MCR-70-457 (Vol II)

FINAL REPORT

SPIN VECTOR CONTROL FOR A SPINNING SPACE STATION

VOLUME II: ANALYTIC MANUAL

Get DRF

By:

T. Hendricks
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November 1970

Prepared under Contract No. NAS8-25247 by

MARTIN MARIETTA CORPORATION
Denver Division
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
George C. Marshall Space Flight Center
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FOREWORD

This document represents Volume I of the final report on NASA Huntsville Contract entitled "Spin Vector Control for a Spinning Space Station". The report is prepared in two volumes:

Volume I - User's Manual

Volume II - Analytical Manual

TABLE OF CONTENTS

ASTRACT

This report constitutes the documentation of a technical study for developing an efficient computer program for a restricted rotational dynamics problem involving multiple bodies connected by movable joints. The configuration is particularly applicable to a space station system with a rotor producing a large spin vector. The computer program is an effective tool for study of control of the spin vector and effects of the spin vector on other functions.

I. INTRODUCTION

The object of this study was to develop a means of investigating the dynamic behavior of a dual spin space station. In particular, the extremely large spin vector of such a system presents control problems as well as having complex and often intuitively unexpected influence on the response of the system to internal and external forces. Emphasis was given to construction of an efficient computer program for simulating this system and functions relating to it.

Basically the program is an application of Russell's momentum method of formulating and solving the equations of motion of a system of rigid bodies connected by movable joints. In the interest of efficiency the method is not applied in full generality. Rather, the problem is restricted to configurations and constraints that are closely related to the actual space station systems that are of interest. Through choices of parameters and optional features in the program, a sufficiently wide range of situations can still be accommodated.

Primary features of the program are:

1. Rotor-stator configuration with single degree of freedom bearing joint.
2. Complete generality in inertial parameters of both rotor and stator.
3. Movable mass on rotor.

4. Nutation damping pendulums on rotor.

5. CMGs on stator.

6. Reaction jets on stator and rotor.

7. Attitude control by reaction jets or CMGs.

8. Transverse rate (nutation) removal by reaction jets.

9. Spin control by reaction jets and/or torque motor.

10. Flexibility of control functions by incorporation of control laws in subroutines.

11. Gravity gradient effects.

12. Docking capabilities.

Although the program was exercised extensively in order to assure proper functioning, no exhaustive study of any problem or aspect of spin vector behavior was made. In the course of program check-out some interesting results were obtained and these are described.

II. DESCRIPTION OF RUSSELL'S METHOD

Equations of motion pertaining to this problem are derived in detail in Appendix B. In this section the general development of these equations into a complete method of solution will be explained.

The application of Newton's law to a configuration of rigid bodies as given in Figure 1 results in equations of motion as follows.

Derivative of Total Angular Momentum

$$\dot{\bar{H}} = \bar{T}_{1_{EXT}} + \bar{T}_{0_{EXT}} \tag{2.1}$$

Derivative of Angular Momentum of Body 1

$$\begin{aligned}\dot{\bar{h}} = \bar{T}_{1_{EXT}} + \bar{T}_{0\rightarrow 1} + \bar{T}_{3\rightarrow 1} + \bar{T}_{4\rightarrow 1} + \bar{d}_{13} \times \bar{F}_{3\rightarrow 1} \\ + \bar{d}_{14} \times \bar{F}_{4\rightarrow 1} + \bar{\ell}_2 \times \bar{F}_{2\rightarrow 1}\end{aligned} \tag{2.2}$$

Derivative of Angular Momentum of Pendulums

$$\dot{\bar{h}}_3 = \bar{T}_{1\rightarrow 3} - \bar{\ell}_3 \times \bar{F}_{13}; \; \dot{\bar{h}}_4 = \bar{T}_{1\rightarrow 4} - \bar{\ell}_4 \times \bar{F}_{14}$$

Since Body 3 and Body 4 are point masses, $h_3 = h_4 = 0$ and $\dot{\bar{h}}_3 = \dot{\bar{h}}_4 = 0$, for the pendulums then

$$\bar{T}_{1\rightarrow 3} = \bar{\ell}_3 \times \bar{F}_{13}; \; \bar{T}_{1\rightarrow 4} = \bar{\ell}_4 \times \bar{F}_{14} \tag{2.3}$$

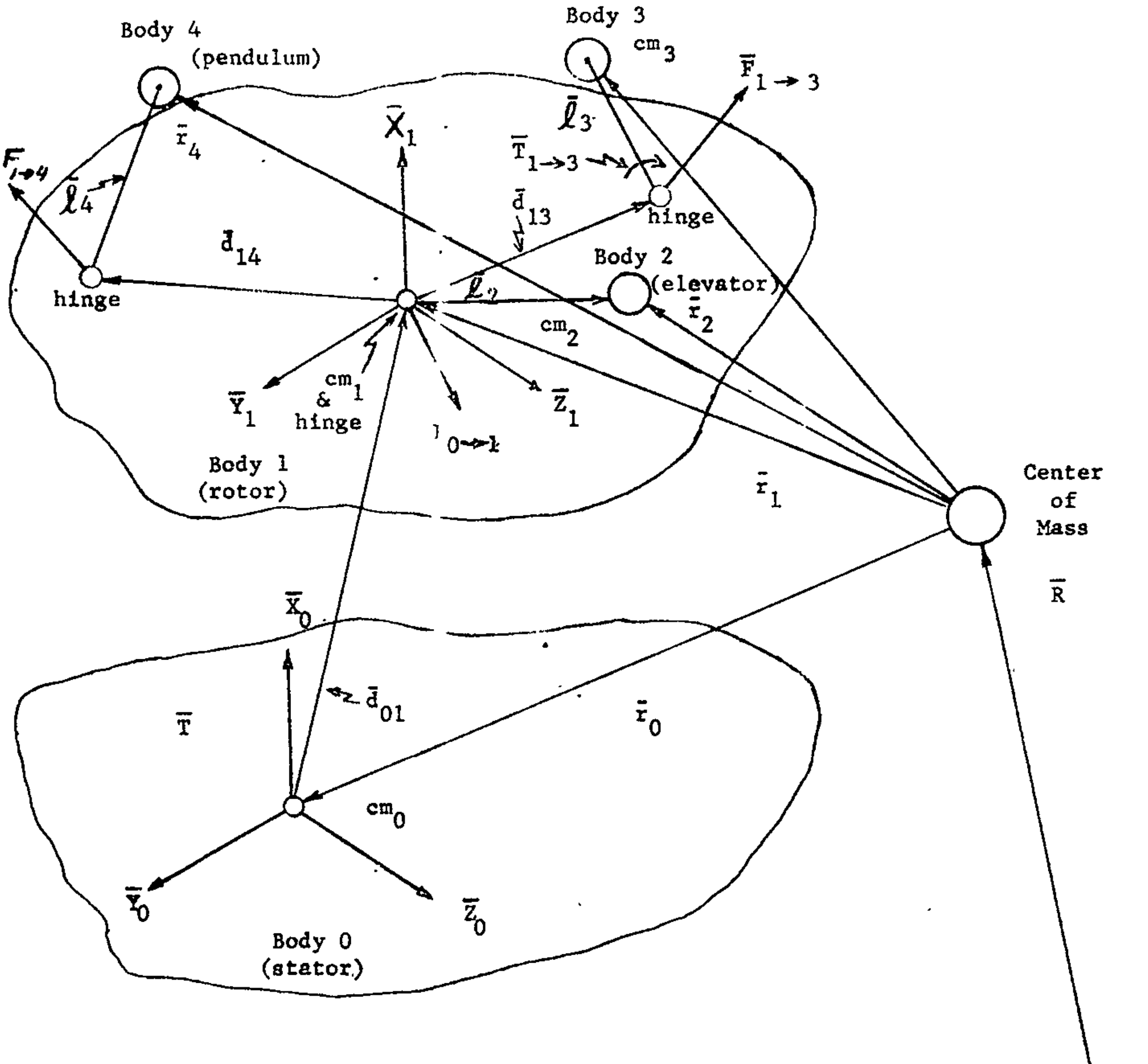


Figure 1 General Body Configuration

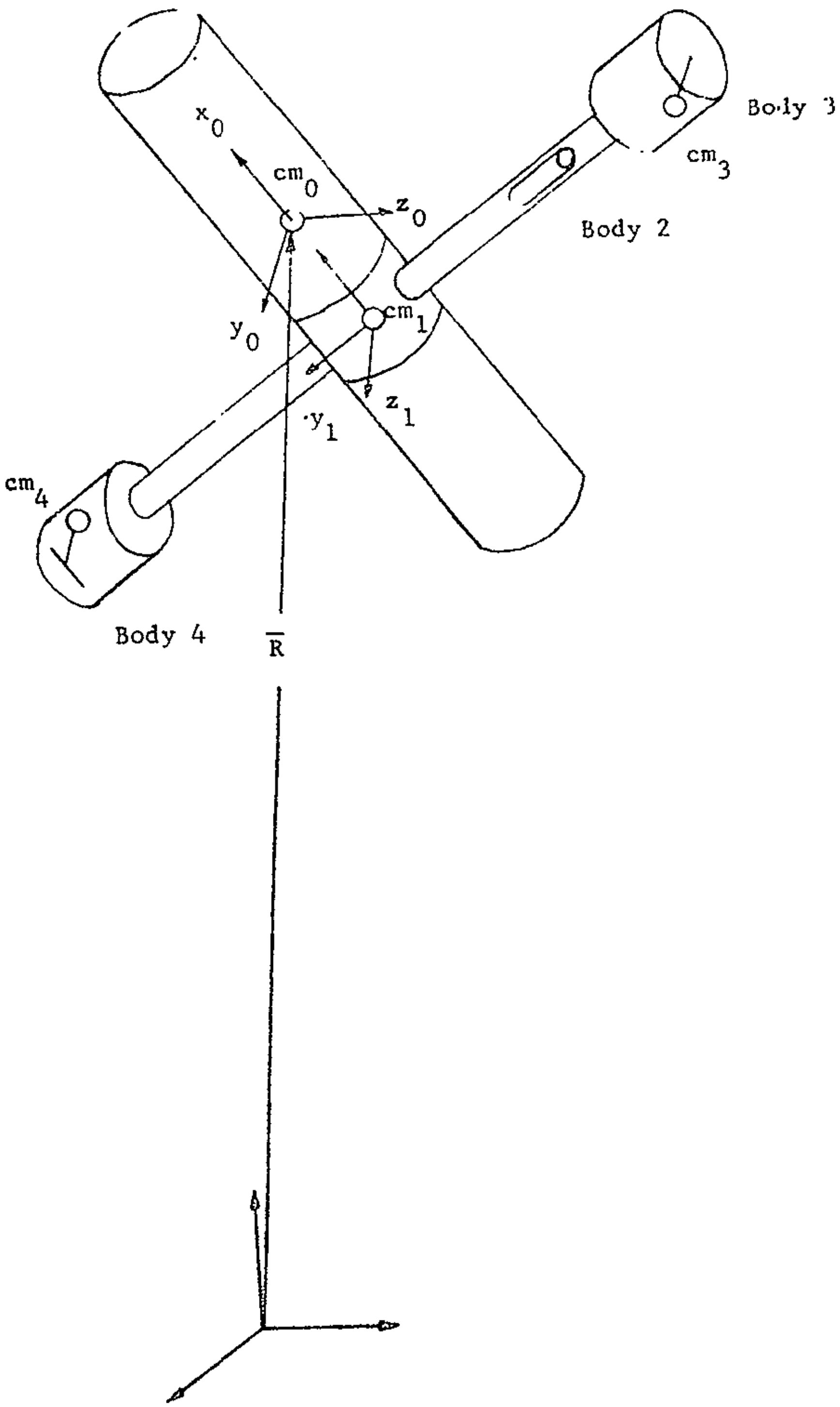


Figure 2 Spacecraft Body Configuration

In the Euler method angular momenta are expressed in terms of angular rates. The derivatives of angular momenta then involve angular accelerations. The integration is carried out on the angular accelerations to obtain angular rates. At each step the angular momenta can be obtained from values of angular velocity and a knowledge of geometric and inertial properties.

In Russell's momentum approach the integration is carried out directly on angular momenta. The angular rates at each step are obtained from the angular momenta.

The momentum method was selected for this program. Relatively simple expressions for derivatives based on Equations 2.1, 2.2, and 2.3 are used in the integration. The problem of obtaining values for angular velocities from values of angular momentum obtained by integration will be resolved later.

In order to carry out the integration it is necessary that terms on the right hand side of the derivative expressions (Equations 2.1, 2.2 and 2.3) be determined entirely by information available at each point. That is, in order to proceed to the next time point it is necessary that the derivatives can be calculated.

The terms $T_{1_{EXT}}$ and $T_{0_{EXT}}$ would have contributions from several sources:

1. Externally applied torques as given forcing functions which are known functions of time.

2. Externally applied control torques depending by some control law on geometry and/or rates. A current complete solution will be supplying geometric and rate information from which these torques can be obtained at each point.

3. Externally applied torques depending on orientation (gravity gradient). Again, the current geometric information will permit calculation of these forces and torques at each point.

The torques acting around the various hinge lines ($T_{0\rightarrow 1}$, $T_{1\rightarrow 3}$, $T_{1\rightarrow 4}$) are all a result of control and/or friction functions. These torques are determined at each point through specific control laws from angle and angular rate data.

The remaining terms ($d_{13} \times F_{3\rightarrow 1}$, $\ell_2 \times F_{2\rightarrow 1}$, $\ell_3 \times F_{1\rightarrow 3}$, etc.) in Equations 2.1, 2.2 and 2.3 are all torques due to interbody forces. These terms will be dealt with in three steps:

1. Forces will be expressed in terms of center of mass accelerations ($\ddot{\bar{r}}_1$, $\ddot{\bar{r}}_2$, $\ddot{\bar{r}}_3$, $\ddot{\bar{r}}_4$) through Newton's law. For example,

$$F_{1\rightarrow 3} = m_3(\ddot{\bar{R}} + \ddot{\bar{r}}_3)$$

The cluster center of mass acceleration is found from the sum of all external forces

$$\ddot{R} = \frac{1}{m} \sum F_{EXT}$$

then

$$\overline{T}_{13} = m_3 \ell_3 \times \ddot{\bar{r}}_3 + \frac{m_3}{m} \ell_3 \times \sum F_{EXT}$$

Through a change of variables the terms involving these accelerations will be replaced by terms involving velocities only. Continuing the same example, define

$$\bar{h}_3' = \bar{\ell}_3 \times m_3 \dot{\bar{r}}_3$$

then

$$\dot{\bar{h}}_3' = \dot{\bar{\ell}}_3 \times m_3 \dot{\bar{r}}_3 + T_{13} - \frac{m_3}{m} \ell_3 \times \sum F_{EXT}$$

The replacement of mass center accelerations by mass center velocities introduces time derivatives of other terms such as ℓ_3, d_{13}, etc.

3. Finally, it is observed that velocities can be expressed in terms of geometric configuration and angular rates. Hence, the torques due to interbody forces are determined by known external forces and torques and the angles and angle rates.

In this modified form (see Appendix B, equations B5.3, B5.5) the derivatives of angular momentum are determined by the known external and interbody torques, the known external forces and by angles and angle rates. A procedure to obtain current values for angular rates from current values of angular momentum is clearly essential. It is also apparent that a concurrent integration of angle rates to obtain geometric features such as gimbal angles and orientation is also necessary in the solution of this problem. The only remaining requirement for determination of derivatives at each step is the means of deriving angle rates from angular momenta.

This procedure will now be described. Angular momentum depends on geometry, inertial properties and angle rates. For the system of rigid bodies in Figure 1 the angular momentum expressions are:

Total Angular Momentum

$$\bar{H} = \bar{\bar{I}}_0 \cdot \bar{\omega}_0 + \bar{\bar{I}}_1 \cdot \bar{\omega}_1 + \bar{r}_0 \times m_0 \dot{\bar{r}}_0 + \bar{r}_1 \times m_1 \dot{\bar{r}}_1 + \bar{r}_2 \times m_2 \dot{\bar{r}}_2$$

$$+ \bar{r}_3 \times m_3 \dot{\bar{r}}_3 + r_4 \times m_4 \dot{\bar{r}}_4$$

"Primed" Angular Momentum of Body 1

$$\bar{h}_1' = \bar{\bar{I}}_1 \cdot \bar{\omega}_1 + m_2 \bar{\ell}_2 \times \dot{\bar{r}}_2 + m_3(\bar{d}_{13} + \bar{\ell}_3) \times \dot{\bar{r}}_3 + m_4(\bar{d}_{14} + \bar{\ell}_4) \times \dot{\bar{r}}_4$$

"Primed" Angular Momentum of Body 3 and Body 4

$$\bar{h}_3' = \bar{\ell}_3 \times m_3 \dot{\bar{r}}_3; \; \bar{h}_4' = \bar{\ell}_4 \times m_4 \dot{\bar{r}}_4$$

The center of mass distance vectors, r_0, r_1, r_2 . . . are expressible in terms of various vector parameters $\bar{d}$'s, $\bar{\ell}$'s (Figure 1 and appendix B equation B2.2). Consequently, the center of mass velocities can be expressed in terms of derivatives of the geometric parameters. ($\dot{\bar{d}}_{01}$, $\dot{\bar{d}}_{13}$, $\dot{\bar{\ell}}_3$, etc.). These derivatives are in turn each determined by the angular rate of the body in which the parameter is defined as a constant. For example, $\dot{\bar{d}}_{01} = \bar{\omega}_0 \times \bar{d}_{01}$. When all these substitutions have been made, the angular momenta will appear as sums of terms with the following properties:

1. All terms are of first degree in angular velocity.
2. Aside from the angular velocity factors, no other factors in any term are explicitly time derivatives That is, the other factors can all be evaluated entirely from a knowledge of the inertial parameter and the current values of all angles.

It is necessary now to recognize that the actual computation must be done on scalar equations for the components of the vector equations that have been used up to this point. Scalar equations will be obtained for the following six components of angular momentum:

1. H_x x component of total $\bar{H}$
2. H_y y component of total $\bar{H}$
3. H_z z component of total $\bar{H}$
4. h'_{1x} component of $\bar{h}'_1$ around the unconstrained x-axis of Body 1
5. g'_3 component of $\bar{h}'_3$ around the unconstrained hinge line of Body 3
6. g'_4 component of $\bar{h}'_4$ around the unconstrained hinge line of Body 4

These six components of angular momentum are expressed as functions of the following six components of angular velocity:

1. ω_{0x} angular velocity of Body 0 around the Body 0 x-axis
2. ω_{0y} angular velocity of Body 0 around the Body 0 y-axis
3. ω_{0z} angular velocity of Body 0 around the Body 0 z-axis
4. Ω_1 angular velocity of Body 1 with respect to Body 0
5. Ω_3 angular velocity of Body 3 with respect to Body 1

6. Ω_4 angular velocity of Body 4 with respect to Body 1

The six scalar equations for the components of angular momentum in terms of the six angular velocity components are tediously long. However, since $\bar{\omega}$'s occurred only in the first degree in the vector equations, these scalar equations are linear in the angle rate components. The six linear algebraic equations will be stated as:

$$\begin{bmatrix} H_x \\ H_y \\ H_z \\ h'_{1x} \\ g'_3 \\ g'_4 \end{bmatrix} = \begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} \omega_{0x} \\ \omega_{0y} \\ \omega_{0z} \\ \Omega_1 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}$$

or $h = M\omega$

Then ω is found from $\omega = M^{-1}h$. The elements of M are calculated (in EMCALC subroutine) entirely from inertial properties and a knowledge of the geometry of the system (angles and components of d_{01}, d_{13}, ℓ_3, etc.). The angles and h are supplied by integration. Hence, the angular velocities can be obtained and the integration loop is closed.

The process can be summarized as follows:

1. At time t_n given h_n, ω_n, θ_{1n}, θ_{3n}, θ_{4n}, calculate $\dot{h}_n$.

2. Up date to t_{n+1}; by integration obtain h_{n+1}, $\theta_{1_{n+1}}$, $\theta_{3_{n+1}}$, $\theta_{4_{n+1}}$.

3. Set up M matrix from constants and $\theta_{1_{n+1}}$, $\theta_{3_{n+1}}$, $\theta_{4_{n+1}}$.

4. Solve $\omega_{n+1} = M^{-1}_{n+1} h_{n+1}$.

5. Return to step 1 for t_{n+1}.

Initially ω and the geometric configuration is given. Starting values for H_x, H_y, H_z and h'_{1x} are computed from geometry and ω in XDOT, the derivative calculation subroutine, which has to be called to supply the initial derivatives. Initial values for g'_3 and g'_4 are obtained by calculating M and using the initial values of ω in the last two equations of $h = M\omega$.

At each time point the orientation of Body 0 with respect to an inertial frame is updated by integration of quaternion parameters. The complete procedure is outlined in the flow chart of Figure 3 and diagrammed in Figure 4 in more detail. Figure 4 also indicates the effects of the moving mass or elevator (Body 2) and the CMGs. The moving mass contributes to h and elements of the M matrix. The CMGs modify certain elements of M and modify H_x, H_y and H_z before the solution for ω.

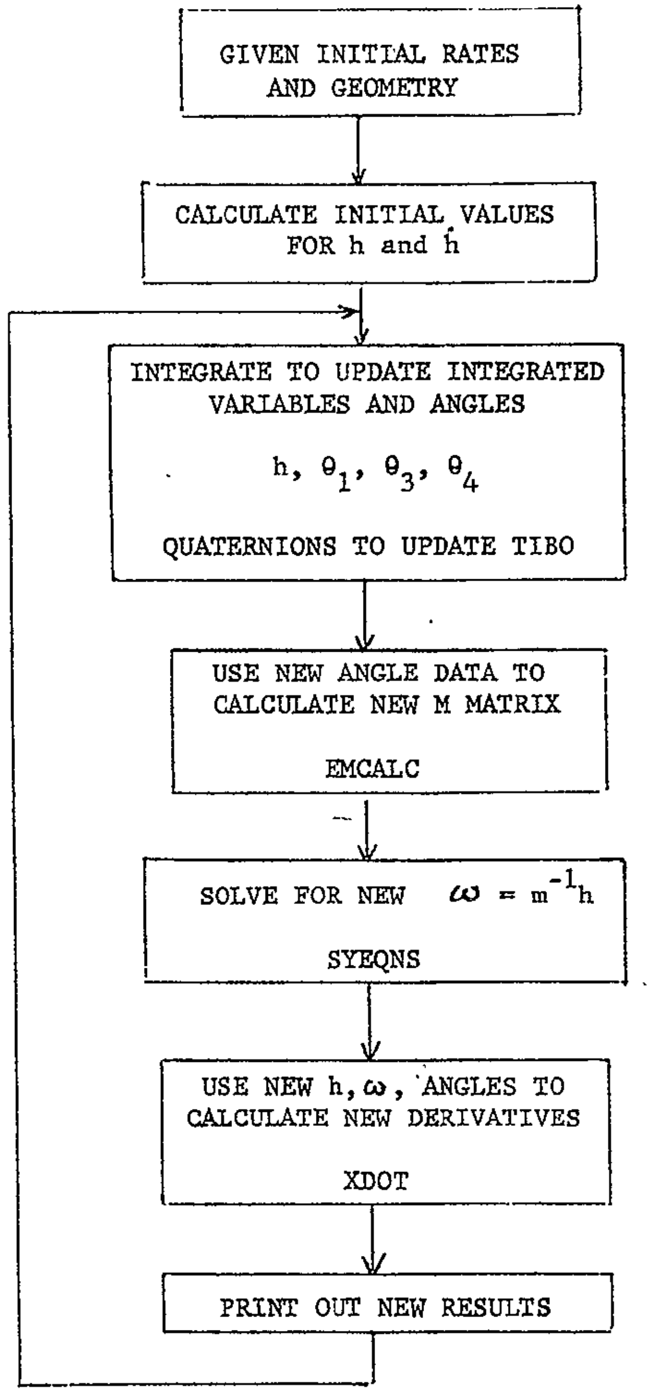


FIGURE 3 PROGRAM FLOW CHART

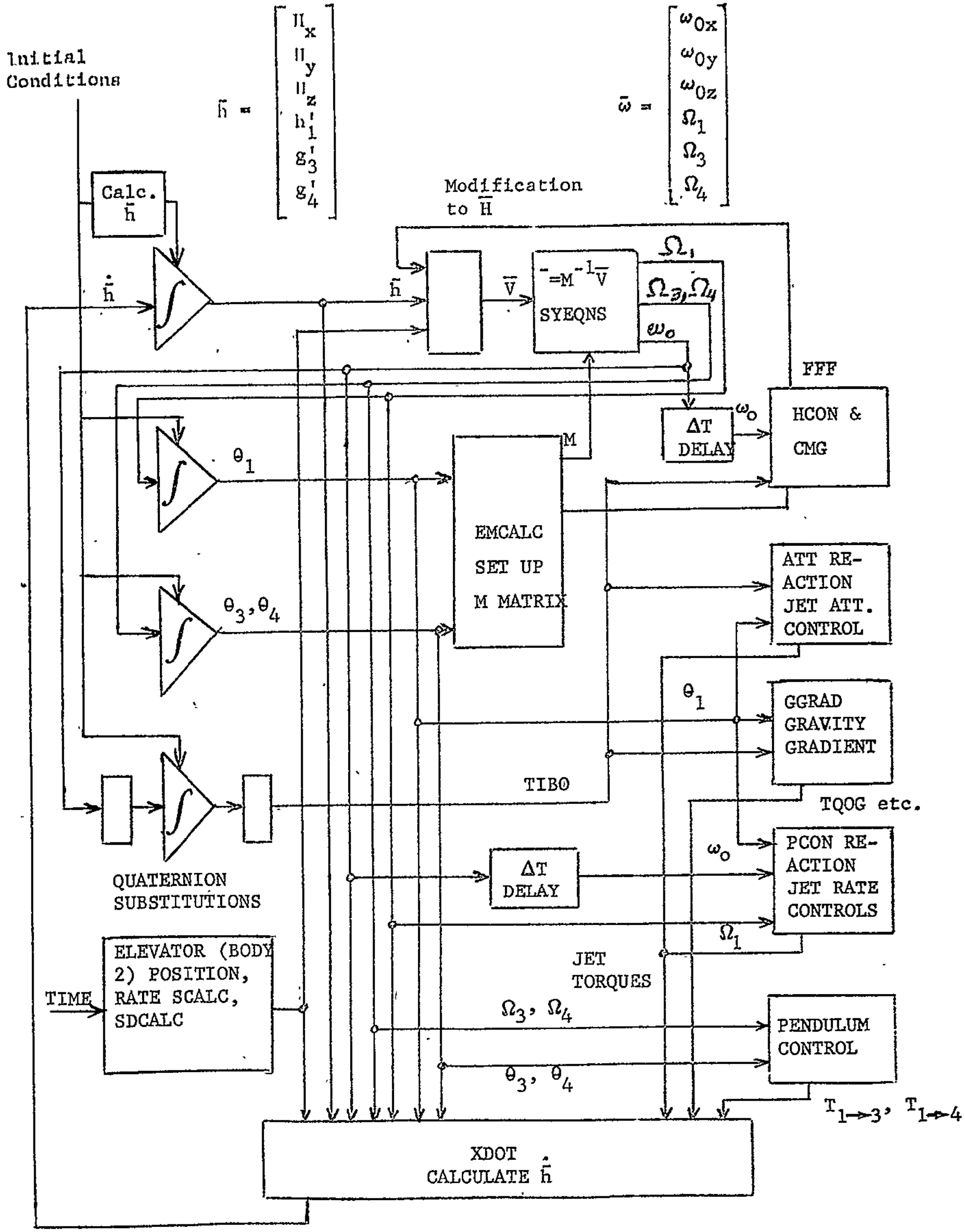


Figure 4 Program Flow Diagram

III. PROGRAM STRUCTURE

A general view of the main program will be presented with reference to the flow chart Figure 5 .

The RCON section initiates the read-in of data for each run and stops the program when the last run has been completed.

The INPUT section reads in and prints out all input data for each run in accordance with the options that are included.

The Initialization section resets initial values of certain variables, evaluates constants in accordance with input data and calculates initial values for integrated variables.

In the Integration section the angular momenta and angle variables are updated in time by appropriate substitutions and use of the FOMS integration subroutine. Docking impulse contributions to angular momentum are also made at this point.

The Quaternion section is logically a part of integration, but involves considerably more substitutions in the employment of quaternion parameters. The FOMS subroutine is used for integration. The result is an updating of the transformation from Body 0 to the inertial frame.

In the CONTM section, control functions by CMGs and reaction jets are activated in accordance with instruction flags in the input data.

In the INVERT section the M matrix is constructed by calling the EMCALC subroutine. Contributions to the angular momentum by terms due to the movable mass and CMGs are made at this point. Then the system of equations $h = M\omega$ is solved for the angular rates by making necessary substitutions and calling the subroutine SYEQNS. There are two alternative routes here depending on whether the pendulums are used. If pendulums are included, a set of six algebraic equations is solved. Otherwise, a set of only four equations is solved.

After the angular rates have been found, time is updated and the derivatives of the angular momentum variables are calculated by calling XDOT. The calculation of these derivatives completes the evaluation of all quantities pertaining to a point in time, and the results are printed out in the OUTPUT section. Again, the options included will determine the printout structure.

Before returning to the integration section for the next step, checks are made to determine if the parametric modifications required by docking must be made and also whether the run is finished.

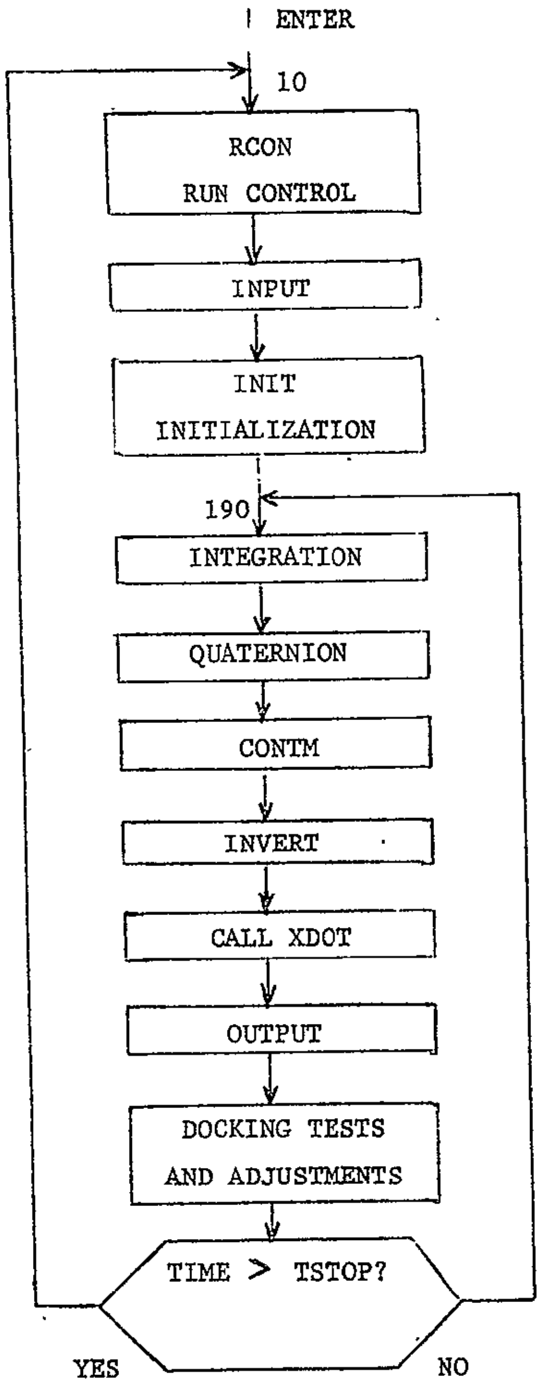


Figure 5 Program Flow Chart

IV. DESCRIPTION OF SPECIAL FEATURES

A. Restrictions and Assumptions

The various features and options of the program are subject to a number of restrictions and assumptions which will be listed in this section. Besides providing a single convenient access to such information, this section will also give an overall view of the capabilities of the program.

1. Body Configurations

Maximum of five bodies - the motion of one body (Body 2, the movable mass) is completely constrained.

The program can be run without the pendulums and/or without the movable mass.

If either the rotor or the stator is to be eliminated, the mass and moments of inertia of the eliminated body may have to be approximated by small values. If these parameters are set to zero, the M matrix may be singular or poorly conditioned.

2. Gimbal Constraints

The joint between Body 0 and Body 1 is a single degree of freedom bearing. The bearing axis is parallel to the x-axis of Body 0 and coincident with the x-axis of Body 1.

The hinges between Body 1 and the pendulums are single degree of freedom. The hinge lines must be parallel to the y-z plane.

3. Translational Motion

The orbit is circular with a radius specified in the input data. The orbital position is obtained by linearly incrementing the orbit angle with time in accordance with a constant angular velocity derived from the radius. Variations in gravitational forces due to non-uniform or non-spherical earth are not accommodated. The orientation of the orbital plane and position with reference to the earth are not specified.

4. Attitude Control by Reaction Jet

With the reaction jet attitude control scheme employed in this program the maximum angle of spin vector reorientation that may be commanded in a single maneuver is 60°.

5. CMG

A maximum of six CMGs can be located on Body 0. Each CMG may be zero DOF (reaction wheel), 1 DOF or 2 DOF.

6. Movable Mass (Body 2, elevator)

Body 2 moves in a straight line in Body 1. The motion is entirely constrained and is determined by the subroutines SCALC and SDCALC.

7. Reaction Jets

Control is limited to the x, y and z axes of Body 0 and the x and y axes of Body 1. Torque on each axis is produced by two oppositely directed pure couples identical in position and strength. One jet control gain (for all four jets) and one distance between jets (for both couples) can be specified for each axis.

8. Second Rotor

The effects of a second rotor can be approximated by a reaction wheel on Body 0. The reaction wheel angular momentum would be adjusted to cancel the angular momentum of the first rotor Body 1. This approximation is in error due to lack of freedom in positioning and due to incorrect distribution of masses.

9. Integration Step Size

The time increment may be selected. The choice depends on rates encountered in the problem and accuracy desired.

B. Movable Mass and Pendulums

The movable mass Body 2 and the pendulums Body 3 and Body 4 are incorporated into the system at the very basic level of the equations of motion. That is, the actual system equations of motion are structured to include these bodies. Consequently, these bodies have already been considered in the detailed discussion of the method and in the derivation of the equations of motion in Appendix B.

Terms due to Body 2, Body 3 and Body 4 appear profusely throughout the expressions for components of angular momentum derivatives and in the formation of elements of the M matrix. When these bodies are absent all of the terms due to them are rendered ineffective by setting in values of zero for appropriate mass and length parameters rather than being eliminated by programming.

C. Attitude Control by Reaction Jet

The basic approach to the problem of attitude control by use of reaction jets will be introduced with reference to Figure 6. Initially the spin vector (x-axis of space station) is aligned with x_I, the x-axis of the inertial frame. It is desired to reorient the spin vector to a direction given by the vector c_x which here lies in the x_I - y_I plane at an angle of α from the x_I axis. The desired reorientation is accomplished by a simple rotation about the z_I axis. However, when a moderate torque is applied briefly to the z_I axis, the path taken by the spin vector will be the nutation cone A which misses the desired orientation by a wide margin.

In principal a "brute force" technique could be employed to rotate the spin vector through the desired angle. A sufficiently large torque applied about the z_I axis would ensure that the predominant motion is essentially a rotation about the z_I axis. As the direction c_x is approached, a large reverse torque is applied to remove the transverse rate. Transverse rates remaining after this cancellation can be removed by the normal propulsion control. Remaining direction error can be reduced by repeating this procedure.

Instead of using this inefficient method the approach employed here minimizes fuel consumption by utilizing the nutational coning motion of the spin vector as part of the control policy. The desired orientation is approached by allowing the spin vector to traverse a segment of free body nutational motion (such as B in Figure 6) resulting from application of certain torques. At the termination of this path segment the spin vector direction is substantially closer to c_x. Residual transverse rates are removed by the normal reaction jet control. The process is repeated until the specified direction is approached as closely as required.

The attitude control subroutine will now be discussed in more detail. Terms are defined in Table I.

A reorientation procedure will not be initiated unless transverse angular rates are below some prescribed minimum ($|WO(2)|$ and $|WO(3)|$ less than .0002). Transverse rate removal is given priority because these rates would interfere with the attitude correction process.

If transverse rates are within tolerance, the spin vector direction is compared to the specified attitude direction vector CA. In this program the attitude test is made on attitude vector direction cosines transformed to BODYO coordinates (CB). If this difference is within tolerance ($< 2^\circ$), no attitude correction by reaction jets will be made.

This correction process is open loop. The vector CB is not updated and no attitude error comparisons are made during the process. The sensed attitude error is used only to establish initial values and process parameters.

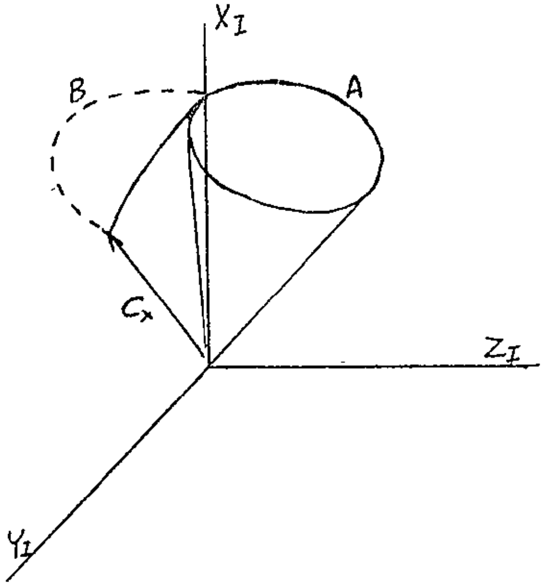
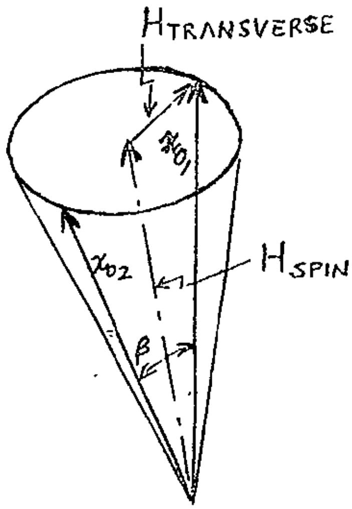


Figure 6



$$\sin\frac{\beta}{2} \approx \frac{H_{TRANSVERSE}}{H_{SPIN}}$$

$$= \frac{AK\ \Delta t_{AK}}{\omega_{SPIN}\ I_{SPIN}}$$

$$AK = \frac{1}{\Delta t_{AK}}\ \omega_{SPIN}\, I_{SPIN}\, \sin\frac{\beta}{2}$$

Figure 7

Process initialization consists of storing the current TIBO as TC, calculating normalized components CA1 and CA2 of the projection of the attitude vector CA in the Y-Z plane of BODYO, and calculating the torque magnitude AK required to produce the correction path.

The required torque AK is estimated from simple approximate geometric and dynamic relationships. In Figure 7 the desired reorientation of the spin vector from x_{01} to x_{02} is a rotation through the angle β. A nutation path of the H vector could pass near the directions x_{01} and x_{02} if the spin and transverse angular momenta are related by $\sin \beta/2 \approx \frac{H_{transverse}}{H_{spin}}$. $H_{transverse}$ will be produced by a torque AK acting for approximately 6 seconds, that is $H_{transverse} = 6$ AK. The required torque is then approximated by

$$AK = 1/6 \; W1(1) * BODY1I(1,1) * \sin \beta/2.$$

The discussion will be continued with reference to Figure 8 a representation of the Y-Z plane of the TCI frame - a frame fixed in inertial space coincident with the BODYO frame at the time that the attitude correction process is initiated. The projection of the desired direction vector on this plane is CAC. CA2 and CA3 are direction cosines of CAC. The projection on the $Y_{TCI} - Z_{TCI}$ plane of the path which the spin vector direction takes due to application of torque AK for the time Δt_{AK} is the arc B. The direction of torque AK is obtained from CA1 and CA2.

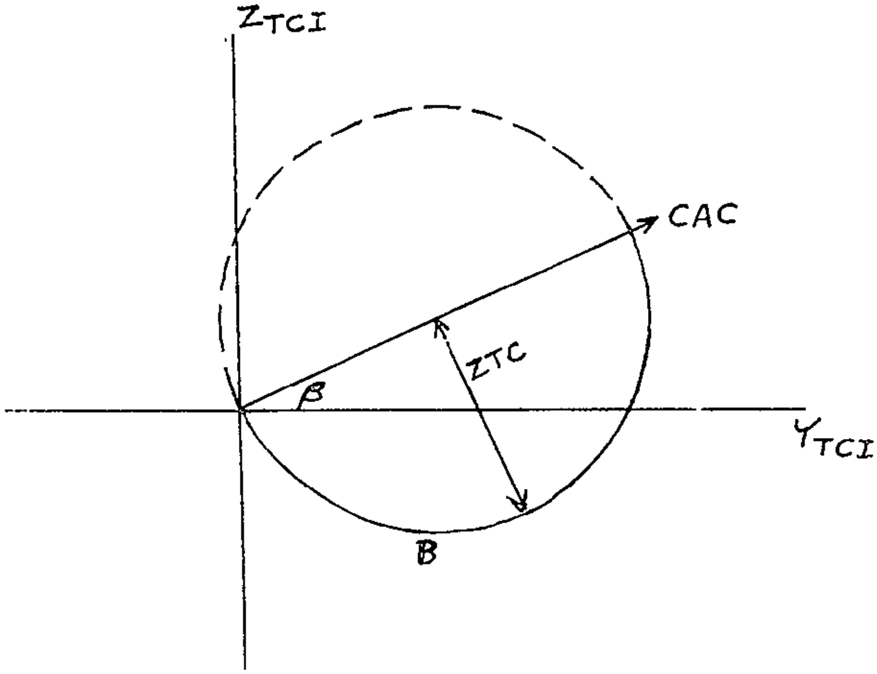


Figure 8

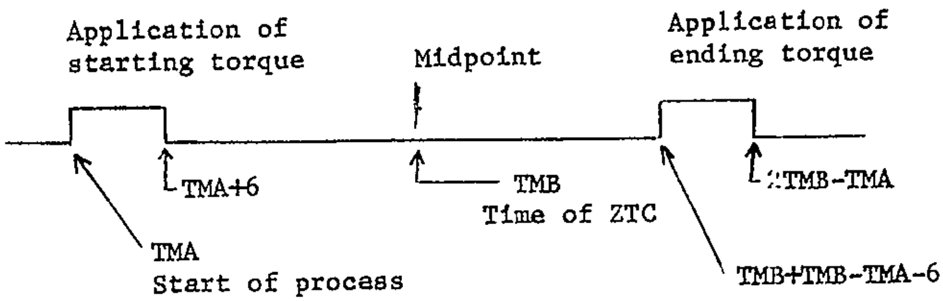


Figure 9

The projected path B shows that the spin vector does pass close to the desired direction CAC, but it is necessary to apply a torque to stop the nutational motion near CAC. The estimate for the time of application of this second torque is obtained by sensing the midpoint of the arc. For this purpose define an angle β between the TCI y-axis and the projection CAC. Note that $\cos\beta$ and $\sin\beta$ are already available as CA2 and CA3. In a frame obtained by rotating TCI through an angle β around the TCI x-axis the z component of the spin vector reaches its maximum excursion ZTC at the midpoint of the arc B. The time at which ZTC is sensed is the basis for timing the terminating torque near CAC. Note that the direction of the terminating torque is the same as for the initiating torque since the direction of travel of the spin vector projection has reversed in the course of traversing the arc B.

At the time 2TMB-TMA when the terminating torque application is complete, the normal propulsion control is restored. After transverse angular rates are again removed, the process is repeated to bring the spin vector successively closer to the specified direction.

The reaction jets for removing transverse angular rates can be located on either the stationary or rotating body as provided for in PCON. The torques required for the correction process itself, however, must be obtained from jets located on the

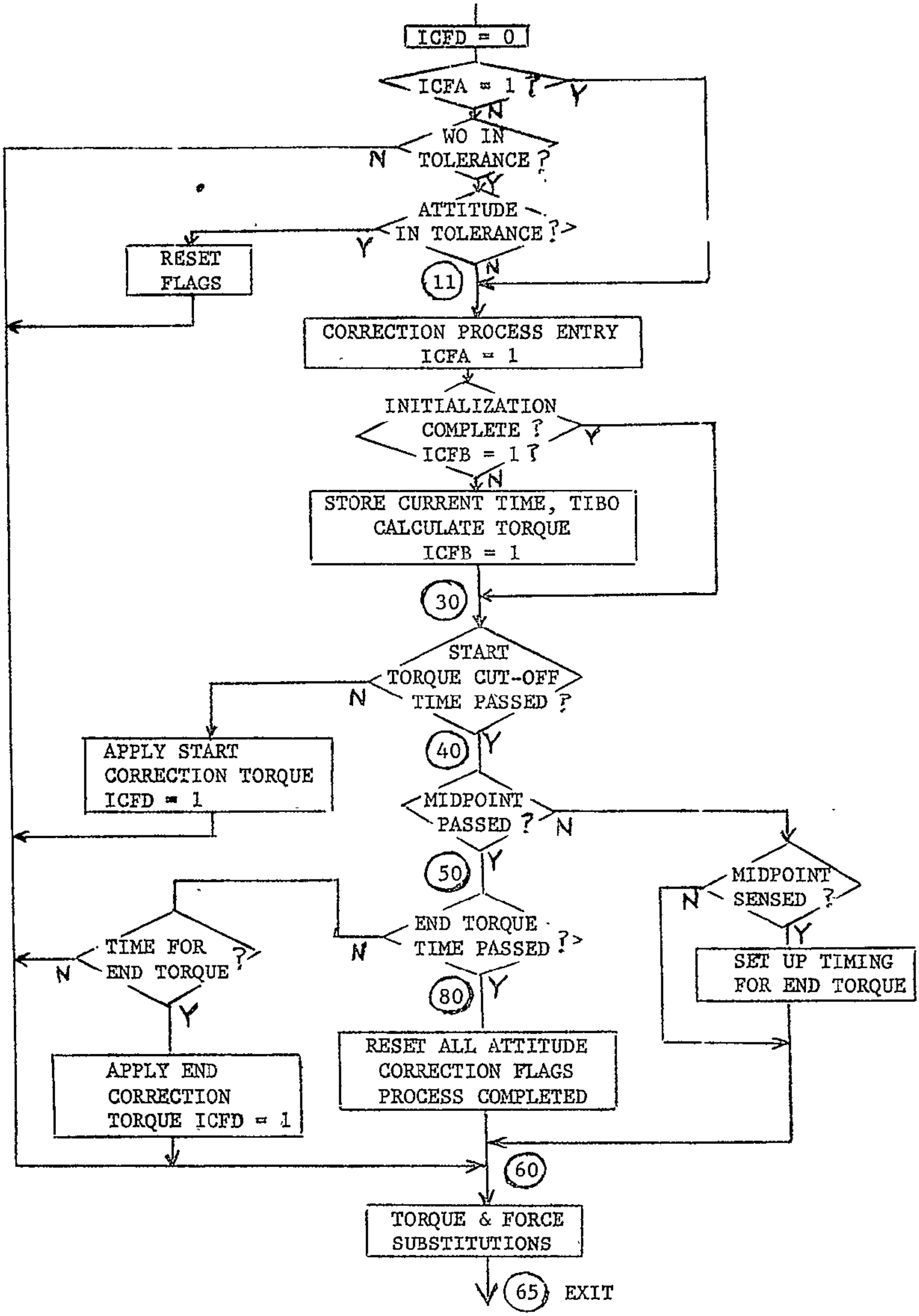


Figure 9 Attitude Control by Reaction Jet Flow Chart

Table I Terminology in Attitude Control Routine

AK	- Calculated estimate of torque for attitude correction
AOJ(12)	- Reaction jet lever arms
CA2	- Direction cosines of projection of attitude
CA3	- vector in YZ plane of Body 0 at start of process
CB(3)	- Attitude vector in Body 0 frame
FAT(8)	- Force equivalents for attitude correction torques at reaction jet lever arms
ICFA	- Attitude correction process flag
ICFB	- Process initialization flag
ICFC	- Midpoint sense flag
ICFD	- Torque application flag
TC	- TIBO at start of process
TMA, TMB, TMC	- Times associated with application of torques
ZTC	- Maximum excursion of spin vector from "diameter" of correction path
ZTCL	- Previous value of ZTC; used in sensing extreme value of ZTC

stationary body. This restriction is necessary because jets on the rotating body will not in general be in the proper position at the time that torque is required.

The attitude control by reaction jet feature was devised to provide large changes in direction of the spin vector. This restriction is consistent with the coarse tolerances and crude approximations applied. It is assumed that fine adjustments to attitude and continuous accurate attitude control would be supplied by CMGs.

The simple system simulated in this program is also limited in the reorientation angle that can be accommodated in a single command (a change in the vector CA). In order to avoid algebraic sign reversals that would disrupt the control process, the commanded change in spin vector direction must be less than 90°. A margin must also be provided for nutational path inaccuracies due to the crude estimates used to establish control process parameters. Accordingly, a maximum spin vector reorientation angle of 60° has been imposed for a command. Of course, attitude changes by larger angles can be accomplished by a succession of several commands.

The simple reaction jet attitude control system presented here is intended to be an example of one of the problems that can be handled by the program. It is recognized that considerable improvement could be made in this system and that other systems may have superior performance.

D. Angular Momentum Control Devices

The math model in this Section gives the angular momentum of the k^{th} CMG about the center of mass (c.m.) of the CMG, expressed in the o^{th} frame. Control commands to the model will be gimbal rate commands $\dot{\emptyset}$ and $\dot{\theta}$, depending on the degrees of freedom possessed by the CMG. These rates are integrated to keep track of $\emptyset$ and θ.

Total mass of the CMG is to be considered as part of body 0 and located at the c.m. of the component. This point mass must be incorporated into the mass and inertias of body 0. The total angular momentum vector H of the CMG is to be expressed in the o^{th} frame and included with all other such quantities on the o^{th} body to constitute the angular momentum used in Russell's equations.

Figure 10 shows the basic configuration necessary to describe the reaction wheel, one degree-of-freedom (DOF) CMG, or two DOF CMG. Considering the two DOF case, the gimbals are defined by $\emptyset$ and θ.

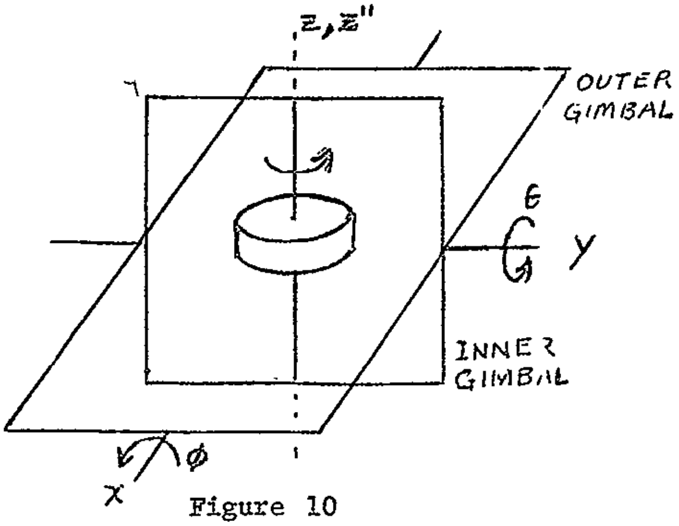


Figure 10

When ϕ and θ are equal to zero, the axes of the wheel are aligned to the zero gimbal frame (C_{ok}). The C_{ok} frame is then related to the o^{th} body frame by the transformation $[C_{ok}B_o]$ or by the inverse relationship $[B_oC_{ok}]$.

Each momentum device will contribute a momentum vector consisting of two parts given as

$$h_{ok} = f_{ok} + E_{ok}\omega_o. \qquad \text{D-1}$$

The first part, f_{ok}, is due to the commanded gimbal rates while the second part, $\dot{E}_{ok}$, is due to the angular velocity of the o^{th} body. As explained in section (II), the ω's of the five main bodies are realized by solving the equation.

$$\bar{\omega} = M^{-1}\bar{h}$$

E_{ok} contributes to the M's and f_{ok} contributes to the $\bar{h}$'s.

Before attempting to derive the equations several variables need to be defined:

1. B_o Body frame for Body 0.
2. C_{ok} Null gimbal frame for k^{th} device in body 0.
3. $[C_{ok}B_o]$ Transformation from body 0 frame to k^{th} null gimbal frame.

4. $[B_oC_{ok}]$ Inverse of $[C_{ok}B_o]$.

5. h_{ok} Total momentum vector of k^{th} device expressed in o^{th} frame.

6. H_k Total momentum vector of k^{th} device expressed in null gimbal frame.

7. H_I Angular momentum of the inner gimbal including mass of the wheel (C_{ok} frame).

8. H_o Angular momentum of the outer gimbal (C_{ok}) frame.

9. H Angular momentum of the wheel (C_{ok}) frame.

10. $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{ok}$ Angular rate of the null gimbal frame.

11. Prime Frame Principle axes of the outer gimbal.

12. Double Prime Principle axes of the inner gimbal.

13. $[OG_{ok}C_{ok}]$ Transformation from null gimbal frame to outer gimbal frame.

14. $[C_{ok}OG_{ok}]$ Inverse of $[OG_{ok}C_{ok}]$.

15. $[IG_{ok}C_{ok}]$ Transformation from null to inner gimbal frame.

16. $[C_{ok}IG_{ok}]$ Inverse of $[IG_{ok}C_{ok}]$.

17. H_{ok} Momentum of the k^{th} wheel. (Assume constant for CMGs).

18. I_o Inertia matrix of outer gimbal.

19. I_1 Inertia matrix of inner gimbal including inertia of wheel.

20.
$$\left[OG_{ok}C_{ok}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & S\phi \\ 0 & -S\phi & C\phi \end{bmatrix}_k$$

21.
$$\left[IG_{ok}C_{ok}\right] \begin{bmatrix} C\theta & S\phi S\theta & -S\theta C\phi \\ 0 & C\phi & S\phi \\ S\theta & -S\phi C\theta & C\phi C\theta \end{bmatrix}_k$$

Two Degree-of-Freedom CMG

For the 2 DOF CMG, the total momentum will consist of three parts

$$H_k = H_I + H_o + H \qquad \text{D-2}$$

Each component of Equation D-2 will be derived separately. They then will be summed and the sum split into two parts; one part a function of ω_o, the other part a function of the gimbal rates and wheel momentum.

The angular velocity of the outer gimbal frame is related to the angular velocity of the body 0 by

$$\left[\omega'\right]_{ok} = \left[OG_{ok}C_{ok}\right] \left[C_{ok}B_o\right] \left[\omega_o\right]. \qquad \text{D-3}$$

Therefore, the inertial rate of the outer gimbal is equal to the angular rate of the outer gimbal ω'_{ok} plus the gimbal rate $\dot{\phi}_{ok}$. The angular momentum of the outer frame H'_{ok} is then given as

$$[H'_o]_{ok} = [I'_o]_{ok} \left\{ [\omega']_{ok} + [\dot{\phi}]_{ok} \right\}$$

expressed in the prime frame and as

$$[H_o]_{ok} = [C_{ok}OG_{ok}] \quad [H'_o]_{ok}$$

expressed in the null gimbal frame. When expanded,

$$[H_o]_{ok} = [C_{ok}OG_{ok}] \quad [I'_o]_{ok} \left\{ [OG_{ok}C_{ok}] \quad [C_{ok}B_o] \quad [\omega_o] + \dot{\phi}_{ok} \right\}. \qquad \text{D-4}$$

The H_I component of Equation D-2 is obtained by applying a similar approach. Angular rate of the inner gimbal is given in terms of the body rate ω_o as

$$[\omega'']_{ok} = [IG_{ok}C_{ok}] \quad [C_{ok}B_o] \quad [\omega_o].$$

The inertial rate of the inner gimbal is given as

$$[\omega'']_{ok} + [IG_{ok}OG_{ok}] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{bmatrix}_{ok}$$

and

$$[H_I'']_{ok} = [I_I''] \left\{ [\omega'']_{ok} + [IG_{ok}OG_{ok}] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{bmatrix}_{ok} \right\}.$$

When expanded and expressed in the null gimbal frame, H_I becomes

$$[H_I]_{ok} = [C_{ok}IG_{ok}] \ [I_I'']_{ok} \left\{ [IG_{ok}C_{ok}] \ [C_{ok}B_o] \ [\omega_o] + [IG_{ok}OG_{ok}] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{bmatrix}_{ok} \right\} \qquad \text{D-5}$$

The third component H, of Equation D-2, can be obtained by expressing the momentum of the wheel in the double primed system and then transforming it to the null gimbal frame. Let the momentum of the wheel be

$$[H'']_{ok} = \begin{bmatrix} 0 \\ 0 \\ H''_z \end{bmatrix},$$

since the spin vector of the wheel is always along the Z" axis. Therefore,

$$[H\omega]_{ok} = [C_{ok} IG_{ok}] \begin{bmatrix} 0 \\ 0 \\ H''_z \end{bmatrix}$$

This can be written as

$$[H\omega]_{ok} = H_z \begin{bmatrix} \sin\theta \\ -\cos\theta \sin\phi \\ \cos\theta \cos\phi \end{bmatrix}_{ok} \qquad \text{D-6}$$

when expressed in the null gimbal frame.

We now have the three components of H_k expressed in the null gimbal frame. At this point we want to divide H_k into the component due to ω_o and the part due to gimbal rates as given in Equation D-2,

$$H_{ok} = E_{ok}\omega_o + f_{ok}.$$

therefore, when the three components of Equation D-2 are added to expressed in the o^{th} frame, the two components are

$$E_{ok} = \left[B_o C_{ok}\right] \left\{ \left[C_{ok} OG_{ok}\right] \left[I'_o\right]_{ok} \left[OG_{ok} C_{ok}\right] \left[C_{ok} B_o\right] \right.$$

$$\left. + \left[C_{ok} IG_{ok}\right] \left[I''_I\right] \left[IG_{ok} C_{ok}\right] \left[C_{ok} B_o\right] \right\}, \qquad \text{D-7}$$

$$F_{ok} = \left[B_o C_{ok}\right] \left\{ \left[C_{ok} OG_{ok}\right] \left[I'_o\right] \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \right.$$

$$\left. + \left[C_{ok} IG_{ok}\right] \left[I''_I\right] \left[IG_{ok} OG_{ok}\right] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{bmatrix} + H_z \begin{bmatrix} S\theta \\ -C\theta S\phi \\ C\theta C\phi \end{bmatrix}_{ok} \right\}. \qquad \text{D-8}$$

One Degree-of-Freedom CMG

Equation D-2reduces to $H_k = H_I + H_W$ for the one degree-of-freedom CMG, and the angle ϕ no longer exists. By setting ϕ and $\dot{\phi}$ equal to zero, Equation D-5 reduces to

$$\left[H_I\right]_{ok} = \left[C_{ok} IG_{ok}\right] \left[I''_I\right]_{ok} \left\{ \left[IG_{ok} C_{ok}\right] \left[C_{ok} B_o\right] \left[\omega_o\right] \right.$$

$$\left. + \left[IG_{ok} OG_{ok}\right] \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}_{ok} \right\} \qquad \text{D-9}$$

and Equation D-6 reduces to

$$\left[H_W\right]_{ok} = H_z \begin{bmatrix} S\theta \\ 0 \\ C\theta \end{bmatrix}_{ok} \qquad \text{D-10}$$

Summing terms and expressing in the oth frame

$$E_{ok} = \left[B_oC_{ok}\right] \left\{ \left[C_{ok}IG_{ok}\right] \left[I_I''\right] \left[IG_{ok}C_{ok}\right] \left[C_{ok}B_o\right] \right\} \qquad \text{D-11}$$

$$f_{ok} = \left[B_oC_{ok}\right] \left\{ \left[C_{ok}IG_{ok}\right] \left[I_I''\right] \left[IG_{ok}C_{ok}\right] \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + H_z \begin{bmatrix} S\theta \\ 0 \\ C\theta \end{bmatrix}_{ok} \right\} \qquad \text{D-12}$$

Reaction Wheel

For the reaction wheel, there are no terms that are a function of the oth body rates. Therefore, E_{ok} is equal to zero. Also, since θ and $\dot{\theta}$ are equal to zero, Equation D-12 reduces to

$$f_{ok} = \left[B_oC_{ok}\right] H_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

E. Integration Method

Components of angular moments, gimbal angles and quaternion variables must be integrated in this program. In the interest of obtaining computational efficiency some attention was directed toward selection of an appropriate routine for updating all these integrated variables.

Three methods were considered.

1. Predictor-Corrector
2. Runge-Kutta
3. Adams

A predictor-corrector method had been used in a previous program (Ref. 3). Some machine-time analysis showed that iterations of the lengthy calculations involved were very time consuming. Consequently, a non-iterative method was sought that would be accurate enough and yet avoid repetitions of the long calculations for each step.

The Runge-Kutta method is non-iterative but does require calculation of derivatives at points between t_n and t_{n+1} in order to obtain the value of the variable at t_{n+1}. In many applications the calculation of derivatives at the intermediate points is not difficult and the improved accuracy easily offsets this minor additional computation.

In the rotational dynamic problems treated by this program, however, the calculation of derivatives is quite involved. The derivatives of angular momentum depend on angular rates. These rates in turn are found by constructing and inverting the matrix which relates angular momenta to angular rates. In terms of this program a derivative calculation would require entering the INVERT BLOCK which calls the EMCALC and SYEQNS subroutines and finally calling the XDOT subroutine. Apparently most of the calculation for each step would have to be repeated for each intermediate derivative. It is doubtful then that the Runge-Kutta method can produce much improvement in speed over the iterative predictor-corrector method.

The Adams method uses derivatives at t_n, t_{n-1}, . . . to integrate to t_{n+1} (Ref. 4). Since derivatives at these times have already been calculated for previous steps, it is necessary only to store them for future calculations.

Comparisons were made between a 4th order Runge-Kutta and a 2nd order Adams method in a rotational dynamics problem. With the same value of Δt the Runge-Kutta was much more accurate due to the higher order of approximation. However, when Δt in the Adams program was reduced so that the accuracy of the two programs became comparable, the Adams program still showed a distinct advantage in speed.

Some higher order Adams integrations were also investigated. An increase in the order improves the accuracy without any appreciable increase in machine time. Serious disadvantages arise though when the use of remote past derivatives produces a delay that leads to poor transient response or even instabilities in some situations. Many problems could be computed more effectively by a higher order Adams integration. However, since this program is to be applied to a wide variety of problems, it is advisable to remain with the simple second order Adams method.

V. DISCUSSION OF RESULTS

A. Nutation Damping Pendulums

Preliminary nutation damping pendulum simulations were made using moderately large pendulums disposed as follows on the rotor.

Length - 10 ft.

Mass - 100 slug

Center of rotor to hinge distance - 100 ft.

Even long runs (up to 1000 sec real time) with these large pendulums failed to show a perceptible decrease in transverse angular rates.

Observation of small changes in transverse rates was hindered by the unsymmetric rotor. Angular rate variations introduced by this lack of symmetry tend to obscure small changes due to other effects.

The possibility that stator pendulums would be more effective than rotor pendulums was considered. With appropriate parameter values and initial conditions this type of run was made using the same pendulums. Again no change in transverse rates was apparent.

Reduction of Δt did not indicate that these results were due entirely to instabilities introduced by computational errors.

In order to exaggerate effects to observable levels, pendulums of these extreme proportions were simulated.

Length - 200 ft.

Mass - 3000 slug

Rotor center to hinge distance - 300 ft.

Under these conditions the pendulums on the rotor produced definite

Table 2 Parameter Values for Computer Runs

$$BOMASS = 2.6 \times 10^4$$

$$BODYOI = \begin{bmatrix} 1.4 \times 10^7 & 0 & 0 \\ 0 & 3.2 \times 10^8 & 0 \\ 0 & 0 & 4.2 \times 10^8 \end{bmatrix}$$

$$B1MASS = 3.8 \times 10^3$$

$$BODY1I = \begin{bmatrix} 6 \times 10^7 & 0 & 0 \\ 0 & 6 \times 10^7 & 0 \\ 0 & 0 & 6 \times 10^6 \end{bmatrix}$$

$$D01 = 0$$

$$SP = .4$$

B. Spin Control and Attitude Hold

There are many techniques available that can be used to maintain the relative rotational rates between bodies 0 and 1 equal to a constant. The spin control loop that is considered in this report is basically a one actuator control loop where the one actuator consists of a direct electric motor drive between the despun hub and the rotor. A control law which drives the torque motor and adaptively compensates for frictional torques has been programmed in Subroutine TORK01 and is given by the following equations:

$$T_{MOTOR} = g_2 \Omega_1 + \xi \qquad 5\text{-}1$$

$$\dot{\xi} = g_3 \ (\Omega_1 - \Omega_{1_{DESIRED}}) \qquad 5\text{-}2$$

with the initial conditions

$$\xi(0) = - g_2 \Omega_1 \qquad 5\text{-}3$$

It is easy to show that the equilibrium occurs when $\Omega_1 = \Omega_{DESIRED}$ and that this solution is independent of any frictional torques. A typical sequence of events follows. Assume that the frictional torque suddenly increases. The result is a decrease in the spin rate Ω_1 and a corresponding increase in ξ, which is coupled via equation 5-1 to give an increase in T_{MOTOR} resulting in an increase in Ω_1 to the desired value. In this manner changes in friction are adaptively compensated.

A simple approximate analysis can be used to show how the gains g_2 and g_3 can be determined. Assuming that bodies 0 and 1 are sufficiently decoupled so that we can write Euler's equation as:

$$I_{1x}\ddot{\theta}_{1x} = (T_{MOTOR} - \text{friction}) \qquad 5\text{-}4$$

$$I_{0x}\ddot{\theta}_{0x} = -(T_{MOTOR} - \text{friction}) \qquad 5\text{-}5$$

recalling that

$$\dot{\theta}_{1x} = \Omega_1 + \dot{\theta}_{0x} \qquad 5\text{-}6$$

Substitution of equations (5-4) and (5-5) into $\ddot{\theta}_{1x} = \dot{\Omega}_1 + \dot{\theta}_{0x}$ yields

$$\dot{\Omega}_1 = -\frac{(T_{MOTOR} - \text{friction})}{I_{0x}} - \frac{(T_{MOTOR} - \text{friction})}{I_{1x}} \qquad 5\text{-}7$$

Differentiating (5-7) and assuming $\frac{d(\text{friction})}{dt} \approx 0$ we get

$$\ddot{\Omega}_1 = -\frac{\dot{T}_{MOTOR}}{I_{0x}} - \frac{\dot{T}_{MOTOR}}{I_{1x}} \qquad 5\text{-}8$$

Differentiating equation (5-1) and substituting into (5-8) gives with some rearranging

$$\ddot{\Omega}_1 + \left[\frac{I_{0x} + I_{1x}}{I_{0x}I_{1x}}\right] g_2\dot{\Omega}_1 + \left[\frac{I_{0x} + I_{1x}}{I_{0x}I_{1x}}\right] g_3\Omega_1 = -\left[\frac{I_{0x} + I_{1x}}{I_{0x}I_{1x}}\right] g_3\Omega_{DESIRED} \qquad 5\text{-}9$$

which is the desired result.

Equation (5-9) is a simple linear second order differential equation. It is now an easy matter to choose the gains g_2 and g_3 to give any desired dynamic response. Figure 11 shows a typical time response of Ω_1. The desired value of Ω_1 was .4 rad/sec while the actual value was .39 rad/sec when the problem was started. After an initial transient the actual and desired Ω_1 are approximately equal.

Once the relative angular rates between bodies 1 and 0 are stabilized to some desired value there remains the problem of orienting body 0 in inertial space. A simple CMG control law was used for this task. Two single degree of freedom

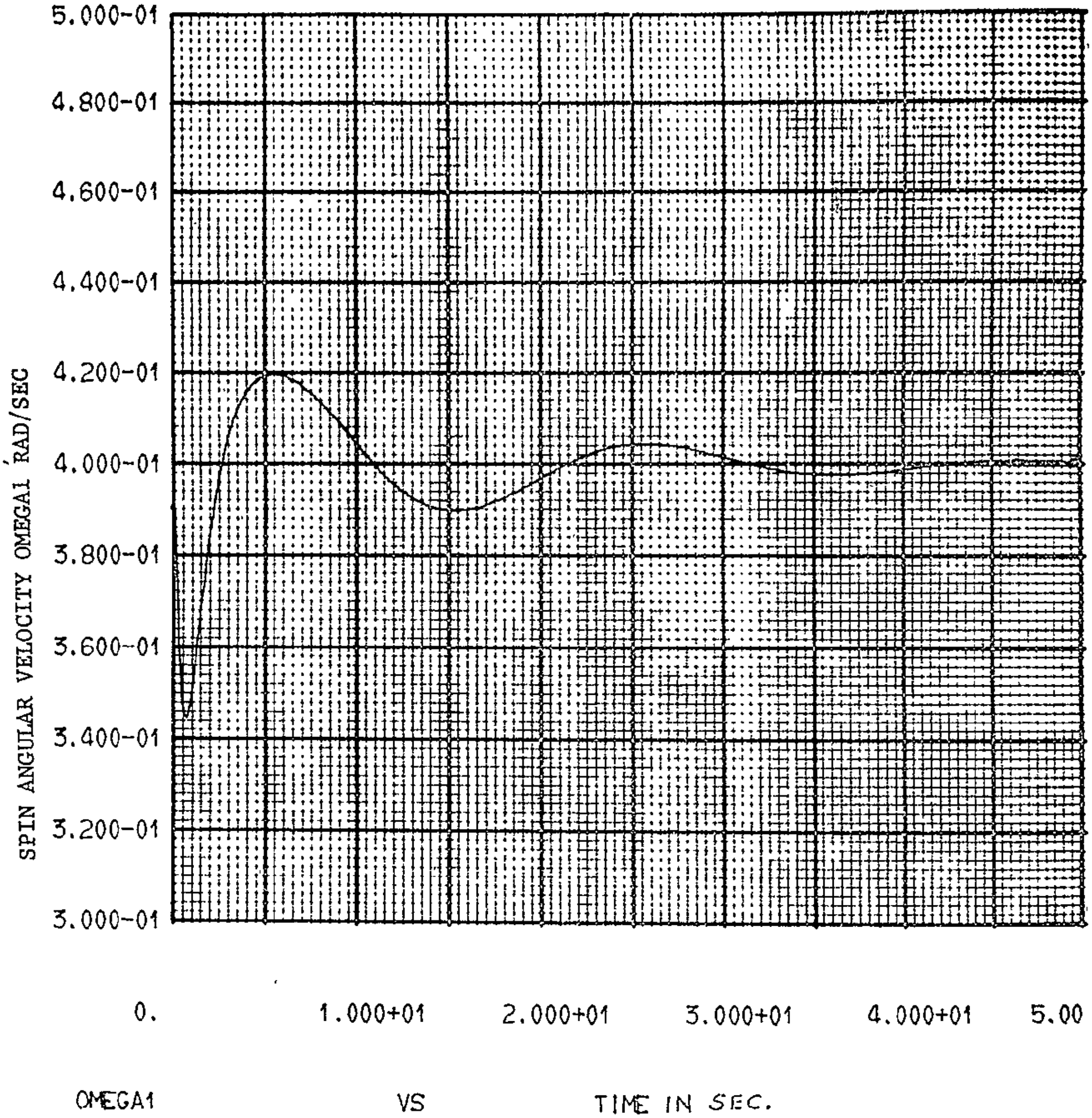
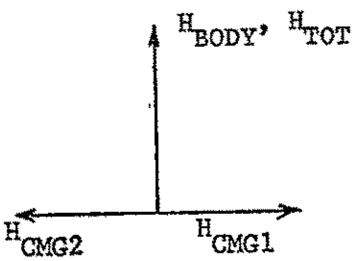
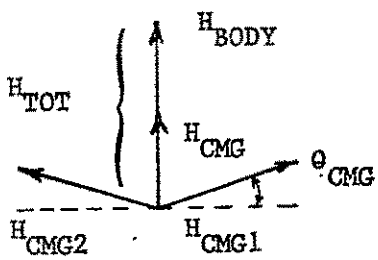


Figure 11 Spin Control Time Response

CMGs were positioned on body 0 so that there respective momentum vectors in the null position are opposite each other and perpendicular to the momentum vector of the body as is shown in the sketch below:





Clearly,

$$H_{TOT} = H_{BODY} + H_{CMG} \tag{5-10}$$

$$H_{CMG} = 2H_{CMG1}\sin\,\theta_{CMG} \tag{5-11}$$

Differentiation of equation (5-11) yields

$$\dot{H}_{CMG} = 2H_{CMG1}\cos\,\theta_{CMG}\,\dot{\theta}_{CMG} \tag{5-12}$$

As a control law for equation 5-12 let

$$\theta_{CMG} = g_4\theta_{0x} + g_5\theta_{0x} \tag{5-13}$$

Using the uncoupled form of Euler's equation and assuming that the only torque acting on body 0 is that due to the CMGs the following equation results

$$I_{0x}\ddot{\theta}_{0x} = 2H_{CMG}\cos\theta_{CMG}\,(g_4\dot{\theta}_{0x} + g_5\theta_{0x}) \qquad 5\text{-}14$$

For small CMG angles $\cos\theta_{CMG} \approx 1$ and equation (5-14) degenerates to

$$I_{0x}\theta_{0x} = 2H_{CMG}\ \ g_4\dot{\theta}_{0x} + g_5\theta_{0x} \qquad 5\text{-}15$$

Equation (5-15) can be used to select the constants g_4 and g_5 so that the dynamic response of θ_{0x} is satisfactory. Using values for g_4, g_5, H_{CMG1}, I_{0x} is shown below in Table 3 a computer run was initiated whose results are shown in figure 12

Table 3

Constant	Value
g_4	-1
g_5	-1
H_{CMG1}	7.0×10^6 ft-lb-sec
I_{0x}	1.4×10^7 slug ft^2

Figure 12 shows how the CMGs repositions the angle θ_{0x} from

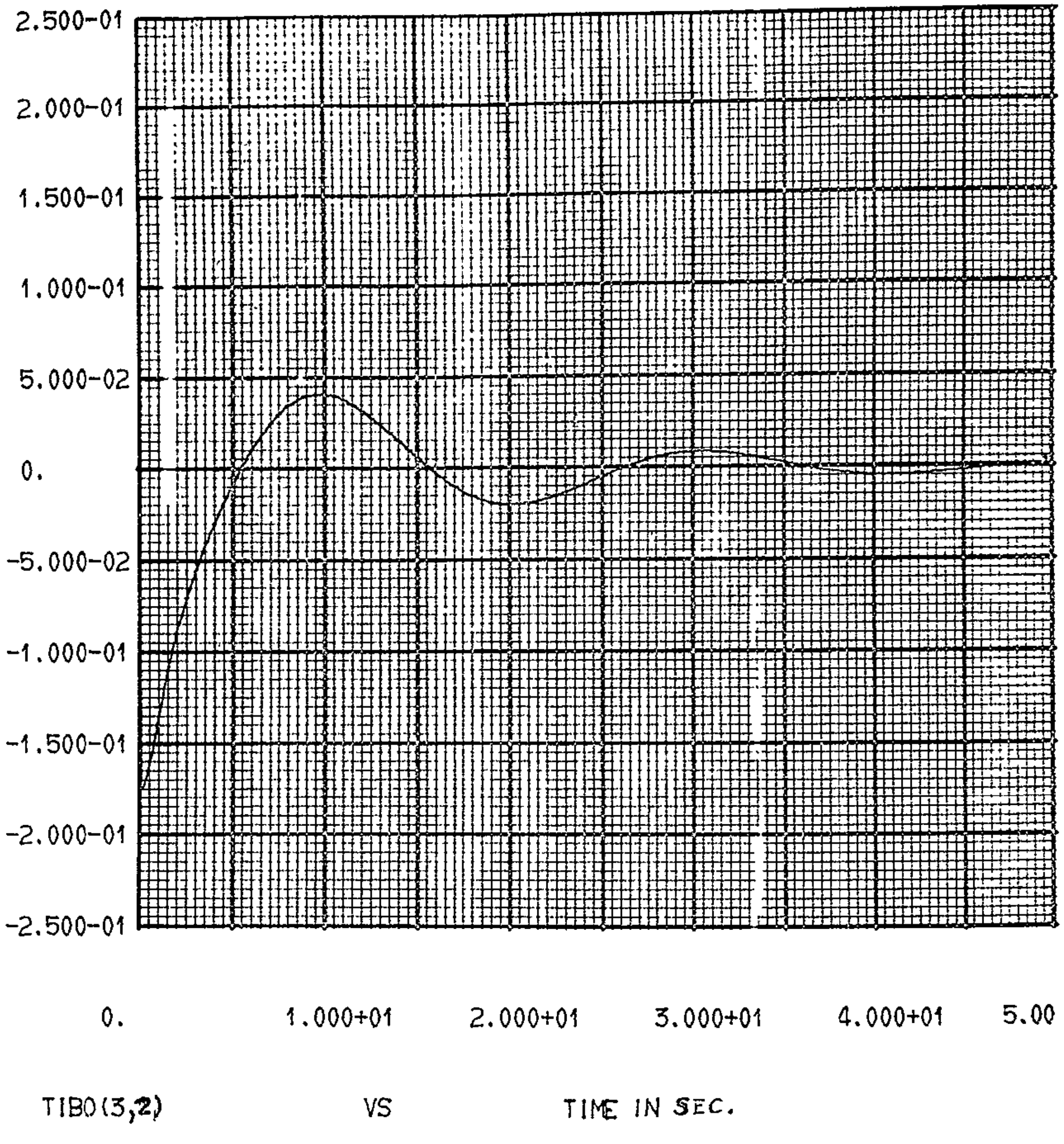


Figure 12 Attitude Hold Time Response

and initial offset value of θ_{0x} = .175 radians to the desired value of $\theta_{0x} = 0$ radians. This is simply a representative run showing how CMGs can be used to reposition or reorient the attitude of body 0.

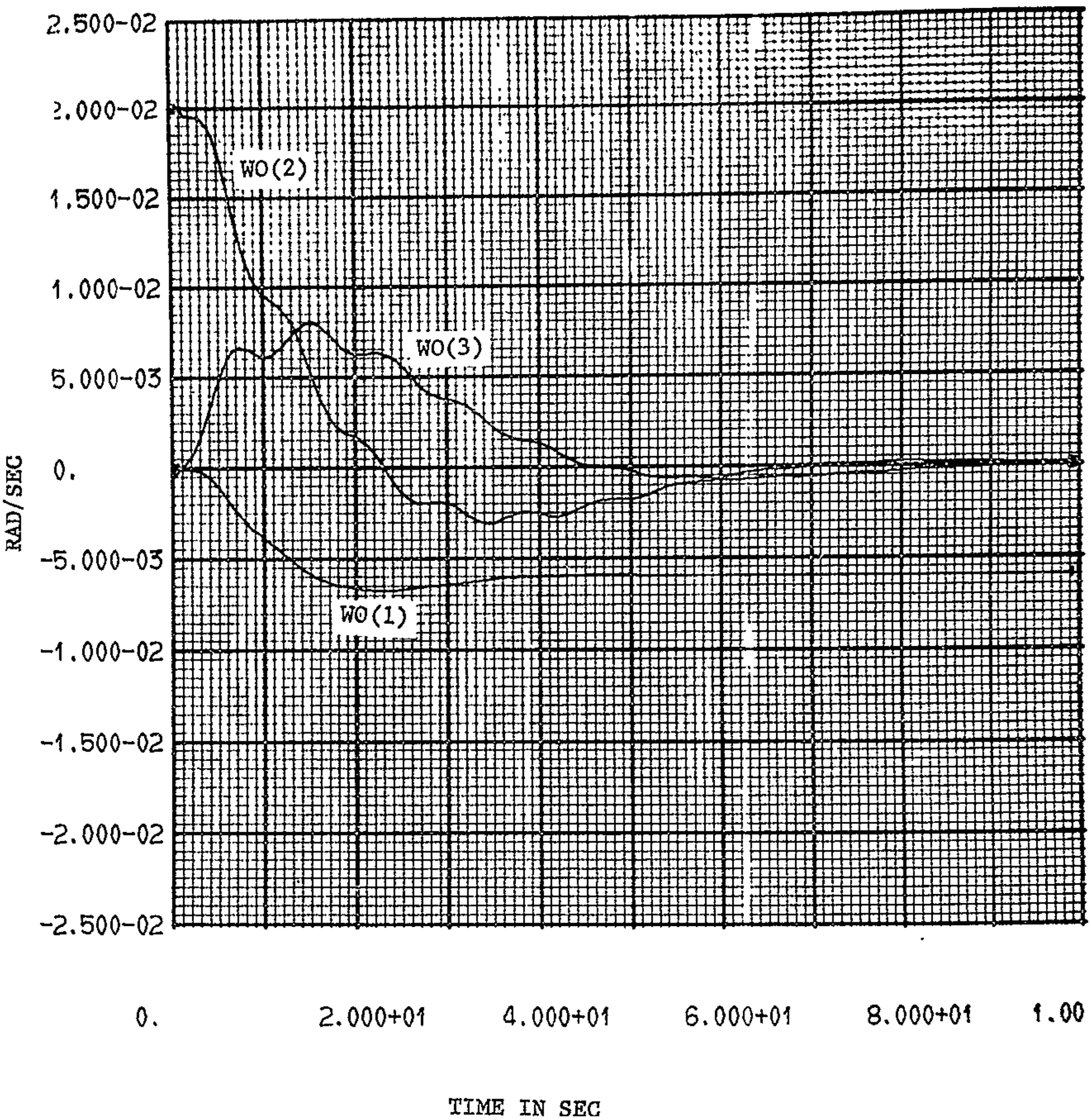
C. Propulsion Control

A demonstration of transverse angular rate reduction by use of reaction jets was made for the standard space station configuration (Table 2). Control gains were - 2 x 10^5 and jet couple arm lengths were 200 ft.

With jets on the stator an initial transverse rate magnitude was reduced by a factor of four in approximately 30 sec. (Figure 13) The transverse rate reduction by stator jets was in good agreement with results predicted by the formula given in the User's Manual description of PCON.

The effect of an identical set of rotor reaction jets is shown in Figure 14 . The transverse rate magnitude was reduced by a factor of four in 100 sec. The rotor jet action is slower because the rotor has jets on only one transverse axis, which due to rotation is at times not oriented properly for effective control. In terms of total impulse required for a given reduction in transverse rates, however, the effectiveness of rotor and stator jets was about the same.

In Figure 13 and Figure 14 it is seen that the stator acquired a small spin axis rate (WO(1)) during the control action. A corresponding change in spin rate of the rotor also occurred. These rates could have been corrected by spin axis jets. Spin axis control was not applied in these runs in order that transverse control comparisons would not be influenced by spin axis control effects.



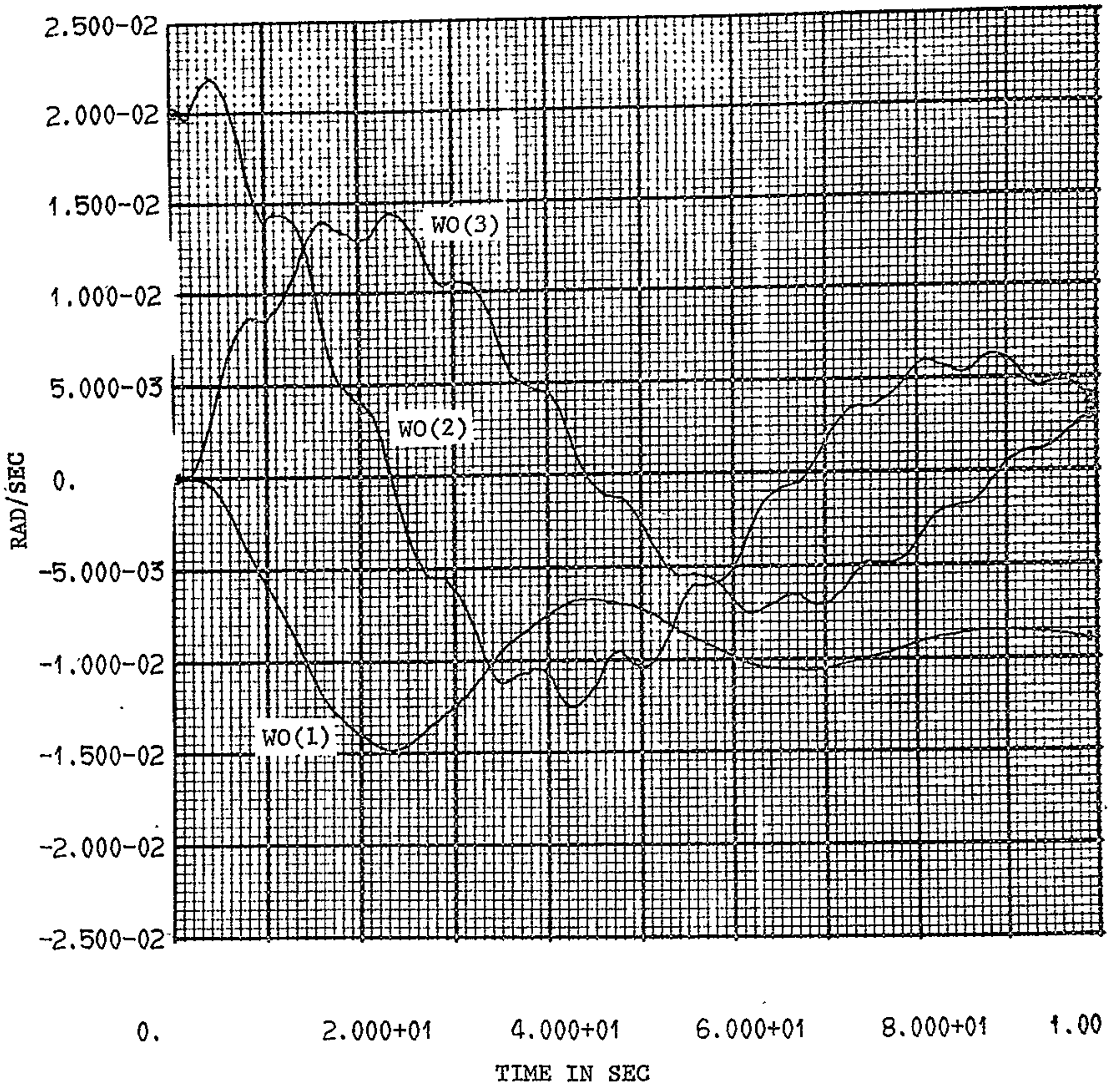
DELTAT = .5 SEC
TIBOI(1,1) = TIBOI(2,2) = TIBOI(3,3) = 1.0
WO(1) = 0   WO(2) = .02 RAD/SEC   WO(3) = 0   (INITIAL)
OMEGA1 = .4 RAD/SEC   (INITIAL)

SEE ALSO TABLE 2

AOJ(1) = 10	CGAIN0(1) = 0
AOJ(2) = 200	CGAIN0(2) = -10^5
AOJ(3) = 200	CGAIN0(3) = -10^5
A1J(1) = 10	CGAIN1(1) = 0
A1J(2) = 10	CGAIN1(2) = 0

Figure 13 PROPULSION CONTROL JETS ON BODY 0

2



DELTAT = .5 SEC

TIBOI(1,1) = TIBOI(2,2) = TIBOI(3,3) = 1.0

WO(1) = 0 WO(2) = .02 RAD/SEC WO(3) = 0 (INITIAL)

OMEGA1 = .4 RAD/SEC (INITIAL)

SEE ALSO TABLE 2

AOJ(1) = 10 CGAINO(1) = 0
AOJ(2) = 10 CGAINO(2) = 0
AOJ(3) = 10 CGAINO(3) = 0
A1J(1) = 10 CGAIN1(1) = 0
A1J(2) = 200 FT CGAIN1(2) = -10^5

Figure 14 PROPULSION CONTROL

JETS ON BODY 1

D. Attitude Control by Reaction Jets

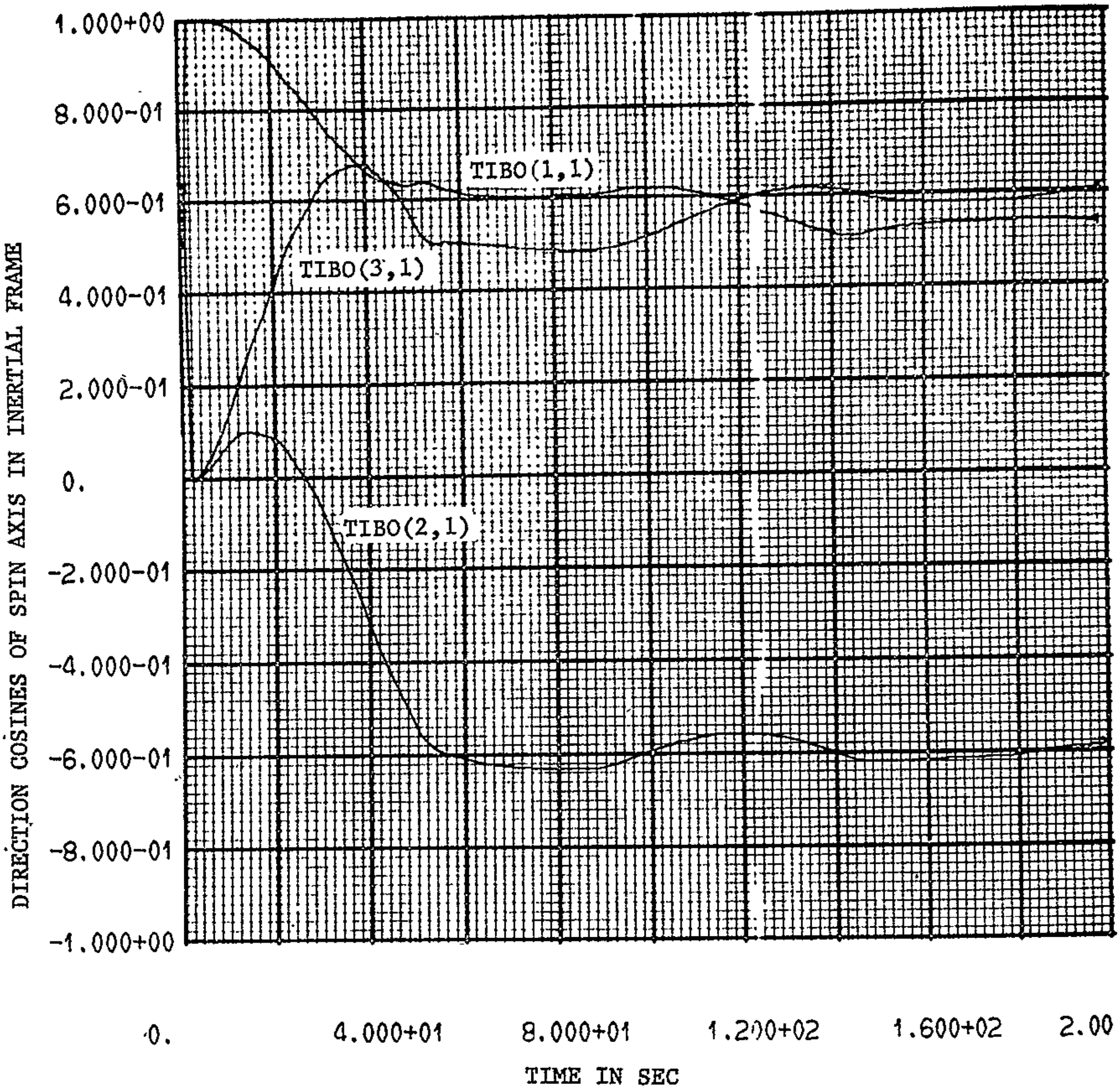
The performance of the reaction jet attitude control function is indicated in Figure 15. Initially the spin axis is aligned with the inertial x-axis (TIBO(1,1) = 1.0, TIBO(2,1) = 0, TIBO(3,1) = 0.) The direction to which the spin axis is to be reoriented has the directions in inertial space CA(1) = .5, CA(2) = - .612, CA(3) = .612.

The first attitude correction segment was completed in about 54 sec. The stator x-axis was rotated to a direction given by the direction cosines TIBO(1,1) = .63, TIBO(2,1) = - .6, TIBO(3,1) = .5. From 54 sec to 90 sec residual transverse rates were removed, and the spin axis direction changed only slightly.

From 90 sec to 144 sec the second correction process occurre The spin axis arrived at a direction in inertial space given by the direction cosines TIBO(1,1) = .52, TIBO(2,1) = - .62, TIBO(3,1) = .6. Subsequent removal of transverse rates occurred until about 174 sec.

A third correction maneuver was initiated, but was not completed by the end of the run.

At the end of the 200 sec run the new direction of the spin axis was within 3^{o} of the specified reorientation direction.



DELTAT = 1.0 SEC
TIBOI(1,1) = TIBOI(2,2) = TIBOI(3,3) = 1.0
WO(M) = 0. (INITIAL)
OMEGA1 = .4 RAD/SEC (INITIAL)
CA(1) = .5 CA(2) = - .612 CA(3) = .612

AOJ(1) = 10	CGAINO(1) = 0
AOJ(2) = 200 FT	CGAINO(2) = - 2 x 10^5
AOJ(3) = 200 FT	CGAINO(3) = - 2 x 10^5
A1J(1) = 10	CGAIN1(1) = 0
A1J(2) = 10	CGAIN1(2) = 0

SEE ALSO TABLE 2

Figure 15 REACTION JET ATTITUDE CONTROL

instability. An initial transverse angular velocity magnitude of .02 rad/sec increased to more than .03 rad/sec in 150 sec. Pendulums on the stator had a definite nutation damping effect. The transverse rate magnitude decreased from .02 rad/sec to .018 rad/sec in 150 sec of real time.

These studies lead to the following conclusions:

1. Reasonably sized pendulum nutation dampers will not be very effective.
2. This program is not an efficient means of studying pendulum nutation dampers unless they are of impractically large proportions.
3. The inertial configuration proposed is unstable with nutation damping pendulums on the rotor.

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APPENDIX A

THE QUATERNION SOLUTION OF THE BODY TO INERTIAL COORDINATE TRANSFORMATION

Consider two sets of coordinates systems whose axes are denoted by X_i, Y_i, Z_i and X_b, Y_b, Z_b. Initially the two sets of axes are coincident. By Euler's theorem any rotation of the X_b, Y_b, Z_b system may be expressed as a rotation through some angle, about some axis. Suppose then that the X_b, Y_b, Z_b set is rotated through an angle (the positive direction of rotation is given by the right hand rule) about an axis whose direction cosines are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ relative to the X_i, Y_i, Z_i set. Let x_b, y_b, z_b be the X_b, Y_b, Z_b coordinates of a vector and let x_i, y_i, z_i be the X_I, Y_I, Z_I coordinates of the same vector. The transformation matrix A of Equation A-1 is required.

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = A \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \tag{A-1}$$

It can be shown[1] that A is given by:

$$A = \begin{bmatrix} 1-2s^2\frac{\mu}{2}s^2\alpha & 2(s^2\frac{\mu}{2}c\alpha c\beta - s\frac{\mu}{2}c\frac{\mu}{2}c\gamma) & 2(c\alpha c\gamma s^2\frac{\mu}{2} + s\frac{\mu}{2}c\frac{\mu}{2}c\beta) \\ 2(s^2\frac{\mu}{2}c\alpha c\beta + s\frac{\mu}{2}c\frac{\mu}{2}c\gamma) & 1-2s^2\frac{\mu}{2}s^2\beta & 2(s^2\frac{\mu}{2}c\beta c\gamma - s\frac{\mu}{2}c\frac{\mu}{2}c\alpha) \\ 2(c\alpha c\gamma s^2\frac{\mu}{2} - s\frac{\mu}{2}c\frac{\mu}{2}c\beta) & 2(s^2\frac{\mu}{2}c\beta c\gamma + s\frac{\mu}{2}c\frac{\mu}{2}c\alpha) & 1-2s^2\frac{\mu}{2}s^2\gamma \end{bmatrix} \tag{A-2}$$

A. C. Robinson, "On the Use of Quaternions in Simulation of Rigid Body Motion", TR58-17, Wright Air Development Center, Wright-Patterson AFB, Ohio, March 1960, pp 5-7.

In order to introduce the quaternion make the following substitutions:

$$e_0 = c\,\frac{\mu}{2},\ e_1 = c\alpha s\,\frac{\mu}{2},\ e_2 = c\beta s\,\frac{\mu}{2},\ e_3 = c\gamma s\,\frac{\mu}{2} \qquad \text{(A-3)}$$

The A matrix becomes

$$A = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2 & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \qquad \text{(A-4)}$$

The quaternion was invented by Hamilton in 1843. The quaternion q is composed of four parts,

$$q = q_0 + i\,q_1 + j\,q_2 + k\,q_3$$

where q_0, q_1, q_2 and q_3 are real numbers and i, j, and k are hyperimaginary numbers satisfying the conditions

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

The conjugate of the quaternion q is:

$q^* = q_0 - i\,q_1 - j\,q_2 - k\,q_3$. The magnitude of q is defined as $|q| = (q^*q)$

Now that the quaternion is defined its use in coordinate transformations will be developed. Define a quaternion V_b by

$$V_b = i\,x_b + j\,y_b + k\,z_b.$$

Now consider the quaternion product $q\,V_b q^*$ where $q = e_0 + ie_1 + je_2 + ke_3$. e_0, e_1, e_2, e_3 are defined by Equation A-3. Then $q\,V_b q^* = (e_0 + ie_1 + je_2 + ke_3)(ix_b + jy_b + kz_b)(e_0 - ie_1 - je_2 - ke_3)$. When this expression is expanded using the defined quaternion operations the following is obtained

$$\begin{aligned} q\,V_B q^* = {} & \left[(e_0^2 + e_1^2 - e_2^2 - e_3^2)\,x_b + 2(e_1e_2 - e_0e_3)\,y_b + 2(e_1e_3 - e_0e_2)\,z_b\right] i \\ & + \left[2(e_1e_2 + e_0e_2)\,x_b + (e_0^2 - e_1^2 + e_2^2 - e_3^2)\,y_b + 2(e_2e_3 - e_0e_1)\,z_b\right] j \\ & + \left[2(e_1e_3 - e_0e_2)\,x_b + 2(e_2e_3 + e_0e_1)\,y_b + (e_0^2 - e_1^2 - e_2^2 + e_3^2)\,z_b\right] k \end{aligned} \quad \text{(A-5)}$$

Now if the A matrix of Equation A-4 is substituted into Equation A-1 and the indicated multiplication is carried out, the following expression is obtained

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} (e_0^2 + e_1^2 - e_2^2 - e_3^2)\,x_b + 2(e_1e_2 - e_0e_3)\,y_b + 2(e_1e_3 + e_0e_2)\,z_b \\ 2(e_1e_2 + e_0e_3)\,x_b + (e_0^2 - e_1^2 + e_2^2 - e_3^2)\,y_b + 2(e_2e_3 - e_0e_1)\,z_b \\ 2(e_1e_3 - e_0e_2)\,x_b + 2(e_2e_3 + e_0e_1)\,y_b + (e_0^2 - e_1^2 - e_2^2 + e_3^2)\,z_b \end{bmatrix} \quad \text{(A-6)}$$

The x_i, y_i, x_i of Equation A-6 are identical to the coefficients of i, j and k in Equation A-5. Hence, the quaternion multiplication qV_Bq^* accomplishes the same transformation as Equation A-1 as long as

$$e_0 = \cos\frac{\mu}{2},\ e_1 = \cos\alpha \sin\frac{\mu}{2},\ e_2 = \cos\beta \sin\frac{\mu}{2}, = \cos\gamma \sin\frac{\mu}{2}.$$

Define the quaternion V_I by

$$V_I = ix_i + jy_i + kz_i.$$

Then

$$V_I = qV_Bq^* = qV_Bq^*$$

Now consider two successive rotations of the X_b, Y_b, Z_b system. Let the first rotation be defined by q_1. Then suppose the X_b, Y_b, Z_b system is again rotated. Let this rotation be defined by q_2 and x_b', y_b', z_b' denote the final position of the moving axes. Note that q_1 must be referenced to X_I, Y_I, Z_I and q_2 must be referenced to X_b, Y_b, Z_b. Let x_i, y_i, z_i; x_b, y_b, z_b; x_b', y_b', z_b' be the X_I, Y_I, Z_I; X_b, Y_b, Z_b; X_b', Y_b', Z_b'; coordinates of the same vector. Then define

$$V_I = ix_i + jy_i + kz_i$$

$$V_B = ix_b + jy_b + kz_b$$

$$V_B' = ix_b' + jy_b' + kz_b'.$$

So that

$$V_I = q_1V_Bq_1^*$$

$$V_B = q_2V_B'q_2^*$$

or

$$V_I = q_1q_2V_B'q_2^*q_1^* = q_1q_2V_B(q_1q_2)^*.$$

Thus the quaternion which defines the transformation from the X_b', Y_b', Z_b' system to the X_1, Y_1, Z_1 system is q_1q_2.

Suppose the angular velocity of the rotating coordinate frame is known in terms of the components of angular velocity along the rotating coordinates. Let P, Q, R be the X_b, Y_b, Z_b components of the angular velocity. Let q(t) define the relationship between the rotating and the fixed coordinates at time t. During time t to t + Δt suppose the moving set undergoes a rotation of $\Delta\mu$ about an axis which has direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ with respect to the moving frame. The quaternion $q(t + \Delta t)$ which defines the transformation between the fixed and moving frames at time $t + \Delta t$ is required.

Now the transformation between the rotating frame at time t and the rotating frame at time $t + \Delta t$ is

$$q_\Delta = \cos \frac{\Delta\mu}{2} + \sin \frac{\Delta\mu}{2} (i\cos\alpha + j\cos\beta + k\cos\gamma).$$

Since q(t) and q_Δ define successive rotations, $q(t + \Delta t)$ must be

$$q(t + \Delta) = q(t)q_\Delta.$$

Then

$$q(t + \Delta t) - q(t) = q(t)q_\Delta - q(t)$$

or

$$q(t + \Delta t) - q(t) = q(t) \left[\cos \frac{\Delta\mu}{2} + \sin \frac{\Delta\mu}{2} (i\cos\alpha + j\cos\beta + k\cos\gamma)\right] - q(t)$$

As $\Delta t \rightarrow 0$, $\Delta\mu \rightarrow 0$ so $\cos \frac{\Delta\mu}{2}$ and $\sin \frac{\Delta\mu}{2} \rightarrow \frac{\Delta\mu}{2}$

$$q(t + \Delta t) - q(t) = q(t) + q(t) \frac{\Delta}{2} (i\cos\alpha + j\cos\beta + k\cos\gamma) - q(t)$$

or

$$\frac{dq(t)}{dt} = \lim_{\Delta t \to 0} \frac{q(t + \Delta t) - q(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{q(t)}{2} \frac{\Delta\mu}{\Delta t} (i\cos\alpha + j\cos\beta + k\cos\gamma).$$

$$\dot{q}(t) = \frac{1}{2} q(t) \ (iP + jQ + kR).$$

So that

$$\dot{e}_0 + i \ _1 + j \ _2 + k \ _3 = \frac{1}{2} (e_0 + ie_1 + je_2 + ke_3) \ (iP + jQ + kR)$$

$$= \frac{1}{2} \ ie_0P + je_0Q + ke_0R - e_1P + ke_1Q$$

$$- je_1R - ke_2P - e_2Q + ie_2R + je_3P - ie_3Q - e_3R$$

$$= -\frac{1}{2} (e_1P + e_2Q + e_3R) + \frac{1}{2} (e_0P + e_2R - e_3Q)i$$

$$+ \frac{1}{2} (e_0Q - e_1R + e_3P)j + (e_0R + e_1Q - e_2P)k$$

Then the differential equations for the four components of the quaternion are:

$$\dot{e}_0 = -\frac{1}{2} (e_1P + e_2Q + e_3R)$$

$$\dot{e}_1 = \frac{1}{2} (e_0P + e_2R - e_3Q)$$

$$\dot{e}_2 = \frac{1}{2} (e_0Q - e_1R + e_3P)$$

$$\dot{e}_3 = \frac{1}{2} (e_0R + e_1Q - e_2P)$$

The initial conditions are $e_0(o) = 1$, $e_1(o) = e_2(o) = e_3(o) = 0$, since the moving and inertial system are coincident at $t = 0$.

Thus, only four linear differential equations need be solved to find the components of the quaternion. The A matrix can then be determined from the components of the quaternion. Referring to Equation A-4

$$a_{11} = e_0^2 + e_1^2 - e_2^2 - e_3^2$$

$$a_{12} = 2(e_1e_2 - e_0e_3)$$

$$a_{13} = 2(e_1e_3 + e_0e_2)$$

$$a_{21} = 2(e_1e_2 + e_0e_3)$$

$$a_{22} = e_0^2 - e_1^2 + e_2^2 - e_3^2$$

$$a_{23} = 2(e_2e_3 - e_0e_1)$$

$$a_{31} = 2(e_1e_3 - e_0e_2)$$

$$a_{32} = 2(e_2e_3 + e_0e_1)$$

$$a_{33} = e_0^2 - e_1^2 - e_2^2 + e_3^2$$

The A matrix computed from the quaternion components will be orthogonal if and only if the quaternion has a magnitude of unity. From the definition of the components of q(t) the magnitude of q(t) should always be unity. However, the numerical intergration of the differential equations for q(t) introduce errors which cause the magnitude of q(t) to differ from unity. q(t) may be kept at unit magnitude by normalization at each intergration step. Normalization is accomplished by dividing each component of q(t) by the square root of the sum of the squares of the components.

APPENDIX B

DERIVATION OF DYNAMICS EQUATIONS

1. The Space Station Configuration

The equations of motion governing the dynamics of a dual spin space station will be derived by the momentum techniques as exploited by Russell [1,2] for an N-body spacecraft. The space station configuration is shown on Figure 1 and consists of a despun (stator) body and a rotating (rotor) body which contains two inverted pendulums and an elevator. Active control will be employed between the rotor and the stator to maintain the correct spin rate of the rotor and overcome the frictional torques between the rotor and the stator. The attitude and nutation control will be achieved by control moment gyros in the stator and a suitable combination of thrusters on both the rotor and the stator

The variables used in the formulation of the dynamics of the space station are defined on Figure 1, and with a few exceptions is basically the symbolism adopted by Russell loc. cit. The major assumption involved in the formulation of the dynamics of the space station is that the two pendulums and elevator are point masses, the inference being that they do not have any moments of inertia. In these cases the basic rotational dynamics for each point mass body is replaced by a static torque balance which results in considerable simplification of the dynamics.

2. Center of Mass Equations

From the definition of the center of mass it follows that

$$m_0\bar{r}_0 + m_1\bar{r}_1 + m_2\bar{r}_2 + m_3\bar{r}_3 + m_4\bar{r}_4 = 0 \qquad \text{B2.1}$$

The geometry of the multiple connected rigid body (see Figure 1) yields:

$$\bar{r}_1 = \bar{r}_0 + \bar{d}_{01}$$

$$\bar{r}_2 = \bar{r}_0 + \bar{d}_{01} + \bar{\ell}_2$$

$$\bar{r}_3 = \bar{r}_0 + \bar{d}_{01} + \bar{d}_{13} + \bar{\ell}_3 \qquad \text{B2.2}$$

$$\bar{r}_4 = \bar{r}_0 + \bar{d}_{01} + \bar{d}_{14} + \bar{\ell}_4$$

Using equations B2.2 in equation B2.1 determines $\bar{r}_0$ in terms of the multiple body vectors as follows,

$$m_0\bar{r}_0 + m_1(\bar{r}_0 + \bar{d}_{01}) + m_2(\bar{r}_0 + \bar{d}_{01} + \bar{\ell}_2) + m_3(\bar{r}_0 + \bar{d}_{01} + \bar{d}_{13} + \bar{\ell}_3)$$
$$+ m_4(\bar{r}_0 + \bar{d}_{01} + \bar{d}_{14} + \bar{\ell}_4) = 0.$$

Defining $m = m_0 + m_1 + m_2 + m_3 + m_4$ the above result can be reduced to:

$$m\bar{r}_0 + (m-m_0)\ \bar{d}_{01} + m_2\bar{\ell}_2 + m_3\ (\bar{d}_{13} + \bar{\ell}_3) + m_4\ (\bar{d}_{14} + \bar{\ell}_4) = 0. \qquad \text{B2.3}$$

Substituting this result into equations B2.2 yields the local center of mass vectors from the total center of mass position in terms of the multiple body vectors.

$$\bar{r}_1 = \frac{m_0}{m}\, d_{01} - \frac{m_2}{m}\bar{\ell}_2 - \frac{m_3}{m}(\bar{d}_{13} + \bar{\ell}_3) - \frac{m_4}{m}(\bar{d}_{14} + \bar{\ell}_4)$$

$$\bar{r}_2 = \frac{m_0}{m}\,\bar{d}_{01} + (1 - \frac{m_2}{m})\bar{\ell}_2 - \frac{m_3}{m}(\bar{d}_{13} + \bar{\ell}_3) - \frac{m_4}{m}(\bar{d}_{14} + \bar{\ell}_4) \qquad \text{B2.4}$$

$$\bar{r}_3 = \frac{m_0}{m}\,\bar{d}_{01} - \frac{m_2}{m}\bar{\ell}_2 + (1 - \frac{m_3}{m})(\bar{d}_{13} + \bar{\ell}_3) - \frac{m_4}{m}(\bar{d}_{14} + \bar{\ell}_4)$$

$$\bar{r}_4 = \frac{m_0}{m}\,\bar{d}_{01} - \frac{m_2}{m}\bar{\ell}_2 - \frac{m_3}{m}(\bar{d}_{13} + \bar{\ell}_3) + (1 - \frac{m_4}{m})(\bar{d}_{14} + \bar{\ell}_4)$$

3. <u>Rectilinear Dynamics</u>

The rectilinear dynamics for the five body cluster depicted on Figure 1 are determined by the following vector differential equations (Newton's Law).

$$m_0(\ddot{\overline{R}} + \ddot{\overline{r}}_0) = \sum_{\alpha} \overline{F}_{0\alpha} + \overline{F}_{1\rightarrow 0}$$

$$m_1(\ddot{\overline{R}} + \ddot{\overline{r}}_1) = \sum_{\beta} \overline{F}_{1\beta} + \overline{F}_{0\rightarrow 1} + \overline{F}_{2\rightarrow 1} + \overline{F}_{3\rightarrow 1} + \overline{F}_{4\rightarrow 1}$$

$$m_2(\ddot{\overline{R}} + \ddot{\overline{r}}_2) = \overline{F}_{1\rightarrow 2} \qquad \text{B3.1}$$

$$m_3(\ddot{\overline{R}} + \ddot{\overline{r}}_3) = \overline{F}_{1\rightarrow 3}$$

$$m_4(\ddot{\overline{R}} + \ddot{\overline{r}}_4) = \overline{F}_{1\rightarrow 4}$$

The external forces acting on the stator (0 body) and the rotor (1st body) are denoted by $\overline{F}_{0\alpha}$ and $\overline{F}_{1\beta}$ respectively. The inter-body forces are represented by $\overline{F}_{\mu\rightarrow\gamma}$, which is interpreted as the the force due to the μ^{th} body acting on the γ^{th} body. Since every force has an equal and opposite reaction this yields the skew relationship.

$$\overline{F}_{\mu\rightarrow\gamma} = -\overline{F}_{\gamma\rightarrow\mu}$$

The vector sum of equations B3.1 yields

$$m\ddot{\overline{R}} + (m_0\ddot{\overline{r}}_0 + m_1\ddot{\overline{r}}_1 + m_2\ddot{\overline{r}}_2 + m_3\ddot{\overline{r}}_3 + m_4\ddot{\overline{r}}_4) = \sum_{\alpha} \overline{F}_{0\alpha} + \sum_{\beta} \overline{F}_{1\beta}$$

which reduces to

$$m\ddot{\bar{R}} = \sum_{\alpha}\bar{F}_{0\alpha} + \sum_{\beta}\bar{F}_{1\beta} \qquad \text{B3.2}$$

by virtue of the center of mass definition (B2.1).

The primary purpose of equations B3.1 is that they are used to eliminate the unknown interbody forces which appear in the rotational dynamics.

4. Rotational Dynamics

The rotational dynamics for the five body cluster are governed by the following vector differential equations,

$$\dot{\bar{h}}_0 = \sum_{\alpha}\bar{a}_{0\alpha} \times \bar{F}_{0\alpha} + \bar{d}_{01} \times \bar{F}_{1\to 0} + \bar{T}_{1\to 0} \qquad \text{B4.1}$$

$$\dot{\bar{h}}_1 = \sum_{\beta}\bar{a}_{1\beta} \times \bar{F}_{1\beta} + \bar{d}_{13} \times \bar{F}_{3\to 1} + \bar{d}_{14} \times \bar{F}_{4\to 1}$$
$$+ \bar{\ell}_2 \times \bar{F}_{2\to 1} + \bar{T}_{0\to 1} + \bar{T}_{3\to 1} + \bar{T}_{4\to 1}. \qquad \text{B4.2}$$

The angular momentum vectors for the 0^{th} body and the 1^{st} body are denoted by $\bar{h}_0$ and $\bar{h}_1$ respectively. Using the same notation as for the interbody forces, the interbody torques are represented by $\bar{T}_{\mu\to\gamma}$, with the equivalent relationship $\bar{T}_{\mu\to\gamma} = - \bar{T}_{\gamma\to\mu}$. The torques due to gravity gradient will be represented by appropriate external forces acting on the bodies of the cluster.

Since we are dealing with point masses for the inverted pendulums, the rotational dynamics degenerate into a static balance of torques.

$$\bar{T}_{1\rightarrow 3} = \bar{\ell}_3 \times \bar{F}_{1\rightarrow 3}$$

$$\bar{T}_{1\rightarrow 4} = \bar{\ell}_4 \times \bar{F}_{1\rightarrow 4} \qquad \text{B4.3}$$

The total angular momentum for the cluster of five bodies is defined by

$$\bar{H} = \bar{h}_0 + \bar{h}_1 + \bar{r}_0 \times m_0\dot{\bar{r}}_0 + \bar{r}_1 \times m_1\dot{\bar{r}}_1 + \bar{r}_2 \times m_2\dot{\bar{r}}_2$$
$$+ \bar{r}_3 \times m_3\dot{\bar{r}}_3 + \bar{r}_4 \times m_4\dot{\bar{r}}_4 \qquad \text{B4.4}$$

and is governed by the differential (Euler's) equation

$$\dot{\bar{H}} = \sum_{\alpha}(\bar{r}_0 + \bar{a}_{0\alpha}) \times \bar{F}_{0\alpha} + \sum_{\beta}(\bar{r}_1 + \bar{a}_{1\beta}) \times \bar{F}_{1\beta} \qquad \text{B4.5}$$

The rotational dynamics of the space station will be governed by the unconstrained components of the vector equations B3.1.(3), B4.2, B4.3 and B4.5. Physically the dynamics of the stator and rotor are given by equations B4.2 and B4.5; and equation B4.5 is chosen rather than equation B4.1 since it avoids some of the constraint problems between the stator and the rotor. The effects of the elevator on the dynamics of the space station are inherent in the interbody forces of equation B3.1(3). The unconstrained component of this equation can be used to control the motion of the elevation, however, for this study we shall ignore the elevator control problem and determine the effects of prescribed elevator motions on the stability of the space station. Finally, the effects of the inverted pendulums are determined by equation B4.3.

For these equations the unknown interbody forces have to be eliminated and the equations for the unconstrained components recast into normal form suitable for numerical integration. This basically is the momentum technique.

5. <u>The Momentum Method</u>

The unknown interbody forces are eliminated from the appropriate rotational dynamic equations by means of the rectilinear dynamic equations B3.1. Treating equation B4.2 first, and eliminating the interbody torques $\overline{T}_{3\rightarrow 1}$ and $\overline{T}_{4\rightarrow 1}$ in addition to the interbody forces yields

$$\begin{aligned}\dot{\bar{h}}_1 = &- \ell_2 \times \ddot{\bar{r}}_2 + m_3(d_{13} + \ell_3) \times \ddot{\bar{r}}_3 - m_4(d_{14} + \ell_4) \times \ddot{\bar{r}}_4 \\ &- \left[m_2\ell_2 + m_3(d_{13} + \ell_3) + m_4(d_{14} + \ell_4)\right] \times \Big(\sum_{\alpha} \frac{F_{0\alpha}}{m} \\ &+ \sum_{\beta} \frac{F_{1\beta}}{m}\Big) + \sum a_{1\beta} \times F_{1\beta} + T_{01}\end{aligned} \qquad \text{B5.1}$$

If we define

$$\bar{h}_1' = \bar{h}_1 + m_2\ell_2 \times \dot{\bar{r}}_2 + m_3(d_{13} + \ell_3) \times \dot{\bar{r}}_3 + m_4(d_{14} + \ell_4) \times \dot{\bar{r}}_4 \qquad \text{B5.2}$$

which represents the angular momentum of the rotor, elevator and inverted pendulums about the hinge point between the rotor and stator, then equation B5.1 can be put into normal form as

$$\frac{dh_1'}{dt} = m_2\dot{\bar{\ell}}_2 \times \dot{\bar{r}}_2 + m_3(\dot{\bar{d}}_{13} + \dot{\bar{\ell}}_3) \times \dot{\bar{r}}_3 + m_4(\dot{\bar{d}}_{14} - \dot{\ell}_4) \times \dot{\bar{r}}_4$$

$$- [m_2\ell_2 + m_3(\bar{d}_{13} + \bar{\ell}_3) + m_4(d_{14} + \ell_4)] \times (\sum_\alpha \frac{F_{0\alpha}}{m}$$

$$+ \sum_\beta \frac{F_{1\beta}}{m}) + \sum a_{1\beta} \times F_{1\beta} + T_{0\rightarrow 1} \qquad \text{B5.3}$$

Equation B4.3(1) for the inverted pendulum reduces to

$$\bar{T}_{1\rightarrow 3} = \bar{\ell}_3 \times m_3(\ddot{\bar{R}} + \ddot{\bar{r}}_3)$$

$$= \bar{\ell}_3 \times m_3\ddot{\bar{r}}_3 + \bar{\ell}_3 \times \frac{m_3}{m}(\sum_\alpha \bar{F}_{0\alpha} + \sum_\beta \bar{F}_{1\beta})$$

Defining

$$\bar{h}_3' = \bar{\ell}_3 \times m_3\dot{\bar{r}}_3 \qquad \text{B5.4}$$

reduces the pendulum equation to

$$\frac{d\bar{h}_3'}{dt} = \dot{\bar{\ell}}_3 \times m_3\dot{\bar{r}}_3 - \bar{\ell}_3 \times \frac{m_3}{m}\left(\sum_{\alpha} \bar{F}_{0\alpha} + \sum_{\beta}\bar{F}_{1\beta}\right) + \bar{T}_{1\rightarrow 3} \qquad \text{B5.5}$$

Similarly by defining

$$\bar{h}_4' = \bar{\ell}_4 \times m_4\dot{\bar{r}}_4 \qquad \text{B5.6}$$

the second pendulum equation becomes

$$\frac{d\bar{h}_4'}{dt} = \dot{\bar{\ell}}_4 \times m_4\dot{\bar{r}}_4 - \bar{\ell}_4 \times \frac{m_4}{m}\left(\sum_{\alpha}\bar{F}_{0\alpha} + \sum_{\beta}\bar{F}_{1\beta}\right) + \bar{T}_{1\rightarrow 4} \qquad \text{B5.7}$$

Therefore, the unconstrained components of equations B5.3, B5.5 and B5.7 together with equation B4.5 represents the rotational dynamics for the space station. The total angular momentum differential equation will be expressed in the stator (0^{th} body) coordinate system, therefore equation B4.5 becomes

$$\frac{d\bar{H}}{dt} + \bar{\omega}_0 \times \bar{H} = \sum_{\alpha}(\bar{r}_0 + \bar{a}_0) \times \bar{F}_0 + \sum_{\beta}(\bar{r}_1 + \bar{a}_1) \times \bar{F}_1$$

If we substitute for $\bar{r}_0$ by equation 2.3 and $\bar{r}_1$ by equation 2.4 then the above reduces to

$$\frac{d\bar{H}}{dt} + \bar{\omega}_0 \times \bar{H} = \sum_{\alpha} [- \frac{(m-m_0)}{m} \bar{d}_{01} - \frac{m_2 \bar{\ell}_2}{m} - \frac{m_3(\bar{d}_{13} + \bar{\ell}_3)}{m} - \frac{m_4(\bar{d}_{14} + \bar{\ell}_4)}{m}$$

$$+ \bar{a}_{0\alpha}] \times \bar{F}_{0\alpha} + \sum_{\beta} [\frac{m_0}{m} \bar{d}_{01} - \frac{m_2 \bar{\ell}_2}{m} - \frac{m_3(\bar{d}_{13} + \bar{\ell}_3)}{m} - \frac{m_4(\bar{d}_{14} + \bar{\ell}_3)}{m}$$

$$+ \bar{a}_{1\beta}] \times \bar{F}_{1\beta} \qquad \text{B5.8}$$

Similarly the $\bar{h}_1'$ angular momentum and the inverted pendulum equations will be expressed in the rotor (body 1) coordinate system, so that equations B5.3, B5.5 and B5.7 become

$$\frac{d\bar{h}_1'}{dt} + \boldsymbol{\omega}_1 \times \bar{h}_1' = m_2 \dot{\bar{\ell}}_2 \times \dot{\bar{r}}_2 + m_3(\dot{\bar{d}}_{13} + \dot{\bar{\ell}}_3) \times \dot{\bar{r}}_3 + m_4(\dot{\bar{d}}_{14} + \dot{\bar{\ell}}_4) \times \dot{\bar{r}}_4$$

$$- [m_2 \bar{\ell}_2 + m_3(\bar{d}_{13} + \bar{\ell}_3) + m_4(d_{14} + \ell_4)] \times (\sum_{\alpha} \frac{F_{0\alpha}}{m} + \sum_{\beta} \frac{F_{1\beta}}{m})$$

$$+ \sum a_{1\beta} \times F_{1\beta} + T_{0\rightarrow 1} \qquad \text{B5.9}$$

$$\frac{d\bar{h}_3'}{dt} + \bar{\omega}_1 \times \bar{h}_3' = \dot{\bar{\ell}}_3 \times m_3 \dot{\bar{r}}_3 - \bar{\ell}_3 \times \frac{m_3}{m} (\sum_{\alpha} \bar{F}_{0\alpha} + \sum_{\beta} \bar{F}_{1\beta}) + \bar{T}_{1\rightarrow 3} \qquad \text{B5.10}$$

and

$$\frac{d\bar{h}_4'}{dt} + \bar{\omega}_1 \times \bar{h}_4' = \dot{\bar{\ell}}_4 \times m_4 \dot{\bar{r}}_4 - \bar{\ell}_4 \times \frac{m_4}{m} (\sum_{\alpha} \bar{F}_{0\alpha} + \sum_{\beta} \bar{F}_{1\beta}) + \bar{T}_{1\rightarrow 4} \qquad \text{B5.11}$$

To develop the unconstrained components of these equations the relative orientations and kinematrics for the cluster of bodies must be defined.

Define $\bar{j} = m_2\bar{\ell}_2 + m_3(\bar{d}_{13} + \bar{\ell}_3) + m_4(\bar{d}_{14} + \bar{\ell}_4)$

Then B5.8 can be written as:

$$\frac{d\bar{H}}{dt} + \omega_0 \times \bar{H} = \frac{1}{m}\left(- \frac{m-m_0}{m}\,\bar{d}_{01} - \bar{j}\right) \times \sum_{\alpha}\bar{F}_{0\alpha}$$

$$+ \frac{1}{m}\left(m_0\bar{d}_{01} - \bar{j}\right) \times \sum_{\beta}\bar{F}_{1\beta}$$

$$+ \sum_{\alpha}\bar{a}_{0\alpha} \times \bar{F}_{0\alpha} + \sum\bar{a}_{1\beta} \times \bar{F}_{1\beta} \qquad \text{B5.12}$$

and B5.9 can be written as:

$$\frac{d\bar{h}'_1}{dt} + \bar{\omega}_1 \times \bar{h}'_1 = m_2\dot{\bar{\ell}}_2 \times \dot{\bar{r}}_2 + m_3(\dot{\bar{d}}_{13} + \dot{\bar{\ell}}_3) \times \dot{\bar{r}}_3 + m_4(\dot{\bar{d}}_{14} + \dot{\bar{\ell}}_4) \times \dot{\bar{r}}_4$$

$$- \frac{1}{m}\bar{j}\left(\sum_{\alpha}F_{0\alpha} + \sum F_{1\beta}\right) + \sum a_{1\beta} \times F_{1\beta} + T_{0\rightarrow 1}$$

which can be written

$$\frac{d\bar{h}'_1}{dt} + \bar{\omega}_1 \times \bar{h}'_1 = \dot{\bar{r}}_1 \times \left[- m_2\dot{\bar{\ell}}_2 - m_3(\dot{\bar{d}}_{13} + \dot{\bar{\ell}}_3) - m_4(\dot{\bar{d}}_{14} + \dot{\bar{\ell}}_4)\right]$$

$$- \frac{1}{m}\bar{j}\left(\sum F_{0\alpha} + \sum F_{1\beta}\right) + \sum a_{1\beta} \times F_{1\beta} + T_{0\rightarrow 1} \qquad \text{B5.13}$$

Apply equations B5.10 and B5.11 to find the components along the unconstrained hinge lines of the pendulums.

Unconstrained component along s_3 axis

$$\frac{d}{dt}(h_{3y}s_{3y} + h_{3z}s_{3z}) + s_{3y}(\omega_{1z}h'_{3x} - \omega_{1x}h'_{3z}) + s_{3z}(\omega_{1x}h'_{3y} - \omega_{1y}h'_{3x})$$

$$= m_3 s_{3y}(\dot{\ell}_{3z}\dot{r}_{3x} - \dot{\ell}_{3x}\dot{r}_{3z}) + m_3 s_{3z}(\dot{\ell}_{3x}\dot{r}_{3y} - \dot{\ell}_{3y}\dot{r}_{3x})$$

$$+ \frac{m_3}{m} s_{3y}\left\{\ell_{3z}(F_{0x} + F_{1x}) - \ell_{3x}(-F_{0y}\sin\theta_1 + F_{0z}\cos\theta_1 + F_{1z})\right\}$$

$$+ \frac{m_3}{m} s_{3z}\left\{\ell_{3x}(F_{0y}\cos\theta_1 + F_{0z}\sin\theta_1 + F_{1y}) - \ell_{3y}(F_{0x} + F_{1x})\right\} \qquad \text{B5.14}$$

Unconstrained component along s_4 axis

$$\frac{d}{dt}(h_{4y}s_{4y} + h_{4z}s_{4z}) + s_{4y}(\omega_{1z}h'_{4x} - \omega_{1x}h_{4z}) + s_{4z}(\omega_{1x}h'_{4y} - \omega_{1y}h_{4z})$$

$$= m_4 s_{4y}(\dot{\ell}_{4z}\dot{r}_{4x} - \dot{\ell}_{4x}\dot{r}_{4z}) + m_4 s_{4z}(\dot{\ell}_{4x}\dot{r}_{4y} - \dot{\ell}_{4y}\dot{r}_{4x})$$

$$+ \frac{m_4}{m} s_{4y}\left\{\ell_{4z}(F_{0x} + F_{1x}) - \ell_{4x}(F_{0y}\sin\theta_1 + F_{0z}\cos\theta_1 + F_{1z})\right\}$$

$$+ \frac{m_4}{m} s_{4z}\left\{\ell_{4x}(F_{0y}\cos\theta_1 + F_{0z}\sin\theta_1 + F_{1y}) - \ell_{4y}(F_{0x} + F_{1x})\right\} \qquad \text{B5.15}$$

APPENDIX C

GRAVITY GRADIENT EFFECTS

In this Appendix a derivation will first be outlined of the expression for the gravity gradient moment about the mass center of a single vehicle segment. The total gravity gradient moment about the mass center of a cluster is regarded as the sum of the individual moments thus computed for the segments, plus a moment due to differential forces acting at the mass centers of those segments. The differential force expression is then derived under the assumption that the entire mass of a segment is located at its center of mass, and results from the difference in position within the gravitational field of the segment's mass center and that of the cluster as a whole.

1. The Gravity Gradient Moment on a Single Segment

Consider Figure C-1, with $\overline{R}_o$ being the radius vector from the center of the earth and $\bar{r}$, that from the center of mass of the body to an increment of mass dm.

The force on dm is then

$$\overline{dF} = \frac{-\ u\ dm\ \overline{R}}{R^3} \tag{C-1}$$

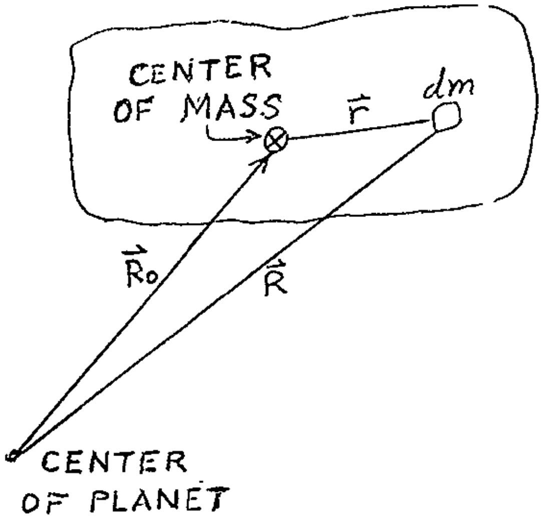

Figure C-1

The resulting moment is

$$\overline{dM} = \bar{r} \times \overline{dF} = \frac{-\,u\ dm\ \bar{r}\times\bar{R}}{R^3} \tag{C-2}$$

Making the substitutions $\bar{R} = \bar{R}_o + \bar{r}$ and

$$R^3 = \left\{(\bar{R}_o + \bar{r}) \cdot (\bar{R}_o + \bar{r})\right\}^{\frac{3}{2}} = R_o^3 \left\{1 + \frac{2\bar{R}_o\bar{r}}{R_o^2} + \frac{r^2}{R_o^2}\right\}^{\frac{3}{2}},$$

we can write the moment as

$$dM = - \frac{u\ dm\ \bar{r} \times (\bar{R}_o + \bar{r})}{R_o^3} \cdot \left(1 + \frac{\overline{2R}_o \cdot \bar{r}}{R_o^3}\right)^{-\frac{3}{2}} . \quad \text{(C-3)}$$

Since $\bar{r} \times (\bar{R}_o + \bar{r}) = \bar{r} \times R_o$ and

$$\left(1 + \frac{\overline{2R}_o \cdot \bar{r}}{R_o^3} + \frac{r^2}{R_o^2}\right)^{-\frac{3}{2}} \approx \left(1 - \frac{\overline{3R}_o \cdot \bar{r}}{R_o^2}\right) \text{ for } r << R_o,$$

we can integrate (C-3) over the entire body to get the approximate expression

$$M = \int_{Bj} - \frac{u\ \bar{r} \times \bar{R}_o}{R_o^3} \left(1 - \frac{\overline{3R}_o \cdot \bar{r}}{R_o^2}\right) dm. \quad \text{(C-4)}$$

Expanding and noting that $\int_{Bj} \bar{r} dm = 0,$

(C-4) can be reduced to the form

$$M = - \frac{3\ u}{R_o^5} \int_{Bj} (\bar{R}_o \times \bar{r})(\bar{r} \cdot \bar{R}_o)\ dm. \quad \text{(C-5)}$$

If this vector equation is written in matrix form, referring all vectors to the body-fixed B_j-frame, and the indicated integration carried ou:, we arrive at the desired expression

$$\overset{(3x1)}{M} = -\frac{3}{R_o^5}\,\overset{(3x3)}{R}\;\overset{(3x3)}{I_j}\;\overset{(3x1)}{[R_o]} \qquad \text{(C-6)}$$

where $[R_o]$ is the matrix expression of R_o in the B_j-frame, and R is the cross product matrix obtained therefrom:

$$R = \begin{bmatrix} 0 & -R_{o_z} & R_{o_y} \\ R_{o_z} & 0 & -R_{o_x} \\ -R_{o_y} & R_{o_x} & 0 \end{bmatrix}$$

and I_j is the conventional moment of inertia matrix of Body J in terms of the B_j-frame. Equation (C-6) is used in the gravity gradient subroutine in the form shown.

2. <u>Differential Force on Body J</u>

Consider Figure C-2, where $\bar{r}_j$ denotes the vector from the mass center of the cluster to that of Body J, having mass m_j.

The total force on Body J is

$$= -\frac{u\, m_j \bar{R}_j}{R_j^3}.$$

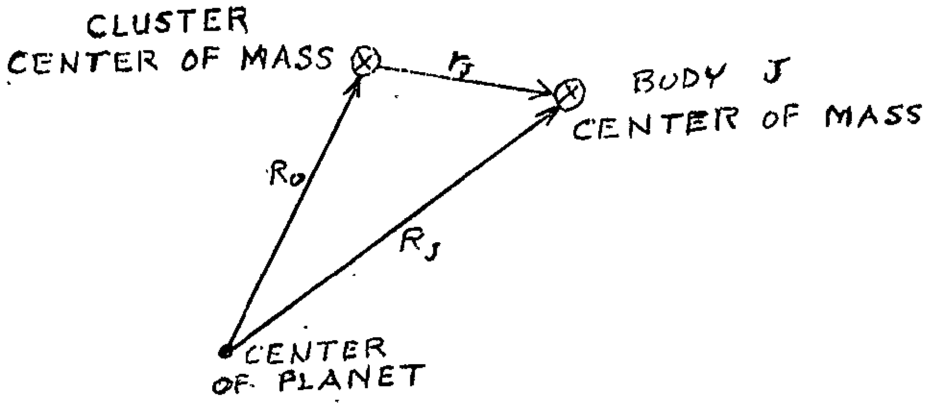

Figure C-2

Following a procedure similar to that of paragraph 1 above, the approximation

$$\overline{F}_j \approx -\frac{u\, m_j}{R_o^3}\left\{\overline{R}_o + \bar{r}_j - \frac{\overline{3R_o} \cdot \bar{r}_j}{R_o^2}\,\overline{R}_o\right\} \qquad \text{(C-7)}$$

is obtained, valid for $r_j \ll R_o$.

Since $-\frac{u\, m_j R_o}{R_o^3}$ is the gravitational force on B_j that would occur if $\bar{r}_j$ were zero, the desired differential force is simply

$$\delta F_j = -\frac{u\, m_j}{R_o^3}\left\{\bar{r}_j - \frac{\overline{3R_o} \cdot \bar{r}_j}{R_o^2}\,\overline{R}_o\right\}. \qquad \text{(C-8)}$$

The matrix equivalent of Equation (C-8) is used in the Gravity

Gradient Subroutine.

The derivation of components of the vector R_o in the frames of Body 0 and Body 1 will be described with reference to Figure C-3.

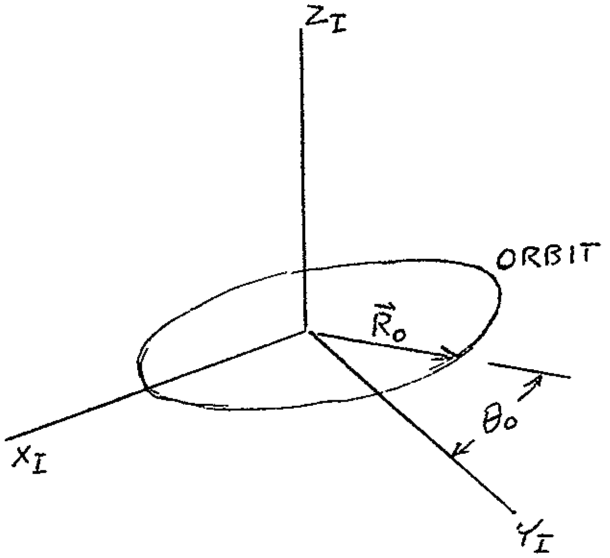

Figure C-3

The translational motion is restricted to a circular orbit in the X_I-Y_I plane. The origin of the I frame coincides with the center of the earth. The orbit angle is measured from the $+ Y_I$ axis. The vector $\overline{R}_o$ is easily expressed in the I frame as

$$\left.\overline{R}_o\right]_{\text{I Frame}} = |R_o| \begin{pmatrix} -s\theta_o \\ c\theta_o \\ 0 \end{pmatrix}$$

Application of the inverse of the TIBO transformation will then give the components in the Body 0 frame. The components of a unit vector $\overline{R}_o/|R_o|$ are designated as DB(1), DB(2) and DB(3) at the beginning of the subroutine. For the Body 1 calculations these components are expressed in Body 1 coordinates through the A transformation.