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ABSTRACT

It has been pointed out by previous authors that the turbulent motion of the ionized interstellar gas will cause a given line of force of the galactic magnetic field to wander in a random manner even though the average field may be quite regular. We derive here an equation that describes the development of the probability density of a given field line as a function of distance along the average field. We demonstrate that a Fokker-Planck type of equation is quite unsuitable in this case and that the answer must be sought in the theory of stationary Gaussian processes. We also derive the distribution function for the "end points" of a field line that starts on the galactic central plane. End points in this context means the point where the field line first wanders so near the surface of the galactic disk that it can no longer confine cosmic-rays.

A DYNAMIC EQUATION FOR STOCHASTIC MAGNETIC FIELD LINES IN THE GALAXY

I. INTRODUCTION

The earth is situated near ($\sim$ 20 pc) the central plane of the galactic disk. The average galactic field ($\sim$ 2-3 μ gauss) appears to lie mainly in the disk and run approximately parallel to the spiral arms (Davis and Greenstein 1951; Hiltner 1956; van de Hulst 1967; Davis and Berge 1968). There are also theoretical arguments that it should lie essentially in the azimuthal direction (Parker 1969, Roberts and Yuan 1970). However, a given flux tube such as the one that passes through the earth and its immediate vicinity will not, in general, follow the pattern of the average field but will wander in a random manner about the galactic disk. This random wandering of a given field line or group of lines has been discussed at length by Jokipii and Parker (1969 a, b) and has been related by them to the problem of cosmic-ray propagation in the galaxy. Briefly the relationship is as follows: the random wondering of the field line to the surface of the galactic disk provides a means whereby a cosmic-ray particle, which at the position of the earth is buried deeply ($\sim 10^7$ cyclotron radii) in the disk, may escape the disk in a time of the order of 10^6 years. At the point where the field line reaches the "surface" the cosmic-ray particles can escape presumably via the bubble blowing instability (Parker 1965). Since the galaxy does not have sharp edges the notion of the surface of the disk is necessarily a bit vague but it is usually assumed that when a field line reaches a point sufficiently far from the central plane (~ 100 pc) the instability will occur and the cosmic-ray particles will escape.

The positions of the ends, for cosmic-ray purposes, of the flux tube passing through the earth are important parameters in any discussion of the cosmic-ray anisotropy, whether from a smoothed out source point of view (Kulsrud and Pearce 1969) or from a statistical source point of view (Jones 1970 a, b). In either case the average cosmic-ray streaming observed at the earth will depend on the position of the earth relative to the end points of its associated flux tube.

Unfortunately a determination of the actual positions of these end points is impossible, at least by present techniques, and we must turn our attention to their probability distribution function. Jokipii and Parker (1969 b) discuss this question and give a rough estimate of where a given field line might end. Their estimate was very crude, however, and was of no value in determining where the maximum likelihood position would be.

In this paper we will consider this end point problem in more detail and obtain a probability distribution function for the end points of the flux tube passing through the earth.

In their analysis Jokipii and Parker (1969 b) determine that a field line whose mean position lies in the central plane of the galactic disk has a probability of being found between z and z + dz above (or below) the central plane given by

$$f(z)\,dz = \exp(-|z|/z_0)\,dz/z_0.$$

The general problem we wish to consider is: given that the field line in question is at $z = 0$ at a point $y = 0$, where y measures the distance along the mean field, what is the probability distribution in z as a function of y. Our specific question is the following: given that the field line is at $z = 0$, $y = 0$, the position of the earth, what is the probability that it continues a distance y down the mean field direction without reaching a distance $|z| = z_1$ from the central plane and thus terminating its role as a cosmic-ray container.

The equation that we shall derive will be a dynamic equation in the sense that it will describe the development of the probability function along the direction y starting with some given form at $y = 0$ in a manner completely analogous to the development in time of dynamic stochastic systems. We shall not consider the actual time development of the stochastic field in this paper. Typical rms velocities of the gas clouds that carry the magnetic fields are ~5 Km/sec which correspond to motions of about 15 pc in a typical cosmic ray lifetime of 10^6 years. We cannot claim that such a motion is truly negligible, however, we would not expect a pronounced effect from ignoring it at the present time.

We shall first, following Jokipii and Parker (1969 a), consider the dynamic equation to be of the Fokker-Planck type. The requirements that the asymptotic probability distribution be of the form exp $(-|z|/z_0)$ requires that the diffusion coefficient be of the form exp $(|z|/z_0)$ (indeed any asymptotic form d(z) with the requirement of no probability current requires a diffusive coefficient of the form $d^{-1}(z)$). It will turn out that because of this and the fundamental assumptions inherent in a Fokker-Planck type equation the results will be quite unsatisfactory. They will indicate a far too rapid transport of the field lines to the boundary. Attempts to patch up this problem will be seen to be also unsatisfactory.

It will turn out that we can construct a model of the wandering of field lines based on the theory of Gaussian processes from which we can extract a dynamic equation that is free of the objectionable characteristics of the Fokker-Planck approach.

II. THE FOKKER-PLANCK APPROACH

The probability that a given field line can be found between z and z + dz above the galactic central plane at a point y along the direction of the mean field will be given by f(z; y) dz where f is the probability density function.[1] We shall assume initially that f satisfies a Fokker-Planck equation of the form.

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[1] We shall adopt the notation of Feller (1966) in distinguishing between a probability density f(x) of a random variable,

$$P(a \le X \le b) = \int_a^b f(x)dx,$$

and its distribution function F(x),

$$P(X \le x) \equiv F(x) = \int_{-\infty}^x f(y)\, dy.$$

The various variables of a multivariate density or distribution function will usually be separated by commas though we will sometimes use a semicolon to indicate that the variables to its right are to be considered as parameters in this particular instance. In a conditional probability density the variables to the right of a vertical slash are the one that are fixed, i.e.,

$$P(x \le X \le x + dx \mid Y = y) = f(x \mid y).$$

$$\frac{\partial f(z,y)}{\partial y} + \frac{\partial}{\partial z}(d_1(z)f(z,y)) - \frac{1}{2}\frac{\partial^2}{\partial z^2}(d_2(z)f(z,y)) = 0 \qquad (1)$$

where $d_1(z) = \langle \delta z \rangle / \delta y$ and $d_2(z) = \langle \delta z^2 \rangle / \delta y$. We require that

$$f(z,y) \rightarrow \exp(-|z|/z_0)/2z_0 \text{ as } y \rightarrow \infty,$$

we therefore have

$$\frac{\partial}{\partial z}[d_1(z)\exp(-|z|/z_0)] - \frac{1}{2}\frac{\partial^2}{\partial z^2}[d_2(z)\exp(-|z|/z_0)] = 0. \qquad (2)$$

The first integral of equation (2) is the probability current or

$$d_1(z)\exp(-|z|/z_0) - \frac{1}{2}\frac{\partial}{\partial z}[d_2(z)\exp(-|z|/z_0)] = j = \text{const}. \qquad (3)$$

j could be an overall constant which would destroy the symmetry of the problem with respect to plus and minus z and would add a decidedly unphysical aspect to the solution. A j which changed sign at z = 0 in order to preserve the symmetry would involve a source or sink of probability at z = 0 and would be equally unphysical. We therefore choose j = 0 everywhere.

From (3) with j = 0 we obtain finally,

$$d_1(z) - \frac{1}{2}d_2'(z) + \frac{1}{2}\frac{\epsilon(z)}{z_0}d_2(z) = 0 \qquad (4)$$

where

$$d_2' = d(d_2)/dz \text{ and } \epsilon(z) = \pm 1 \text{ for } z \gtrless 0$$

respectively. If we divide the galactic magnetic field into an average field $\underset{\sim}{B}_0$ and a small fluctuating part $\underset{\sim}{B}_1$ we have (Jokipii and Parker 1969 a)

$$d_1(z) = <\delta z>/\delta y = <B_1>/B_0$$

$$d_2(z) = <\delta z^2>/\delta y \approx 2L <B_{1z}^2>/B_0^2$$

where L is the correlation length of the random field $\underset{\sim}{B}_1$. Since we must have

$$<\underset{\sim}{\nabla} \cdot \underset{\sim}{B}_1> = \underset{\sim}{\nabla} \cdot <\underset{\sim}{B}_1> = 0.$$

this gives

$$\frac{\partial <B_{1z}>}{\partial z} = - \frac{\partial <B_{1y}>}{\partial y}$$

(we are considering two dimensions only but similar considerations would apply to the three dimensional case). If we wish our problem to be stationary, i.e. all statistical parameters are independent of y we must require

$$\frac{\partial <B_{1z}>}{\partial z} = - \frac{\partial <B_{1y}>}{\partial y} = 0 \text{ and } <B_{1z}> = \text{const}.$$

Once again we require $\langle B_1 \rangle = 0$ to preserve symmetry and we are left with

$$d_2'(z) = \frac{\epsilon(z)}{z_0} d_2(z) + d_2(z) - \nu \exp(|z|/z_0)$$

where

$$\nu = [\langle \delta z^2 \rangle / \delta y]_{z=0} \, .$$

We have arrived at the equation

$$\frac{\partial f(z,y)}{\partial y} - \frac{\nu}{2} \frac{\partial^2}{\partial z^2} [\exp(|z|/z_0) \, f(z,y)] = 0 \tag{6}$$

with the initial condition $f(z, 0) = \delta(z)$ as our fundamental equation. If we make the substitution

$$f(y, z) = \exp(-|z|/z_0) \, u(z,y)$$

we have the equation

$$\exp(-|z|/z_0) \frac{\partial u}{\partial y} - \frac{\nu}{2} \frac{\partial^2 u}{\partial z^2} = 0 \tag{7}$$

with the initial condition

$$u(z)_{y=0} = \delta(z)$$

and asymptotic condition

$$u(z)_{y \to \infty} \to (2z_0)^{-1} .$$

Equation (7) may be solved by the standard techniques of constructing a Green function after Laplace transforming the y variable. The solution is

$$u(|z|,y) = \frac{1}{2z_0} \sum_{n=0}^{\infty} \frac{J_0[\beta_n \exp(-|z|/z_0)]}{J_0[\beta_n]} \exp(-\nu\beta_n^2 y/8z_0^2) \qquad (8)$$

where J_0 (x) is the Bessel function of order zero that is regular at x = 0 and β_n is the nth root of the equation J_1 (x) = 0 with $\beta_0 = 0$. It is easy to see that the effective length for relaxation of the initial probability distribution is $y_0 = 8 z_0^2 \nu^{-1}$.

We now wish to use the dynamic equation (6) to compute the probability $F_1(y; z_1)$ that a field line has reached the point y without ever having made an excursion in $|z|$ as great as z_1. If one thinks a moment about standard particle diffusion with totally absorbing walls one will realize that after a time t all those particles that are left are just those that have survived a time t without ever having reached the absorbing wall. We therefore wish a solution of equation (6) with the boundary condition

$$f_1(z,y) = 0 \text{ for } z = \pm z_1$$

and then

$$F_1(y;z_1) = \int_{-z_1}^{z_1} f_1(z,y)\, dz.$$

The distribution function for the endpoints is

$$F_2(y;z_1) = 1 - F_1(y;z_1)$$

and since the endpoint density is

$$f_2(y;z_1) = \partial F_2/\partial y$$

we have using (6)

$$f_2(y;z_1) = -\frac{\partial}{\partial y}F_1(y;z_1) = -\int_{-z_1}^{z_1}\frac{\partial f_1(z,y)}{\partial y}\,dz$$

$$= \frac{\nu}{2}\int_{-z_1}^{z_1}\frac{\partial^2}{\partial z^2}\left[\exp(|z|/z_0)\,f_1(z,y)\right]dz \qquad (9)$$

$$= \frac{\nu}{2}\left(\frac{\partial u}{\partial z}\bigg|_{z_1} - \frac{\partial u}{\partial z}\bigg|_{-z_1}\right) = \nu\,\frac{\partial u}{\partial|z|}\bigg|_{z_1}.$$

Once again using standard techniques we find that $f_2(y;z_1)$ is given by

$$f_2(y;z_1) = \frac{\nu}{2\pi z_0^2}\sum_{n=1}^{\infty}\frac{\exp[-\nu\beta_n'^2 y/8z_0^2]}{[d\bar{D}/d\eta]_{\eta=\beta_n'}} \qquad (10)$$

where the β_n' are the zeroes of the function

$$\bar{D}(\eta;z_1) \equiv J_1(\eta)\,Y_0(\eta x_1) - Y_1(\eta)\,J_0(\eta x_1)$$

where Y_m is the mth order Bessel function of the second kind and $x_1 \equiv \exp(-z_1/2z_0)$.

One can now compute the distribution function of the end points i.e., the probability that the position of endpoint Y is less than or equal to y:

$$P(Y \le y) = F_2(y;\ z_1)$$

However, since this approach will turn out to be unsatisfactory we will not waste time examining $F_2(y; z_1)$ in detail but will take a simpler approach.

It is easy to see that the various moments $\langle (y/y_0)^m \rangle$ are given by

$$\begin{aligned} \langle (y/y_0)^m \rangle &\equiv \int_0^\infty (y/y_0)^m\, f_2(y;\ z_1)\, dy \\ &= \frac{4}{\pi}\, m! \sum_{n=1}^{\infty} \frac{1}{(\beta_n'^2)^{m+1}\, [d\bar{D}/d\eta]_{\eta=\beta_n'}} \end{aligned} \qquad (11)$$

and they are, of course, functions of z_1/z_0.

In Table I we have listed various values of z_1/z_0 and the corresponding values of $\langle y/y_0 \rangle$ or the mean length of a field line.

One notices immediately that the mean lengths of a field line are considerably less than the characteristic length y_0 even for an absorbing wall as much as four characteristic heights z_0 away. To evaluate the length $y_0 \equiv 8 z_0^2/\nu$ we note that

$$\nu \approx 2L \langle B_{1z}^2 \rangle / B_0^2 |_{z=0}. \text{ But } \langle B_{1z}^2 \rangle / B_0^2 |_{z_0} \approx 2\, z_0^2/L^2$$

so

$$y_0 \approx 2L$$

This says that for $z_1/z_0 = 2.2$ (the case considered by Jokipii and Parker 1969 b) an average field line will reach one of the surfaces $z = \pm z_1$ in a distance $\langle y \rangle = .65L$.

Jokipii and Parker's estimate was about ten times this or 6L. Furthermore this says that an average field line has a slope $2.2\ z_0/.65L \approx 3.4\ z_0/L$ whereas we initially assumed a mean slope of $\approx z_0/L$ so we have an internal contradiction as well as an external one.

The difficulty arises because we have a situation that violates the basic assumption underlying all diffusion type equations. The assumption is that there exist at least two scales in the problem: a microscopic scale of the forces driving the system (the random field B_1 is the force in our problem), and a macroscopic scale on which the forces and the response of the system are completely uncorrelated. The diffusion type equation describes the behavior of the system on the macroscopic scale only. We can see that this is not the case here; the microscopic and macroscopic scales are both characterized by the correlation length L. It is this violation of the two scale requirement that causes the Fokker-Planck approach to yield self contradictory results.

One could attempt to patch up the Fokker-Planck equation by replacing the constant diffusion coefficient $\nu = 2L < B_{1z}^2 > / B_0^2$ by a generalization that is y dependent,

$$\nu(y) = (2/B_0^2) \int_0^y C_{zz}(y')\, dy'$$

where $C_{zz}(y)$ is the zz component of the random field correlation tensor, i.e.

$$C_{ij}(\underset{\sim}{y}) = <B_i(\underset{\sim}{x})\, B_j(\underset{\sim}{x} + \underset{\sim}{y})>$$

Using $\nu(y)$ in equation (1) is equivalent to replacing the variable y with the variable η where $\eta(y) = (\nu(\infty))^{-1} \int_0^y \nu(y')\, dy'$ in the original solution of equation (1).

This is only an <u>ad hoc</u> procedure and cannot be considered completely satisfactory. It does, however, modify the rapid initial dispersion of probability, the dispersion is proportional to y rather than $y^{1/2}$ for small y. The overall effect is small though and only increases the mean field line length by an addition of approximately L/2.

We should, therefore, turn to an entirely different approach to the problem, one in which there is no need to postulate two length scales, a microscopic scale and a macroscopic scale. Such a procedure can be found in the theory of stationary, Gaussian processes which we shall discuss in the next section.

III. THE GAUSSIAN PROCESS APPROACH

The random wandering of the galactic magnetic field is produced by the turbulent motion of the gas to which the field is frozen. We shall, therefore, turn our attention to a particular model of this underlying interstellar gas. This model will not necessarily be a correct picture of the turbulent galactic gas, however, it is at least a physically realizable model and from it we can calculate, without any mathematical approximations, the statistical properties of the magnetic field. This means that while the results we obtain for the field line wandering may not correspond exactly with reality, they will at least be physically realizable and will contain no unphysical properties introduced via some mathematical approximation scheme. As we have seen in the last section not all methods of generating dynamic equations for probability densities can make this claim. The hope is, of course, that our results will constitute a reasonable approximation to the situation that exists in our galaxy.

Consider the following two dimensional situation. A quiescent gas has a uniform magnetic field $B\hat{\underset{\sim}{e}}_y$ embedded in it. The field lines may be labeled by the value of their z coordinate at the y = 0 plane. Now imagine that the gas is displaced in the z direction by an amount s which varies from one point to another depending on the value of the y coordinate thus s = s(y). We now consider an ensemble of such situations and let s(y) be a random function on this ensemble. We shall assume s(y) to have the following properties: it is expressible as a Fourier series

$$s(y) = \sum_i s_i \cos (k_i y + \phi_i).$$

and the phase angles ϕ_i are random variables, uniformly distributed between 0 and 2π. From these properties it can be shown (Rice, 1944, 1945) that s(y) is a stationary, Gaussian processes. This means that the multivariate probability density for the N quanitities $x_1, x_2 \ldots x_N$ where the x_i are values of the function s or any of its derivatives with respect to y taken at N points y_1, $y_2 \ldots y_n$ (where some or all of the y_i may be identical) is given by the N dimensional Gaussian density

$$f(\underset{\sim}{x})\, d\underset{\sim}{x} = (2\pi)^{-N/2}\, |M|^{-1/2} \exp\left(-\frac{1}{2}\, \bar{\underset{\sim}{x}}\, M^{-1}\, \underset{\sim}{x}\right) d\underset{\sim}{x} \qquad (12)$$

where $\bar{\underset{\sim}{x}}$ and $\underset{\sim}{x}$ are the row and column vectors respectively with components $x_1, x_2 \ldots x_N$ and M is the matrix $M_{ij} = \langle x_i x_j \rangle$. The process is stationary because the quantities $\langle x_i x_j \rangle$ are functions of the relative separation only. From this it is evident that the probability density function is a Gaussian rather than an exponential as inferred by Jokipii and Parker (1969 b). Of course, what is observed is the velocity distribution of the interstellar gas clouds which is approximately an exponential function of cloud velocity (Spitzer 1968). The position and velocity distributions will have the same form only if the dynamic (in the time development sense) system is a linear one and it is tempting to argue that non-linearities could cause the velocity distribution to be exponential even though the position distribution is Gaussian.

This argument is dangerous, however, for if the system is appreciably non-linear the Fourier modes will not be normal modes in the dynamical sense and even if they are started with random phases mode coupling can produce correlations in the phases as time advances. So it appears that we must simply acknowledge that this model departs somewhat from observation and hope that the departure is not too serious.

From equation 12 the distribution of probability for a field line passing through $y = 0$, $z = 0$ may be readily obtained. For if a field line passes through $y = 0$, $z = 0$, at y we have $z(y) = s(y) - s(0) = \Delta(y)$. If we define the correlation function $\psi(y)$ as

$$\langle s(x)\, s(x+y)\rangle = \langle s(0)\, s(y)\rangle \equiv \langle s^2\rangle\, \psi(y)$$

the probability density for $s_0 = s(0)$ and $s_1 = s(y)$ is given by

$$f(s_0, s_1)\,ds_0\,ds_1 = \frac{\exp\left[-\frac{1}{2}\,(s_0^2 + s_1^2 - 2 s_0 s_1 \psi)/\langle s^2\rangle(1-\psi^2)\right]}{2\pi \langle s^2\rangle\, [1-\psi^2]^{1/2}}\,ds_0\,ds_1 \tag{13}$$

where

$$\langle s_0^2\rangle = \langle s_1^2\rangle \equiv \langle s^2\rangle \text{ and } \psi \equiv \psi(y)$$

If we express s_1 as $s_1 = s_0 + \Delta$ in equation (13) we may then integrate over all values of s_0 holding Δ fixed to obtain the probability density for Δ

$$f(\Delta)d\Delta = \frac{\exp\left[-\frac{1}{2}\Delta^2/<s^2>(1-\psi)\right]}{[2\pi <s^2>(1-\psi)]^{1/2}} d\Delta \tag{14}$$

We see immediately that this is not the same as the probability density for s_1 under the condition that s_0 is zero. This would be given by

$$\begin{aligned} f(s_1 | s_0)_{s_0=0} &= [f(s_1, s_0)/f(s_0)]_{s_0=0} \\ &= \frac{\exp\left[-\frac{1}{2} s_1^2/<s^2>(1-\psi^2)\right]}{[2\pi <s^2>(1-\psi^2)]^{1/2}} \end{aligned} \tag{15}$$

and can be seen to differ from expression (14) in containing the factor $(1-\psi^2)$ rather than $(1-\psi)$. The reason that expression (14) must be used rather than (15) is that the knowledge that a field line passes through the point $y = 0$, $z = 0$ does not tell us that $s_0 = 0$; we do not know the value of s_0 and hence must average over all possible values.

Our next problem is to find the probability that a field line has not reached a point z_1 above or below the central plane in a distance y along the mean field. Unfortunately the absorbing wall analogy will not serve us in this instance. There is a dynamic equation governing the development of the density in equation (14). It is easy to verify that (14) is the Green function for the equation

$$\frac{\partial f(\Delta, y)}{\partial y} + <s^2>\psi' \frac{\partial^2 f(\Delta, y)}{\partial \Delta^2} = 0 \tag{16}$$

with the initial condition $f(\Delta, 0) = \delta(\Delta)$ and where $\psi' = d\psi/dy$. In the appendix we show that the diffusion equation (16) may be derived directly using the formalism of Kubo (1961). It can be seen that equation (16) is equivalent to a standard diffusion equation with diffusion coefficient $<S^2>$ and a "time" variable $(1 - \psi)$ which varies only between 0 and 1. This means that the diffusion only turns on for a finite time (actually length y in our case) of order L after which nothing more happens. In the absorbing wall problem a field line has a finite probability of reaching the wall in a distance of order L but if it doesn't reach the wall by this distance there is a good chance that it never will.

The reason for this difficulty is that equation (16) describes a process that is decidedly non-Markovian; the evolution of the probability density $f(\Delta, y)$ depends not just on the immediate form of f but on how far f has developed from its <u>initial</u> form $f(\Delta, 0)$, e.g. on its past history. It can be seen that if one starts with an initial $f(\Delta, 0) = \delta(\Delta)$ the solution relaxes asymptotically to the form of equation (14). However, this is not a stable solution for if one starts with this form for $f(\Delta, 0)$ equation (16) will relax it further to a Gaussian with a dispersion twice as large as before. So we see that the analogy with particle diffusion completely breaks down here and we must turn to another method for solving this problem.

The answer is found in the solution of the zero crossing problem as given by Rice (1945). Rice shows that for a random function z(y) with a probability density for z and $z' \equiv dz/dy$ at a given y, $f(z, z'; y)$ the probability that the curve crosses the zero axis with a positive slope between y and $y + dy$ is given by

$$dy \int_0^\infty z' \, f(0, z' ; y) \, dz' . \tag{17}$$

Expression (17) may be immediately generalized to our case to determine the probability density for a field line displacement to cross the $z = z_1$ axis with positive slope (since the first crossing must be outward we are only interested in outward crossings). We have

$$f(y ; z_1) \, dy = dy \int_0^\infty \Delta' \, f(z_1, \Delta' ; y) \, d\Delta' . \tag{18}$$

The probability density $f(z_1, \Delta' ; y)$ may be derived as follows: Starting with the trivariate Gaussian density for s_0, s_1, and s_1',

$$f(s_0, s_1, s_1' ; y) = (2\pi)^{-3/2} \left(|M|\right)^{-1/2} \exp - \frac{1}{2} \bar{x} M^{-1} x$$

where

$$\bar{x} = (s_0, s_1, s_1')$$

and

$$M = \langle s^2 \rangle \begin{pmatrix} 1 & \psi & \psi' \\ \psi & 1 & 0 \\ \psi' & 0 & -\psi''_{(0)} \end{pmatrix}$$

we may express $s_1 = s_0 + \Delta$ and $s_1' = \Delta'$. We may then integrate over all possible values of the s_0 in a straightforward though tedious calculation to obtain

$$f(\Delta, \Delta'; y) = (2\pi)^{-1}(\langle s^2 \rangle D)^{-1} \exp\left[\frac{-\Delta'^2(1-\psi)}{\langle s^2 \rangle D} - \frac{\Delta'\Delta\psi'}{\langle s^2 \rangle D} + \frac{\Delta^2\psi''_{(0)}}{2\langle s^2 \rangle D}\right] \tag{19}$$

where

$$D = [-2\psi''_{(0)}(1-\psi) - (\psi')^2]$$

Expression (19) may now be inserted into expression (18) and the indicated integration may be performed analytically to obtain

$$f(y;\delta) = \frac{D^{1/2}\exp\left[\frac{2\psi''_{(0)}\delta^2}{D}\right]}{4\pi(1-\psi)}$$

$$-\frac{\exp\left[\frac{-\delta^2}{(1-\psi)}\right]}{4\pi^{1/2}(1-\psi)^{3/2}}\,\delta\psi'\left[1-\mathrm{erf}\left(\frac{\delta\psi'}{2(1-\psi)D^{1/2}}\right)\right] \tag{20}$$

where

$$\delta = z_1/2\langle s^2 \rangle^{1/2}$$

To proceed beyond this point an explicit form must be chosen for the correlation function $\psi = \psi(y)$. This function is not known for the motions of the gas in our galaxy so we shall have to choose one as an ad hoc procedure. This should not be a serious disadvantage since we shall see that the main features of the function in expression (20) do not depend on the detailed form of $\psi(y)$ but can be understood on the basis of the scale of the correlation function, or correlation length L.

If the correlation length is defined by

$$L = \int_0^\infty \psi(y)\, dy$$

we may choose a Gaussian as a typical form for $\psi(y)$ to obtain

$$\psi(y) = \exp(-\pi y^2/4L^2) = \exp(-x^2).$$

With this choice we may now write the probability (per unit x) as an explicit function of x

$$f(x;\delta) = \frac{\exp(-4\delta^2/D(x))}{2\pi[1-\exp(-x^2)]} D^{1/2}(x) + \frac{x\delta \exp(-x^2)\exp[-\delta^2/1-\exp(-x^2)]}{\pi^{1/2}[1-\exp(-x^2)]^{3/2}}$$

$$\times \left[1 - \mathrm{erf}\left(\frac{-2x\delta\exp(-x^2)}{[1-\exp(-x^2)]^{1/2} D^{1/2}(x)}\right)\right] \qquad (21)$$

Where we have multiplied expression (20) by a factor of 2 to allow for the fact that we are interested in the two surfaces $z = \pm z_1$. In Figure 1 we have plotted the function $f(x, \delta)$ for three different values of δ. The choice $\delta = 0.8$ was made for purposes of comparison with the results of Jokipii and Parker (1969 b) for there is the same unconditional probability of finding a given field line outside of this distance from the central plane as there was for finding it beyond one scale height ($\sim$130 pc) in their model. The major features of these curves are readily understood, for values of $x \gg 1$ ($y \gg L$) the flat portion

$$f(x;\ \delta) \simeq (2\pi)^{-1} \exp(-\delta^2)$$

simply reflects the fact that initial conditions have become forgotten and the probability has relaxed to its asymptotic state.

The peak at $x = \delta$ can be seen to be due entirely to the ballistic propagation of initial conditions. At $y = 0$ the slope Δ' has a probability density function

$$f(\Delta')d\Delta' = [2\pi\langle s^2\rangle \psi''(0)]^{-1/2} \exp(\Delta'^2/2\langle s^2\rangle\psi''(0))\, d\Delta'. \qquad (22)$$

If we assume that the field line continues with the same slope Δ' until it interests the surface $z = z_1$ it will interest at $y = y_1 \equiv z_1/\Delta'$. Substituting $\Delta' = z_1/y_1$ into expression (22) we obtain a probability density for y_1, using $\psi''(0) = \pi/2L^2$

$$f(y_1)dy_1 = \frac{z_1}{y_1^2} \frac{\exp(-z_1^2 L^2/\pi y_1^2 \langle s^2\rangle)dy_1}{[\pi^2 \langle s^2\rangle /L^2]^{1/2}}$$

$$= \pi^{-1/2} (\delta/x_1^2) \exp(-\delta^2/x_1^2)\, dx_1. \qquad (23)$$

It is easy to see that this expression fits the initial part of the curves in Figure 1, rather well at least up to the maximum at $x_1 = \delta$. Of course as δ gets larger the peak gets smaller because the ballistic propagation of initial conditions washes out as the field line has to go a distance of the order of L or further before interesting the plane $z = z_1$.

The probability expressed by the density $f(x; \delta)$, expression (21), is the probability that the field line will cross the surfaces $z = \pm z_1$ between x and x + dx with no concern as to whether or not it is the first crossing since x = 0. Since the probability that the field line has not crossed the surfaces anywhere between 0 and x is given by

$$\exp\left(-\int_0^x f(x';\delta)\, dx'\right)$$

the probability that a field line crosses the surfaces between x and x + dx for the first time is given by

$$\frac{dF}{dx} = f(x; \delta) \exp\left(-\int_0^x f(x'; \delta)\, dx'\right). \qquad (24)$$

We call this expression dF/dx since the integral F is the distribution function for the "end point" X of the field line, i.e.,

$$P(X \leq x) = F(x) = 1 - \exp\left(-\int_0^x f(x'; \delta)\, dx'\right). \qquad (25)$$

dF/dx and F(x) are plotted in Figures 2 and 3 respectively. In Figure 3 we have marked the values of x for which F(x) = 1/2 for the three values of δ. For δ = 0.8, the case corresponding to that considered by Jokipii and Parker, we see that $x_{1/2}$ is slightly less than 4 or in other words the field line has a 50% chance of getting out after about 4 correlation lengths. This is somewhat less than the estimate of 6 correlation lengths made by Jokipii and Parker (1969 b). The discrepancy originates in the fact that the previous authors asked a somewhat different question, namely how many observations, each one a correlation length apart, must be made for there to be a 50% chance of catching the field line outside the surfaces $z = \pm z_1$. It is clear that this question does not count those cases where the field line goes out and comes back in between successive observations and hence overestimates the distance required for the field line to pass through the critical surfaces for the first time. All in all the discrepancy does not appear to be a large one and it is a bit surprising that the results are as close as they are considering the rough nature of the original estimate.

IV. CONCLUSION

We have seen that the model of a Gaussian process for the galactic gas motions allows us to make exact calculations about the probability density of the galactic magnetic field. Whereas the model does not exactly represent reality the fact that no mathematical approximations need to be made assures us that our results are at least physically realizable and presumably not too far from the truth. With this approach we have been able to calculate the distribution function for the "end points" of the field line passing through the earth, end points meaning with respect to cosmic ray confinement. The point $x_{1/2}$ of 50% escape probability was seen to be roughly consistent with a previous estimate. More important a probability density function was obtained which should prove useful in investigating the stochastic aspects of galactic cosmic rays.

APPENDIX

We start with an initial density of field lines $f(z, y)$ (for one field line $f(z, y) = \delta(z - z_0)$ where z_0 is the position of the field line). If the material carrying the field lines is displaced in the z direction as a function of position y as $s(y)$ the density function becomes a function of y as

$$f(z, y) = f_0 (z - s(y) + s(0)) \qquad \text{(A-1)}$$

where $f_0(z)$ is the density function at $y = 0$. This may be written as a Taylor expansion using the Taylor expansion operator as

$$f(z, y) = \exp\left[- (s(y) - s(0)) \frac{\partial}{\partial z}\right] f_0(z) \qquad \text{(A-2)}$$

where expanding the exponential in a power series of its argument will generate the conventional Taylor series.

We may now take an ensemble average of (A-2) over the ensemble of possible displacements $s(y)$ assuming that the initial $f_0(z)$ is statistically independent from $s(y)$ we have

$$\langle f(z, y)\rangle = \left\langle \exp\left[- (s(y) - s(0)) \frac{\partial}{\partial z}\right]\right\rangle \langle f_0(z) \rangle \qquad \text{(A-3)}$$

It has been shown (Kubo 1961) that if $s(y)$ represents a Gaussian process the average of the exponential operator greatly simplifies and we have

$$\left\langle \exp\left[-(s(y) - s(0))\frac{\partial}{\partial z}\right]\right\rangle = \exp\left[\frac{1}{2}\langle(s(y) - s(0))^2\rangle\frac{\partial^2}{\partial z^2}\right] \quad \text{(A-4)}$$

assuming $\langle s(y)\rangle = 0$. Expanding

$$\langle(s(y) - s(0))^2\rangle = \langle s^2(y) + s^2(0) - 2s(y)\,s(0)\rangle = 2\langle s^2\rangle(1-\psi(y)) \quad \text{(A-5)}$$

so

$$\langle f(z,y)\rangle = \exp\left[\langle s^2\rangle(1-\psi)\frac{\partial^2}{\partial z^2}\right]\langle f_0(z)\rangle . \quad \text{(A-6)}$$

If we differentiate equation (A-6) with respect to y we can obtain the differential equation for which (A-6) is the solution.

$$\frac{\partial\langle f\rangle}{\partial y} = -\langle s^2\rangle\frac{\partial\psi}{\partial y}\frac{\partial^2}{\partial z^2}\exp\left[\langle s^2\rangle(1-\psi)\frac{\partial^2}{\partial z^2}\right]\langle f_0\rangle = -\langle s^2\rangle\psi'\frac{\partial^2}{\partial z^2}\langle f\rangle . \quad \text{(A-7)}$$

We can immediately see that this is just equation (16) whose solution is expression (14). If we had taken ensemble averages over only that subset of the ensemble for which $s(0) = 0$ we would have obtained a more complicated differential equation, one that is satisfied by expression (15) as we would expect.

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FIGURE CAPTIONS

Figure 1. Outward crossing rate per unit $x = \sqrt{\pi}\ Y/2L$ of field line crossing surfaces $z = \pm z_1$. Curves for three values of $\delta = z_1/2 < s^2 >^{1/2}$ are shown as functions of x.

Figure 2. Outward first crossing rate of field line. Probability density per unit $x = \sqrt{\pi}\ y/2L$ for three values of $\delta = z_1/2 < s^2 >^{1/2}$.

Figure 3. Probability that first crossing point is less than $x = \sqrt{\pi}\ y/2L$ for three values of $\delta = z_1/2 < s^2 >^{1/2}$.

Table I

Mean Length of Field Line

z_1/z_0	$\langle y/y_0 \rangle$
0.5	.027
1.0	.092
2.0	.284
2.2	.326
4.0	.754

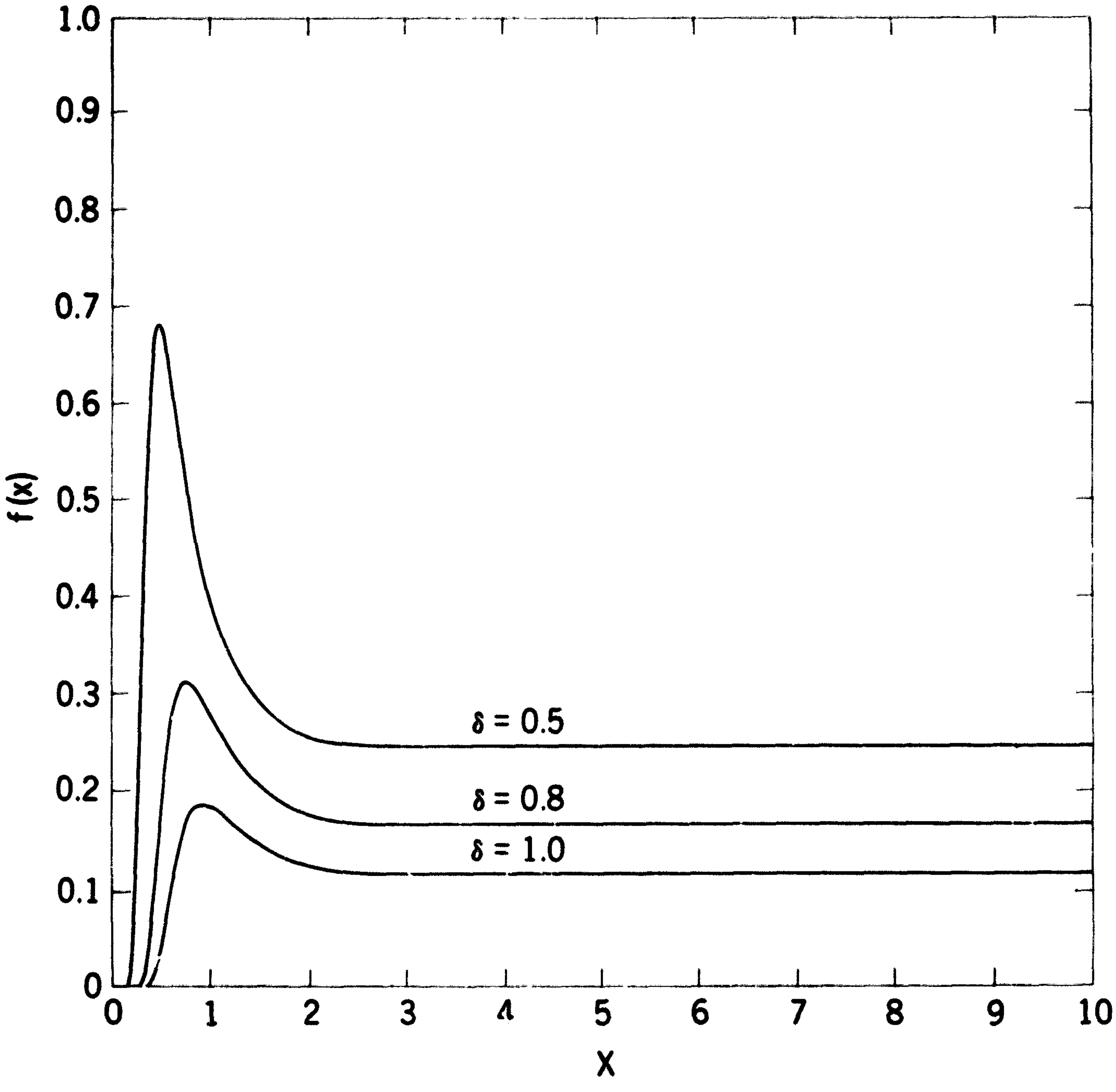

Figure 1. Outward crossing rate per unit $x = \sqrt{\pi}y/2L$ of field line crossing surfaces $z = \pm z_1$. Curves for three values of $\delta = z_1/2 < s^2 >^{1/2}$ are shown as functions of x.

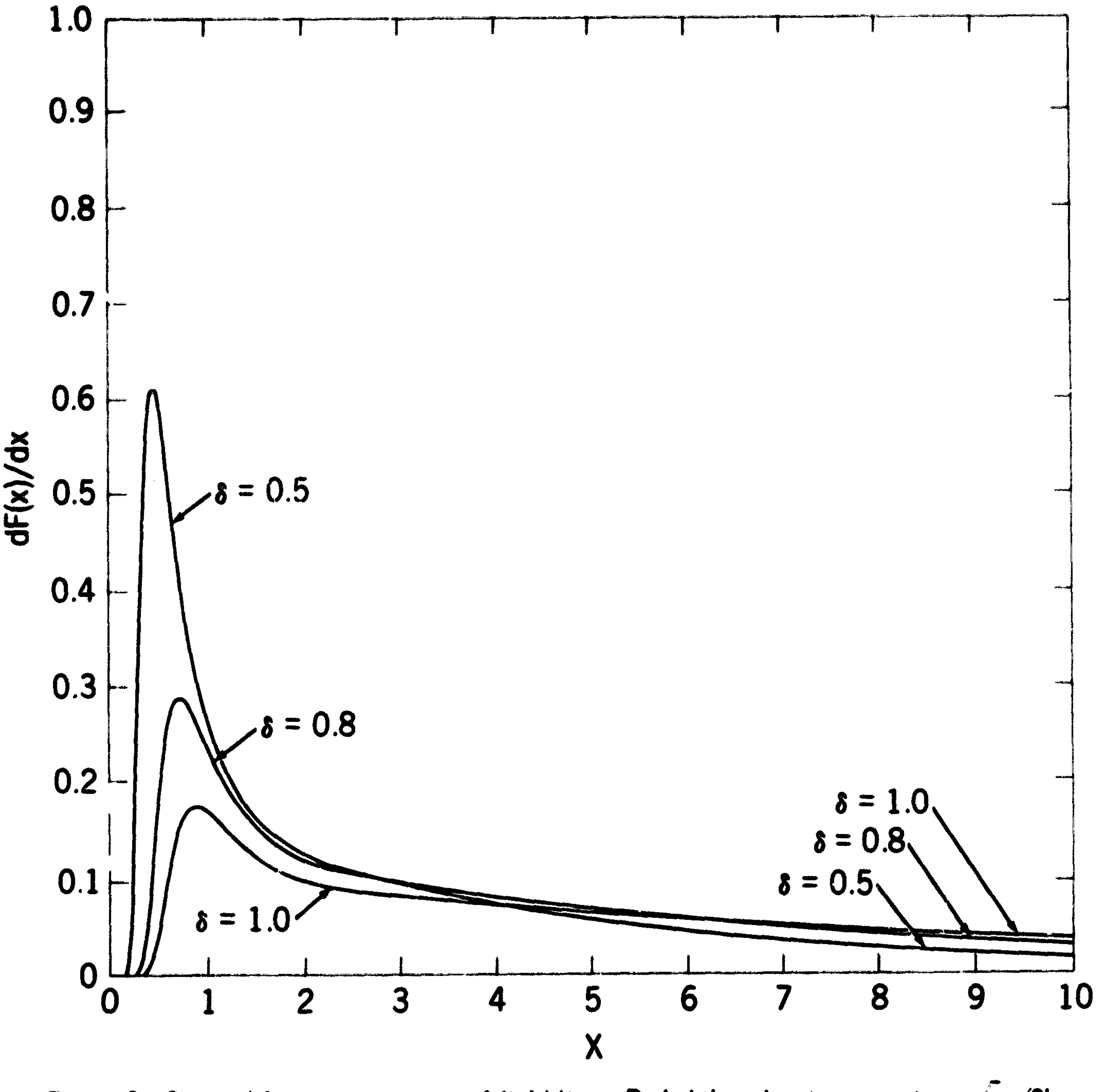

Figure 2. Outward first-crossing rate of field line. Probability density per unit $x = \sqrt{\pi}\, y / 2L$ for three values of $\delta = z_1 / 2 < s^2 >^{1/2}$.

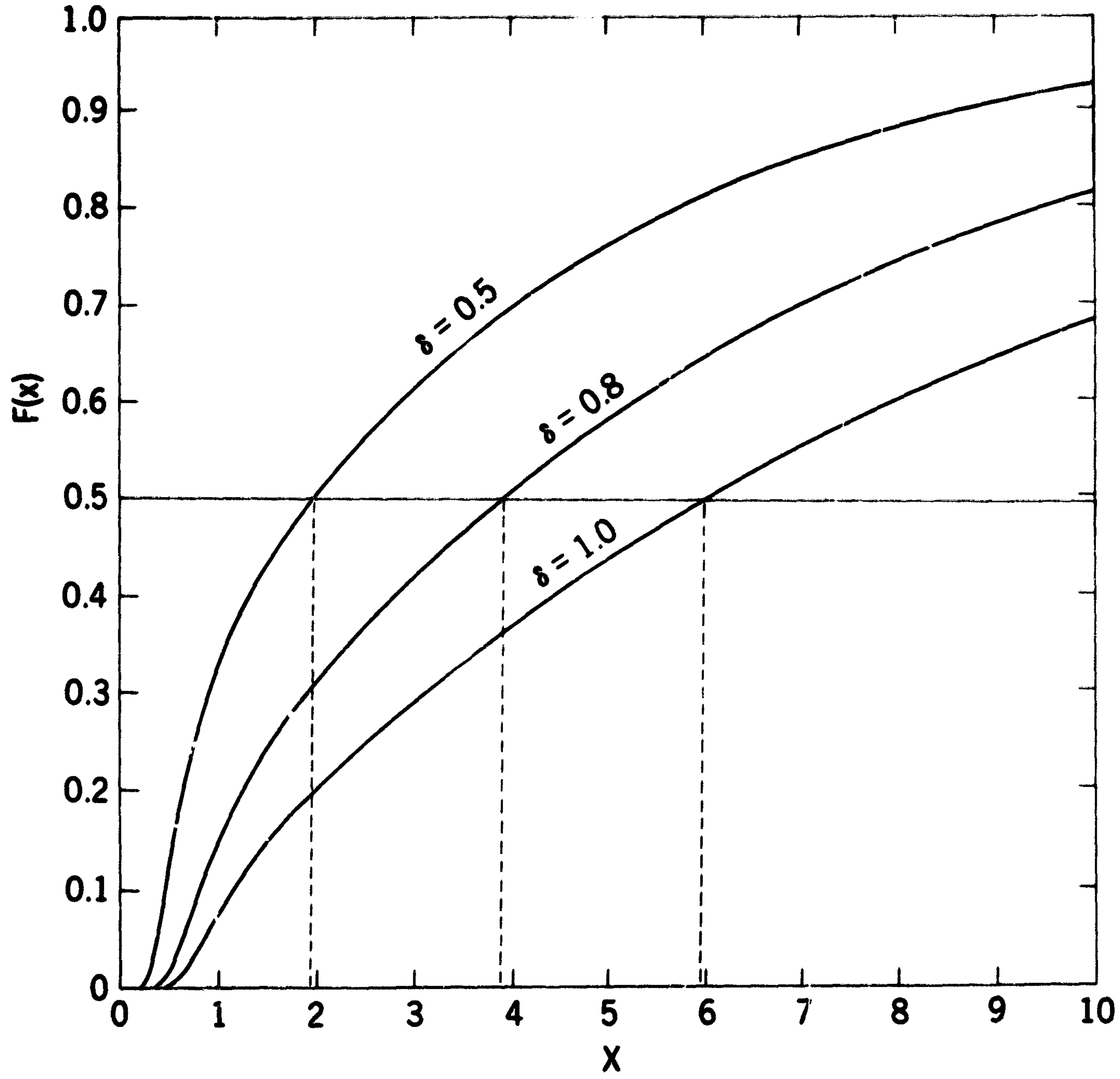

Figure 3. Probability that first crossing point is less than $x = \sqrt{\pi}y/2L$ for three values of $\delta = z_1/2 < s^2 >^{1/2}$.