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**DETERMINATION OF STATISTICS FOR ANY
ROTATION OF AXES OF A BIVARIATE
NORMAL ELLIPTICAL DISTRIBUTION**

By Lee W. Falls and Harold L. Crutcher
Aero-Astroynamics Laboratory

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16. ABSTRACT <p>A method is presented for the determination of the estimates of means, variances, covariances, and correlations for any rotation of axes of a bivariate normal elliptical distribution. This problem arises in connection with vehicle launches at Cape Kennedy, Florida, when winds normal and parallel to the vehicle flight path are launch constraints. A vector wind data sample is resolved into wind components in a rectangular coordinate system, and the bivariate normal elliptical distribution theory is applied to the data sample of component winds. However, theoretical inferences are required for any arbitrary firing azimuth of the launch vehicle. This requires a rotation of axes of the bivariate normal surface and the computation of new statistics after rotation.</p>					
* U. S. Dept. of Commerce, NOAA, Environmental Data Service, National Climatic Center, Asheville, North Carolina.					
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DETERMINATION OF STATISTICS FOR ANY ROTATION OF AXES
OF A BIVARIATE NORMAL ELLIPTICAL DISTRIBUTION

SUMMARY

A method is presented for the determination of the estimates of means, variances, covariances, and correlations for any rotation of axes of a bivariate normal elliptical distribution. This problem arises in connection with vehicle launches at Cape Kennedy, Florida, when winds normal and parallel to the vehicle flight path are launch constraints. A vector wind data sample is resolved into wind components in a rectangular coordinate system, and the bivariate normal elliptical distribution theory is applied to the data sample of component winds. However, theoretical inferences are required for any arbitrary firing azimuth of the launch vehicle. This requires a rotation of axes of the bivariate normal surface and the computation of new statistics after rotation.

I. INTRODUCTION

In 1846, Bravais [1] presented the theory of normal frequency distributions in two and more variables. Most modern statistical textbooks include some discussion of such distributions. Outstanding contributions are notable in the works of Bertrand [2], Czuber [3], Pearson [4], Kluyver [5], and Student [6].

Further developments and some practical applications of these distributions with particular regard to geophysical problems are presented by Crutcher and Baer [7], Crutcher and Moses [8], and Groenewoud, Hoaglin, Vitalis, and Crutcher [9]. Anderson [10] and Hald [11] treat the subject in general form. This report presents a method for the determination of the estimates of means, variances, covariances, and correlations for any rotation of axes of a bivariate normal elliptical distribution. Although the procedure can be extended to the case for more than two variables, this case is not discussed here.

This problem arises in the determination of probabilities of successful launches at Cape Kennedy, Florida. For example, winds exceeding some given value normal and parallel to the flight path may be defined as launch constraints. Those winds normal to the flight path are designated as right and left cross winds, while those parallel to the flight path

are called head and tail winds. A vector wind data sample is resolved into wind components in a rectangular coordinate system and the bivariate normal elliptical distribution theory is applied to the data sample of component winds. However, theoretical inferences are required for any arbitrary firing azimuth of the launch vehicle. Making such inferences requires a rotation of the axes of the bivariate normal surface and the computation of new statistics after rotation.

II. THE BIVARIATE NORMAL ELLIPTICAL DISTRIBUTION

Let x_1 and x_2 be normally distributed variables with parameters (ξ_1, σ_1) and (ξ_2, σ_2) . ξ_1 and ξ_2 are the respective means, while σ_1 and σ_2 are the respective standard deviations. Let ρ be the correlation coefficient, which is a measure of the dependence between x_1 and x_2 . Now, the equation of the two-dimensional normal density function takes the form

$$p(x_1, x_2) = (2\pi \sigma_1 \sigma_2 [1 - \rho^2]^{1/2})^{-1} \exp \left\{ -(2[1 - \rho^2])^{-1} \left[\left(\frac{x_1 - \xi_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x_1 - \xi_1}{\sigma_1} \right) \left(\frac{x_2 - \xi_2}{\sigma_2} \right) + \left(\frac{x_2 - \xi_2}{\sigma_2} \right)^2 \right] \right\}.$$

Given the variables x_1 and x_2 , we calculate the estimates of the bivariate normal parameters. Denote the estimates as \bar{x}_1 , \bar{x}_2 , S_{x_1} , S_{x_2} , and $r_{x_1 x_2}$. A special case of the problem described here is to determine the estimates of the parameters where a rotation of the axes through an angle α reduces the correlation of components to zero. This case is discussed in Crutcher and Baer [7], and Groenewoud, Hoaglin, Vitalis, and Crutcher [9]. From the statistics in the (x_1, x_2) space, we must determine the statistics for any rotation of the axes through an arbitrary angle α in the following way.

Let (y_1, y_2) be the new space after rotation with orthogonal axes y_1 and y_2 .

$$Y_1 = X_1 \cos \alpha + X_2 \sin \alpha \quad (1)$$

$$Y_2 = X_2 \cos \alpha - X_1 \sin \alpha, \quad (2)$$

where α is the angle of rotation, and x_1 and x_2 are the original axes along which the values of X_1 and X_2 are measured. Denote the sample variances of (x_1, x_2) as $S_{x_1}^2$ and $S_{x_2}^2$, respectively. The covariance of (x_1, x_2) is

$$S_{x_1 x_2} = r_{x_1 x_2} S_{x_1} S_{x_2}.$$

For the special case when $S_{x_1 x_2}$ or $r_{x_1 x_2}$ is equal to zero, the variance-covariance matrix is

$$\begin{bmatrix} S_{x_1}^2 & S_{x_1 x_2} \\ S_{x_2 x_1} & S_{x_2}^2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda I$$

or

$$\begin{bmatrix} S_{x_1}^2 - \lambda & S_{x_1 x_2} \\ S_{x_2 x_1} & S_{x_2}^2 - \lambda \end{bmatrix} = 0.$$

Setting the determinant of this matrix equal to zero and solving for the two roots of the quadratic equation is essentially the rotation of axes through an angle α to where components along the new axes are uncorrelated. Thus, the roots λ_1 and λ_2 are the variances along the major and minor axes, respectively, of the bivariate distribution and the covariances (correlations) are zero. From these data, variances and covariances (correlations) along any set of new axes can be determined by rotating the axes from this basic position. Then the matrix will be

$$\begin{bmatrix} S_{y_1}^2 & S_{y_1 y_2} \\ S_{y_2 y_1} & S_{y_2}^2 \end{bmatrix}$$

where the new space is (y_1, y_2) and y_1 and y_2 are the axes.

The statistics in the (y_1, y_2) space may be derived as follows: Let $\bar{Y}_1, \bar{Y}_2, S_{y_1}^2, S_{y_2}^2, S_{y_1 y_2}, r_{y_1 y_2}$ be the means, variances, covariance, and correlation coefficient in the (y_1, y_2) space. Taking mathematical expectations of equations (1) and (2), we have for the new means

$$\bar{Y}_1 = \bar{X}_1 \cos \alpha + \bar{X}_2 \sin \alpha \quad (3)$$

$$\bar{Y}_2 = \bar{X}_2 \cos \alpha - \bar{X}_1 \sin \alpha. \quad (4)$$

Subtracting equation (3) from equation (1), we have

$$(Y_1 - \bar{Y}_1) = (X_1 - \bar{X}_1) \cos \alpha + (X_2 - \bar{X}_2) \sin \alpha, \quad (5)$$

and subtracting equation (4) from equation (2), we have

$$(Y_2 - \bar{Y}_2) = (X_2 - \bar{X}_2) \cos \alpha - (X_1 - \bar{X}_1) \sin \alpha. \quad (6)$$

Let

$$x_1 = (X_1 - \bar{X}_1)$$

$$x_2 = (X_2 - \bar{X}_2)$$

$$y_1 = (Y_1 - \bar{Y}_1)$$

$$y_2 = (Y_2 - \bar{Y}_2)$$

be the deviations from the respective means. Now, equations (5) and (6) become

$$y_1 = x_1 \cos \alpha + x_2 \sin \alpha \quad (7)$$

$$y_2 = x_2 \cos \alpha - x_1 \sin \alpha. \quad (8)$$

Squaring equations (7) and (8) gives

$$y_1^2 = x_1^2 \cos^2 \alpha + x_2^2 \sin^2 \alpha + 2x_1 x_2 \cos \alpha \sin \alpha \quad (9)$$

$$y_2^2 = x_2^2 \cos^2 \alpha + x_1^2 \sin^2 \alpha - 2x_1 x_2 \cos \alpha \sin \alpha. \quad (10)$$

Now, taking expected values of equations (9) and (10), we have for the new variances in the (y_1, y_2) space

$$S_{y_1}^2 = S_{x_1}^2 \cos^2 \alpha + S_{x_2}^2 \sin^2 \alpha + 2S_{x_1 x_2} \cos \alpha \sin \alpha \quad (11)$$

and

$$S_{y_2}^2 = S_{x_2}^2 \cos^2 \alpha + S_{x_1}^2 \sin^2 \alpha - 2S_{x_1 x_2} \cos \alpha \sin \alpha. \quad (12)$$

To derive the covariance, $S_{y_1 y_2}$, in the (y_1, y_2) space, we multiply equation (7) by equation (8) and obtain

$$y_1 y_2 = x_1 x_2 (\cos^2 \alpha - \sin^2 \alpha) + \cos \alpha \sin \alpha (x_2^2 - x_1^2). \quad (13)$$

Taking the mathematical expectation of equation (13) gives the covariance in the (y_1, y_2) space

$$S_{y_1 y_2} = S_{x_1 x_2} (\cos^2 \alpha - \sin^2 \alpha) + \cos \alpha \sin \alpha (S_{x_2}^2 - S_{x_1}^2). \quad (14)$$

Finally, the correlation coefficient in the (y_1, y_2) space is defined as

$$r_{y_1 y_2} = \frac{S_{y_1 y_2}}{S_{y_1} S_{y_2}}. \quad (15)$$

Equations (3), (4), (11), (12), (14), and (15) give the required estimates of the statistics in the (y_1, y_2) space.

A computer program whose input is the statistics in the (x_1, x_2) space and any arbitrary angle of rotation of the bivariate normal surface is available upon request. The output of the program includes the required statistics in the (y_1, y_2) space.

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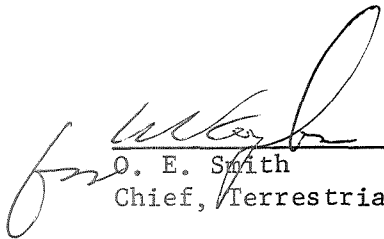
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APPROVAL

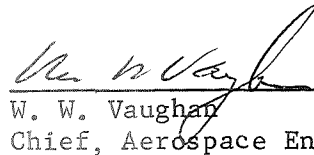
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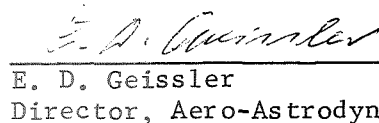
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O. E. Smith
Chief, Terrestrial Environment Branch



W. W. Vaughan
Chief, Aerospace Environment Division



E. D. Geissler
Director, Aero-Astrodynamic Laboratory