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REALIZATION THEORY OF RATIONAL TRANSFER MATRICES

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1. INTRODUCTION

In this memorandum we discuss the algebraic connection between a rational transfer matrix $H(\lambda)$ and its realization by a time-invariant dynamic system, or triple of constant matrices (C,A,B). It is shown that both $H(\lambda)$ and its matrix realizations can be represented by a suitable morphism of modules over the ring of polynomials in λ. Morphisms of this type are the fundamental algebraic objects in the theory of finite, constant linear systems.

The discussion to follow was stimulated by results of Kalman, Gilbert and Youla (documented in Section 8) and makes no important claim to originality. However, our development differs in some respects from that of the authors cited and is presented as a basis for research on certain problems of multivariable system synthesis (e.g., decoupling) which have not so far been treated from a module-theoretic point of view.

As algebraic prerequisite we draw only on the elementary theory of modules as given, for example, by MacLane and Birkhoff [11].

2. NOTATION

R^n is real n-dimensional vector space; $R=R^1$; $R[\lambda]$ is the ring of polynomials over R in a single transcendental element λ; $\underline{R}^n[\lambda]$ is the free $R[\lambda]$-module generated by R^n. All modules which arise have $R[\lambda]$ as associated ring. Dual modules are denoted by a prime, e.g., $(\underline{R}^n[\lambda])'$. In general, script light-face will denote vector spaces over R; script bold-face, modules over $R[\lambda]$. Morphisms of R-vector spaces will be called maps and denoted by A,B,... Morphisms of modules are printed in bold-face $\underline{S},\underline{T},\ldots$ A map $B:R^m \to R^n$ has a natural extension to a morphism of free modules $\underline{B}:\underline{R}^m[\lambda]\to\underline{R}^n[\lambda]$ as follows: If $\{u_i, i\epsilon\underline{m}\}$ is a basis for R^m then the u_i also serve as a basis (set of independent free generators) for $\underline{R}^m[\lambda]$, and we define $\underline{B}u_i=Bu_i$, $i\epsilon\underline{m}$, regarding R^n as a subset of $\underline{R}^n[\lambda]$. Finally, if $\alpha,\theta\epsilon R[\lambda]$, $[\theta]_\alpha$ is the residue class of $\theta(\text{mod }\alpha)$ in the quotient ring $R[\lambda]/\alpha(\lambda)R[\lambda]$.

3. TRANSFER MATRICES AND MORPHISMS

A transfer matrix (TX) is any (finite) rational matrix of the form

$$H(\lambda) = \alpha(\lambda)^{-1}\,\hat{H}(\lambda) \tag{3.1}$$

where $\alpha(\lambda)\epsilon R[\lambda]$ and $\hat{H}(\lambda)$ is a polynomial matrix with $\deg \hat{H} < \deg \alpha$. $H(\lambda)$ is reduced if the GCD of $\alpha(\lambda)$ and the elements of $\hat{H}(\lambda)$ is 1. Every $p\times m$ TX determines a morphism of modules as follows. Let $\{u_j, j\epsilon\underline{m}\}$ be a basis for R^m and $\{y_i, i\epsilon\underline{p}\}$ a basis for R^p. The elements

$$y_i(\lambda) = y_i + \alpha(\lambda)\underline{R}^p[\lambda],\ i\epsilon\underline{p}$$

form an independent set of generators for the torsion module $\underline{R}^p[\lambda]/\alpha(\lambda)\underline{R}^p[\lambda]$. Define

$$\underline{H}:\underline{R}^m[\lambda] \to \underline{R}^p[\lambda]/\alpha(\lambda)\underline{R}^p[\lambda] \tag{3.2}$$

by the rule

$$\underline{H}u_j = \sum_{i=1}^{p}\hat{H}_{ij}(\lambda)y_i(\lambda),\ j\epsilon\underline{m} \tag{3.3}$$

where $\hat{H}_{ij}(\lambda)$ is the (i,j)th element of $\alpha(\lambda)H(\lambda)$. A morphism of $R[\lambda]$-modules of type (3.2) will be called a transfer morphism (TM).

Every TM determines a TX as follows. With generators u_j, $y_i(\lambda)$ chosen as before, the evaluations (3.3) of the $\hat{\underline{H}}u_j$ determine polynomials $\hat{H}_{ij}(\lambda)$ uniquely, mod $\alpha(\lambda)$. In particular let $\hat{H}_{ij}(\lambda)$ be the unique member of its residue class with deg $\hat{H}_{ij}$ < deg α, and let $\hat{H}(\lambda)$ be the matrix of $\hat{H}_{ij}(\lambda)$. Then $H(\lambda) = \alpha(\lambda)^{-1} \hat{H}(\lambda)$ is the TX determined by $\underline{H}$.

Let $\underline{H}$ be a TM as in (3.2). Clearly

$$\text{Ker } \underline{H} \supset \alpha(\lambda)\underline{R}^m[\lambda]$$

Let $\underline{I}$ be the subset of polynomials $\beta \varepsilon R[\lambda]$ such that

$$\text{Ker } \underline{H} \supset \beta(\lambda)\underline{R}^m[\lambda]$$

Clearly $\underline{I}$ is an ideal of $R[\lambda]$, so that $\underline{I} = \alpha_o(\lambda)R[\lambda]$ for a polynomial $\alpha_o \varepsilon R[\lambda]$. In general $\alpha_o | \alpha$. Taking α and α_o to be monic, we say that the TM $\underline{H}$ is <u>reduced</u> if $\alpha_o = \alpha$. It is easily seen that $\underline{H}$ is a reduced TM if and only if the TX determined by $\underline{H}$ is reduced. Obviously, each TX is equal (as a rational matrix) to a unique reduced TX with monic denominator polynomial. To avoid ambiguity we regard a p×m TX as a pair $(\alpha(\lambda), \hat{H}(\lambda))$ with $\alpha(\lambda)$ monic; then the corresponding TM is uniquely determined after choice of bases in R^p and R^m.

4. DYNAMIC SYSTEMS

For our purposes a <u>dynamic system</u> (DS) is a triple of maps (C,A,B)

$$C: R^n \to R^p, \; A: R^n \to R^n, \; B: R^m \to R^n \tag{4.1}$$

which can be thought of as coming from the equations $\dot{x} = Ax + Bu$, $y = Cx$. Every DS determines a TX by the rule

$$H(\lambda) = C(\lambda - A)^{-1}B \tag{4.2}$$

where we have chosen bases for R^m, R^n, R^p and used the same symbols for the corresponding matrices.

Let $T:R^n \to R^n$ be an automorphism of R^n. Relative to fixed bases in R^m, R^n, R^p the DS $(CT, T^{-1}AT, T^{-1}B)$ determines the same TX as (C,A,B). DS which are related in this way will be called equivalent ($\backsim$).

With C,A,B as in (4.1), let $\underline{C},\underline{A},\underline{B}$ be their natural extensions to morphisms of free modules

$$\underline{C}:\underline{R}^n[\lambda] \to \underline{R}^p[\lambda],\ \underline{A}:\underline{R}^n[\lambda] \to \underline{R}^n[\lambda],\ \underline{B}:\underline{R}^m[\lambda] \to \underline{R}^n[\lambda]$$

If $\alpha(\lambda)\varepsilon R[\lambda]$ is divisible by the minimal polynomial of A, the morphism

$$\alpha(\lambda)(\lambda-\underline{A})^{-1}:R^n[\lambda] \to R^n[\lambda]$$

is defined in the natural way. Let $\underline{P}_\alpha$ be the canonical projection

$$\underline{P}_\alpha:\underline{R}^p[\lambda] \to \underline{R}^p[\lambda]/\alpha(\lambda)\underline{R}^p[\lambda]$$

We say that (C,A,B) realizes the TM $\underline{H}$ if $\alpha(\lambda)$ is divisible by the minimal polynomial of A, and

$$\underline{H} = \underline{P}_\alpha\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{B} \qquad (4.3)$$

It is clear that (C,A,B) realizes $\underline{H}$ if and only if the TX of $\underline{H}$ satisfies (4.2); and that equivalent DS realize the same TM. There is defined in this way a function from the quotient class $\{DS\}/\backsim$ to the class of TM.

5. CONSTRUCTION OF A CANONICAL REALIZATION

Write

$$\underline{U} = \underline{R}^m[\lambda],\ \underline{Y} = \underline{R}^p[\lambda]/\alpha(\lambda)\underline{R}^p[\lambda]$$

and let $\underline{H}:\underline{U}\to\underline{Y}$ be a TM. Let

$$\underline{M} = \underline{U}/\text{Ker } \underline{H} \tag{5.1}$$

Lemma 4.1

There is a unique integer n, and a map $A:R^n \to R^n$, such that

$$\underline{R}^n[\lambda]/(\lambda-\underline{A})\underline{R}^n[\lambda] \approx \underline{M} \tag{5.2}$$

Proof

Let $\bar{u}(\lambda) = u(\lambda)+\text{Ker } \underline{H} \in \underline{M}$. Since $\alpha(\lambda)\underline{R}^m[\lambda] \subset \text{Ker } \underline{H}$,

$$\alpha(\lambda)\bar{u}(\lambda) = \alpha(\lambda)u(\lambda) + \text{Ker } \underline{H} = \text{Ker } \underline{H} = 0;$$

so that $\underline{M}$ is a torsion module of finite type over the principal ideal domain $R[\lambda]$. By known structure theorems ([11],Ch. 10, Ths. 5,6) there exist unique monic polynomials $\alpha_1,\ldots,\alpha_k$ such that $\alpha_1|\alpha$, $\alpha_2|\alpha_1,\ldots,$ $\alpha_k|\alpha_{k-1}$, $\deg \alpha_k \geq 1$, and

$$\underline{M} \approx \bigoplus_{i=1}^{k} R[\lambda]/\alpha_i(\lambda)R[\lambda] \tag{5.3}$$

Let

$$n = \sum_{i=1}^{k} \deg \alpha_i$$

By the rational canonical structure theorem ([11], Ch. 10, Th. 8) there is a map $A:R^n \to R^n$, unique up to similarity equivalence, such that

$$\underline{R}^n[\lambda]/(\lambda-\underline{A})\underline{R}^n[\lambda] \approx \bigoplus_{i=1}^{k} R[\lambda]/\alpha_i(\lambda)R[\lambda]; \tag{5.4}$$

the $\alpha_i(\lambda)$ are the invariant polynomials of A. By (5.3) and (5.4) the result follows. ∎

We shall say that a realization (C,A,B) of $\underline{H}$ is canonical if (5.2) holds.

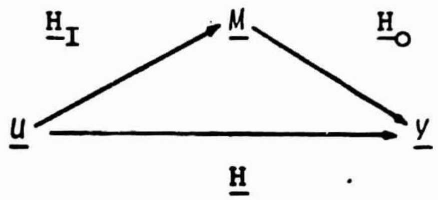
It is convenient in the following to define $\underline{M}$ concretely as an isomorphic copy of the module on the right side of (5.4), namely

$$\underline{M} = \bigoplus_{i=1}^{k} \beta_i(\lambda)R[\lambda]/\alpha(\lambda)R[\lambda] \qquad (5.5)$$

where $\beta = \alpha/\alpha_i$, $i\varepsilon\underline{k}$. $\underline{H}$ can now be exhibited as a product

$$\underline{H} = \underline{H}_O\underline{H}_I, \qquad (5.6)$$

where $\underline{H}_I$ is the canonical projection of $\underline{U}$ on $\underline{M}$ and $\underline{H}_O$ is the induced monomorphism from $\underline{M}$ to $\underline{Y}$, as in the diagrams below.



(D1)

$$\underline{U} \xrightarrow{\underline{H}_I} \underline{M} \longrightarrow 0 \qquad (D2)$$

$$0 \longrightarrow \underline{M} \xrightarrow{\underline{H}_O} \underline{Y} \qquad (D3)$$

The subscripts I,O refer to "input" and "output". D2 and D3 are exact sequences, i.e., $\underline{H}_I$ is an epimorphism and $\underline{H}_O$ is a monomorphism. It will be seen in Section 6 that "epic" and "monic" have the system – theoretic meaning of "controllable" and "observable".

Write

$$\underline{X} = \underline{R}^n[\lambda]/(\lambda-\underline{A})\underline{R}^n[\lambda] \qquad (5.7)$$

and $\underline{A} = (\lambda-\underline{A})\underline{R}^n[\lambda]$. It is easy to check that elements $\bar{x}\varepsilon\underline{X}$ have the representation

$$\bar{x} = x + \underline{A}, \tag{5.8}$$

where $x\varepsilon R^n$ is uniquely determined by $\bar{x}$. Furthermore, if $x(\lambda) = x + (\lambda-\underline{A})x^*(\lambda)$ for $x\varepsilon R^n$, $x^*(\lambda)\varepsilon\underline{R}^n(\lambda)$, then

$$\lambda x(\lambda) = (\underline{A}+(\lambda-\underline{A}))x(\lambda) \ \varepsilon \ Ax+(\lambda-\underline{A})\underline{R}^n[\lambda],$$

so that

$$\lambda\bar{x} = Ax + \underline{A} \tag{5.9}$$

Next, since $\underline{X} \approx \underline{M}$, (5.3) implies

$$\underline{X} \approx \bigoplus_{i=1}^{k} \underline{X}_i \tag{5.10}$$

where $\underline{X}_i$ is a cyclic module with generator

$$\bar{g}_i = g_i + \underline{A} \tag{5.11}$$

and $\bar{g}_i$ has minimal annihilating polynomial $\alpha_i(\lambda)$.

We can now exhibit explicitly an isomorphism $\underline{T}:\underline{M}\to\underline{X}$. Write

$$\underline{M}_i = \beta_i(\lambda)R[\lambda]/\alpha(\lambda)R[\lambda], \ i\varepsilon\underline{k}$$

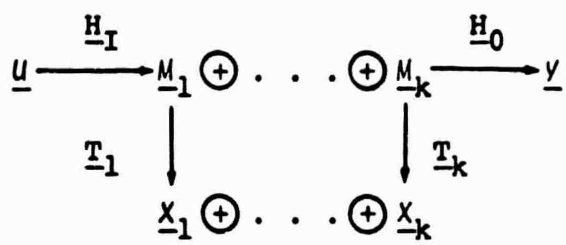
and let e_i be the generator of $\underline{M}_i$, i.e., $e_i = [\beta_i]_\alpha$. Define

$$\underline{T}_i : \underline{M}_i \to \underline{X}_i; \ e_i \longmapsto \bar{g}_i \qquad i\varepsilon\underline{k} \tag{5.12}$$

and finally

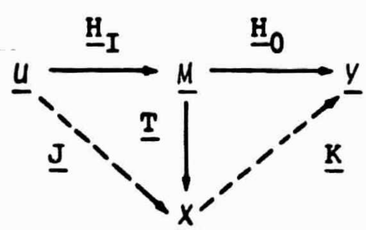
$$\underline{T} = \bigoplus_{i=1}^{k} \underline{T}_i$$

The situation is now summarized by the diagram:



(D4)

The remainder of the realization procedure amounts simply to filling in the diagonals so that D4 commutes:



(D5)

Define $\underline{J}:\underline{U} \to \underline{X}$ as the morphism

$$\underline{J} = \underline{T}\,\underline{H}_I \tag{5.13}$$

Let $\{u_i, i\varepsilon\underline{m}\}$ be a basis for R^m, hence of $\underline{U}$. Then

$$\begin{aligned} \underline{J}u_i &= \underline{T}(\underline{H}_I u_i) \\ &= b_i + \underline{A} \end{aligned}$$

for uniquely determined vectors $b_i \varepsilon R^n$. Thus $\underline{J}$ determines a unique map $B:R^m \to R^n$ by the rule

$$Bu_i = b_i,\ i\varepsilon\underline{m} \tag{5.14}$$

The morphism $\underline{K}:\underline{X} \to \underline{Y}$ is defined by $\underline{KT} = \underline{H}_o$, or

$$\underline{K} = \underline{H}_o\underline{T}^{-1}$$

$$= \underline{H}_o \; (\underline{T}_1^{-1} \oplus \ldots \oplus T_k^{-1}) \qquad (5.15)$$

Observe that T_i^{-1} maps each element $x(\lambda)\varepsilon\underline{X}_i$ into its corresponding residue class in $\underline{M}_i$; i.e., if $\bar{x} = \xi(A)g_i+\underline{A} \; \varepsilon \; \underline{X}_i$, then

$$\underline{T}_i^{-1}\bar{x} = [\xi(\lambda)]_\alpha \qquad (5.16)$$

To determine an explicit expression for the $\underline{T}_i$ we make a digression on cyclic maps. Let $G:R^n \to R^n$ be cyclic, with minimal polynomial $\gamma(\lambda)$ and generator $g\varepsilon R^n$. Introduce the element $h'\varepsilon(R^n)'$ conjugate to g, defined by

$$h'g = \ldots = h'G^{n-2}g = 0; \; h'G^{n-1}g = 1 \qquad (5.17)$$

Let

$$\gamma(\lambda) = \lambda^n - (c_1+c_2\lambda+\ldots+c_n\lambda^{n-1})$$

for real numbers c_i; and define polynomials

$$\gamma^{(i)}(\lambda) = \lambda^{n-i} - (c_{i+1}+c_{i+2}\lambda+\ldots+c_n\lambda^{n-i-1}), \; i\varepsilon\underline{n} \qquad (5.18)$$

so that $\gamma^{(i)}(\lambda) = \lambda\gamma^{(i+1)}(\lambda) - c_{i+1}$, $i\varepsilon\underline{n-1}$, and $\gamma^{(n)}(\lambda) = 1$. For the natural extension $\underline{G}:\underline{R}^n[\lambda] \to \underline{R}^n[\lambda]$ it is easy to verify

$$\gamma(\lambda) = (\lambda-\underline{G}) \sum_{i=1}^{n} \gamma^{(i)}(\lambda)\underline{G}^{i-1}$$

and we define

$$\gamma(\lambda)(\lambda-\underline{G})^{-1} = \sum_{i=1}^{n} \gamma^{(i)}(\lambda)\underline{G}^{i-1} \tag{5.19}$$

With $\underline{h}'$ the natural insertion of h' in $(\underline{R}^n[\lambda])'$, define the functional $\underline{F}:\underline{R}^n[\lambda] \to R[\lambda]$ by

$$\underline{F} = \underline{h}'\gamma(\lambda)(\lambda-\underline{G})^{-1} \tag{5.20}$$

Lemma 5.2

For every $\xi \varepsilon R[\lambda]$

$$\underline{F}\xi(G)g = \xi(\lambda) \bmod \gamma(\lambda) \tag{5.21}$$

If $\hat{\underline{F}} \varepsilon (\underline{R}^n[\lambda])'$ has the property (5.21) then $\hat{\underline{F}}-\underline{F} = \gamma(\lambda)\tilde{\underline{F}}$ for some $\tilde{\underline{F}} \ \varepsilon \ (\underline{R}^n[\lambda])'$.

Proof

To verify (5.21) note from (5.17), (5.19) and (5.20),

$$\underline{F}g = \sum_{i=1}^{n} \gamma^{(i)}(\lambda)h'G^{i-1}g = \gamma^{(n)}(\lambda) = 1;$$

$$\begin{aligned}\underline{F}A^{i+1}g &= \underline{F}[\lambda\underline{G}^i-(\lambda-\underline{G})\underline{G}^i]g \\ &= \lambda\underline{F}G^ig - \alpha(\lambda)h'G^ig \\ &= \lambda\underline{F}G^ig \bmod \gamma(\lambda);\end{aligned}$$

and (5.21) follows by induction on i. The uniqueness (mod γ) of the functional determined by (5.21) is implied by the fact that $\{G^{i-1}g, i\varepsilon\underline{n}\}$ is a basis for $\underline{R}^n[\lambda]$. ∎

Now let $X_i \subset R^n$ be the subspace

$$X_i = \{x : x \in R^n, x + \underline{A} \in \underline{X}_i\}$$

Then X_i is A-cyclic with minimal polynomial $\alpha_i(\lambda)$ and (by (5.11)) generator g_i. Let $\underline{X}_i[\lambda]$ be the free module generated by X_i. Then the morphism

$$\alpha_i(\lambda)(\lambda - \underline{A})^{-1} : \underline{X}_i[\lambda] \to \underline{R}^n[\lambda]$$

is well-defined. Next, write $n_i = \dim X_i$ (=deg α_i) and define $\underline{h}_i' \in (\underline{X}_i[\lambda])'$ by

$$\underline{h}_i' g_i = \ldots = \underline{h}_i' \underline{A}^{n_i - 2} g_i = 0, \qquad \underline{h}_i' \underline{A}^{n_i - 1} g_i = 1$$

Finally, let $\underline{F}_i : \underline{X}_i[\lambda] \to R[\lambda]$ be the functional given by

$$\underline{F}_i x(\lambda) = \underline{h}_i' \alpha_i(\lambda)(\lambda - \underline{A})^{-1} x(\lambda), \qquad x(\lambda) \in \underline{X}_i[\lambda] \tag{5.22}$$

Lemma 5.3

The morphism $\underline{T}_i^{-1} : \underline{X}_i \to \underline{M}_i$ is given by

$$\underline{T}_i^{-1}(\bar{x}) = [\beta_i(\lambda) \underline{F}_i x]_\alpha \tag{5.23}$$

for $\bar{x} = x + \underline{A} \in \underline{X}_i$.

Proof

Since $x \in X_i \subset \underline{X}_i[\lambda]$, (5.23) makes sense. Now apply Lemma 5.2, with X_i in place of R^n, the restriction of A to X_i in place of G, and $\gamma(\lambda) = \alpha_i(\lambda)$. ▌

It remains to compute $\underline{K}$ from (5.15) and (5.23). Let $\{y_i, i \in \underline{p}\}$ be a basis for R^p, so that

$$y_i(\lambda) = y_i + \alpha(\lambda) R^p[\lambda], \quad i \in \underline{p}$$

is an independent set of generators for the torsion module $\underline{Y}$. Applying $\underline{H}_o$ to the generators $e_j=[\beta_j]_\alpha$ of $\underline{M}$, there results

$$\underline{H}_o e_j = \sum_{i=1}^{p} \tilde{\eta}_{ij}(\lambda) y_i(\lambda),\ j\varepsilon\underline{k} \tag{5.24}$$

for unique (mod α) polynomials $\tilde{\eta}_{ij}$. It will be shown that $\beta_j | \tilde{\eta}_{ij}$. In fact with $\alpha_j=\alpha/\beta_j$, we have $\alpha_j e_j=0\ \varepsilon\ \underline{M}_i$, so that

$$\begin{aligned} 0 &= \underline{H}_o(\alpha_j e_j) \\ &= \alpha_j \underline{H}_o(e_j) \\ &= \alpha_j(\lambda) \sum_{i=1}^{p} \tilde{\eta}_{ij}(\lambda) y_i(\lambda) \end{aligned}$$

Since the $y_i(\lambda)$ are independent generators, $\alpha_j(\lambda)\tilde{\eta}_{ij}(\lambda) = 0$ (mod $\alpha(\lambda)$), i.e.,

$$\alpha_j(\lambda)\tilde{\eta}_{ij}(\lambda)\ \varepsilon\ \alpha(\lambda)R[\lambda]$$

or $\tilde{\eta}_{ij}(\lambda)\ \varepsilon\ \beta_j(\lambda)R[\lambda]$, and the assertion follows; so

$$\underline{H}_o e_j = \beta_j(\lambda) \sum_{i=1}^{p} \eta_{ij}(\lambda) y_i(\lambda),\ j\varepsilon\underline{k} \tag{5.25}$$

for suitable $\eta_{ij}\varepsilon R[\lambda]$. Also, by (5.21),

$$h_j' \eta_{ij}(\underline{A})\alpha_j(\lambda)(\lambda-\underline{A})^{-1} g_j = \eta_{ij}(\lambda),\ \text{mod}\ \alpha_j,\ i\varepsilon\underline{p},\ j\varepsilon\underline{k} \tag{5.26}$$

Now let $x+\underline{A}\varepsilon\underline{X}$, so that $x = \sum_{j=1}^{k} x_j$ with $x_j \varepsilon X_j$, $j\varepsilon\underline{k}$, uniquely determined by x. Let $\{y_i', i\varepsilon\underline{p}\}$ be the basis dual to $\{y_i, i\varepsilon\underline{p}\}$ in $(R^p)'$; define the map $C:R^n \to R^p$ by the rule

$$y_i'Cx = \sum_{j=1}^{k} h_j'\eta_{ij}(A)x_j, \quad i\varepsilon\underline{p},\ x\varepsilon R^n; \tag{5.27}$$

and let $y_i'\varepsilon(R^p[\lambda])'$, $\underline{C}:\underline{R}^n[\lambda] \to \underline{R}^p[\lambda]$ be the corresponding natural extensions. Then

$$\underline{K}(x+\underline{A}) = \underline{H}_o(\underline{T}^{-1}(x+\underline{A}))$$

$$= \underline{H}_o \sum_{i=1}^{k}[\underline{h}_j'\alpha_j(\lambda)(\lambda-\underline{A})^{-1}x_j]e_j$$

$$= \sum_{j=1}^{k}[\underline{h}_j'\alpha_j(\lambda)(\lambda-\underline{A})^{-1}x_j]\beta_j(\lambda)\sum_{i=1}^{p}\eta_{ij}(\lambda)y_i(\lambda)$$

(by (5.25))

$$= \sum_{i=1}^{p}\sum_{j=1}^{k}[\underline{h}_j'\eta_{ij}(\underline{A})\beta_j(\lambda)\alpha_j(\lambda)(\lambda-\underline{A})^{-1}x_j]y_i(\lambda)$$

(by (5.26))

$$= \sum_{i=1}^{p}[y_i' \sum_{j=1}^{k}\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x_j]y_i(\lambda)$$

(by (5.27))

$$= \sum_{i=1}^{p}[y_i'\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x]y_i(\lambda)$$

$$= \underline{P}_\alpha\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x \tag{5.28}$$

for all $x+\underline{A}\ \varepsilon\ \underline{X}$.

To complete the discussion let $u_i\varepsilon R^m$ be a free generator of $\underline{U}$. Then

$$\underline{H}u_i = \underline{K}\,\underline{J}u_i$$

$$= \underline{K}(Bu_i+\underline{A})$$

$$= \underline{P}_\alpha \underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}\, Bu_i \qquad \text{(by (5.28))}$$

$$= \underline{P}_\alpha \underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}\, \underline{B}u_i, \quad i\epsilon \underline{m};$$

so that

$$\underline{H} = \underline{P}_\alpha \underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{B};$$

i.e., (C,A,B) is a realization of $\underline{H}$.

We have shown that a canonical realization of a TM always exists. It will now be shown to have important special properties.

6. PROPERTIES OF CANONICAL REALIZATIONS

Let (C,A,B) be a DS. The pair (A,B) is controllable if

$$\sum_{i=1}^{n} \text{Range}\,(A^{i-1}B) = R^n; \tag{6.1}$$

and (C,A) is observable if

$$\bigcap_{i=1}^{n} \text{Ker}\,(CA^{i-1}) = 0 \tag{6.2}$$

These well known concepts, due to Kalman, are fundamental. The DS (C,A,B) will be called complete if (A,B) is controllable and (C,A) is observable. It is easily checked that completeness is a class property relative to equivalence (~).

From a practical viewpoint it is of interest to realize a TM in such a way that the state space R^n is of minimal dimension. A realization with this property is minimal.

The central result of realization theory is

Theorem 6.1

Let the DS (C,A,B) be a realization of the TM H. The following properties of (C,A,B) are equivalent: (i) (C,A,B) is canonical; (ii) (C,A,B) is complete; (iii) (C,A,B) is minimal. Furthermore, there is a bijection between the class of reduced TM and the quotient class of complete DS modulo equivalence (~).

In the following, the notation of Section 5 will be used freely.

Lemma 6.1

Let $\underline{S}: X \to X$ be a morphism and define the function $S: R^n \to R^n$ by the rule: $\underline{S}(x + \underline{A}) = \grave{x} + \underline{A}$ implies $Sx = \grave{x}$. Then S is a map and $SA = AS$. If $\underline{S}$ is an automorphism, so is S.

Proof

Since $x, \grave{x} \in R^n$ are uniquely determined by the elements $x + \underline{A}$, $\grave{x} + \underline{A} \in X$, the function S is well-defined and is clearly a map. Recalling (5.9),

$$\begin{aligned}\underline{S}(Ax + \underline{A}) &= \underline{S}(\lambda(x + \underline{A})) \\ &= \lambda\underline{S}(x + \underline{A}) \\ &= \lambda(\grave{x} + \underline{A}) \\ &= A\grave{x} + \underline{A}\end{aligned}$$

so that $S(Ax) = A\hat{x} = A(Sx)$. The second statement of the lemma is obvious. ▮

Lemma 6.2

If (C,A,B) is a canonical realization, it is complete.

Proof

Suppose first that (C,A,B) is constructed as in Section 5, i.e., B is given by (5.14) and C by (5.27). As for controllability, since $\underline{H}_I$ is epic there are polynomials $\rho_{ij}(\lambda)$ $(i\varepsilon\underline{k}, j\varepsilon\underline{m})$ such that

$$e_i = \underline{H}_I\left[\sum_{j=1}^{m} \rho_{ij}(\lambda)u_j\right] = \sum_{j=1}^{m} \rho_{ij}(\lambda)\underline{H}_I(u_j), \qquad i\varepsilon\underline{k}$$

Therefore

$$g_i + \underline{A} = \underline{T}e_i$$

$$= \sum_{j=1}^{m} \rho_{ij}(\underline{A})\underline{T}(\underline{H}_I u_j)$$

$$= \sum_{j=1}^{m} [\rho_{ij}(A)b_j + \underline{A}];$$

i.e.,

$$g_i = \sum_{j=1}^{m} \rho_{ij}(A)b_j \qquad i\varepsilon\underline{k}$$

Since any $x\varepsilon R^n$ can be written

$$x = \sum_{i=1}^{k} \xi_i(A)g_i$$

for suitable $\xi_i \varepsilon R[\lambda]$, controllability is proved.

For observability, since $\underline{H}_o$ is monic and $\underline{T}$ is an isomorphism, $\underline{H}_o\underline{T}^{-1}$ is monic, so that

$$\bar{x} = x + \underline{A} \;\varepsilon\; \underline{X} \text{ and } \underline{H}_o\underline{T}^{-1}\,\bar{x} = 0 \text{ imply } \bar{x} = 0 \tag{6.3}$$

By (5.28) and (6.3), $\underline{H}_o\underline{T}^{-1}\,\bar{x}=0$ if and only if the polynomials

$$\underline{y}_i'\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x = 0, \quad \text{mod } \alpha(\lambda),\ i\varepsilon\underline{p}$$

or

$$\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x = 0 \tag{6.4}$$

in $\underline{R}^p[\lambda]$. By use of (5.18) and (5.19) it is seen that (6.4) is true if and only if $CA^{i-1}x=0$, $i\varepsilon\underline{n}$; that is,

$$x\varepsilon R^n \text{ and } CA^{i-1}x=0,\ i\varepsilon\underline{n},\ \text{imply } x=0,$$

so that (C,A) is observable.

Now let $(\hat{C},A,\hat{B})$ be any canonical realization, i.e., (5.2) holds for A. Define morphisms

$$\underline{\hat{J}}:\underline{U}\to\underline{X};\ u\mapsto \hat{B}u + \underline{A},\ u\varepsilon R^m \tag{6.5}$$

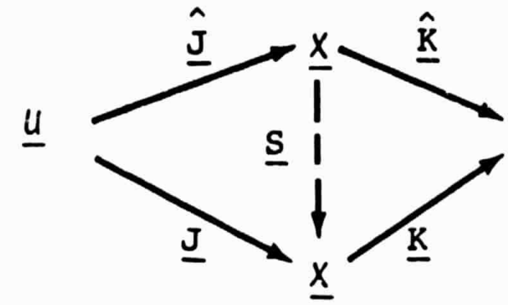
$$\underline{\hat{K}}:\underline{X}\to\underline{Y};\ x + \underline{A}\mapsto \underline{P}_\alpha\hat{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x, \quad x + \underline{A}\ \varepsilon\ \underline{X} \tag{6.6}$$

Then $\underline{H}=\underline{P}_\alpha\underline{\hat{C}}\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{\hat{B}}$, so that

$$\begin{aligned}\underline{H}u_i &= \underline{P}_\alpha\underline{\hat{C}}\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{\hat{B}}u_i\\ &= \underline{P}_\alpha\underline{\hat{C}}\alpha(\lambda)(\lambda-\underline{A})^{-1}\hat{B}u_i\\ &= \underline{\hat{K}}(\hat{B}u_i+\underline{A})\\ &= \underline{\hat{K}}\underline{\hat{J}}u_i \qquad\qquad i\varepsilon\underline{m}\end{aligned}$$

and therefore $\underline{H}=\underline{\hat{K}}\underline{\hat{J}}$.

Next determine B,C according to (5.14) and (5.27), so that (C,A,B) is complete, and $\underline{H}=\underline{K}\underline{J}$, where $\underline{J},\underline{K}$ are given by (5.13) and (5.15). It will be shown that there exists an automorphism $S:X\to X$ such that diagram D6 commutes. Let $\underline{\hat{K}}_1:\underline{X}\to\underline{X}/\text{Ker }\underline{\hat{K}}$ be the



(D6)

canonical projection, so that $\hat{\underline{K}}=\hat{\underline{K}}_2\hat{\underline{K}}_1$, where $\hat{\underline{K}}_2:\underline{X}/\text{Ker }\hat{\underline{K}} \to \underline{X}$ is monic. Since $\underline{K}$ is also monic (by the construction of Section 5), there exist submodules $\hat{\underline{X}}_o \approx \underline{X}/\text{Ker }\hat{\underline{K}}$ and $\underline{X}_o \approx \underline{X}$ such that $\hat{\underline{X}}_o \approx \underline{X}_o \approx \underline{H}\underline{U}$. Since $\underline{J}:\underline{U}\to\underline{X}$ is epic and $\underline{H}\underline{U} = \underline{K}(\underline{J}\underline{U})$ it follows that $\underline{H}\underline{U} = \underline{K}\underline{X}$, so that $\underline{X}_o \approx \underline{X}$. As in the proof of Lemma 6.1, an isomorphism $\underline{Q}:\underline{X}_o\to\underline{X}$ induces a map $Q:X_o\to X$ of the corresponding vector spaces; it is immediately verified that Q is an isomorphism, so that $X_o = X$, hence $\underline{X}_o = \underline{X}$, and $\underline{X}\approx\underline{X}/\text{Ker }\hat{\underline{K}}$. If $\hat{C}$ is the vector subspace

$$\hat{C} = \{x : x \in R^n,\ x+\underline{A} \in \text{Ker } \hat{\underline{K}}\}$$

then, as before, an isomorphism $\underline{Q}:\underline{X}\to\underline{X}/\text{Ker }\hat{\underline{K}}$ induces an isomorphism of vector spaces $Q:R^n\to R^n/\hat{C}$, which implies $\hat{C}=0$, and so Ker $\hat{\underline{K}} = 0$. Thus $\hat{\underline{K}}$ is monic, so there is an automorphism $\underline{S}$ of $\underline{X}$ such that $\underline{K}\underline{S} = \hat{\underline{K}}$. Then $\hat{\underline{K}}\hat{\underline{J}} = \underline{H} = \hat{\underline{K}}(\underline{S}^{-1}\hat{\underline{J}})$; $\hat{\underline{K}}$ monic implies $\hat{\underline{J}}=\underline{S}^{-1}\underline{J}$; and D6 commutes, as asserted.

By Lemma 5.1, $\underline{S}$ induces an automorphism $S:R^n\to R^n$ such that $SA=AS$. Clearly $\hat{B}=S^{-1}B$; and $\underline{K}\underline{S} = \hat{\underline{K}}$ implies

$$\underline{P}_\alpha \underline{C}\underline{S}\alpha(\lambda)(\lambda-\underline{A})^{-1} = \underline{P}_\alpha \underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{S}$$

$$= \underline{P}_\alpha \hat{\underline{C}}\alpha(\lambda)(\lambda-\underline{A})^{-1}.$$

By (5.28),

$$\underline{y}_i'\underline{C}\underline{S}\alpha(\lambda)(\lambda-\underline{A})^{-1}x = \underline{y}_i'\hat{\underline{C}}\alpha(\lambda)(\lambda-\underline{A})^{-1}x \in R[\lambda]$$

for $i\in\underline{p}$ and $x\in R^n$. Hence equality holds for $x(\lambda)\in\underline{R}^n[\lambda]$; setting

$x(\lambda) = (\lambda - \underline{A})x_j (j \epsilon \underline{n})$ for a basis $\{x_j, j \epsilon \underline{n}\}$ of R^n, there follows $y_i' CS = y_i' \hat{C}$, $i \epsilon \underline{p}$, and so $CS = \hat{C}$. Thus $(\hat{C},\hat{A},\hat{B}) = (CS, S^{-1}AS, S^{-1}B) \sim (C,A,B)$ and therefore (C,A,B) is complete. ∎

Lemma 6.3

Any two canonical realizations of a TM are equivalent.

Proof

If (C_1,A_1,B_1) and (C_2,A_2,B_2) are canonical realizations of $\underline{H}:\underline{U}\to\underline{Y}$, there are integers n_1,n_2 such that

$$\underline{R}^{n_i}[\lambda]/(\lambda-\underline{A}_i)\underline{R}^{n_i}[\lambda] \approx \underline{U}/\text{Ker } \underline{H}$$

By the rational canonical structure theorem $n_1=n_2=n$, say, and there is an R-vector space automorphism S of R^n such that $SA_2=A_1S$. Thus

$$(C_2,A_2,B_2) = \left(C_2, S^{-1}A_1S, B_2\right) \sim \left(C_2S, A_1, S^{-1}B_2\right)$$

By the method of proof of Lemma 6.2, there are maps B,C such that

$$\left(C_2S, A_1, S^{-1}B_2\right) \sim (C,A,B)$$

and

$$(C_1,A_1,B_1) \sim (C,A_1,B);$$

and the assertion follows. ∎

Lemma 6.4

If (C,A,B) is a complete realization of a TM $\underline{H}$, then it is canonical.

Proof

In the notation of Section 5, it must be shown that

$$\underline{U}/\text{Ker } \underline{H} \approx \underline{X}$$

For this it is enough to exhibit an epimorphism $\underline{S}:\underline{U}\to\underline{X}$ such that Ker $\underline{S}$ = Ker $\underline{H}$. Define $\underline{S}$ by the rule $\underline{S}u_i = \underline{B}u_i + \underline{A}$, $i\varepsilon\underline{m}$. Since (A,B) is controllable, $\underline{S}$ is epic. Let $u(\lambda)\varepsilon$Ker $\underline{S}$, so that $\underline{B}u(\lambda) = (\lambda-\underline{A})u^*(\lambda)$ for some $u^*(\lambda)\varepsilon\underline{R}^n[\lambda]$. Since (C,A,B) is a realization,

$$\underline{H}u(\lambda) = \underline{P}_\alpha\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}((\lambda-\underline{A})u^*(\lambda))$$

$$= \underline{P}_\alpha\underline{C}\alpha(\lambda)u^*(\lambda)$$

$$= 0$$

and therefore Ker $\underline{S} \subset$ Ker $\underline{H}$.

For the reverse inclusion, suppose

$$\underline{P}_\alpha\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x(\lambda) = 0$$

for some $x(\lambda)\varepsilon\underline{R}^n[\lambda]$. Since $x(\lambda)=x+(\lambda-\underline{A})x^*(\lambda)$ for some $x\varepsilon R^n$, there follows

$$\underline{P}_\alpha\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x = 0$$

and therefore

$$\underline{C}\alpha(\lambda)(\lambda-\underline{A})^{-1}x = 0$$

From this it follows by observability that x=0, i.e., $x(\lambda)\varepsilon(\lambda-\underline{A})\underline{R}^n[\lambda]$. Thus if $u(\lambda)\varepsilon$ Ker $\underline{H}$ then $\underline{B}u(\lambda) = 0$, so that $\underline{S}u(\lambda) = \underline{B}u(\lambda) + \underline{A} = 0$ in $\underline{X}$. Hence Ker $\underline{H} \subset$ Ker $\underline{S}$, which completes the proof. ▮

Lemma 6.5

A realization is minimal only if it is complete.

Proof

Let (C,A,B) be a realization of $\underline{H}$, with A a map of R^n. Write

$$E = \sum_{i=1}^{n} \text{Range } (A^{i-1}B);$$

define $A_1:E\to E$, $A_1 = A|E$; and $B_1:R^n\to E$, $u\mapsto Bu$. Choose $F\subset R^n$ arbitrarily such that $E\oplus F = R^n$; let $P:R^n\to E$ be the projection on E along F; and define $C_1:E\to R^p$ by the condition $C_1P = CP$. We claim that (C_1,A_1,B_1) realizes $\underline{H}$. If $u(\lambda)\in\underline{R}^m[\lambda]$ then

$$\alpha(\lambda)(\lambda-\underline{A}_1)^{-1}\ \underline{B}_1u(\lambda) = \alpha(\lambda)(\lambda-\underline{A})^{-1}\ \underline{B}u(\lambda)\ \in\ \underline{E}[\lambda]$$

where $\underline{E}[\lambda]$ is the free module generated by E. Since $\underline{P}\underline{E}[\lambda]\subset\underline{E}[\lambda]$,

$$\underline{C}\alpha(\lambda)(\lambda-\underline{A})\ \underline{B}u(\lambda) = \underline{C}_1\alpha(\lambda)(\lambda-\underline{A}_1)^{-1}\ \underline{B}_1u(\lambda)$$

and the assertion follows.

From this we conclude that (C,A,B) is a minimal realization only if $E=R^n$, i.e., (A,B) is controllable. Similarly, by introducing as new state space the factor space

$$R^n \Big/ \bigcap_{i=1}^{n} \text{Ker } (CA^{i-1})$$

it can be shown that (C,A,B) is minimal only if (C,A) is observable. ∎

Lemma 6.6

A canonical realization is minimal.

Proof

Any canonical realization has state space R^n determined uniquely by the module $U/\text{Ker } \underline{H}$. Now a minimal realization always exists (because a realization does); by Lemma 6.5 it is complete; hence by Lemma 6.4, canonical; and the result follows. ∎

Proof of Theorem 6.1

The first assertion follows immediately by Lemmas 6.2-6.6. For the second, if two complete DS realize the same TM then (Lemma 6.4) they are canonical, hence (Lemma 6.3) equivalent. On the other hand every TM has a canonical, hence complete realization. Finally, let (C,A,B) be complete and realize the reduced TM $\underline{H}$, i.e.,

$$\underline{H} = \underline{P}_\alpha C\alpha(\lambda)(\lambda-\underline{A})^{-1}\underline{B}$$

It will be shown that $\alpha(\lambda)$ is uniquely determined as the minimal polynomial of A. Since (C,A,B) is canonical, $\underline{H} = \underline{KJ}$ with $\underline{K}$, $\underline{J}$ defined as in Section 5. Let α_1 be the minimal polynomial of A. If

$$u(\lambda) = \alpha_1(\lambda)v(\lambda) \in \alpha_1(\lambda)\underline{R}^m[\lambda]$$

with $v(\lambda) = \sum_{i=1}^{m} \theta_i(\lambda)u_i$, then

$$\begin{aligned}
\underline{J}u(\lambda) &= \alpha_1(\lambda)\underline{J}v(\lambda) \\
&= \alpha_1(\lambda)\left[\sum_{i=1}^{m} \theta_i(A)b_i + \underline{A}\right] \\
&= \alpha_1(A) \sum_{i=1}^{m} \theta_i(A)\, b_i + \underline{A} \\
&= 0;
\end{aligned}$$

therefore $\alpha_1(\lambda)\underline{R}^m[\lambda] \subset \text{Ker } \underline{H}$. Since $\underline{H}$ is reduced, $\alpha | \alpha_1$; and by definition of a realization, $\alpha_1 | \alpha$. ∎

7. COMPUTATION OF A CANONICAL REALIZATION

We indicate briefly how the construction of Section 5 can be performed explicitly. Start with the p×m transfer matrix

$$H(\lambda) = \alpha(\lambda)^{-1} \hat{H}(\lambda)$$

As a polynomial matrix, $\hat{H}(\lambda)$ can be factored by elementary row and column operations, into the form

$$\hat{H}(\lambda) = \hat{P}(\lambda)\ D(\lambda)\ \hat{Q}(\lambda)$$

where $\hat{P}(p\times p)$ and $\hat{Q}(m\times m)$ are invertible polynomial matrices (i.e., have R-valued, nonzero determinant) and $D(\lambda) = \text{diag}\ [\delta_1(\lambda),\ldots,\delta_r(\lambda),0,\ldots 0]$ is a p×m polynomial matrix with the diagonal shown (of length min(p,m)), and zeros elsewhere. The δ_i can be chosen such that $\delta_1|\delta_2,\ldots\delta_{r-1}|\delta_r$, and are then unique up to factors in R. Then

$$H(\lambda) = \alpha(\lambda)^{-1} \sum_{i=1}^{r} \hat{p}_i(\lambda)\delta_i(\lambda)\hat{q}_i(\lambda)$$

$$= \sum_{i=1}^{r} \frac{\hat{p}_i(\lambda)\hat{\delta}_i(\lambda)\hat{q}_i(\lambda)}{\alpha_i(\lambda)}$$

$$= \sum_{i=1}^{k} \frac{p_i(\lambda)q_i(\lambda)}{\alpha_i(\lambda)} + E(\lambda)$$

where $\hat{p}_i,\hat{q}_i$ are respectively the i^{th} column, row of $\hat{P},\hat{Q}$; $\alpha_i = \alpha/\text{GCD}(\delta_i,\alpha)$; $\hat{\delta}_i = \delta_i/\text{GCD}(\delta_i,\alpha)$; p_i,q_i are the remainders after division of $\hat{p}_i\hat{\delta}_i$, $\hat{q}_i$, respectively, by α_i; E is a polynomial matrix; and terms in the summation have been relabelled to retain only α_i of positive degree (i.e., $k \le r$).

Suppose deg $\alpha_i = n_i$. Let A_i be a cyclic $n_i \times n_i$ matrix in standard form

$$A_i \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 0 & 1 \\ a_{i1} & . & . & . & . & . & . & a_{in_i} \end{bmatrix}$$

where the a_{ij} are given by

$$\alpha_i(\lambda) = \lambda^{n_i} - \sum_{j=1}^{n_i} a_{ij}\lambda^{j-1}$$

From $\alpha_i(\lambda)$ compute the polynomials $\alpha_i^{(j)}(\lambda)$ $(j \in \underline{n_i})$ as in (5.18), and define n_i-vectors

$$p_i^*(\lambda) = \text{col}\left(1, \lambda, \ldots, \lambda^{n_i - 1}\right)$$

$$q_i^*(\lambda) = \text{row}\left(\alpha_i^{(1)}(\lambda), \ldots, \alpha_i^{(n_i)}(\lambda)\right)$$

Then it is easily checked that

$$\alpha_i(\lambda)(\lambda - A_i)^{-1} = p_i^*(\lambda) q_i^*(\lambda) + \alpha_i(\lambda) E_i(\lambda) \qquad (7.1)$$

where $E_i(\lambda)$ is a polynomial matrix.

Next compute real matrices B_i $(n_i \times m)$ and C_i $(p \times n_i)$ such that

$$C_i p_i^*(\lambda) = p_i(\lambda), \quad q_i^*(\lambda) B_i = q_i(\lambda) \quad (i \epsilon k);$$

C_i and B_i always exist. Finally, let

$$A = \text{diag}\,[A_1,\dots,A_k]$$

$$B = \begin{bmatrix} B_1 \\ \cdot \\ \cdot \\ \cdot \\ B_k \end{bmatrix}, \quad C = [C_1,\dots,C_k]$$

It is a straightforward exercise in matrix algebra to check that the foregoing construction mimics the abstract procedure of Section 5. In particular it is helpful to note that we can take $g_i = \text{col}(0,\dots,0,1)$ and $h_i' = \text{row}\,(1,0,\dots,0)$ in each $n_i \times n_i$ block.

8. BIBLIOGRAPHICAL NOTE

In [1] Kalman described the canonical structure of a dynamic system (relative to controllability and observability) which we exploit in proving Lemma 6.5. Gilbert gave in [2] a realization procedure for a transfer matrix having simple poles; the computation, via residue matrices, is straightforward. Then Kalman [3] observed that, in general, a minimal realization could be computed by starting with an arbitrary realization and reducing to a complete realization, as in [1]; the equivalence of minimality and completeness was announced but not proved. In [4] Kalman gave the first general prescription for construction of a minimal realization, identifying the state space dimension as the (McMillan) degree of the transfer matrix; a realization was computed from a prime-factor (i.e., Jordan) decomposition of the state space. The first published proof of the equivalence of minimality and completeness, and the fact that minimal realizations are equivalent ($\backsim$), is due to Youla [5], who worked directly with the differential equation for the impulse matrix; the technique is more analytical than algebraic. The module-theoretic viewpoint appears in Kalman's fundamental paper [6]; the approach is via an input-output description in the spirit of automata theory, with the states identified as Nerode equivalence classes. As shown in [6], there immediately result the modular structure of the state space, and the crucial factoring of the input-output map, of which our diagram D1 is a modified version. Suggestive remarks on the system-theoretic meaning of homology sequences can be found in the expository paper [7]. In [8] Kalman gave an alternative concrete realization procedure by way of rational canonical decomposition of the state space; this method is summarized in our Section 7, where the formula (7.1) is the "main lemma" of [8]. We note that in [8] the term "canonical" is used to mean what we call "complete". The algebraic relation between non-minimal realizations is discussed in [9]. Finally Ch. 10 f [10] contains a systematic exposition of much of the work just cited, together with a presentation of the B. L. Ho algorithm for minimal realization of an impulse response matrix.

In the present memorandum our approach follows [10] in taking as "input" module the free module $\underline{R}^m[\lambda]$, but differs in the choice of "output" module as $\underline{R}^p[\lambda]/\alpha(\lambda)\underline{R}^p[\lambda]$ rather than the free module of formal power series over R^p (see definition (2.1) of [10], Ch. 10). This set-up seems better matched to the "finiteness" of the transfer matrix as a purely algebraic object (with no particular interpretation attached to it as an input-output map), builds torsion in right at the start, and perhaps results in a more symmetric role for the C matrix. The formal definition of a transfer morphism is new. As a final detail, the definition (5.17) of a "conjugate" generator, and the useful formula (5.21), do not seem to have appeared before explicitly.

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