_1 and E_2 differ by a phase of 90 deg, then

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = E_0 (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)\, e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)} \quad ,$$

and the wave is circularly polarized. The "+" sign is for left circularly polarized waves or is said to have positive helicity. The negative sign is for right circularly polarized waves or negative helicity.

For polychromatic radiation, the harmonic plane wave solution has the form

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \sum_{\lambda=1}^{2} \int dk\, E_{o\,n}\, \hat{\epsilon}_n(k)\, e^{i(\underset{\sim}{k} \cdot \underset{\sim}{r} - \omega t)} \tag{26}$$

where $E_{o\,n}$ are amplitude functions determined by $\underset{\sim}{E}(\underset{\sim}{r}, t)$ at a specified time, and $\hat{\epsilon}_n$ is the polarization vector.

Poynting Vector

For a periodic field such as the electromagnetic wave, the energy crossing a unit area per unit time is given by the Poynting vector

$$\underset{\sim}{S} = \underset{\sim}{E} \times \underset{\sim}{H}$$

The time-average flux of energy is given by the real part of the complex Poynting vector:

$$\boxed{\underset{\sim}{S} = \frac{1}{2} (\underset{\sim}{E} \times \underset{\sim}{H}^*)} \tag{27}$$

Because

$$\underset{\sim}{H} = \frac{\underset{\sim}{B}}{\mu} = \sqrt{\frac{\epsilon}{\mu}}\, \hat{n} \times \underset{\sim}{E}$$

by equation (22) for plane waves, then

$$\underset{\sim}{S} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}}\, |E|^2\, \hat{n} \quad .$$

The energy density $\mathcal{E}$ is

$$\mathcal{E} = \frac{|\underset{\sim}{S}|}{c}$$

The Poynting vector is in the direction of propagation and can be regarded as a measure of the intensity of the wave at a point. The Poynting vector gives the flux density at the surface, L, which is called by Van de Hulst the intensity, "I",

$$\underset{\sim}{S} = c\,\mathcal{E}\,k = \hat{k}\,L = \hat{k}\,\text{“I”}$$

Electromagnetic Potentials [2]

Vector potentials are often introduced to simplify problem solving. According to equation (2),

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad ;$$

then the field of $\underset{\sim}{B}$ is always solenoidal. Since

$$\nabla \cdot (\nabla \times \underset{\sim}{v}) \equiv 0$$

$\underset{\sim}{B}$ can be represented as the curl of another vector $\underset{\sim}{A}_0$:

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}_0 \tag{28}$$

However, $\underset{\sim}{A}_0$ is not uniquely defined by equation (28) because $\underset{\sim}{B}$ is equal also to the curl of some vector $\underset{\sim}{A}$,

$$\boxed{\underset{\sim}{B} = \nabla \times \underset{\sim}{A}} \quad , \tag{29}$$

where

$$\underset{\sim}{A} = \underset{\sim}{A}_0 - \nabla \psi \qquad ;$$

Since

$$\nabla \times (\nabla \psi) \equiv 0$$

ψ is any scalar function of position. If $\underset{\sim}{B}$ is replaced in

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

by either equations (18) or (29), then

$$\nabla \times \left(\underset{\sim}{E} + \frac{\partial \underset{\sim}{A_0}}{\partial t} \right) = 0$$

or

$$\nabla \times \left(\underset{\sim}{E} + \frac{\partial \underset{\sim}{A}}{\partial t} \right) = 0$$

Thus the fields of the vector

$$\underset{\sim}{E} + \frac{\partial \underset{\sim}{A_0}}{\partial t}$$

or

$$\underset{\sim}{E} + \frac{\partial \underset{\sim}{A}}{\partial t}$$

are irrotational and equal to gradients of scalar functions ϕ and ϕ_0:

$$\boxed{\underset{\sim}{E} = -\nabla \phi_0 - \frac{\partial \underset{\sim}{A_0}}{\partial t}} \tag{31}$$

and

$$\underset{\sim}{E} = -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t} \tag{32}$$

The functions ϕ and ϕ_0 are related by

$$\phi = \phi_0 + \frac{\partial \psi}{\partial t}$$

To show this, recall that $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A} = \underset{\sim}{A_0} - \nabla\psi$$

and from equations (31) and (32),

$$\nabla\phi_0 + \frac{\partial \underset{\sim}{A_0}}{\partial t} = \nabla\phi + \frac{\partial \underset{\sim}{A}}{\partial t} = \nabla\phi + \frac{\partial \underset{\sim}{A_0}}{\partial t} - \nabla \frac{\partial \psi}{\partial t}$$

$$\nabla\phi_0 = \nabla\phi - \nabla\frac{\partial \psi}{\partial t}$$

or

$$\nabla\phi = \nabla\phi_0 + \nabla\frac{\partial \psi}{\partial t}$$

and finally,

$$\phi = \phi_0 + \frac{\partial \psi}{\partial t}$$

The functions $\underset{\sim}{A}$ are vector potentials of the field, and the ϕ's are scalar potentials. Using $\underset{\sim}{D} = \epsilon \underset{\sim}{E}$ and $\underset{\sim}{B} = \mu \underset{\sim}{H}$, then

$$\underset{\sim}{D} = -\epsilon\left(\nabla \phi + \frac{\partial \underset{\sim}{A}}{\partial t}\right) \qquad \underset{\sim}{H} = \frac{1}{\mu} \nabla \times A$$

Equations (1) and (4) of Maxwell's equations yield:

$$\nabla \times \nabla \times \underset{\sim}{A} + \mu\epsilon \nabla \frac{\partial \phi}{\partial t} + \mu\epsilon \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = \mu \underset{\sim}{J}$$

$$\nabla^2 \phi + \nabla \cdot \frac{\partial \underset{\sim}{A}}{\partial t} = -\frac{1}{\epsilon} \rho$$

Imposing the Lorentz condition,

$$\boxed{\nabla \cdot \underset{\sim}{A} + \mu\epsilon \frac{\partial \phi}{\partial t} = 0} \quad , \tag{33}$$

we then have

$$\nabla^2 \psi - \mu\epsilon \frac{\partial^2 \psi}{\partial t^2} = \nabla \cdot \underset{\sim}{A_0} + \mu\epsilon \frac{\partial \phi_0}{\partial \epsilon} \tag{34}$$

where ϕ_0 and $\underset{\sim}{A_0}$ are particular solutions. ϕ and $\underset{\sim}{A}$ are defined by

$$\nabla \times \nabla \times \underset{\sim}{A} - \nabla\nabla \cdot \underset{\sim}{A} + \mu\epsilon \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = \mu \underset{\sim}{J}$$

$$\boxed{\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon} \rho} \tag{35}$$

Now, using $\nabla \times \nabla \times \underset{\sim}{A} = \nabla \nabla \cdot \underset{\sim}{A} - \nabla \cdot \nabla \underset{\sim}{A}$,

$$\boxed{\nabla^2 \underset{\sim}{A} - \mu\epsilon \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = -\mu \underset{\sim}{J}} \quad . \tag{36}$$

For the homogeneous case

$$\underset{\sim}{D} = -\nabla \times \underset{\sim}{A'} \qquad\qquad \underset{\sim}{H} = -\nabla \phi' - \frac{\partial \underset{\sim}{A'}}{\partial t}$$

$$\underset{\sim}{E} = -\frac{1}{\epsilon} \nabla \times \underset{\sim}{A'} \qquad\qquad \underset{\sim}{B} = -\mu\left(\nabla \phi' + \frac{\partial \underset{\sim}{A}}{\partial t}\right)$$

the wave equations for these potentials are:

$$\nabla^2 \underset{\sim}{A'} - \mu\epsilon \frac{\partial^2 \underset{\sim}{A'}}{\partial t^2} = 0 \tag{37}$$

$$\nabla^2 \phi' - \mu\epsilon \frac{\partial^2 \phi'}{\partial t^2} = 0 \tag{38}$$

with the condition that

$$\boxed{\nabla \cdot \underset{\sim}{A'} + \mu\epsilon \frac{\partial \phi'}{\partial t} = 0} \quad . \tag{39}$$

Retarded Potentials

Starting from equations (35) and (36)

$$\nabla^2 \underset{\sim}{A} - \mu\epsilon \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = -\mu \underset{\sim}{J} \tag{40}$$

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon} \rho \quad , \tag{41}$$

a solution of the inhomogeneous linear equations for an initial value problem can be represented as the sum of the solution of these equations without the right-hand side (RHS) and a particular solution of these equations with the RHS. To find the particular solution, we divide the whole space into infinitely small regions and determine the field produced by the changes located in one of these volume elements. Because of the linearity of the field equations, the actual field will be the sum of the fields produced by all such elements, i.e., an integral [5].

The charge de in a given volume element is, generally, a function of time. If we choose the origin of coordinates as the center of the volume element, then the charge density is $\rho = de(t)\,\delta\,(R)$ where $\underset{\sim}{R}$ is the distance from the origin and δ represents the Dirac delta functions. Thus, we must solve the equation

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon}\, de(t)\, \delta(\underset{\sim}{R}) \tag{42}$$

Everywhere $\delta(\underset{\sim}{R}) = 0$ we have

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = 0$$

For the case of central symmetry, i.e., ϕ is a function of $\underset{\sim}{R}$ only, we have

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \phi}{\partial R} \right) - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = 0$$

Assuming $\phi = \chi(\underset{\sim}{R}, t)/R$; then

$$\frac{\partial^2 \chi}{\partial R^2} - \mu\epsilon \frac{\partial^2 \chi}{\partial t^2} = 0$$

But this is the equation for plane waves, whose solution has the form

$$\chi = f_1\left(t - \frac{R}{v}\right) + f_2\left(t + \frac{R}{v}\right)$$

where $v = 1/\sqrt{\mu\epsilon}$. Since we want only a particular solution of the equation, it is sufficient to choose only one of the functions f_1 and f_2. Usually it is convenient to take $f_2 = 0$, since this physically represents a wave that is observed before it is generated. Then, everywhere except at the origin, ϕ has the form,

$$\phi = \frac{\chi\left(t - \frac{R}{v}\right)}{R} \tag{43}$$

So far, the function χ is arbitrary; we now choose it so that we also obtain the correct value for the potential at the origin. We must select χ so that at the origin equation (42) is satisfied. This is easily done, noting that as $R \to 0$ the potential increases to infinity, and, therefore, its derivatives with respect to the coordinates increase more rapidly than its time derivative. Consequently as $R \to 0$ we can, in equation (42), neglect $\mu\epsilon\partial^2\phi/\partial t^2$ compared with $\nabla^2\phi$. Then equation (42) goes over to the familiar equation leading to Coulomb's law, i.e., from

$$\nabla^2\phi = -\frac{1}{\epsilon}\, de(t)\, \delta(\underset{\sim}{R})$$

and since

$$\nabla^2\left(\frac{1}{R}\right) = -4\pi\,\delta(\underset{\sim}{R}) \quad ,$$

we have

$$\phi = \frac{de(t)}{4\pi\epsilon R}$$

Thus, near the origin equation (43) must go over into the Coulomb's law, from which it follows that $\chi(t) = de(t)/4\pi\epsilon$, that is,

$$\phi = \frac{de\left(t - \frac{R}{v}\right)}{4\pi\epsilon R}$$

For an arbitrary distribution of charges $\rho(x, y, z, t)$, one can find the solution to equation (41). Let $de = \rho\, dV$ and integrate over the whole space. To this solution of the inhomogeneous equation we can still add the solution ϕ_0 of the homogeneous equation. Thus, the general solution has the form:

$$\phi(x, y, z, t) = \frac{1}{4\pi\epsilon} \int_V \frac{1}{R}\, \rho\left(x', y', z', t - \frac{R}{v}\right) dV' + \phi_0 \tag{44}$$

where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ and $dV' = dx'\, dy'\, dz'$. R is the distance from the volume element dV' to the "field point" at which we determine the potential. Similarly,

$$\underset{\sim}{A}(x, y, z) = \frac{\mu}{4\pi} \int_V \frac{\underset{\sim}{J}\left(x', y', z', t - \frac{R}{v}\right) dV'}{R} + \underset{\sim}{A}_0 \tag{45}$$

Without ϕ_0 and A_0 the above are the "retarded potentials." The volume integrals in equations (44) and (45) represent the contributions from charge distributions contained in the volume element. A_0 and ϕ_0 represent the contributions from fields whose origin is external to the system.

The solution of the wave equation can be accomplished "directly" with Green's functions, G, where G satisfies [5]

$$\nabla^2 G - \mu\epsilon \frac{\partial^2 G}{\partial t^2} = \delta(R)\, \delta(t)$$

and the solution to this equation can be obtained by Fourier transforms.

Dipole Radiation [2]

In the following, the radiation from an electric dipole is discussed. Assume that the distribution of charges changes little during the time r'/c where r' refers to the position of the source point. Let the radiation of the system have periods of order T. Let a be the order of magnitude of the system. The time r'/c is of the order a/c. So that the distribution of charges in the system shall not change significantly during this time, it is necessary that $a/c << T$; but $cT = \lambda$, the wavelength, hence, a condition for the following discussion is

$$a << \lambda \tag{46}$$

Assume that

$$\rho(\tilde{r}, t) = \rho(\tilde{r})\, e^{-i\omega t} \tag{47}$$

$$\tilde{J}(\tilde{r}, t) = \tilde{J}(\tilde{r})\, e^{-i\omega t} \tag{48}$$

and that the electromagnetic potentials and fields have the same time dependence. Since the time-dependence factor is the same in all terms, it can be dropped. Then from equations (45) and (48) the retarded vector potential is

$$\tilde{A}(\tilde{r}) = \frac{\mu}{4\pi} \int \tilde{J}(\tilde{r}') \frac{e^{ik|\tilde{r} - \tilde{r}'|}}{|\tilde{r} - \tilde{r}'|}\, d^3 r' \quad , \tag{49}$$

where $|\tilde{r} - \tilde{r}'| = \tilde{R}$, $k = \omega/c$, and $d^3 r'$ = infinitesimal volume element. Confining the source to a region that is small compared with the wavelength and considering $r >> a$ (Fig. 1), then

$$|\tilde{r} - \tilde{r}'| \simeq r - \hat{n} \cdot \tilde{r}' + \ldots\ldots$$

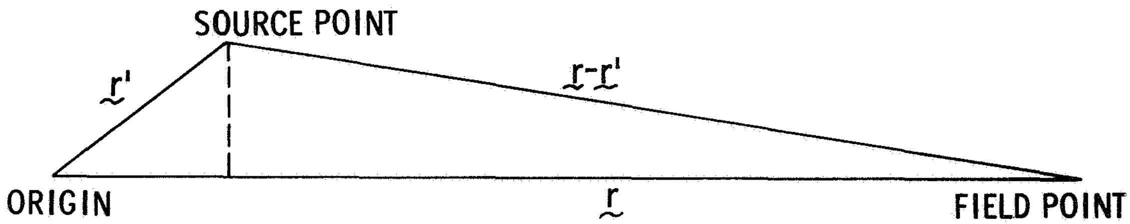


Figure 1. Geometry for the source point, field point, and origin.

$\hat{n}$ is a unit vector in the direction of $\underset{\sim}{r}$, and

$$\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu e^{ikr}}{4\pi r} \int \frac{\underset{\sim}{J}(\underset{\sim}{r}') \, e^{-ik(\hat{n}\cdot\underset{\sim}{r}' + \ldots)} \, d^3 r'}{\left[1 - \frac{\hat{n}\cdot\underset{\sim}{r}'}{r} + \ldots\right]} \tag{50}$$

If $r >> a$ and $a << \lambda$ we can expand the exponential:

$$\frac{e^{-ik\hat{n}\cdot r' + \ldots}}{1 - \frac{\hat{n}\cdot r'}{r} + \ldots} = 1 + \left(\frac{1}{r} - ik\right)(\hat{n}\cdot\underset{\sim}{r}') + \frac{1}{2}\left(\frac{2}{r^2} - \frac{2ik}{r} - k^2\right)(\hat{n}\cdot\underset{\sim}{r}')^2 + \ldots .$$

The m^{th} term of the vector potential becomes

$$A_m = \frac{\mu e^{ikr}}{4\pi r} \frac{(-ik)^m}{m!}\left(1 + \frac{a_1}{ikr} + \ldots + \frac{a_m}{(ikr)^m}\right)\int \underset{\sim}{J}(\underset{\sim}{r}')(\hat{n}\cdot\underset{\sim}{r}')^m \, d^3 r'$$

where a_j are integers. In the far field where $r >> a$ we have

$$\lim_{kr\to\infty} A_m \to \frac{\mu e^{ikr}}{4\pi r} \frac{(-ik)^m}{m!} \int \underset{\sim}{J}(\underset{\sim}{r}')(\hat{n}\cdot\underset{\sim}{r}')^m \, d^3 r' \tag{51}$$

Taking m = 0, equation (51) becomes

$$\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu e^{ikr}}{4\pi r} \int \underset{\sim}{J}(\underset{\sim}{r}') \, d^3 r'$$

Integrating by parts, the integral becomes

$$\int d^3 r' \, \underset{\sim}{J}(\underset{\sim}{r}') = - \int \underset{\sim}{r}' (\nabla' \cdot \underset{\sim}{J}) \, d^3 r' = -i\omega \int \underset{\sim}{r}' \, \rho(\underset{\sim}{r}') \, d^3 r'$$

The last step comes from the continuity equation

$$- \frac{\partial \rho}{\partial t} = \nabla \cdot \underset{\sim}{J}$$

or

$$i\omega\rho = \nabla \cdot \underset{\sim}{J}$$

Hence, the vector potential is

$$\boxed{\underset{\sim}{A}(\underset{\sim}{r}) = \frac{-i\omega\mu e^{ikr}}{4\pi r} \underset{\sim}{p}} \tag{52}$$

where $\underset{\sim}{p}$ the electric dipole moment is defined as

$$\boxed{\underset{\sim}{p} = \int \underset{\sim}{r}' \, \rho(\underset{\sim}{r}') \, d^3 r'} \tag{53}$$

The electric dipole fields are

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \frac{-\mu\omega k}{4\pi} (\hat{n} \times \underset{\sim}{p}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right)$$

$$\underset{\sim}{E} = - \frac{i}{\omega \mu \epsilon} \nabla \times \underset{\sim}{B} = -i \frac{\omega}{k^2} \nabla \times \underset{\sim}{B}$$

$$= \frac{k^2}{4\pi\epsilon} (\hat{n} \times \underset{\sim}{p}) \times \hat{n} \frac{e^{ikr}}{r} + \frac{1}{4\pi\epsilon} \left[3\hat{n}(\hat{n} \cdot \underset{\sim}{p}) - \underset{\sim}{p}\right] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr}$$

In the far-field zone,

$$\boxed{\begin{aligned} \underset{\sim}{B} &= -\omega k \mu \frac{e^{ikr}}{4\pi r} (\underset{\sim}{p} \times n) \\ \underset{\sim}{E} &= -\frac{\omega}{k} \underset{\sim}{B} \times \hat{n} = \frac{e^{ikr}}{4\pi\epsilon r} k^2 \left((\underset{\sim}{p} \times \hat{n}) \times \hat{n}\right) \end{aligned}} \quad . \tag{54}$$

SCATTERING THEORY

Scattering Amplitude

The scattering of plane waves by a particle is completely described by a knowledge of the fields at every point in space. Generally, the observer is interested in knowing the fields at his location. He desires to know the scattered fields at large distances from the scatterer where the scattered waves are less complex and appear spherical. In this region the amplitude of the scattered waves decreases inversely with the distance r from the particle, and the phase has a simple e^{+ikr} dependence. The scattering amplitude defines the strength and phase of this outgoing spherical wave. Since the amplitude and phase of the scattered radiation differ in various directions from the scatterer, the scattering amplitude $\mathscr{A}$ is a function of position. Since the electric field is transverse at large distances, we write the asymptotic form in the two-component notation:

$$\underset{\sim}{E}(\underset{\sim}{r}) \underset{r\to\infty}{\longrightarrow} \underset{\sim}{E_0}\, e^{i\underset{\sim}{k} \cdot \underset{\sim}{r}} \quad \underset{\sim}{\mathscr{A}} \quad (\theta, \phi) \frac{e^{ikr}}{r} \tag{55}$$

The scattering amplitude is most conveniently described by a scattering matrix which allows the polarization directions to be handled in a simple manner.

Figure 2 shows the coordinates used to describe the direction of the incident and scattered radiation along with its polarizations. The x, y, and z axes are fixed to the scatterer but chosen so that the z axis is along the direction of the incident radiation. The unit vector $\hat{n}$ is in the direction of the incident radiation having the propagation

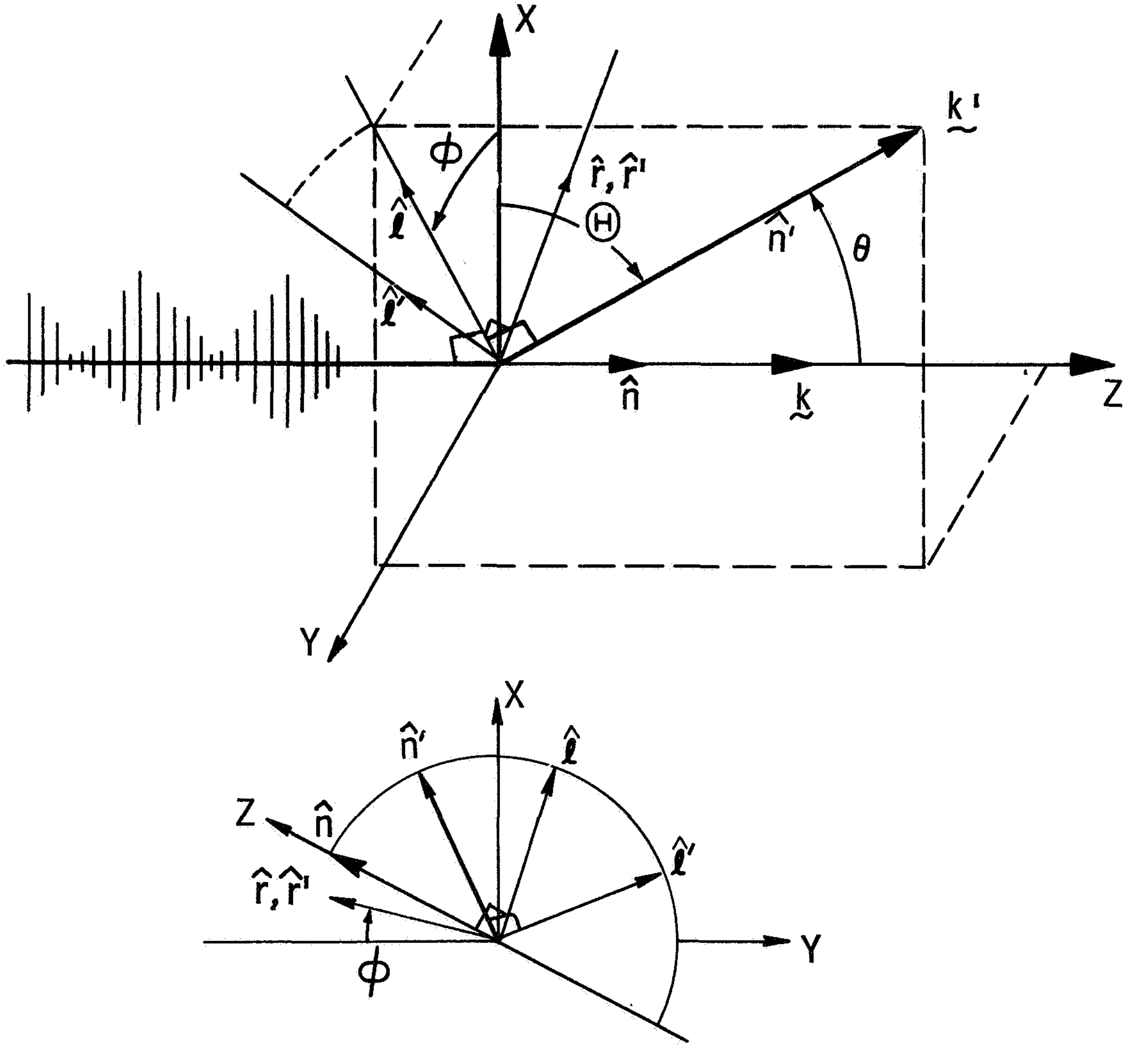


Figure 2. Geometry for the scattering theory.

vector $\underset{\sim}{k}$, and the unit vector $\hat{n}'$ is similarly defined for the scattered radiation having the propagation vector $\underset{\sim}{k}'$ [6]. The vectors $\hat{n}$ and $\hat{n}'$ (or $\underset{\sim}{k}$ and $\underset{\sim}{k}'$) define the scattering plane, and the angle between $\hat{n}$ and $\hat{n}'$ is the scattering angle θ $(0 \leqslant \theta \leqslant 180)$.

The components of the electric field are referred to the $\hat{r}$ - $\hat{\ell}$ system for the incident radiation and the $\hat{r}'$ - $\hat{\ell}'$ system for the scattered radiation. These systems simplify the discussion of polarization direction. The unit vector $\hat{r}$ is perpendicular to the scattering plane, and the x-axis of the particle makes an angle ϕ with $\hat{\ell}$. The unit vector $\hat{\ell}$, which is in the scattering plane, is chosen so that $\hat{r} \times \hat{\ell} = \hat{n}$, and the unit vector $\hat{\ell}'$ is chosen so that $\hat{r}' \times \hat{\ell}' = \hat{n}'$. The unit vectors $\hat{r}$ and $\hat{r}'$ are the same. The quantities E_{0r} and $E_{0\ell}$ are the amplitudes of the components of the electric field in the incident beam along the two directions defined by $\hat{r}$ and $\hat{\ell}$. Likewise, E_r and E_ℓ are the electric field components of the scattered radiation along $\hat{r}'$ and $\hat{\ell}'$.

The scattering amplitude tensor which relates the incident and scattered waves is defined by

$$\boxed{\underset{\sim}{E}_{sca} = \underset{\sim}{\mathscr{A}}(\theta, \phi) \frac{e^{ikr}}{r} = \underset{=}{S}(\theta, \phi) \underset{\sim}{E}_0 \frac{e^{+ikr}}{r}} \tag{56}$$

or

$$\begin{pmatrix} E_\ell \\ E_r \end{pmatrix} = \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{0\ell} \\ E_{0r} \end{pmatrix} \frac{e^{ikr}}{r}$$

The four scattering amplitude functions S_1, S_2, S_3, and S_4 are all functions of θ and ϕ and, in general, are complex. As stated above, the components (E_ℓ, E_r) and $(E_{0\ell}, E_{0r})$ are referred to different sets of coordinates, i.e., $(\hat{\ell}', \hat{r}')$ and $(\hat{\ell}, \hat{r})$. Hence the scattering amplitude tensor is a function of $\underset{\sim}{k}$ and $\underset{\sim}{k}'$, i.e.,

$$\underset{=}{S}(\theta, \phi) = \underset{=}{S}(\underset{\sim}{k}, \underset{\sim}{k}') \quad .$$

The linear relation implied by the S-matrix (scattering matrix) between $\underset{\sim}{E}$ and $\underset{\sim}{E}_0$ is a consequence of the linearity of Maxwell's equation. Because $\underset{\sim}{E}$ and $\underset{\sim}{E}_0$ are vectors, $\underset{=}{S}(\underset{\sim}{k}, \underset{\sim}{k}')$ must be a tensor as shown. For no scattering, $\underset{=}{S}$ reduces to a unit matrix.

Differential Cross Section

The differential scattering cross section $d\sigma/d\Omega$ is the ratio of the flux scattered from the object per unit solid angle to the incident flux on the object per unit area.

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{energy scattered/unit time/unit solid angle}}{\text{energy incident/unit time/unit area}}} \quad . \tag{57}$$

In terms of Poynting's vector, $\underset{\sim}{S}$, for the incident and scattered fluxes, we have

$$\frac{d\sigma}{d\Omega} = \left| \frac{\underset{\sim}{S}_{sca}}{\underset{\sim}{S}_{inc}} \right| R^2 = \left| \frac{L_{sca}}{L_{inc}} \right| R^2$$

Scattering by Electric Dipoles (Rayleigh Scattering)

We now consider an electric dipole at the origin of a Cartesian coordinate system upon which a monochromatic plane wave impinges from along the negative z-axis (Fig. 2).

If the electric dipole is generated by electrons and if the incident wave is a harmonic plane wave of the form,

$$\underset{\sim}{E} = \underset{\sim}{E}_0 \; \hat{i} \, e^{ikz - i\omega t}$$

then the equation of motion is

$$m\ddot{\underset{\sim}{r}} = e\,\underset{\sim}{E}\,(z = 0)$$

Since $\underset{\sim}{p} = e\,\underset{\sim}{r}$, the electric dipole moment is found from

$$\ddot{\underset{\sim}{p}} = \frac{e^2}{m}\,\underset{\sim}{E}\,(z = 0) \quad ,$$

to be

$$\underset{\sim}{p} = -\frac{e^2}{\omega^2 m}\,E_0\,\hat{i}$$

Note that the use of $\underset{\sim}{E}$ at $z = 0$ in the equation of motion implies that $r << \lambda$, i.e., the dipole approximation.

In a dielectric in the presence of an externally applied electric field, the molecular dipoles are stereteched and become oriented. A net electric moment is produced in the direction of the applied field. The dipole moment per unit volume for an isotropic dielectric medium is related to the external electric field by

$$\underset{\sim}{p} = \alpha \underset{\sim}{E}$$

where α $[C^2 m/N]$ is the proportionality constant, the polarizability, and is dependent on the frequency of $\underset{\sim}{E}$. From the discussion on dipole radiation the radiation fields for an electric dipole are [eq. (54)]

$$\underset{\sim}{E} = -\frac{\omega}{k} \underset{\sim}{B} \times \hat{n} = \frac{e^{ikr}}{4\pi \epsilon r} k^2 \left((\underset{\sim}{p} \times \hat{n}) \times \hat{n} \right)$$

$$\underset{\sim}{B} = -\frac{\omega \mu k e^{ikr}}{4\pi r} \underset{\sim}{p} \times \hat{n}$$

where $\underset{\sim}{E}$, $\underset{\sim}{B}$, and $\underset{\sim}{p}$ were assumed to have a harmonic time dependence.

Since the incident electric field is along the z-axis

$$\underset{\sim}{E}_{inc} = E_0 \, e^{ikz - i\omega t} \, \hat{i}$$

or

$$\underset{\sim}{E}_{inc} = (E_{0r} \hat{r} + E_{0\ell} \hat{\ell}) \, e^{ikz - i\omega t}$$

where $E_{0r} = E_0 \sin \phi$ and $E_{0\ell} = E_0 \cos \phi$. Hence

$$\underset{\sim}{B}_{sca} = - \frac{\omega \mu k e^{ikr}}{4\pi r} \left(\alpha \underset{\sim}{E}_{inc} \times \hat{n}'\right)$$

where we have used

$$\underset{\sim}{p} = \alpha \underset{\sim}{E}_{inc} \Big|_{z=0} \quad .$$

Then

$$\underset{\sim}{B}_{sca} = - \frac{\omega \mu k}{4\pi r} \alpha e^{ikr} \left(E_{0r} \hat{r} \times \hat{n}' + E_{0\ell} \hat{\ell} \times \hat{n}'\right) \tag{58}$$

Now since

$$\hat{r} = \hat{r}' \qquad \hat{r} \times \hat{n}' = - \hat{\ell}'$$

$$\hat{\ell} \times \hat{n}' = \hat{r} \cos\theta \qquad \hat{\ell} \times \hat{n} = \hat{r}$$

where we have used $\xi = \angle\, (\hat{\ell}, \hat{n}')$ and $\cos\theta = \sin\xi$. Therefore,

$$\underset{\sim}{B}_{sca} = - \frac{\omega \mu k \alpha e^{ikr}}{4\pi r} \left(E_{0r} (- \hat{\ell}') + E_{0\ell} (\hat{r} \cos\theta)\right)$$

and

$$\underset{\sim}{E}_{sca} = - \frac{\omega}{k} \underset{\sim}{B}_{sca} \times \hat{n}'$$

$$= \frac{\alpha e^{ikr} k^2}{4\pi \epsilon r} \Big(E_{0r} (- \hat{n}' \times \hat{\ell}') + E_{0\ell} \cos\theta \, (\hat{n}' \times \hat{r}') \Big)$$

$$= \frac{\alpha e^{ikr} k^2}{4\pi \epsilon r} \left(E_{0r} \hat{r}' + E_{0\ell} \cos\theta \, \hat{\ell}'\right) \tag{59}$$

Hence,

$$\begin{pmatrix} E_\ell \\ E_r \end{pmatrix} = \underset{\sim}{E}_{sca} = \frac{\alpha k^2 e^{ikr}}{4\pi\epsilon r} \begin{pmatrix} \cos\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_{0\ell} \\ E_{0r} \end{pmatrix}$$

$$= \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{0\ell} \\ E_{0r} \end{pmatrix} \frac{e^{ikr}}{r} \tag{60}$$

where

$$\boxed{\begin{aligned} S_2 &= \frac{\alpha k^2 \cos\theta}{4\pi\epsilon} \\ S_1 &= \frac{\alpha k^2}{4\pi\epsilon} \end{aligned}} \tag{61}$$

The time-averaged Poynting vector for the scattered wave is

$$\underset{\sim}{S}_{sca} = \frac{1}{2} (\underset{\sim}{E}_{sca} \times \underset{\sim}{H}^*_{sca})$$

In the far radiation zone, $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are perpendicular to each other and to the direction of propagation. For a harmonic wave under these conditions,

$$\underset{\sim}{H}^* = \sqrt{\frac{\epsilon}{\mu}} \; \hat{n} \times \underset{\sim}{E}^* \qquad ;$$

thus

$$\underset{\sim}{S}_{sca} = \frac{1}{2} \left(\underset{\sim}{E}_{sca} \times (\hat{n}' \times \underset{\sim}{E}^*_{sca}) \right) \sqrt{\frac{\epsilon}{\mu}} \qquad ,$$

or

$$\underset{\sim}{S}_{sca} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} (E_{or}^2 \hat{n}' + E_{o\ell}^2 \cos^2\theta \, \hat{n}') \qquad (62)$$

Therefore, the energy flux densities $L = 1/2 \sqrt{\epsilon/\mu}\, E^2$ for the incoming and scattered waves are related by

$$L_{sca} = \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} (L_{inc_r} + \cos^2\theta \, L_{inc_\ell}) \qquad (63)$$

If $\phi = 0$ deg, then

$$L_{sca} = \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} \cos^2\theta \, L_{inc} \quad ;$$

if $\phi = 90$ deg,

$$L_{sca} = \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} L_{inc} \quad ,$$

and if $\phi = 45$ deg or the incident radiation is unpolarized,

$$L_{sca} = \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} \frac{1}{2} (1 + \cos^2\theta) L_{inc} \qquad (64)$$

For natural incident radiation, this is the same as averaging over all polarization angles,

$$L_{sca} = \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} \left(\frac{L_{inc}}{2\pi} \int_0^{2\pi} \sin^2\phi d\phi + L_{inc} \frac{\cos^2\theta}{2\pi} \int_0^{2\pi} \cos^2\phi d\phi \right)$$

$$= \frac{a^2 k^4}{(4\pi\epsilon)^2 r^2} \frac{1}{2} (1 + \cos^2\theta) L_{inc}$$

This scattering is called Rayleigh scattering. The differential scattering cross section becomes

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{L_{sca}\, r^2}{L_{inc}} = \frac{\alpha^2\, k^4}{(4\pi\epsilon)^2}\; \frac{1}{2}\; (1 + \cos^2\theta)} \quad . \tag{65}$$

Rayleigh-Gans Scattering [7]

Assume that each volume element in some finite volume gives Rayleigh scattering and does so independently of the other volume elements. The waves scattered in a given direction by all these elements interfere because of the different positions of the volume elements. To calculate the interference effects we have to refer the phases of all scattered waves to a common origin of coordinates and then add the complex amplitudes (Fig. 3).

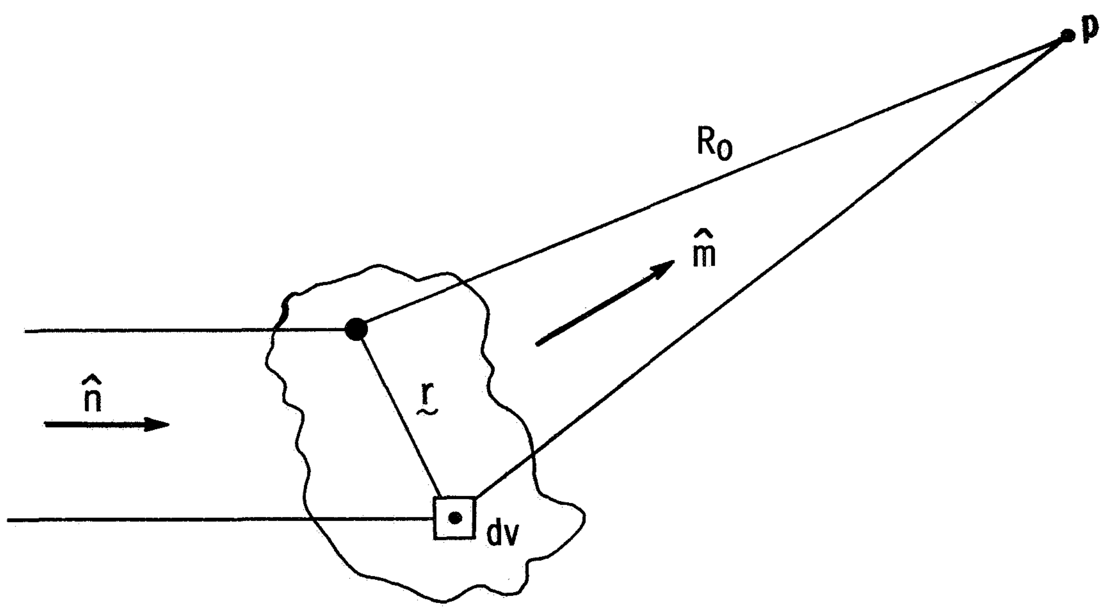


Figure 3. Scattering by a finite volume.

Consider the volume element in Figure 3. If $\underset{\sim}{p}$ is the dipole moment per unit volume, the magnetic induction for the scattered field at point P is

$$\underset{\sim}{B} = - \frac{\omega \mu k e^{ikR}}{4\pi R} \; (\underset{\sim}{p}\, dV \times \hat{m}) \quad .$$

Integrating over the volume, this expression becomes

$$\underset{\sim}{B} = - \frac{\omega \mu k}{4\pi} \int \frac{e^{ikR}}{R} \, dV \, (\underset{\sim}{p} \times m) \tag{66}$$

With $\hat{n}$, a unit vector in the incident direction, and $\hat{m}$, a unit vector in the scattered direction, the phase of e^{ikR} (Fig. 4), i.e., kR is $k\left[R_0 + \underset{\sim}{r} \cdot (\hat{m} - \hat{n})\right]$. The wave "2" has to go $(\underset{\sim}{r} \cdot \hat{n})$ farther to get to the scatterer, but for wave "1" the scatterer is $(\underset{\sim}{r} \cdot \hat{m})$ farther in front of the origin, R_0 is large compared with the body size. Hence,

$$\underset{\sim}{B} = (\underset{\sim}{p} \times \hat{m}) \left(- \frac{\omega \mu k}{4\pi}\right) \frac{e^{ikR_0}}{R_0} \int e^{ik\underset{\sim}{r} \cdot (\hat{m} - \hat{n})} dV$$

which differs from Rayleigh scattering by a factor of the form $\int e^{i\delta} \, dV$, where $\delta = k\underset{\sim}{r} \cdot (\hat{m} - \hat{n})$. We now evaluate the normalized factor

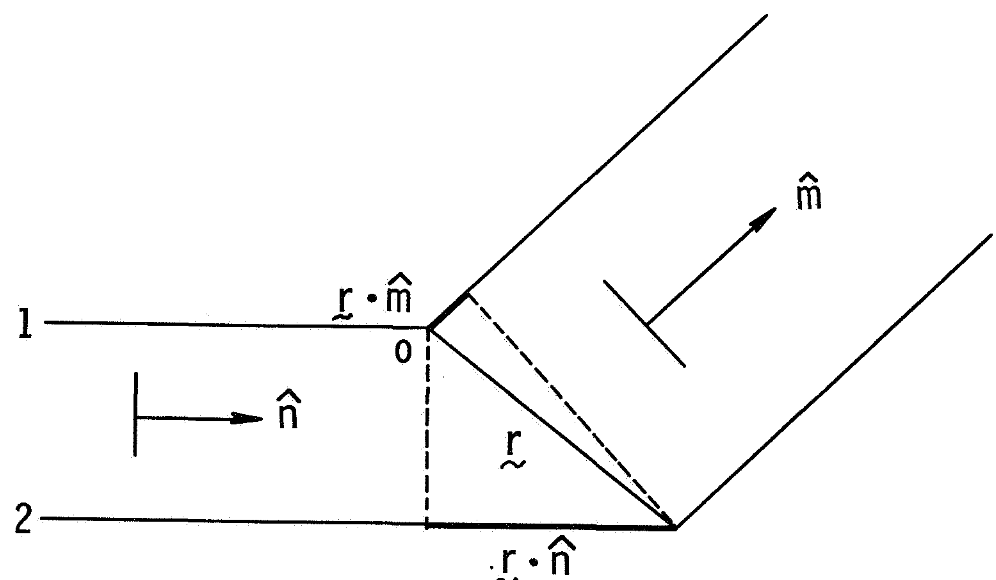


Figure 4. Phase change due to scattering.

$$R(\theta,\phi) = \frac{1}{V}\int e^{i\delta}\, dV \tag{67}$$

for a cylindrically shaped body. As seen from Figure 5 the vector $\hat{m} - \hat{n}$ has the length $2\sin(\theta/2)$ along the bisectrix of the direction $\hat{m}$ and $-\hat{n}$. Hence,

$$\delta = s\, 2\, k \sin \frac{\theta}{2} \tag{68}$$

where

$$s = \underset{\sim}{r} \cdot \frac{(\hat{m} - \hat{n})}{|\hat{m} - \hat{n}|} \tag{69}$$

Hence,

$$R(\theta,\phi) = \frac{1}{V}\int D\, e^{ik\, s \sin \frac{1}{2}\theta}\, ds \tag{70}$$

where we are integrating over "slices" perpendicular to the bisectrix, and each slice has an area D and thickness ds.

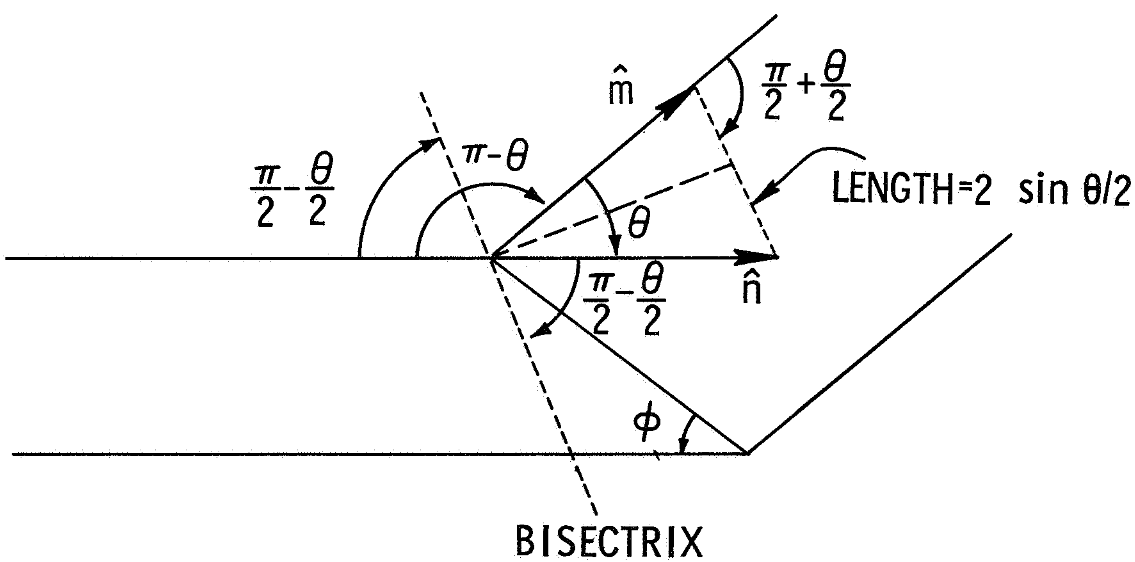


Figure 5. Geometry of Rayleigh-Gans scattering.

We have assumed that the phase shifts of the wavelets passing through the medium are small, i.e.,

$$2\,ka\,|m - 1| << 1 \qquad (71)$$

where a is a length of the order of the size of the particle, and m = the index of refraction.

Circular Cylinders of Finite Length

The function $R(\theta, \phi)$ can be calculated for the circular cylinder of finite length. Let its length by ℓ and its diameter 2a, and let the phase shift be small for a ray traversing the cylinder in any direction. The orientation of the cylinder with respect to the incident wave is arbitrary (Fig. 6).

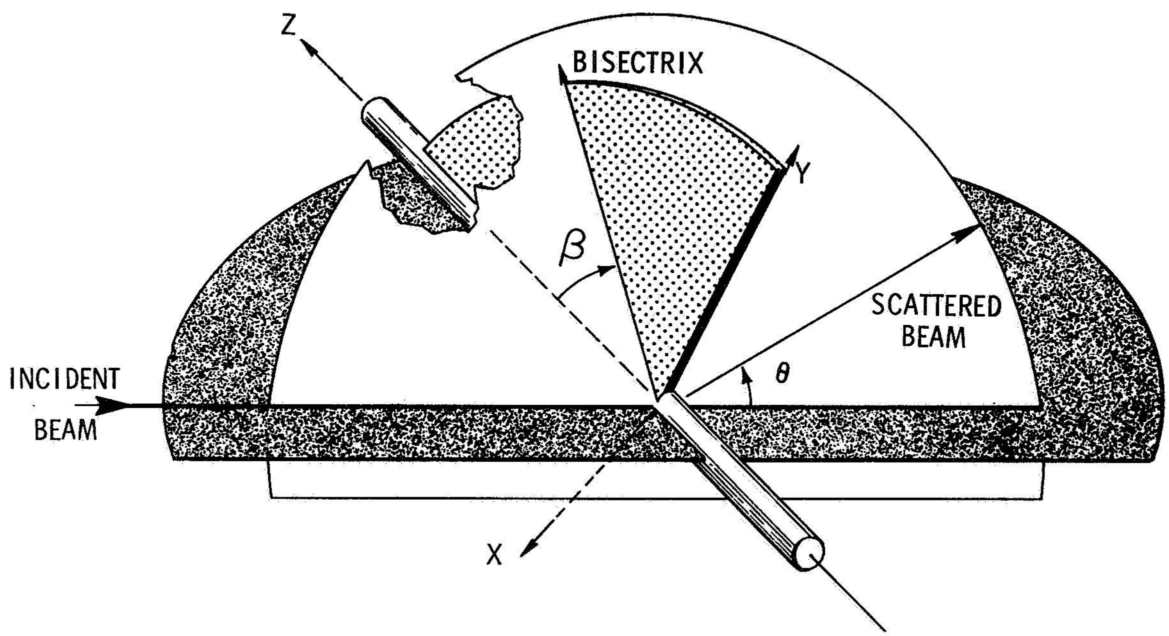


Figure 6. Geometry of scattering by a cylinder.

The volume integration will be performed by using circular slices which are perpendicular to the cylinder axis since this is easier than using slices perpendicular to the bisectrix.

To calculate $e^{ik\underset{\sim}{r} \cdot (\hat{m} - \hat{n})}$ let $\underset{\sim}{r} = \hat{i}\, x + \hat{j}\, y$. Fix the axis of the cylinder so that the bisectrix lies in the y - z plane. Then, $\hat{m} - \hat{n} = a_1 \hat{j} + a_2 \hat{k}$ and

$$\underset{\sim}{r} \cdot (\hat{m} - \hat{n}) = y\, a_1$$

Since $|\hat{m} - \hat{n}| = 2 \sin \theta/2$ and is in the same direction as the bisectrix,

$$a_1 = 2 \sin \frac{\theta}{2} \cos\left(\frac{\pi}{2} - \beta\right)$$

or

$$a_1 = 2 \sin \frac{\theta}{2} \sin \beta$$

where β is the angle between the cylinder axis and the bisectrix, and

$$\underset{\sim}{r} \cdot (\hat{m} - \hat{n}) = 2\, y \sin \frac{\theta}{2} \sin \beta$$

With this relation and with the phase referred to the center of the disk,

$$R_D(\theta, \phi) = \frac{1}{D} \int e^{ik\underset{\sim}{r} \cdot (\hat{m} - \hat{n})} dD$$

becomes

$$R_D(\theta,\phi) = \frac{1}{D} \int_{-a}^{+a} \int_{-\sqrt{a^2 - y^2}}^{+\sqrt{a^2 - y^2}} e^{2iky \sin \frac{\theta}{2} \sin \beta} \, dx \, dy$$

$$= \frac{2}{D} \int_{-a}^{+a} e^{2iky \sin \frac{\theta}{2} \sin \beta} \sqrt{a^2 - y^2} \, dy \qquad (72)$$

We can make a change of variables by letting

$$y = a\omega$$

and

$$v = a2k \sin \frac{\theta}{2} \sin \beta$$

then

$$R_D(\theta,\phi) = \frac{2}{D} \int_{-1}^{+1} e^{iv\omega} a^2 \sqrt{1 - \omega^2} \, d\omega$$

$$= \frac{2a^2}{\pi a^2} \int_{-1}^{+1} e^{iv\omega} \sqrt{1 - \omega^2} \, d\omega$$

$$= \frac{2}{\pi} \left[\int_{0}^{1} e^{iv\omega} \sqrt{1 - \omega^2} \, d\omega - \int_{0}^{-1} e^{iv\omega'} \sqrt{1 - \omega'^2} \, d\omega' \right]$$

Let $\omega' = -\omega$ in the second integral

$$R_D(\theta,\phi) = \frac{2}{\pi} \int_0^1 e^{iv\omega}\sqrt{1-\omega^2}\, d\omega + \int_0^1 e^{-iv\omega}\sqrt{1-\omega^2}\, d\omega \quad ,$$

or

$$R_D(\theta,\phi) = \frac{4}{\pi} \int_0^1 \left(\frac{e^{iv\omega} + e^{-iv\omega}}{2}\right) \sqrt{1-\omega^2}\, d\omega$$

and hence

$$R_D(\theta,\phi) = \int_0^1 \cos v\omega \sqrt{1-\omega^2}\, d\omega$$

$$= F(v) = \frac{2}{v} J_1(v) \qquad (73)$$

where $J_n(v)$ is a Bessel function of order n and

$$F(v) = \frac{4}{\pi} \int_0^1 \cos v\omega \sqrt{1-\omega^2}\, d\omega \qquad (74)$$

The phases of the disk have been referred to the center of the disk. In considering the whole cylinder, the phase can be referred to the center of the cylinder. This introduces another phase factor given by

$$kz\,\hat{k}\cdot(\hat{m}-\hat{n})$$

But since

$$(\hat{m} - \hat{n}) = a_1 \hat{j} + a_2 \hat{k}$$

the phase is $k\ z a_2$ where $a_2 = \cos\beta$. This phase effect is given by

$$R_L(\theta,\phi) = \frac{1}{\ell} \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} e^{i2kz \sin\frac{\theta}{2} \cos\beta} \, dz \tag{75}$$

Let

$$\mu = \ell k \sin\frac{\theta}{2}\cos\beta \quad , \; z = t\,\frac{\ell}{2} \quad , \quad dz = \frac{\ell}{2}\, dt$$

then

$$R_L(\theta,\phi) = \frac{1}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} e^{i\mu t}\, dt\, \frac{\ell}{2} = \frac{1}{2} \int_0^1 e^{i\mu t}\, dt + \frac{1}{2} \int_0^1 e^{-i\mu t}\, dt$$

$$= \int_0^1 \cos\mu t\, dt$$

Thus we can say

$$\boxed{E(\mu) = \int_0^1 \cos\mu t\, dt = \left(\frac{\pi}{2\mu}\right)^{1/2} J_{1/2}(\mu)} \quad . \tag{76}$$

The final phase factor is the product of the two phase factors, i.e.,

$$R(\theta,\phi) = F\left(2\,ka \sin\frac{\theta}{2} \sin\beta\right) E\left(k\ell \sin\frac{\theta}{2} \cos\beta\right) .$$

The angle β and be related to the angle α between the incident light and the z-axis the angle $\theta/2$ and the azimuth angle ϕ between the incident light – z-axis plane and the incident light-scattered light plane by (Fig. 7)

$$\cos\beta = -\cos\alpha \sin\frac{\theta}{2} + \sin\alpha \cos\frac{\theta}{2} \cos\phi \tag{77}$$

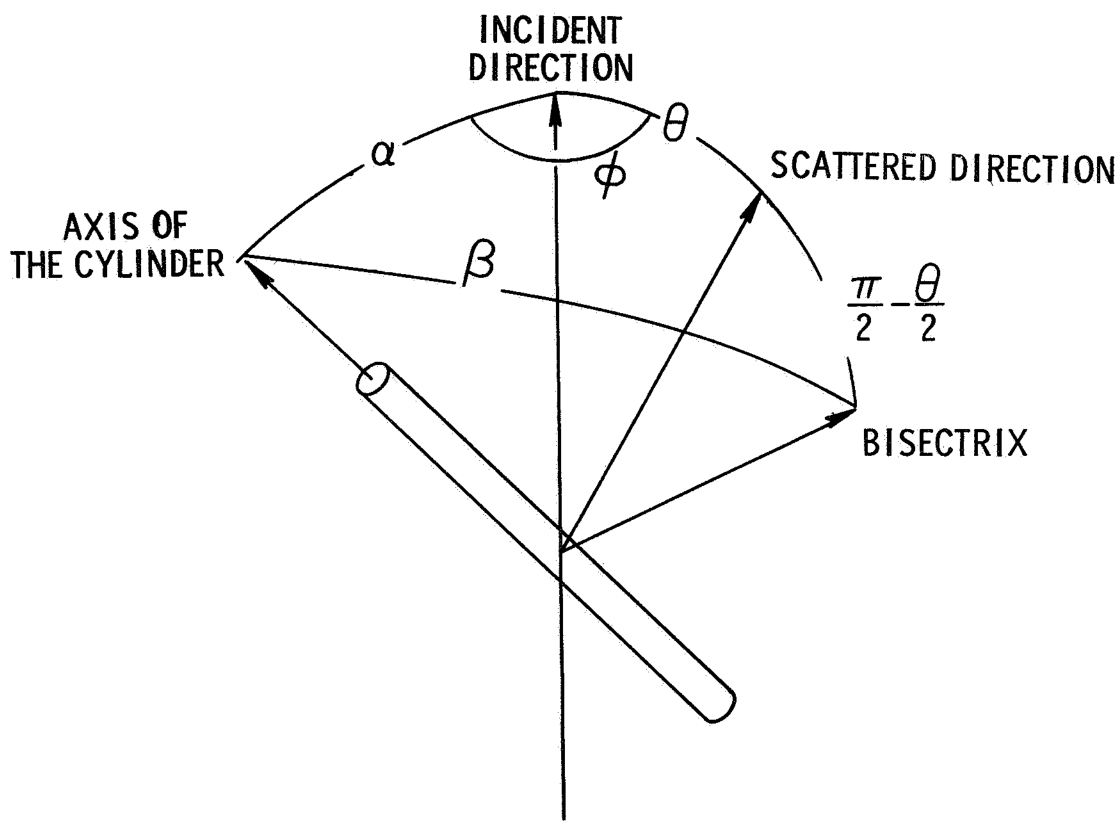


Figure 7. Scattering angles.

To find the scattering amplitude tensor we follow the steps as in the Rayleigh scattering section to obtain:

$$\underset{\sim}{B} = -\frac{\omega\mu k}{4\pi} (\underset{\sim}{p} \times \hat{m}) \frac{e^{ikR_0}}{R_0} V R(\theta, \phi)$$

$$\underset{\sim}{E} = \frac{\omega^2\mu}{4\pi} \left((\underset{\sim}{p} \times \hat{m}) \times \hat{m} \right) \frac{e^{ikR_0}}{R_0} V R(\theta, \phi) \tag{78}$$

$$= \frac{k^2 (aV)}{4\pi \epsilon_0} \frac{e^{ikR_0}}{R_0} (E_{0r} \hat{r}' + E_{0\ell} \cos\theta \, \hat{\ell}') R(\theta, \phi)$$

Hence,

$$S_1 = \frac{(aV) k^2}{4\pi \epsilon_0} R(\theta, \phi)$$

$$S_2 = \frac{(aV) k^2}{4\pi \epsilon_0} \cos\theta \, R(\theta, \phi) \tag{79}$$

We note that since $R(\theta, \phi)$ is independent of polarization effects the polarization will be that of a Rayleigh scattering particle:

$$P(\theta) = \frac{|S_1|^2 - |S_2|^2}{|S_1|^2 - |S_2|^2} = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} = \frac{\sin^2\theta}{1 + \cos^2\theta} \tag{80}$$

The differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = \frac{(aV)^2 k^4}{(4\pi \epsilon_0)^2} R^2(\theta, \phi) \frac{1}{2} (1 + \cos^2\theta) \tag{81}$$

The appendix gives a computer program listing for calculating the functions F and E and the scattering intensity functions. Also included, but not derived in this paper, is a phase factor for spheres G. Table 2 tabulates some values of these factors. Figure 8 shows them for comparison.

TABLE 2. THE DIFFRACTION FUNCTIONS

U	E(U)	F(U)	G(U)	U	E(U)	F(U)	G(U)
.00	1.00000	1.00000	1.00000	5.00	-.19178	-.13103	-.05705
.10	.99833	.99875	.99900	5.10	-.18153	-.13219	-.06453
.20	.99335	.99501	.99601	5.20	-.16990	-.13201	-.07083
.30	.98507	.98879	.99103	5.30	-.15703	-.13055	-.07598
.40	.97355	.98013	.98409	5.40	-.14310	-.12791	-.08002
.50	.95885	.96907	.97522	5.50	-.12828	-.12416	-.08300
.60	.94107	.95567	.96446	5.60	-.11273	-.11940	-.08498
.70	.92031	.93999	.95185	5.70	-.09661	-.11374	-.08599
.80	.89670	.92211	.93745	5.80	-.08010	-.10725	-.08611
.90	.87036	.90211	.92131	5.90	-.06337	-.10005	-.08539
1.00	.84147	.88010	.90351	6.00	-.04657	-.09223	-.08389
1.10	.81019	.85619	.88411	6.10	-.02986	-.08389	-.08168
1.20	.77670	.83048	.86321	6.20	-.01340	-.07513	-.07882
1.30	.74120	.80311	.84089	6.30	.00267	-.06606	-.07537
1.40	.70389	.77421	.81723	6.40	.01821	-.05676	-.07141
1.50	.66500	.74392	.79235	6.50	.03310	-.04734	-.06699
1.60	.62473	.71237	.76633	6.60	.04720	-.03787	-.06219
1.70	.58333	.67972	.73928	6.70	.06043	-.02846	-.05707
1.80	.54103	.64613	.71132	6.80	.07266	-.01918	-.05169
1.90	.49805	.61174	.68256	6.90	.08383	-.01012	-.04612
2.00	.45465	.57672	.65310	7.00	.09386	-.00134	-.04041
2.10	.41105	.54123	.62306	7.10	.10267	.00709	-.03463
2.20	.36750	.50542	.59256	7.20	.11023	.01509	-.02883
2.30	.32422	.46945	.56172	7.30	.11650	.02262	-.02306
2.40	.28144	.43349	.53064	7.40	.12145	.02963	-.01737
2.50	.23939	.39768	.49946	7.50	.12507	.03607	-.01182
2.60	.19827	.36217	.46827	7.60	.12736	.04190	-.00644
2.70	.15829	.32711	.43719	7.70	.12833	.04709	-.00127
2.80	.11964	.29265	.40632	7.80	.12802	.05163	.00365
2.90	.08250	.25892	.37579	7.90	.12645	.05549	.00829
3.00	.04704	.22604	.34568	8.00	.12367	.05866	.01262
3.10	.01341	.19414	.31609	8.10	.11974	.06114	.01661
3.20	-.01824	.16334	.28712	8.20	.11472	.06293	.02025
3.30	-.04780	.13374	.25886	8.30	.10870	.06403	.02352
3.40	-.07516	.10543	.23139	8.40	.10174	.06447	.02640
3.50	-.10022	.07850	.20479	8.50	.09394	.06426	.02890
3.60	-.12292	.05304	.17913	8.60	.08540	.06343	.03099
3.70	-.14320	.02910	.15447	8.70	.07620	.06200	.03269
3.80	-.16102	.00675	.13088	8.80	.06647	.06002	.03400
3.90	-.17635	-.01397	.10840	8.90	.05629	.05751	.03491
4.00	-.18920	-.03302	.08708	9.00	.04579	.05451	.03544
4.10	-.19958	-.05038	.06697	9.10	.03507	.05108	.03560
4.20	-.20752	-.06602	.04809	9.20	.02423	.04726	.03541
4.30	-.21306	-.07995	.03046	9.30	.01338	.04310	.03488
4.40	-.21627	-.09217	.01411	9.40	.00264	.03865	.03403
4.50	-.21723	-.10269	-.00095	9.50	-.00791	.03395	.03288
4.60	-.21602	-.11154	-.01473	9.60	-.01816	.02907	.03146
4.70	-.21275	-.11876	-.02721	9.70	-.02802	.02405	.02979
4.80	-.20753	-.12437	-.03842	9.80	-.03740	.01895	.02790
4.90	-.20050	-.12845	-.04836	9.90	-.04622	.01381	.02580

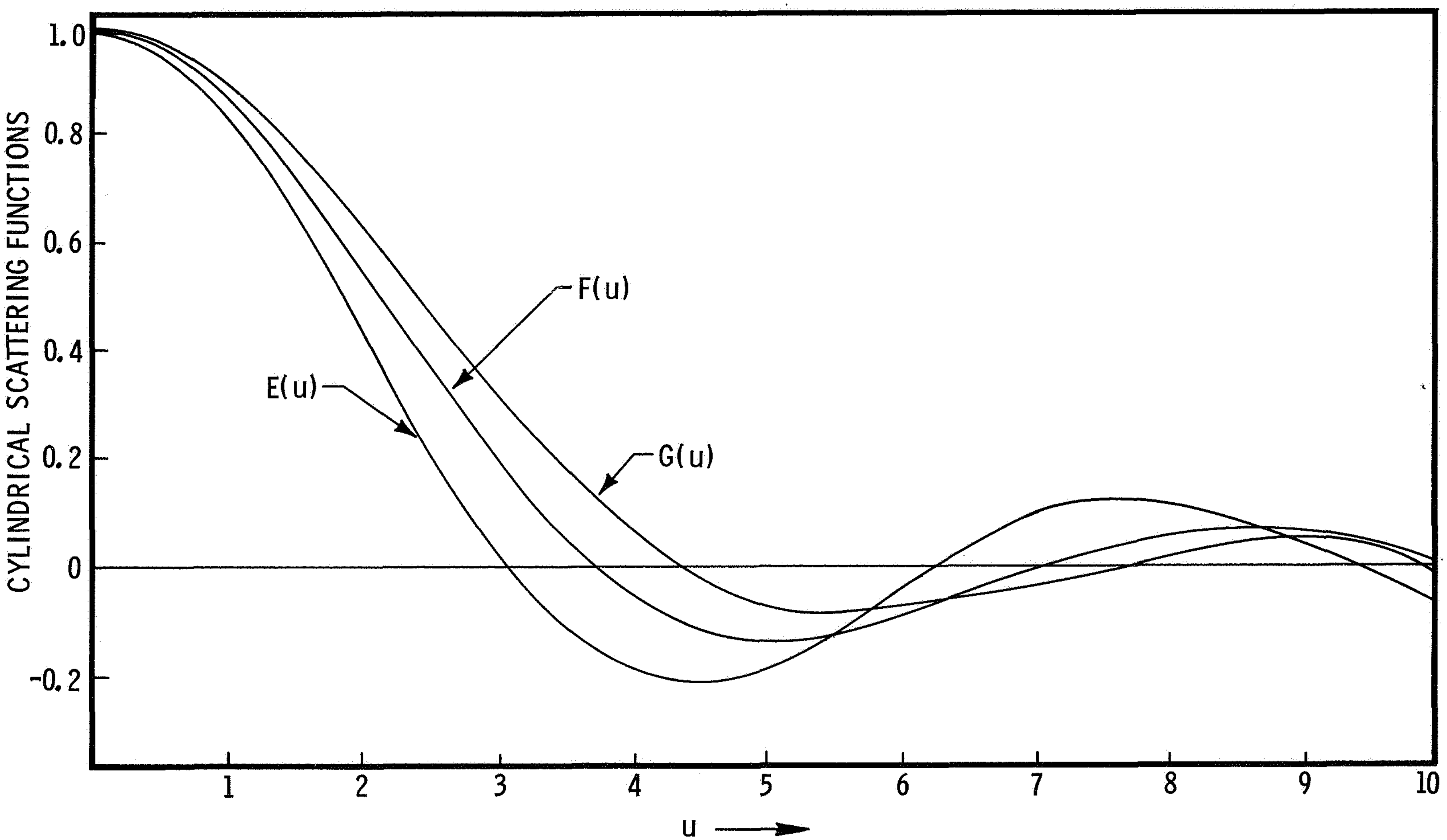


Figure 8. Cylindrical functions.

Randomly Oriented Cylinders [7]

To obtain the differential scattering cross section in the Rayleigh-Gans domain for randomly oriented cylinders, an integration over the solid angle of the orientation of the cylinder is performed:

$$\overline{R^2} = \frac{1}{4\pi} \int R^2(\theta, \phi)\, d\Omega \tag{82}$$

where $\overline{R^2}$ is an average cylindrical function for random orientation. The values θ and ϕ are considered fixed. The solid-angle integration reduces to an integration over β. For the case of thin rods $(R \to E)$,

$$\overline{R^2} = \int_0^1 E^2\,(z \cos\beta)\, d \cos\beta$$

$$= \frac{1}{z} \int_0^{2z} \frac{\sin\omega}{\omega}\, d\omega - \left(\frac{\sin z}{z}\right)^2 \tag{83}$$

where $z = k\ell \sin 1/2\,\theta$. The first term is the sine integral. For the case of disks,

$$\overline{R^2} = \int_0^1 F(z \sin\beta)\, d\cos\beta = \frac{2}{z^2}\left[\, 1 - F(2z)\,\right] \tag{84}$$

where $z = 2ka \sin 1/2\,\theta$.

Lorentz-Lorenz Formula [8]

The relation between polarizability and refractive index will now be derived.

An external electric field stretches and orients molecular dipoles in a dielectric. The amount of polarization $\underset{\sim}{P}$ is equal to the average electric dipole moment per unit volume. The polarization induces a field which modifies the relation between the intensity $\underset{\sim}{E}$ and the displacement vector $\underset{\sim}{D}$. Hence,

$$\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \tag{85}$$

The total polarization is caused by (1) polarization by stretching P_1 and (2) polarization by dipole orientation P_2. The polarization caused by elastically "stretched" molecules is given by

$$\underset{\sim}{P}_1 = N \alpha \underset{\sim}{E}' \tag{86}$$

where N = the number of molecules per unit volume, α = proportionality constant (polarizability), and $\underset{\sim}{E}'$ is the effective internal electric field for the dipoles.

The polarization caused by orienting the permanent dipole moments can be approximated by

$$\underset{\sim}{P}_2 = \frac{1}{3} \frac{p^2 N \underset{\sim}{E}'}{kT} \tag{87}$$

where p is the molecular dipole moment, k is the Boltzmann constant, and T is the temperature. But for an oscillating field the relation becomes

$$\underset{\sim}{P}_2 = \frac{Np^2}{3kT} \left(\frac{1}{1 + i\omega\tau} \right) e^{i\omega t} \underset{\sim}{E}'$$

where τ is the relaxation time for the dipoles, and τ is related to the fractural torque on the molecules and the temperature.

The Clausius-Mosetti theory gives the value of the internal field for a dipole as a function of the external field $\underset{\sim}{E}$ and the polarization $\underset{\sim}{P}$. Consider a cavity within a dielectric. The local field in the cavity is made up of two parts: (1) the external field $\underset{\sim}{E}$ and (2) the field produced by the polarization within the cavity. This second field can be calculated knowing that for a small area dA on the surface of a cavity the induced charge density is $\underset{\sim}{P} \cos\theta$. For a spherical cavity the amount of charge induced in a spherical zone is $2\pi r^2 \underset{\sim}{P} \sin\theta \cos\theta \, d\theta$, and the field at the center of the cavity is $1/2 \, \epsilon_0 \underset{\sim}{P} \cos^2\theta \sin\theta \, d\theta$. Integrating over all the charges gives the total field as $\underset{\sim}{P}/3 \epsilon_0$. Hence, the internal field is

$$\underset{\sim}{E}' = \underset{\sim}{E} + \frac{\underset{\sim}{P}}{3\,\epsilon_0} \tag{88}$$

Using $\underset{\sim}{D} = \underset{\sim}{P} + \epsilon_0\,\underset{\sim}{E} = \epsilon\,\underset{\sim}{E}$, the Clausius-Mosetti relation is obtained:

$$\underset{\sim}{E}' = \frac{\underset{\sim}{P}}{\epsilon - \epsilon_0} + \frac{\underset{\sim}{P}}{3\,\epsilon_0}$$

$$\boxed{\underset{\sim}{E}' = \left(\frac{\epsilon + 2\,\epsilon_0}{\epsilon - \epsilon_0}\right) \frac{\underset{\sim}{P}}{3\,\epsilon_0}} \tag{89}$$

For polarization due to elastic stretching alone,

$$\frac{\underset{\sim}{P}}{\underset{\sim}{E}} = 3\,\epsilon_0 \; \frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} = N\alpha$$

or in terms of the refractive index, $m = \sqrt{\epsilon/\epsilon_0}$,

$$\boxed{N\alpha = 3\,\epsilon_0 \left(\frac{m^2 - 1}{m^2 + 2}\right)} \qquad \text{(Lorentz-Lorenz)} \tag{90}$$

Including permanent dipoles gives the following expression:

$$\frac{m^2 - 1}{m^2 + 2} = \frac{N}{3\,\epsilon_0} \left[\alpha + \frac{1}{3} \frac{p^2}{kT} \frac{1}{1 + i\omega\tau} \right]$$

showing the variation of the complex refractive index with temperature and frequence. For very high frequencies the second term in the bracket is negligible. Then the Rayleigh-Gans differential scattering cross section in terms of the refractive index is

$$\frac{d\sigma}{d\Omega} = k^4 V^2 \left(\frac{m - 1}{2\pi}\right)^2 \frac{1}{2} \left(1 + \cos^2\theta\right) R^2\,(\theta, \phi) \tag{91}$$

where

$$N\alpha = 3\,\epsilon_0 \; \frac{m^2 - 1}{m^2 + 2} \;\xrightarrow[m \to 1]{}\; 2\,\epsilon_0\,(m - 1) \tag{92}$$

Scattering by Infinite Cylinders

In conclusion we will consider the specific case of a thin infinite cylinder. According to Cooke and Kerker [9], the radiation scattered from an infinite cylinder lies on the surface of a cone whose apical angle is twice the complement of the tilt angle. From Rayleigh-Gans scattering by cylinders $(ka << 1)$, the expression for the scattered intensity is given by [7]:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\,V^2\,(m - 1)^2}{8\,\pi^2}\;(1 + \cos^2\theta)\cdot E^2\left(k\ell \sin\frac{\theta}{2}\;\cos\beta\right) \tag{93}$$

For infinite cylinders $(\ell \to \infty)$ E becomes significant only when $\cos\beta = 0$, which includes $\theta = 0$, the forward scattering direction. This yields by equation (77)

$$\cos\alpha\,\sin\left(\left(\frac{1}{2}\right)\theta\right) = \sin\alpha\,\cos\left(\left(\frac{1}{2}\right)\theta\right)\cos\phi \tag{94}$$

Using this restriction on α in the expression

$$\cos\gamma = \cos\alpha\,\cos\theta + \sin\alpha\,\sin\theta\,\cos\phi$$

or

$$\cos\gamma = \cos\alpha\left(2\cos^2\frac{\theta}{2} - 1\right) + \sin\alpha\left(2\sin\frac{\theta}{2}\;\cos\frac{\theta}{2}\right)\cos\phi \tag{95}$$

where γ is the angle between the cylinder axis and the scattering direction, gives the basic result that the scattered light is confined to the surface of a cone of apical angle 2α i.e., $\gamma = \alpha$.

Theoretically, one can show the formation of the scattering cone of the infinite cylinder by varying the length of a finite cylinder. Figure 9 shows the normalized scattering differential cross section

$$\frac{\frac{d\sigma(\theta)}{d\Omega}}{\frac{d\sigma(\theta=0)}{d\Omega}} = E^2\left(k\ell \sin\frac{\theta}{2}\cos\beta\right)\cdot F^2\left(2\,ka \sin\frac{\theta}{2}\sin\beta\right) \tag{96}$$

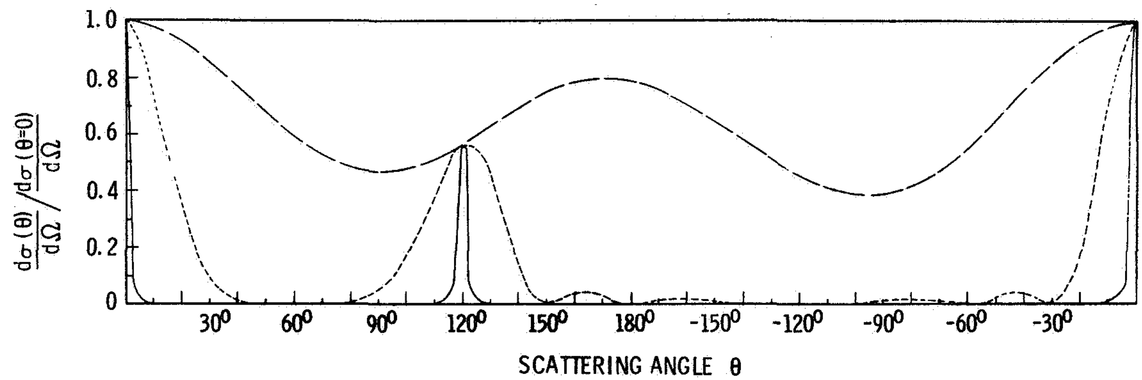


Figure 9. The effect of varying the length ℓ of a finite cylinder of radius a = 0.03 μm and refractive index m = 1.33 (—— ℓ = 0.1 μm, ---- ℓ = 1 μm, —— ℓ = 10 μm).

versus the scattering angle for the particular case where the scattered beams lie in the plane defined by the incident beam and the axis of the cylinder (Fig. 10). The cylinder of radius a = 0.3 μm has a tilt angle of 60 deg; hence, the scattered beams with scattering angles of 60 deg and -120 deg coincide with the axis of the cylinder. For a length ℓ = 0.1 μm (ℓ/a $\simeq$ 3), the peak due to the scattering cone has not appeared. A wide scattering cone peak is seen for the case ℓ = 1 μm (ℓ/a $\simeq$ 33) at 0 deg and 120 deg. For ℓ = 10 μm (ℓ/a $\simeq$ 333), the scattering pattern approaches that of an infinite cylinder with a scattering cone having an apical angle of 120 deg. The value of the normalized scattering differential cross section at θ = 120 deg is nonzero, and not 1 as a result of a finite radius, i.e.,

$$\frac{1}{2}\,F^2\left(2\,ka \sin\frac{\theta}{2}\right)\left(1+\cos^2\theta\right) = 0.561$$

The cone can easily be seen with a milliwatt helium-neon laser (0.6328 μm) and a slender wire [10]. FIgure 11 shows the geometry of the experiment and photographs of the scattering pattern. Both the 25.4-μm-diam copper wire and the 1034-μm-diam wire shown

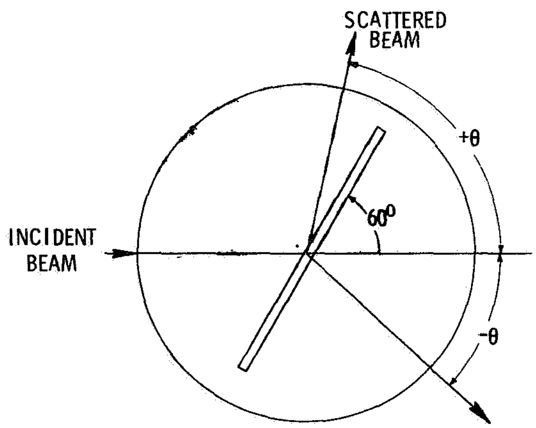
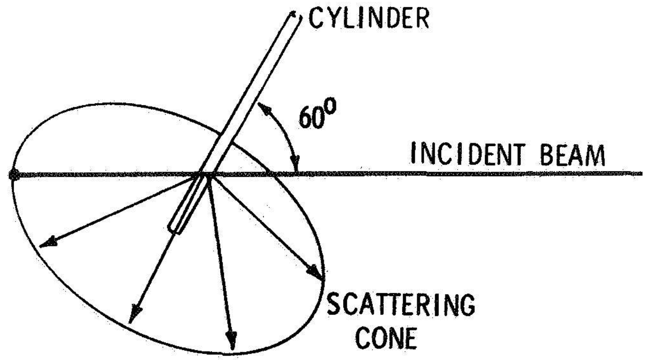


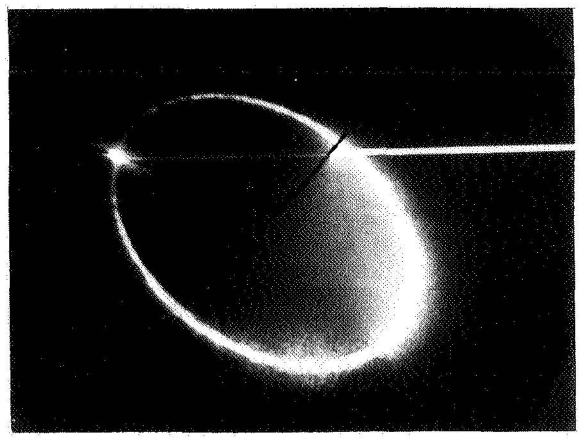
Figure 10. Cylinder scattering in the plane defined by the cylinder axis and incident beam.

have a tilt angle of approximately 60 deg.[1] The incident beam is made visible through the use of smoke. The scattering cone cross section is observed on a plane normal to the axis of the cylinder. Note that the incident beam lies on the cone in the forward scattering direction. The thick wire is visible because of the light scattered by the smoke. The black patch seen in the photograph of the thin wire was used to reduce the amount of diffuse light coming from the screen. Cooke and Kerker [9] explain that the irregularities of the diffuse circles for the thick wire are caused by contamination on the wire, e.g., dust. The fine concentric circles which appear in the thin wire case are probably caused by irregularities on the surface of the wire.

1. Although Rayleigh-Gans theory is not applicable, the scattering geometry is equivalent to the Rayleigh-Gans case.



a



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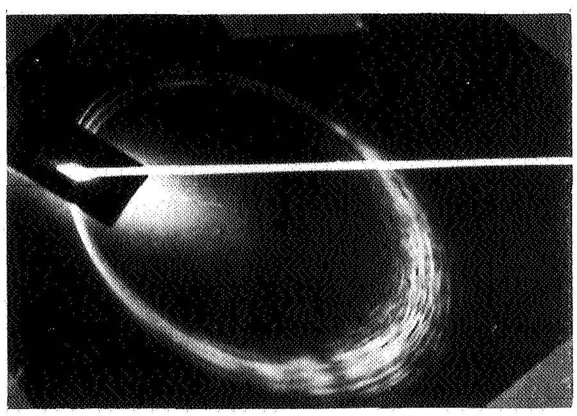

c

Figure 11. a. The experiment geometry.
b. Scattering from thick wire.
c. Scattering from thin wire.

APPENDIX

COMPUTER PROGRAM

A listing of a computer program to calculate I_1 and I_2 for Rayleigh-Gans scattering for a cylinder and a disk is presented, followed by a sample output. The input/output information is given on the comment cards at the beginning of the program. A flow chart is presented in Figure A-1. For reference:

$$I_1 = k^6 V^2 \left(\frac{m - 1}{2\pi}\right)^2 R^2 (\theta,\phi) \tag{A-1}$$

$$I_2 = I_1 \cos^2 \theta \tag{A-2}$$

$$I = \frac{(I_1 + I_2)}{2} \tag{A-3}$$

$$P = \frac{I_1 - I_2}{I_1 - I_2} \tag{A-4}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{I_1 + I_2}{2 k^2} \tag{A-5}$$

```
@RUN,P CYLIND,400980,GAGARYHST561,3,90
@FOR,ISF MAIN
C******************************
C      THIS PROGRAM COMPUTES I1 AND I2 FOR RAYLEIGH-GAN SCATTERING
C      SPHERE G=(3(SINU-UCOSU)/U3)
C      DISK    F=2BESS1U/U
C      ROD     E=SINU/U
C      CYLINDER E*F
C      ELLIPSOIDS--USE G(U) WITH THE RADIUS REPLACED BY OC
C                        (SEE VAN DE HULST PAGE 93 )
C      INPUT******************
C      NI=NUMBER OF RUNS
C      INDEX1=1=SPHERE
C            =2=CYLINDER(DISK OR ROD AS LIMITS)
C      INDEX2=1= NO INTEGRATION OF BETA
C            =2=INTEGRATION OVER ORIENTATION
C      FM =REFRACTIVE INDEX (REAL)
C      A =RADIUS (MICRONS)
C      WAVE =WAVELENGTH (MICRONS)
C      FL=LENGTH OF CYLINDER
C      BETA=ANGLE BETWEEN CYLINDER AXIS AND THE
C            BISECTRIX (OF SCAT DIRECT AND THE AXIS OF CYLINDER)
C      OUTPUT*****************
C      ALP =ALPHA
C      XI1 =I1
C      XI2 =I2
C      P =POLARIZATION
C      THETA =SCATTERING ANGLE(0-180 DEG, STEP=5)
C******************************
   14 FORMAT(31H ***RAYLEIGH-GANS SCATTERING***,//)
    5 FORMAT(18H REFRACTIVE INDEX=,F7.3,/)
    6 FORMAT(17H PARTICLE-RADIUS=,F7.3,8H LENGTH=,F7.3,/)
    7 FORMAT(12H WAVELENGTH=,F7.3,/)
    8 FORMAT(7H ALPHA=,F7.3,/)
    9 FORMAT(1H0,11H SCAT ANGLE,7X,20H INTENSITY FUNCTIONS,19X,
     123H DEGREE OF POLARIZATION, //,11H THETA(DEG),7X,3H I1,12X,3H I2,
     26X,8H I1+I2/2,13X,2H P,/)
   10 FORMAT(1H1)
   11 FORMAT(1H ,I5,4X,3F15.6,4X,F9.6)
   20 FORMAT(8H INDEX1=,I3,22H  1=SPHERE  2=CYLINDER)
   21 FORMAT(8H INDEX2=,I3,9H  1=BETA,F5.2,25H  2=INTEGRATION OVER BETA)
    1 FORMAT(I3)
   22 FORMAT(2I3,2F10.6)
      READ(5,1)NI
      NN=1
      PI=3.1415927
      PI2=2.*PI
    2 CONTINUE
      READ(5,22)INDEX1,INDEX2,BETA,FL
      BETA=BETA*PI/180.
      READ(5,3)FM,A,WAVE
    3 FORMAT(3F10.3)
      FK=PI2/WAVE
      GO TO (23,24),INDEX1
   23 VOL=(4.*PI*A**3)/3.
      GO TO 25
   24 VOL=(PI*A**2)*FL
   25 CONTINUE
      A3=1./3.
C     ALP FOR CYLINDER IS CALCULATED FOR  EQUIVALENT SPHERE
```

```
      ALP=PI2*(3.*VOL/(4.*PI))**A3/WAVE
      CHI=(FK**6*VOL**2)*((FM-1.)/(2.*PI))**2
      WRITE(6,10)
      WRITE(6,14)
      WRITE(6,5)FM
      WRITE(6,6)A,FL
      WRITE(6,7)WAVE
      WRITE(6,8)ALP
      WRITE(6,20)INDEX1
      GO TO (26,27),INDEX1
   27 WRITE(6,21)INDEX2,BETA
   26 CONTINUE
      WRITE(6,9)
      THETA=.0
      DO 4 I=1,37
      Z1=2.*FK*A*SIN(THETA/2.)
      U1=Z1*SIN(BETA)
      Z2=FK*FL*SIN(THETA/2.)
      U2=Z2*COS(BETA)
      U=2.*ALP*SIN(THETA/2.)
      GO TO (28,29),INDEX1
   28 ETA=(G(U))**2
      GO TO 30
   29 GO TO (31,32),INDEX2
   31 ETA=(F(U1)*E(U2))**2
      GO TO 30
   32 CALL RBAR2(Z1,Z2,ETA)
   30 CONTINUE
      ZETA=(COS(THETA))**2
      XI1=CHI*ETA
      XI2=XI1*ZETA
      XI12=(XI1+XI2)/2.
      P=(XI1-XI2)/(XI1+XI2)
      NTH=5*I-5
      WRITE(6,11)NTH,XI1,XI2,XI12,P
      THETA=THETA+5.*PI/180.
    4 CONTINUE
      NN=NN+1
      IF (NN-NI)2,2,13
   13 CONTINUE
      STOP
      FUNCTION G(U)
      EPS=.000001
      IF (ABS(U)-EPS)302,302,301
  302 G=1.
      GO TO 303
  301 G=(SIN(U)-U*COS(U))/(U**3)
      G=G*3.
  303 CONTINUE
      RETURN
      FUNCTION E(U)
      EPS=.000001
      IF (ABS(U)-EPS)312,312,311
  312 E=1.
      GO TO 313
```

```
311 E=SIN(U)/U
313 RETURN
    FUNCTION F(U)
    EPS=.000001
    IF (ABS(U)-EPS)322,322,321
322 F=1.
    GO TO 323
321 F=BSSL(U,1)
    F=F*2./U
323 RETURN
    FUNCTION BSSL(X,N)
    D=.000001
    BJ=0.
    IF(N)10,20,20
 10 IER=1
    RETURN
 20 IF(X)30,30,31
 30 IER=2
    RETURN
 31 IF(X-15.)32,32,34
 32 NTEST=20.+10.*X-X**2/3.
    GO TO 36
 34 NTEST=90.+X/2.
 36 IF(N-NTEST)40,38,38
 38 IER=4
    RETURN
 40 IER=0
    N1=N+1
    BPREV=.0
    IF(X-5.)50,60,60
 50 MA=X+6.
    GO TO 70
 60 MA=1.4*X+60./X
 70 MB=N+IFIX(X)/4+2
    MZERO=MAX0(MA,MB)
    MMAX=NTEST
100 DO 190 M=MZERO,MMAX,3
    FM1=1.0E-28
    FM=.0
    ALPHA=.0
    IF (M-(M/2)*2)120,110,120
110 JT=-1
    GO TO 130
120 JT=1
130 M2=M-2
    DO 160 K=1,M2
    MK=M-K
    BMK=2.*FLOAT(MK)*FM1/X-FM
    FM=FM1
    FM1=BMK
    IF (MK-N-1)150,140,150
140 BJ=BMK
150 JT=-JT
    S=1+JT
160 ALPHA=ALPHA+BMK*S
```

```
      BMK=2.*FM1/X-FM
      IF (N)180,170,180
  170 BJ=BMK
  180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
      BSSL=BJ
      IF (ABS(BJ-BPREV)-ABS(D*BJ))200,200,190
  190 BPREV=BJ
      IER=3
  200 RETURN
      END
@FOR,IS RBAR2
      SUBROUTINE RBAR2(Z1,Z2,ETA)
C     INTEGRATES OVER ORIENATION OF THE CYLINDERS
      DIMENSION Y(16),A(16)
      CALL INT16(Y,A)
      ETA=.0
      DO 1 I=1,16
      F1=(Y(I))
      U1=Z1*SQRT(1.-F1*F1)
      U2=Z2*F1
      ET=(F(U1)*E(U2))**2
      ETA=ET*A(I)+ETA
    1 CONTINUE
      RETURN
      FUNCTION G(U)
      EPS=.000001
      IF (ABS(U)-EPS)302,302,301
  302 G=1.
      GO TO 303
  301 G=(SIN(U)-U*COS(U))/(U**3)
      G=G*3.
  303 CONTINUE
      RETURN
      FUNCTION E(U)
      EPS=.000001
      IF (ABS(U)-EPS)312,312,311
  312 E=1.
      GO TO 313
  311 E=SIN(U)/U
  313 RETURN
      FUNCTION F(U)
      EPS=.000001
      IF (ABS(U)-EPS)322,322,321
  322 F=1.
      GO TO 323
  321 F=BSSL(U,1)
      F=F*2./U
  323 CONTINUE
      RETURN
      FUNCTION BSSL(X,N)
      D=.000001
      BJ=0.
      IF(N)10,20,20
   10 IER=1
```

```
      RETURN
   20 IF(X)30,30,31
   30 IER=2
      RETURN
   31 IF(X-15.)32,32,34
   32 NTEST=20.+10.*X-X**2/3.
      GO TO 36
   34 NTEST=90.+X/2.
   36 IF(N-NTEST)40,38,38
   38 IER=4
      RETURN
   40 IER=0
      N1=N+1
      BPREV=.0
      IF(X-5.)50,60,60
   50 MA=X+6.
      GO TO 70
   60 MA=1.4*X+60./X
   70 MB=N+IFIX(X)/4+2
      MZERO=MAX0(MA,MB)
      MMAX=NTEST
  100 DO 190 M=MZERO,MMAX,3
      FM1=1.0E-28
      FM=.0
      ALPHA=.0
      IF (M-(M/2)*2)120,110,120
  110 JT=-1
      GO TO 130
  120 JT=1
  130 M2=M-2
      DO 160 K=1,M2
      MK=M-K
      BMK=2.*FLOAT(MK)*FM1/X-FM
      FM=FM1
      FM1=BMK
      IF (MK-N-1)150,140,150
  140 BJ=BMK
  150 JT=-JT
      S=1+JT
  160 ALPHA=ALPHA+BMK*S
      BMK=2.*FM1/X-FM
      IF (N)180,170,180
  170 BJ=BMK
  180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
      BSSL=BJ
      IF (ABS(BJ-BPREV)-ABS(D*BJ))200,200,190
  190 BPREV=BJ
      IER=3
  200 RETURN
      END
@FOR,IS INT16
      SUBROUTINE INT16(Y,A)
      DIMENSION Y(16),A(16)
C     16 POINT GAUSSIAN SUBROUTINE
```

```
C     Y AND A MUST BE DIMENSION IN CALLING PROGAM
      Y(1)= .95012510E-01
      Y(2)= 0.28160355
      Y(3)= 0.45801678
      Y(4)= 0.61787624
      Y(5)= 0.75540441
      Y(6)= 0.86563120
      Y(7)= 0.94457502
      Y(8)= 0.98940093
      A(1)= 0.18945061
      A(2)= 0.18260341
      A(3)= 0.16915652
      A(4)= 0.14959599
      A(5)= 0.12462897
      A(6)= .95158512E-01
      A(7)= .62253524E-01
      A(8)= .27152459E-01
      DO 8 I=9,16
      II=17-I
      Y(I)=-Y(II)
    8 A(I)=A(II)
      RETURN
      END
@XQT
```

RAYLEIGH-GANS SCATTERING

REFRACTIVE INDEX= 1.300

PARTICLE-RADIUS= .100 LENGTH=100.000

WAVELENGTH= .500

ALPHA= 11.417

INDEX1= 2 1=SPHERE 2=CYLINDER
INDEX2= 1 1=BETA, 1.57 2=INTEGRATION OVER BETA

SCAT ANGLE	INTENSITY FUNCTIONS			DEGREE OF POLARIZATION
THETA(DEG)	I1	I2	I1+I2/2	P
0	88601.623047	88601.623047	88601.623047	.000000
5	88335.746094	87664.736328	88000.241211	.003813
10	87544.115234	84904.336914	86224.225586	.015308
15	86244.449219	80467.165039	83355.806641	.034654
20	84465.475586	74584.890625	79525.182617	.062122
25	82245.773437	67556.167969	74900.970703	.098060
30	79632.217773	59724.162598	69678.189453	.142857
35	76678.210937	51451.852051	64065.031250	.196881
40	73441.691406	43097.353516	58269.522461	.260379
45	69983.120117	34991.560547	52487.340332	.333333
50	66363.496094	27419.798584	46891.646973	.415252
55	62642.520996	20608.758789	41625.639648	.504902
60	58876.954590	14719.239868	36798.097168	.600000
65	55119.285645	9844.647095	32481.966309	.696920
70	51416.681641	6014.610291	28715.645752	.790546
75	47810.275391	3202.682007	25506.478516	.874437
80	44334.748047	1336.856812	22835.802246	.941458
85	41018.243164	311.579933	20664.911377	.984922
90	37882.488770	.000000	18941.244385	1.000000
95	34943.179687	265.432404	17604.305908	.984922
100	32210.480225	971.264244	16590.872070	.941458
105	29689.675781	1988.830307	15839.252930	.874437
110	27381.871826	3203.069519	15292.470581	.790546
115	25284.722656	4516.006653	14900.364624	.696920
120	23393.163086	5848.288818	14620.725952	.600000
125	21700.095459	7139.110352	14419.602905	.504902
130	20197.029053	8344.923218	14270.976074	.415252
135	18874.645996	9437.319946	14155.982910	.333333
140	17723.308105	10400.461060	14061.884521	.260379
145	16733.472900	11228.325806	13980.899292	.196881
150	15896.047119	11922.032349	13909.039673	.142857
155	15202.678955	12487.383545	13845.031250	.098060
160	14645.975708	12932.719727	13789.347656	.062122
165	14219.686890	13267.146606	13743.416748	.034654
170	13918.827026	13499.121460	13708.974243	.015308
175	13739.775757	13635.405762	13687.590698	.003813
180	13680.336426	13680.336426	13680.336426	.000000

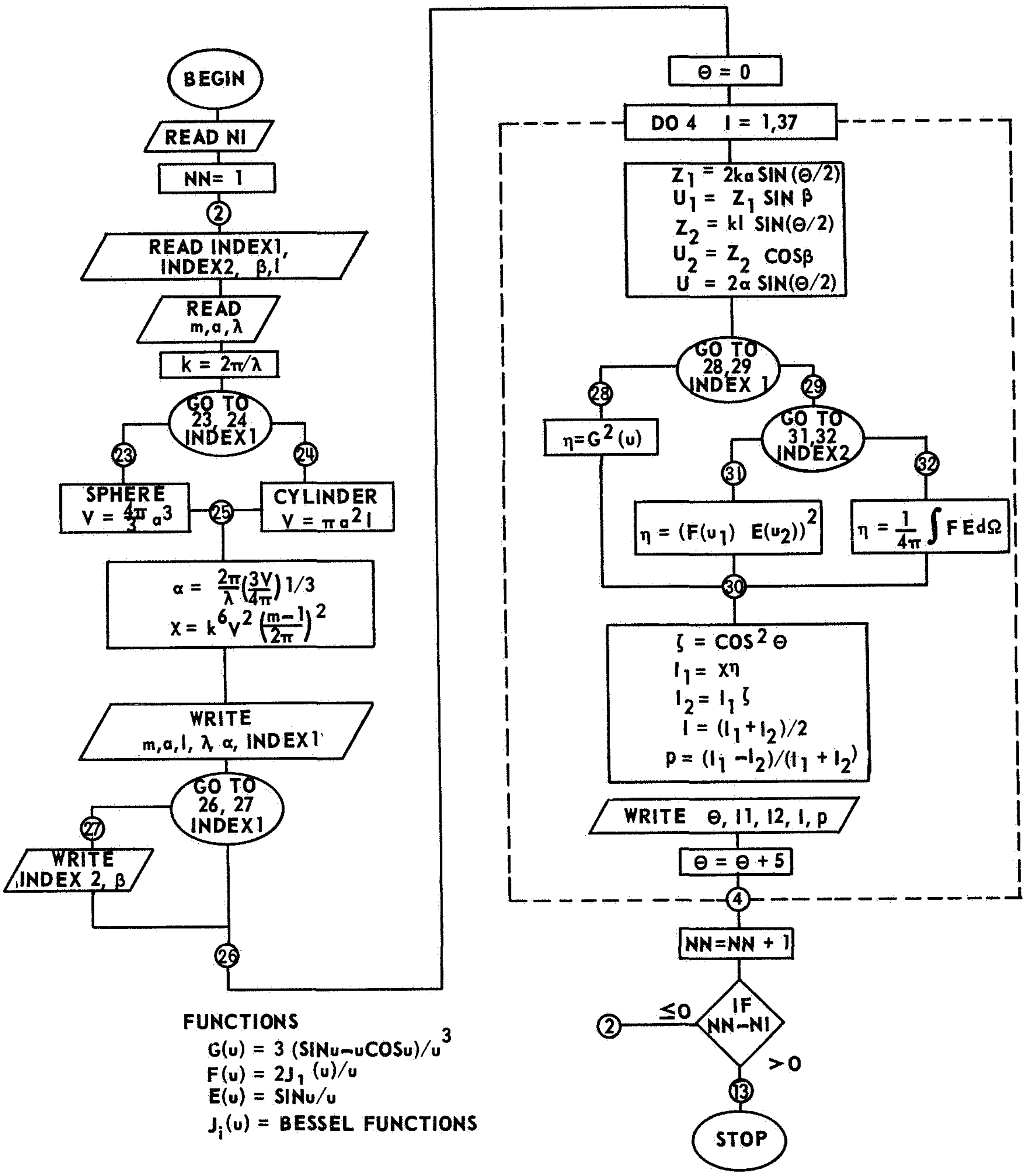

Figure A-1. Flow chart for Rayleigh-Gans scattering program.

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ELECTROMAGNETIC SCATTERING BY CYLINDERS – AN INTRODUCTION

By G. A. Gary and P. D. Craven

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This document has also been reviewed and approved for technical accuracy.

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