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A COMPILATION OF RESULTS PERTAINING TO THE BEHAVIOR OF PHASE LOCKED LOOPS

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I. ABSTRACT

The intent of this work is to provide a summary of some specific results pertaining to what is already known regarding the behavior of Phase Lock Loops:

This work will hopefully serve persons in a systems engineering or management role in readily accessing results which are capable of characterizing existing or envisioned P.L.L. designs.

By focusing on what has been done in this area, this work should bring out what needs to be done in situations which are characterized by constraints which have heretofore been ignored.

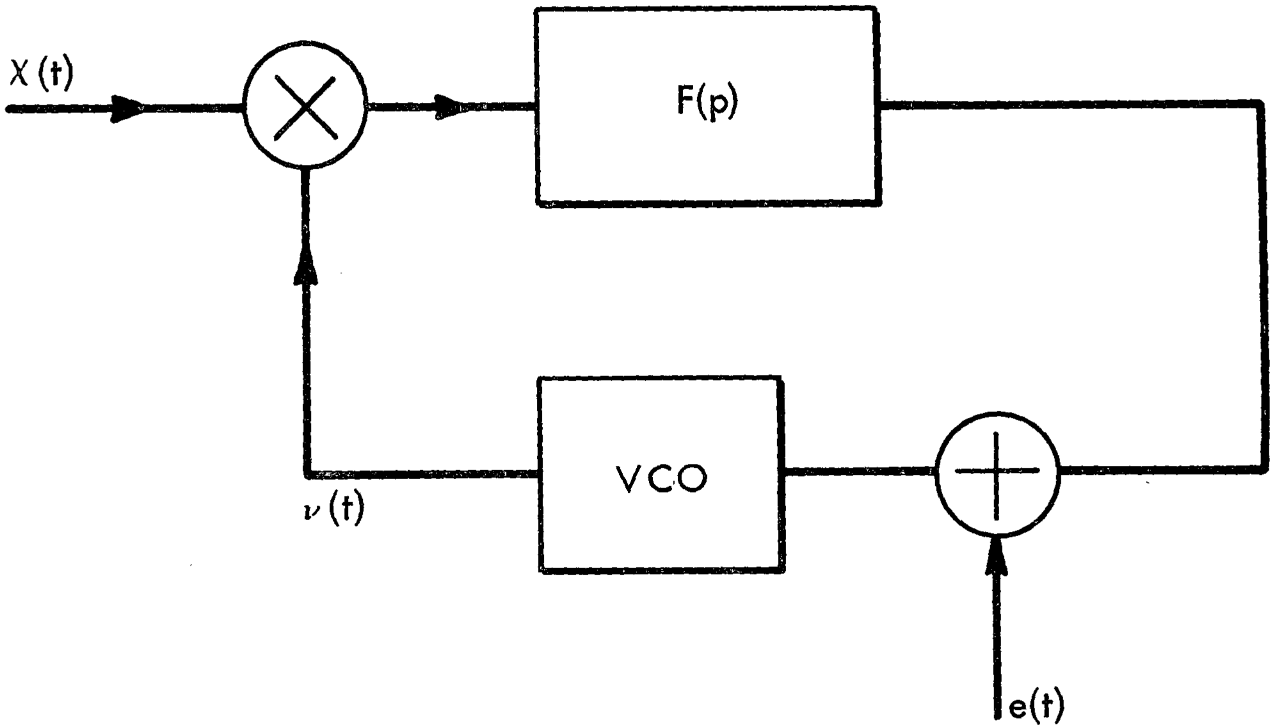
II. INTRODUCTION

Basically a phase locked loop is an electronic servo-mechanism that operates as a coherent detector by continuously correcting the frequency of its local oscillator as a function of the phase error between the incoming signal and that of the local oscillator (VCO), see Figure 1.

Analysis of this relatively simple looking feedback loop has occupied the interest and pocketbooks of more authors and government sponsors than perhaps any other single electronic system. The reason for this stems from its utility in tracking, synchronization and demodulation as well as the mathematical challenge of solving nonlinear, stochastic differential equations.

Because of the mathematical rigor required in the analysis of the P.L.L., analytical effort thus far have developed in steps; wherein each step often embodies different simplifying assumptions. An example of this is that some author's treat the noiseless case for second order loops with constant frequency offset, whereas others may look at a linearized version of the P.L.L. and include only some of the effects of noise.

In addition to keeping track of the results and restrictive assumptions generated by the above mentioned stepwise analyses, one is faced with the additional problem of coping with the evolved segmentation of the entire problem into two phases namely, acquisition and tracking.



$$\chi(t) = A\sqrt{2}\sin\left(\omega_o t + \theta(t)\right) + \eta(t)$$

$$\nu(t) = K_1\sqrt{2}\cos\left(\omega_o t + \hat{\theta}(t)\right)$$

e(t): VCO TUNING VOLTAGE

Fig. 1.

In reality any design would seek to optimize the performance of the P.L.L. with respect to both acquisition and tracking however this is not possible due to the variance in mathematical complexity required of the analyses as well as the conflicting requirements of both phases of operation.

In the interest then, of readily accessing the abundance of work already done, in addition to focusing on the limiting assumptions of each of these results, the following compilation of results is presented.

Following a statement of the overall analysis and design objective, results are presented in a format aimed at clearly identifying: working terminology, inherent assumptions and references for each result presented.

III. ANALYSIS AND DESIGN OBJECTIVE

Referring to Figure 2 one optimization problem which is of theoretical interest is:

$$\min_{g(\cdot),\ \{s(t)\},\ F(p)} G\left\{F(p),\ \{s(t)\},\ g(\cdot)\right\} = w_1 \sigma_\phi^2 + w_2 t_{a\phi}$$

i.e. to minimize the weighted sum of variance of the phase error and acquisition time over all choices of nonlinearity, loop filter and signalling sets. An additional option of swept or fixed VCO could be included in the above optimization problem.

The problem stated above is clearly horrendous. Even the limited steady state version in the absence of initial detuning and for a specified signal set i.e.

$$\min_{F(p),\ g(\ )} G\left\{F(p),\ g(\cdot)\right\} = \sigma_\phi^2$$

has been attacked by only one author [4] in which he obtains the result that in order to minimize the mean square phase error in the presence of zero detuning on should choose $g(\phi) = \text{sign}\left[\sin\phi\right]$ and

$$F(p) = \frac{1 + \tau_2 p}{1 + \tau_1 p}$$

i.e. a second order tracker is optimal when g( ) is specified as above.

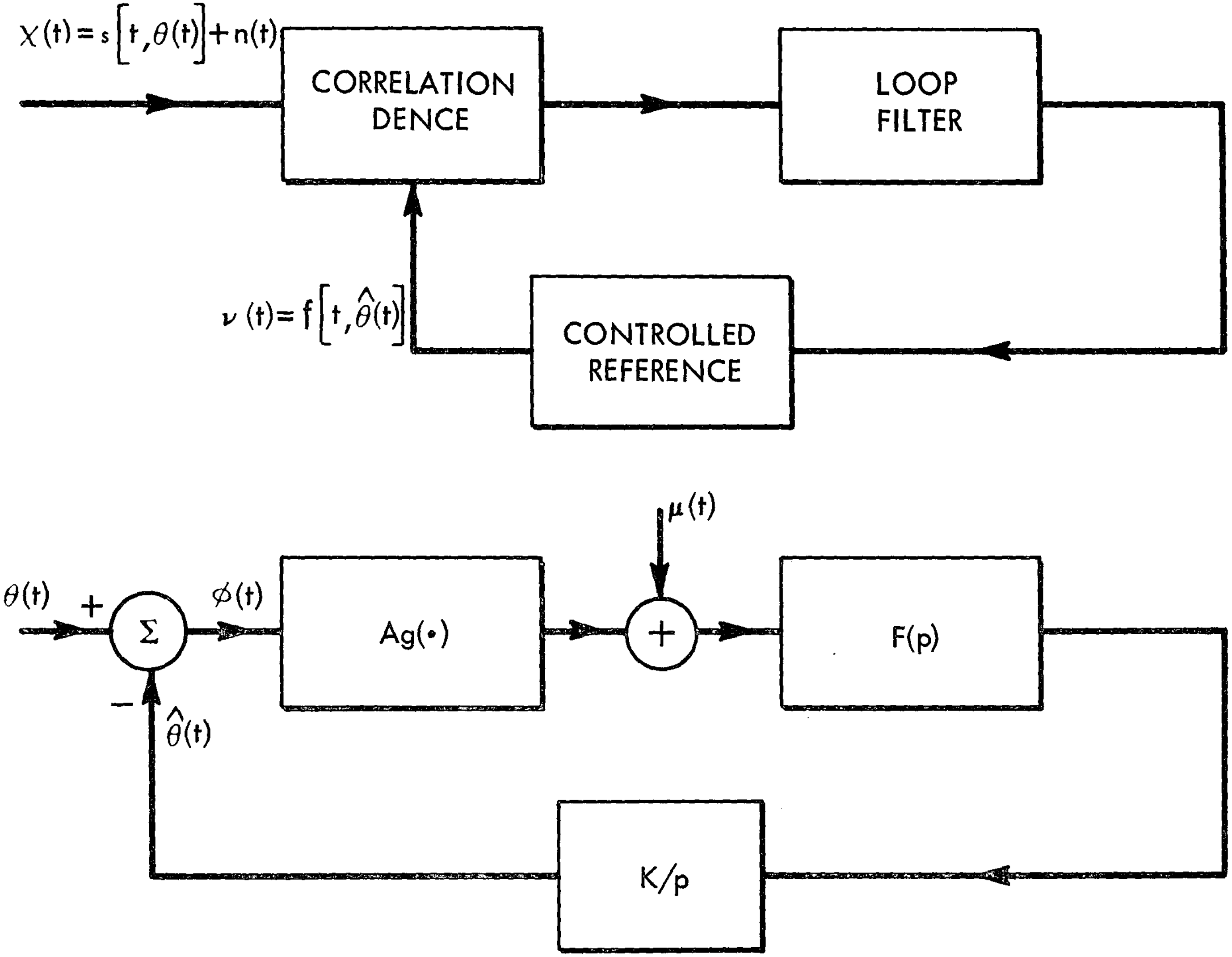


Fig. 2. Generalized tracking loop and equivalent model.

For large signal to noise ratio the optimal combination of $g(\phi)$ and $F(p)$, affords a reduction of phase jitter in the amount of $10 \log_{10}(\rho/2)$ over the conventional second order P.L.L. for which $g(\phi) = \sin\phi$. For smaller S/N the gain decreases.

In practice however one is not generally concerned with the complex optimization problem stated above. The signal set $\{S(t)\}$ is often determined from constraints on power, bandwidth or system complexity, the nonlinearity g( ) is chosen to "match" S(t) in a way which will produce stable lock points and the form of the loop filter F(p) is chosen to minimize the steady state phase or frequency error for a given input.

The remaining problem is generally partitioned into acquisition and tracking in which one searches for a set of design parameters which will yield satisfactory acquisition time while simultaneously providing good tracking behavior. By assigning a time for acquisition based on total communication time available one can then determine, on the basis of available S/N at receiver, the width of the loop bandwidth required. With this information the designer then needs to determine the steady state tracking performance of the P.L.L. configuration. If this performance is unacceptable, in a sense that he requires a narrower loop bandwidth to minimize the effects of noise, then he must allocate more time to acquisition in turn penalizing the available time for data transmission. It is possible to reduce acquisition time by sweeping the local oscillator thru the zone of uncertainty, perhaps in conjunction with a stepwise reduction in loop bandwidth, with the provision that the maximum sweep speed be compatable with available S/N within the loop and sufficient decay time be allowed for transients generated by commutating the loop bandwidth.

The following sections concern themselves with the performance characteristics of particular P.L.L. configuration, hence any further consideration of the overall optimization problem stated above will be ignored.

IV. TERMINOLOGY

The following is intended to serve as definition of terms and symbols used throughout the literature on the subject of P.L.L., many of which are used in this text:

(1) Acquisition: Practically speaking, the loop has acquired when the phase or frequency error remains satisfactorally small over a specified period of time.

(2) Acquisition Time: The time in seconds to achieve acquisition as in (1).

(3) Difference Phase Process: The difference between the input phase process and the estimate of same generated by the VCO. This difference, in the absense of modulation, is expressed herein as a power series ie.

$$\theta d(t) = \sum_{h=0}^{\infty} \theta^{(n)} \frac{t^n}{n!} \quad ; \quad t \geq 0.$$

Thus*: $\theta^{(0)}$ = initial phase uncertainty

$\theta^{(1)}$ = initial frequency uncertainty

$\theta^{(2)}$ = initial frequency rate uncertainty

etc.

(4) Pull in Range: Range of parameters of the difference phase process for which the loop will surely acquire even after slipping cycles.

(5) Steady State: The loop is in the steady state when it has acquired the carrier in the sense specified in (1).

(6) Cycle Slipping: Denotes the condition when the phase error, due to noise, slips out of a (2π) zone of uncertainty. Cycle slipping generally continues with time, in the same way as a diffusion process. Slipping generally occurs asymmetrically in all but the special case when the detuning is zero as discussed in Ref (1).

(7) Threshold: Given that the loop has acquired, the threshold of the loop is expressed as either the minimum signal to noise ratio or the maximum RMS phase error for which the loop will not lose lock (start slipping cycles).

(8) Loop Filter: The block denoted F(p) in Figure 1, where "p" is the conventional Heaviside operator notation. Loop filters frequently encountered in practice, which are used herein are:

$F_1\ (p) = 1.$ First order Loop.

$F_2\ (p) = \dfrac{1 + \tau_2 p}{1 + \tau_1 p}$ Second order Loop with possive integrator.

$F_2^*\ (p) = \dfrac{1 + \tau\ p}{\tau_1 p}$ Second order Loop with high gain active integrator.

(9) $\theta(t)$: input phase process

(10) $\hat{\theta}(t)$: VCO estimate of input phase

(11) $\tilde{\phi}(t)$: Moving phase error process where:

$$\tilde{\phi}(t) = \theta(t) - \hat{\theta}(t)$$

(12) $\phi(t)$: Phase error process reduced modul 2π

(13) g( ): Loop nonlinearity depending on choice of phase comparator.

(14) e(t): VCO tuning voltage. Used to compensate for a known steady state error and/or for sweeping (usually linearly) through the initial zone of frequency uncertainty.

(15) δlock: The number of radians which adequately characterizes the phase lock condition ie if $\phi < \delta$lock for some time interval, then the loop is said to have acquired phase lock.

(16) A: RMS level of the input carrier.

(17) n(t): Additive input noise process taken to be, stationery, white, Gaussian noise with one sided spectral density No watts/Hz.

(18) W_n: Loop natural frequency: Table 1 relates W_n to loop filter parameters.

(19) ζ: Loop damping. Table 1 relates ζ to loop filter parameters.

(20) $B_L^{(1,2)}$: onesided loop bandwidth for first or second order loops respectively.

$$B_L = \int_0^\infty |H(jw)|^2\, df$$ where H(jw) is the transfer function of the overall loop.

(21) K: Overall loop gain where:

$$K = Km \cdot K_{VCO} \cdot K_1 \cdot F(O) \qquad \text{(rad/volt-sec)}$$

where: Km = multipler gain $(\text{volts})^{-1}$
$K_{V\dot{C}O}$ = VCO constant (rad/volt-sec)
K_1 = RMS gain of VCO (volts)
F(O) = Loop filter DC gain

(22) $t_{a\phi}$: Time to acquire phase lock, sec.

(23) $t_{a\dot{\phi}}$: Time to acquire frequency lock, sec.

(24) $\overline{S}$: Average number of cycles slipped per unit time.

(25) T_N : mean time to lose lock.

(26) P_{st} : Probability of losing lock in time "t".

(27) ϕss: Steady state phase error (no noise).

(28) ϕss: Steady state frequency error (no noise).

(29) $\overline{\phi}$: Mean phase error reduced mod 2π.

(30) $\overline{\dot{\phi}}$: Mean rate of o (t).

(31) $I_v(x)$: Imaginary bessel function of order v and argument x.

(32) σ_ϕ^2: "Phase jitter" or variance of phase error process reduced mod. 2π.

(33) T: Expected time between successive phase jumps.

(34) (SNR)L: Signal to noise ratio in loop bandwidth.

(35) ζ: Used interchangeably with (SNR)L.

Table 1

The following table is included to relate the overall loop parameters to the individual parameters of the specific loop filter being employed.

Loop Filter	Natural frequency	Loop Damping	Loop Bandwidth
F_1 (p)	Not applicable	not applicable	$B_L^{(1)} = \frac{AK}{4}$ one sided
F_2 (p)	$Wn^2 = \frac{AK}{\tau_1}$	$\zeta = \frac{1}{2}\left(\frac{AK}{\tau_1}\right)^{1/2}\left(\tau_2 + \frac{1}{AK}\right)$ $r = 4\zeta^2$	$W_L^1 \approx \frac{r+1}{2\tau_2}$ when $r\tau_1 \gg \tau_2$
F_2^* (p)	$Wn^2 = \frac{AK}{\tau_1}$	$\zeta = \frac{\tau_2}{2}\left(\frac{AK}{\tau_1}\right)^{1/2}$	$B_L^{(2)} = \frac{Wn}{2}\left(\tau + \frac{1}{4\zeta}\right)$ one sided

* All derivatives, unless otherwise indicated, are with respect to time.

V. RESULTS

A. Index of Results:

EQUATION NUMBER	NATURE OF RESULT	Bibliography REF. ID.
(1)	General describing equation.	(2)
(2)	Describing equation given no noise and $g(\phi) = \sin\phi$.	(*)
(3)	Describing equation for a restricted phase difference process.	(*)
(4)	Same as (3) for a second order loop.	(*)
(5)	Pull in range for first order loop (no noise).	(2)
(6)	Pull in range for passive second order loop (no noise).	(2)
(7)	Pull in range for high gain second order loop (no noise).	(2)
(8)	Upperbound on VCO sweep rate (no noise).	(2)
(9)	Steady state phase error for first order loop (no noise).	(2)
(10)	Steady state frequency error (no noise) for first order loop.	(2)
(11)	Steady state phase error (no noise) for passive second order loop.	(2)
(12)	Steady state phase error (no noise) for active second order loop.	(2)

(13)	Steady state phase error (no noise) for active second order loop with swept VCO.	(2)
(14)	Phase acquisition time (no noise) for first order loop.	(2)
(15)	Frequency acquisition time (no noise) for active second order order loop.	(2)
(16)	Phase acquisition time for active second order loop.	(2)
(17)	Simulated acquisition in noise.	(3)
(18)	An approximate upperbound on VCO sweep rate in noise.	(3)
(19)	"Rules of Thumb" regarding second order loops.	(13,) (14)
(20)	Probability density of reduced phase error process for a first order loop.	(8)
(21)	Mean steady state phase error in noise for first order loop.	(8)
(22)	Probability density of reduced phase error process for second order loop.	(1)
(23)	Mean steady state phase error in noise for second order loop.	(1)
(24)	Mean steady state frequency error in noise for second order loop.	(1)
(25)	An approximation to (22)	(8)

(26)	A worst case estimate of frequency acquisition time in noise using (17).	(*)
(27)	Phase jitter for first order loop from Linear Theory.	(2)
(28)	Phase jitter for first order loop from Nonlinear Theory.	(8)
(29)	A probability bound on the absolute phase error for second order loops.	(1)
(30)	Phase jitter for passive second order loops.	(1)
(31)	Average number of cycle slips per unit time.	(1)
(32)	Mean time to lose lock.	(1)
(33)	Expected time interval between successive cycle slips.	(1)
(34)	Probability of "k" phase jumps in "t" seconds.	(1)
(35)	Probability of losing lock.	(1)
(36)	Bit error probability using a correlation detector.	(S)
(37)	Approximation to (36) for slowly varying phase error process.	(S)
(38)	Approximation to (36) for rapildy varying phase error process.	(S)

B. Describing equations:

General Equation:

$$\left\{\theta_d(t) - K_{VCO}\frac{de}{dt}\right\} = \phi(t) + AK\frac{F(p)}{p}\left[g(\phi) + \frac{n(t)}{A}\right] \quad (1)$$

Describing Equation given no noise and sinusoidal nonlinearity.

$$\left\{\dot{\theta}_d(t) - K_{VCO}\frac{de}{dt}\right\} = \dot{\phi} + AKF(p)\sin\phi \quad (2)$$

Describing Equation for an unmodulated carrier and a restricted phase difference process.

Given: $\theta_d(t) \cong [\theta^{(0)} + \theta^{(1)} t]$

$F(p) = F_1(p)$: First order loop

$$\left\{\theta^{(1)} - K_{VCO}\, e\right\} = \dot{\phi} + AK\sin\phi \quad (3)$$

Given: $\theta_d(t) \approx \left[\theta^{(0)} + \theta^{(1)} t + \theta^{(2)} \frac{t^2}{2}\right]$

$F(p) = F_2(p)$: Second order loop

$$\theta^{(2)}\left[\tau_1 + t\right] + \left[\theta^{(1)} - eK_{VCO}\right] = \tau_1\ddot{\phi} + \dot{\phi}\left[1 + \tau_2 AK\cos\phi\right] + AK\sin\phi. \quad (4)$$

C. Results pertaining to Acquisition Behavior

1. No noise:
   Pull in Range of First Order Loop:

   Given: $\theta^{(2)} = 0.$, $F(p) = F_2(p)$, fixed VCO.

$$\theta^{(1)} < AK \quad (5)$$

All $\theta^{(0)}$

Pull in Range of Second Order Loop:

Given: $\theta^{(2)} = 0.$, $F(p) = F_2(p)$, fixed VCO.

$$\left|\theta^{(1)}\right| < 2\left[\left(\frac{AK}{\tau_1}\right)\cdot\left(1+\frac{1}{2}AK\tau_2\right)\right]^{1/2} \text{; all } \theta^{(0)}$$
$$= \sqrt{2}\left[2\zeta\omega_n AK - \omega_n^2\right]^{1/2} \quad (6)$$

Given: $\theta^{(3)} = 0.$, $F(p) = F_2^*(p)$, fixed VCO.

$$\theta^{(2)} < \frac{AK}{\tau_1} = \omega_n^{\ 2} \quad (7)$$

All $\theta^{(1)}$ and $\theta^{(0)}$

Given: $\theta^{(3)} = 0.$, $F(p) = F_2^{\ *}(p)$, swept VCO.

$$\left|K_{VCO}\frac{de}{dt}\right| < \left[\frac{AK}{\tau_1} - \theta^{(2)}\right] = \omega_n^{\ 2} - \theta^{(2)} \quad (8)$$

All $\theta^{(1)}$ and $\theta^{(0)}$.

Steady State Errors (No noise)

Given: $\theta^{(2)} = 0.$, $F(p) = F_1(p)$, fixed VCO.

$$\phi_{ss} = \sin^{-1}\left[\frac{\theta^{(1)}}{AK}\right] \text{rad.} \quad (9)$$

$$\boxed{\phi_{ss} = 0.} \qquad (10)$$

<u>Note:</u> $\phi_{ss} \to \infty$ if $\theta^{(2)} \neq 0.$

<u>Given:</u> $\theta^{(2)} = 0.$, $F(p) = F_2(p)$, fixed VCO.

$$\boxed{\phi_{ss} = \sin^{-1}\left[\frac{\theta^{(1)}}{AK}\right] \text{rad.}} \qquad (11)$$

<u>Given:</u> $\theta^{(3)} = 0.$, $F(p) = F_2{}^*(p)$, fixed VCO.

$$\boxed{\phi_{ss} = \sin^{-1}\left[\frac{\tau_1 \theta^{(2)}}{AK}\right] = \sin^{-1}\left[\frac{\theta^{(2)}}{\omega_n{}^2}\right] \text{rad.}} \qquad (12)$$

<u>Given:</u> $\theta^{(2)} = 0.$, $F(p) = F_1(p)$ or $F_2(p)$, swept VCO.

$$\boxed{\phi_{ss} = \sin^{-1}\left[\frac{\theta^{(1)} - e\,K_{VCO}}{AK}\right] \text{rad.}} \qquad (13)$$

<u>Acquisition Time</u> (No noise)

<u>Given:</u> $\theta^{(2)} = 0.$, $F(p) = F_1(p)$, fixed VCO.

$$\boxed{t_{a\phi} = \frac{\sqrt{1 + \left[\frac{\theta^{(1)}}{4B_L^{(1)}}\right]^2}}{2B_L^{(1)}} \cdot \ln\left(\frac{2}{\zeta_{LOCK}}\right) \text{ sec.}} \qquad (14)$$

<u>Given:</u> $\theta^{(2)} = 0.$, $F(p) = F_{2*}(p)$, fixed VCO.

$$\boxed{t_{a\dot{\phi}} \approx \frac{1}{\tau_2}\left[\frac{\theta^{(1)}\tau_1}{AK}\right]^2 = \frac{\left[\theta^{(1)}\right]^2}{2\zeta\,\omega_{n^3}} \text{ sec.}} \qquad (15)$$

$$\boxed{\underset{\text{2nd order}}{t_{a\phi}} \approx t_{a\dot{\phi}} + t_{a\phi}\ \text{(first order)}} \qquad (16)$$

2. Noise Included

In the presence of noise the entire acquisition problem including determination of pull in range, acquisition time, steady state errors and in fact the term acquisition has meaning only in a probabalistic sense. All of the above deterministic quantities computed for no noise, now become random variables and/or random processes that are characterized by joint probability density functions.

Because of the nature of the problem coupled with the mathematical difficulty of dealing with nonlinear equations with stochastic inputs, analytical results for the transient or acquisition phase have not been determined. Analytical results are available however, pertaining to "mean" steady state errors. Results pertaining to the acquisition problem are available only for special simulated situations. Some of the more complete simulation work available [Ref. 3] is presented here in graphical form in fig 3. These results give an indication of the probability of acquisition as a function of S/N, VCO sweep rate, loop damping and loop bandwidth for a second order loop when $\theta = 0$. To apply these results in or system configuration (Figure 1) let:

$$\beta_{no} = 2B_L^{(2)} \tag{17}$$

$$\left(\frac{S}{N}\right)_{OUT} = (SNR)_L .$$

Empirically derived upperbound on VCO Sweep Rate in Noise:
Based on (17) for 90% probability of acquisition:

$$\boxed{R_{max} = \frac{\left[1 - (SNR)_L^{-1/2}\right]\omega_{n^2}}{1 + d} \quad Hz/sec^2} \tag{18}$$

where: $(SNR)_L$ = signal to noise ratio in loop.

$$d = \text{exo}\left(-\zeta\pi/\sqrt{1 - \zeta^2}\right); \ \zeta < 1$$

$$= 0; \ \zeta \geq 1.$$

Note: $\left\{ \underset{SNR \to \infty}{\text{lm}} \{R_{max}\} \right\} = \omega_{n^2}.$

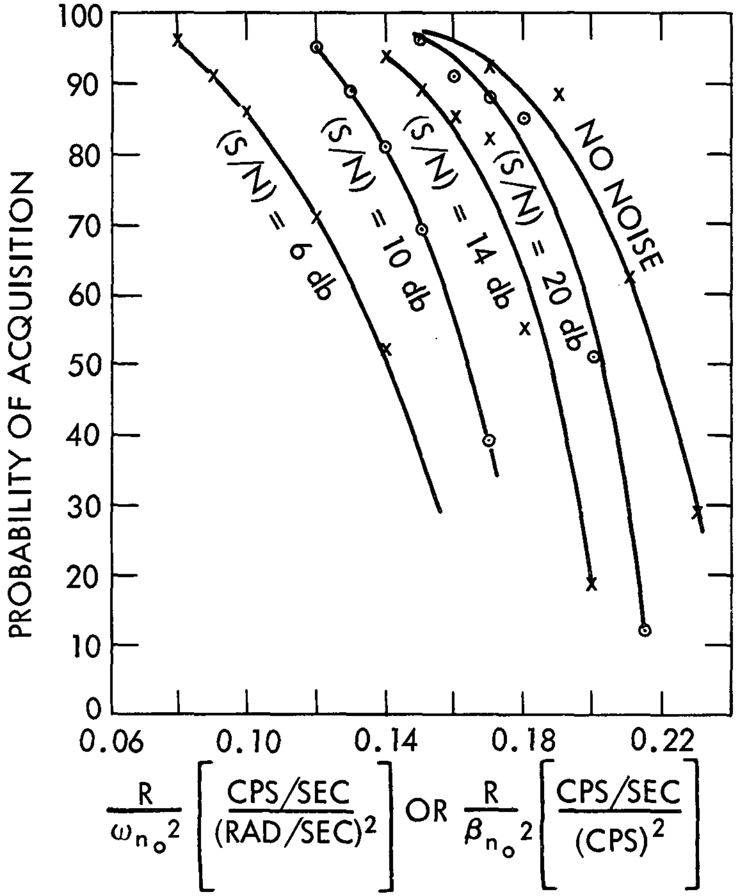


Fig. 3a Probability of acquisition vs normalized sweep rate ($\zeta = 1/2$).

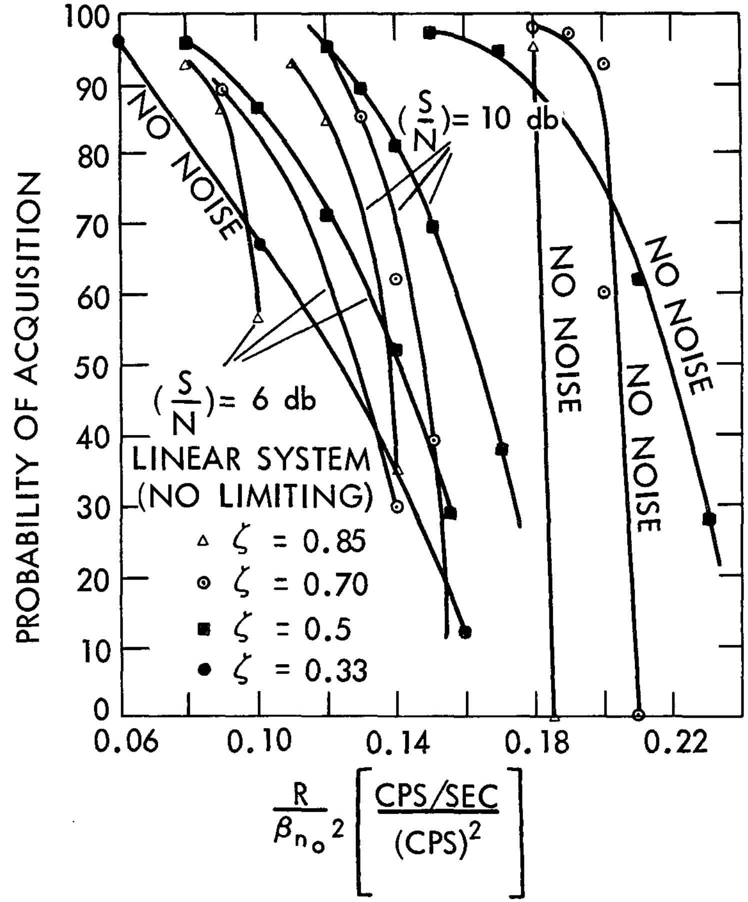


Fig. 3b Probability of acquisition vs normalized sweep rate for various damping factors.

Some "Rules of Thumb" regarding second order loop with noise. Although rigorous results pertaining to acquisition parameters in noise are sketchy it should be mentioned that some widely used "rules of thumb" regarding acquisition behavior dependence on (SNR)L for second order loops are as follows. Ref (13)

1. It is nearly impossible to acquire frequency lock if

$$(SNR)L \leq 0 \text{ DB.} \tag{19}$$

2. Generally a (SNR)L of 8 DB. is considered acceptable in order to acquire frequency lock as long as the carrier is within the required pull in range. The smaller $\theta^{(1)}$ the lower the required (SNR)L for acquisition.

3. The loop generally loses lock, once acquired, if (SNR)L falls below (1.34 - 0) DB.

Steady State Errors (Noise Included)

In the presence of noise one is generally concerned about the mean and variance of both phase and frequency errors over some period of time while the loop is in the steady state.

The following results, some exact and others approximate or asymptotic, were derived using nonlinear analysis techniques and concern the reduced phase error process ϕ (t). Obtaining these results was possible because the reduced phase error process is stationary with bounded variance and submits to the Fokker Plank method Ref (8).

For the tracking application the statistics of the moving phase process ϕ(t) are of interest and are discussed in a later section.

Given: $F(p) = F_1(p); \ \theta^{(1)} = 0.$, fixed VCO.

$$\boxed{p(\phi) = \frac{\exp(\alpha \cos\ )}{2\pi I_o(\alpha)}; -\pi \leq \phi \leq \pi} \tag{20}$$

Where: $\alpha = \dfrac{A^2}{N_o B_L^{(1)}}$; also see Figure 4

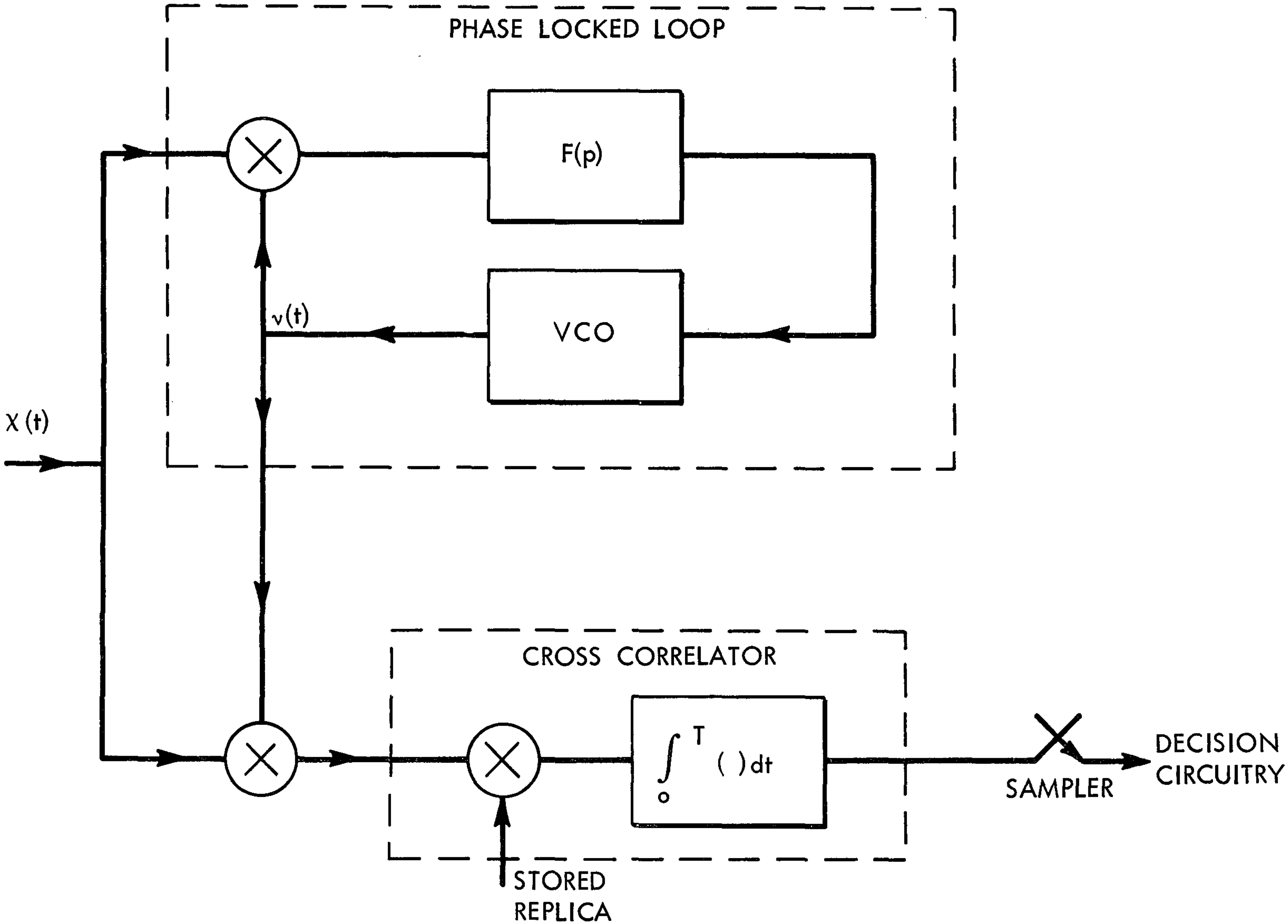


Fig. 4. Coherent receiver/data detector mechanization.

$$\boxed{\overline{\phi}_{ss} = 0.} \tag{21}$$

Given: $F(p) = F_2(p)$; $\theta^{(2)} = 0.$, fixed VCO.

$$\boxed{p(\phi) = \frac{\exp\left[\beta\phi + \alpha\cos\phi\right]}{4\pi^2 e^{-\pi\beta} \cdot \left|I_{j\beta}(\alpha)\right|^2} \cdot \int_{\phi}^{\phi+2\pi} \exp\left[-\beta x - \alpha\cos x\right]dx \; ; \left|\phi\right| < \pi} \tag{22}$$

$$\boxed{\overline{\phi} = \frac{\sin h\pi\beta}{\pi\left|I_{j\beta}(\alpha)\right|^2} \cdot \sum_{m=1}^{\infty}{}^{1}_{1} \frac{m\, I_m(\alpha)}{m^2+\beta^2}\left[\frac{I_o(\alpha)}{m} + \frac{I_m(\alpha)}{4m} + \sum_{\substack{K=1\\K\neq m}}^{\infty}{}^{1}_{1} \frac{2m(-1)^K I_K(\alpha)}{m^2 - K^2}\right],} \tag{23}$$

$$\boxed{\overline{\dot{\phi}} = \theta^{(1)} - AK\overline{\sin\phi}} \tag{24}$$

Where: $r = AK\dfrac{\tau_2^{\,2}}{\tau_1} = 4\zeta^2$: damping measure

$$F_1 = \frac{\tau_2}{\tau_1}$$

$$\beta = \left(\frac{r+1}{r}\right)^2 \cdot \frac{\rho}{2W_L} \cdot \left[\theta^{(1)} - AK(1 - F_1)\overline{\sin\phi}\right].$$

$$\alpha = \left(\frac{r+1}{r}\right)\rho - \frac{1 - F_1}{r\sigma_G{}^2}$$

ζ : loop damping factor.

$W_L \approx \dfrac{(r+1)}{2\tau_2}$ when $r\tau_1 >> \tau_2$: loop bandwidth

Note: $W_L \approx B_L^{(2)}$ when max Loop transfer coincides with incoming carrier frequency.

$$\rho = \frac{A^2}{N_O W_L} \quad : \text{S/N in Loop bandwidth}$$

$$\overline{\sin\phi} = \text{Im}\left\{\frac{I_{1-j\beta}(\alpha)}{I_{-j\beta}(\alpha)}\right\}.$$

$$G = \sin\phi - \overline{\sin\phi}$$

$$\sigma_G^2 = E_\phi\left[(G \quad \bar{G})^2\right].$$

An approximate result for (22)

Given: $F(p) = F_2{}^*(p)$, large $\bar{\alpha}$

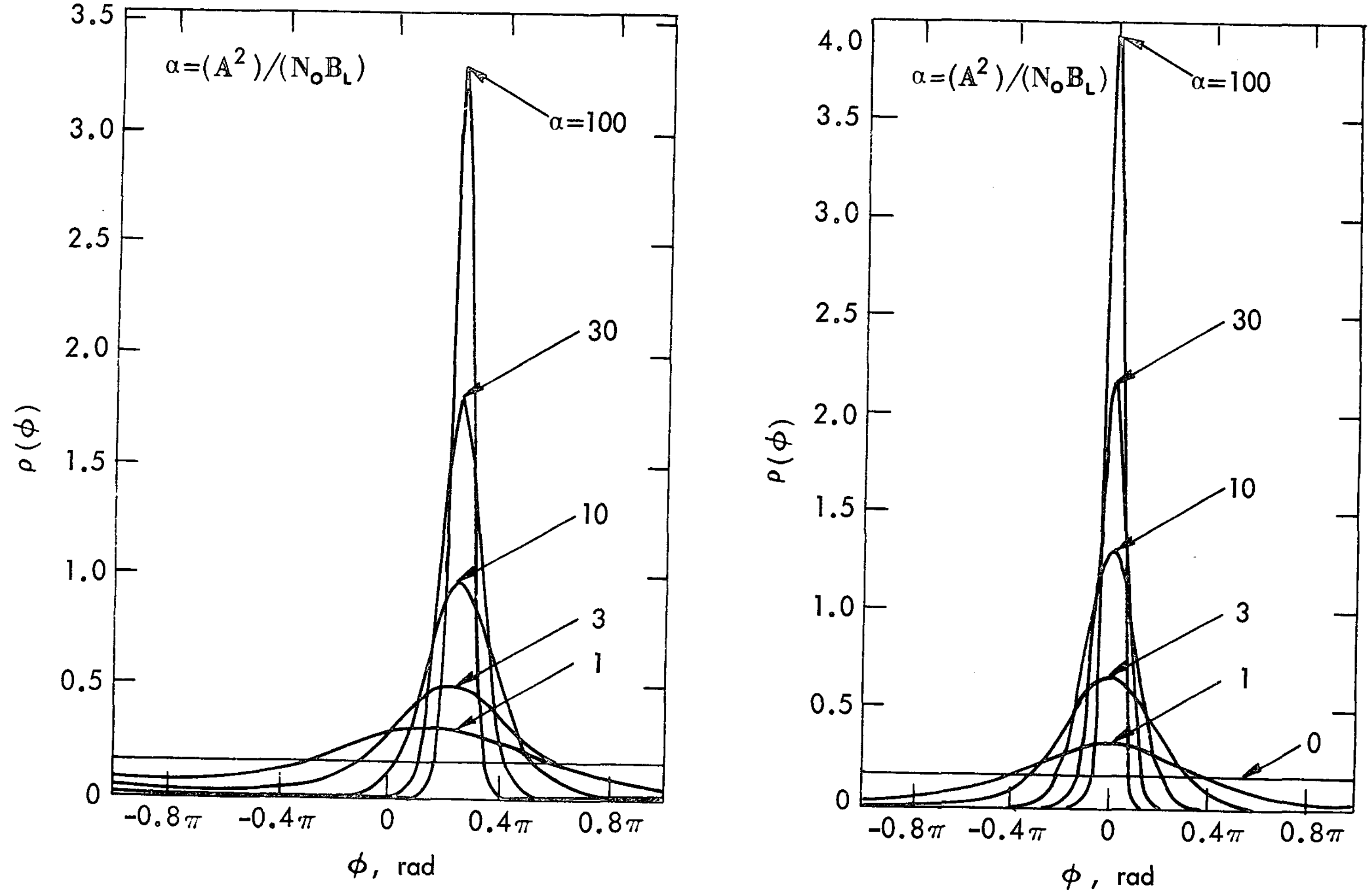
$$p(\phi) \approx \frac{\exp(\bar{\alpha}\cos\phi)}{2\pi I_o(\bar{\alpha})} \quad ; \text{ also see Figure 5} \tag{25}$$

where: $\bar{\alpha} = \dfrac{A^2}{N_O B_L^{(2)}}$

Acquisition Time (In noise)

As already mentioned, exact analytical results for acquisition time in noise are not available. It is possible nowever to use the simulated results presented earlier to obtain a worst case approximation to the acquisition time for a given probability of acquisition. This can be accomplished by dividing the maximum allowable sweep rate into the known initial zone of uncertainty, i.e.

$$t_{a\phi} \approx \frac{\Delta f}{R_{90}} \tag{26}$$



(a) First-order loop steady-state probability densities for $(\omega-\omega_o)/AK = \sin(\pi/4)$.

(b) First-order loop steady-state probability densities for $\omega = \omega_o$.

Fig. 5. Steady-state pdf - first-order loop. (Reprinted from Ref. [8] page 1744)

D. Results Pertaining to Tracking Behavior:

Since tracking with no noise is of little interest, all of the results presented in this section are concerned with the effects of noise.

Steady State Errors: See (21) and (23).

Phase Jitter: Variance of Phase error process

Given: $F(p) = F_1(p)$, $\sin\theta \approx \theta$ (linear theory),

$$\sigma_\phi^2 = \frac{1}{2(SNR)_L} = \frac{B_L^{(1)} N_O}{A^2} \tag{27}$$

SNR → ∞

Given: $F(p) = F_1(p)$. (nonlinear theory.)

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{I_n(\alpha)}{n^2 I_o(\alpha)} \tag{28}$$

Where: $\alpha = A^2 / N_O B_L^{(1)}$

Note: $\lim \sigma_\phi^2 = \pi^2/3$.

$$\text{Prob}\left[\,|\phi| < \phi_1\right] = \frac{\phi_1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{I_n(\alpha)\sin n\phi_1}{n I_o(\alpha)} \tag{29}$$

$0 < \phi_1 < \pi$

Given: $F(p) = F_2(p)$, nonlinear theory

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n I_n(\alpha)}{n^2 I_o(\alpha)} \text{ ; also see Figure 6} \tag{30}$$

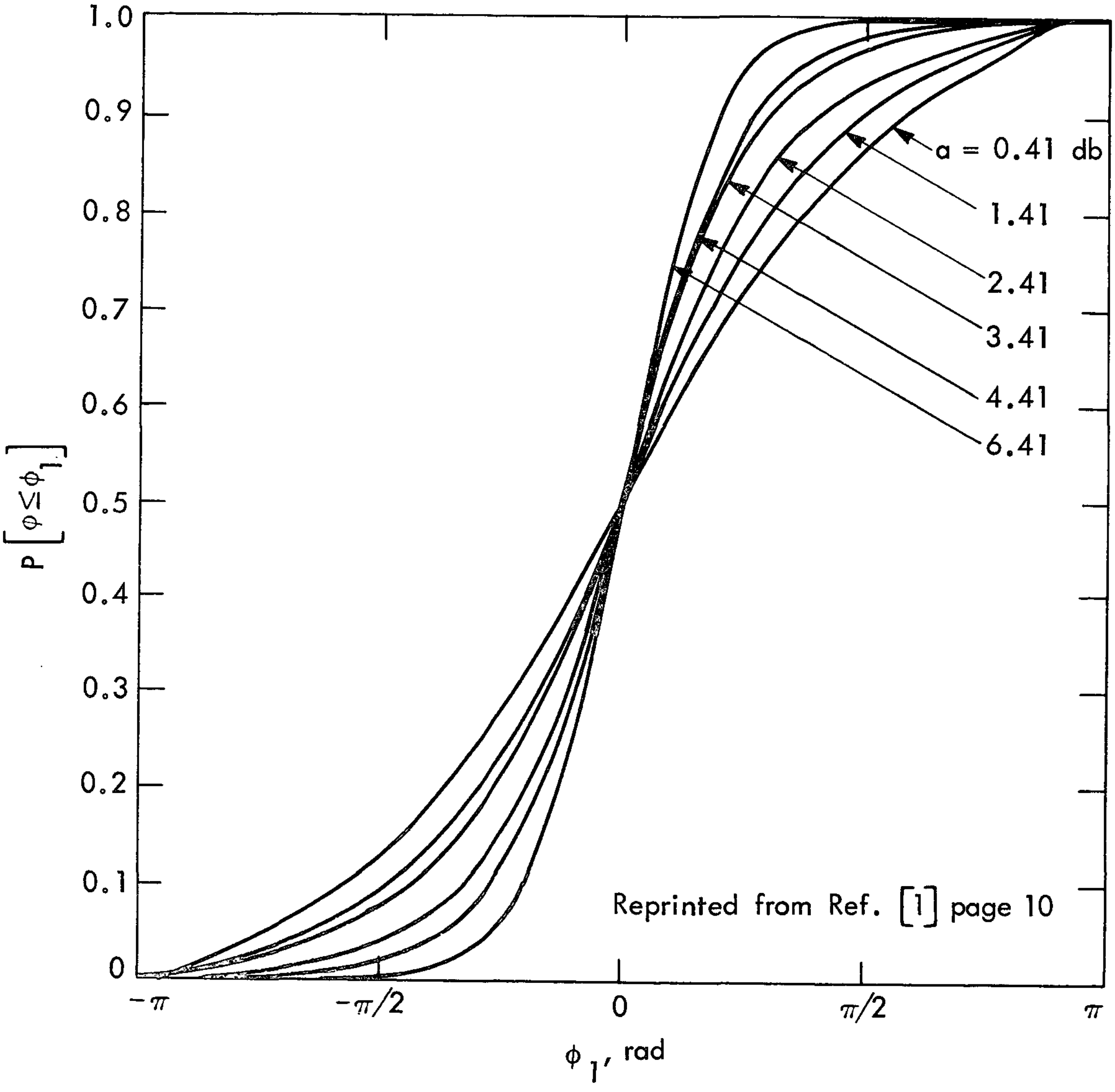


Fig. 6. Cumulative distribution of the measured phase error – second order loop.

Cycle Slipping: All of the following results were obtained for second order loops by using nonlinear methods as applied to the moving phase error process.

Average number of cycle slips per unit time.

$$\bar{S} = \frac{1}{4\pi^2} \cdot \left[\frac{4W_L r^2}{\rho (r + 1)^2} \cdot \frac{\cosh \pi\beta}{\left| I_{j\beta} (\alpha) \right|^2} \right] \text{ ; see FIG 7} \qquad (31)$$

See (24) for symbol definitions.

Mean time to loss of lock.

Given: $F(p) = F_2*(p), \zeta = .707$

$$T_{AV} \approx \frac{2}{W_n} \exp \left[\pi \, (SNR)_L \right] \quad \text{sec.} \qquad (32)$$

Expected time interval between seccessive slips.

$$\Delta T \approx \frac{\pi^2 \rho (r + 1)^2}{r^2 W_L} \cdot \frac{\left| I_{j\beta} (\alpha) \right|}{\cosh \pi\beta} \quad \text{sec.} \qquad (33)$$

See (24) for symbol definition.

Probability of "k" phase jumps in "t" seconds.

$$p(S) \approx \frac{(\bar{S}t)^k \cdot \exp (\bar{S}t)}{k!} \quad \text{: Poisson.} \qquad (34)$$

Probability of losing lock.

$$p(S \geq 1) \approx 1 - \exp (\bar{S}t) . \qquad (35)$$

E. Results Pertaining to Detection:

If one uses a P.L.L. in conjunction with a single correlation detector Figure 4 (a mechanization which is optimal for binary antipodal PSK), the effect of the phase error process on detection error rates is important. The following results Ref (5) are available.

Bit Error Probability (BEP) using a correlation detector.

$$P_E = \int_{-\pi}^{\pi} p(\phi) \operatorname{erfc}\left[\sqrt{2\rho}\ \cos\phi\right] d\phi \qquad (36)$$

Given: Slowly varying phase process

$$P_E \approx \frac{1}{2}\left[1 - \frac{\sqrt{\rho}}{\pi} e^{-\rho/2} \sum_{n=1}^{\infty} \epsilon_K (-1)^K I_K(\rho/2) \cdot \left\{ \mathrm{Re}\frac{[b_{2K+1-j\beta}]}{2K+1} - \mathrm{Re}\frac{[b_{2K-1-j\beta}]}{2K-1} \right\}\right]. \qquad (37)$$

Where: $b_{2K\pm1-j\beta} = \frac{I_{2K\pm1-j\beta}(\alpha)}{I_{-j\beta}(\alpha)}$; also see Figure 8

Given: Rapidly varying phase process

$$P_E \approx \operatorname{erfc}\left[\sqrt{2\rho}\ \overline{\cos\phi}\right] \quad \text{; also see Figure 10} \qquad (38)$$

Note: For the rapidly varying phase case, the BEP can be made arbitrarily small by increasing S/N however an irreducible error results for the slowly varying phase case which depends on the loop stress β and the parameter α. This is expected since effects of rapid variations may be averaged out over the detection interval.

VI. EXAMPLE PROBLEM

A. Introduction: The problem posed concerns acquisition and tracking an unmodulated, CW carrier with the use of a Phase Lock Loop. The system parameters denoted below as (given) are representative of synchronous satellite communication links constrained by a required margin in the power budget and onboard oscillator instabilities of one part in 10^6 operating at 1.6945 GHz.

B. Problem Statement:

Given: received carrier/noise spectral density: $C/kT = 31$ DB Hz

required margin
for power budget: M = 6 DB

total initial frequency
offset due to doppler &
oscillator drift after
calibration procedures: $\Delta f = \pm 500$ Hz

initial doppler
rate offset: $\theta^{(2)} \approx 0$

Specify a Phase Lock Loop design which:

1. Is capable of acquiring the carrier frequency in minimum time.
2. Has satisfactory tracking performance characteristics.

C. Solution

1. Design Approach: In order to achieve minimum acquisition time we shall choose the maximum allowable loop bandwidth which results in satisfactory tracking performance.

2. Since $\theta^{(2)} \cong 0$ we can choose a second order loop in order to minimize the steady state frequency error. Further we will assume the availability of a good active integrator corresponding to the loop filter designated as $F_2^*(p)$ in Table 1.

3. Initially we shall select the maximum loop bandwidth as consistent with available S/N and the "rule of thumb" stated in Equation (19). Specifically:

   $$31 - 8 - \log_{10} B_L^{(2)} = 6$$

   or

   $$B_L^{(2)} \cong 50 \text{ Hz. one sided}$$

4. If we now take $\zeta = 1/\sqrt{2}$, a value generally considered a wise compromise for obtaining jointly satisfactory performance with respect to acquisition and tracking, we obtain the loop natural frequency via Table 1; i.e.

   $$W_n = \frac{2B_L^{(2)}}{(\zeta + 1/4\zeta)} \approx 100 \text{ Hz.}$$

5. For a fixed VCO, we obtain from Equation (15):

$$t_{a\dot{\phi}} \approx \frac{(500)^2}{2\left(\frac{1}{\sqrt{2}}\right)(100)^3} = .178 \text{ sec.}$$

This calculation using $\theta^{(1)} = 500$ presumes that the VCO can be tuned, apriori, to the nominal incoming carrier frequency with good accuracy. If this is not the case one can increase $\theta^{(1)}$ to perhaps $2\Delta f$ and resubstitute into Equation (15).

6. For a swept VCO: we see from Equation (18) that:

$$R_{\dot{max}} = \frac{\left[1 - \frac{1}{\sqrt{8}}\right] W_n{}^2}{1 + \frac{1}{23}}$$

corresponding to 90% probability of acquisition.

$$R_{max} = .632\, W_n{}^2 = 6.32 \text{ KHz/sec}$$

Using this result an upperbound on the time to frequency acquisition can be obtained by using Equation (26)

$$t_{a\dot{\phi}} \leq \frac{\theta^{(1)}}{R_{max}} = \frac{500}{6.32 \times 10^3} = .079 \text{ sec.}$$

For higher probability of acquisition one could sweep slower for which the probability of acquisition can be obtained from (17).

Tracking Performance

7. Computing $r = 4\zeta^2 = 2$ and interpolating from Figure 7 we have: Mean squared phase error

$$\sigma_\phi^2\Big|_{\rho = 8} << .4 \text{ rad.}$$

8. From Figure 8 we see that for = 8 DB then:

$$\frac{2\bar{S}}{W_L} \approx 0. \quad \text{or } \bar{S} \approx 0.$$

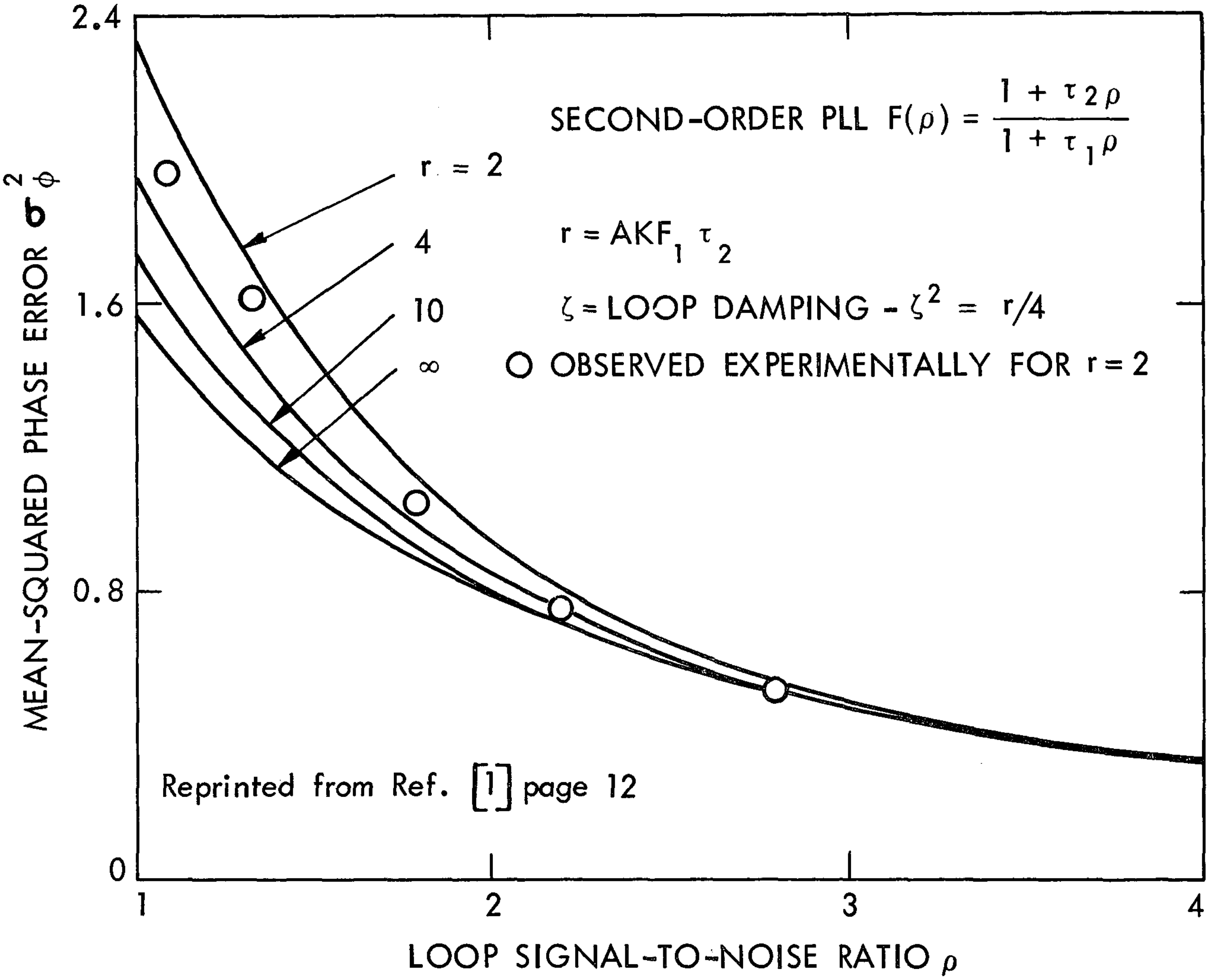


Fig. 7. Variance of the phase error vs loop signal-to-noise ratio for various values of r, for $\rho > 3$ db, $\alpha \simeq \rho$.

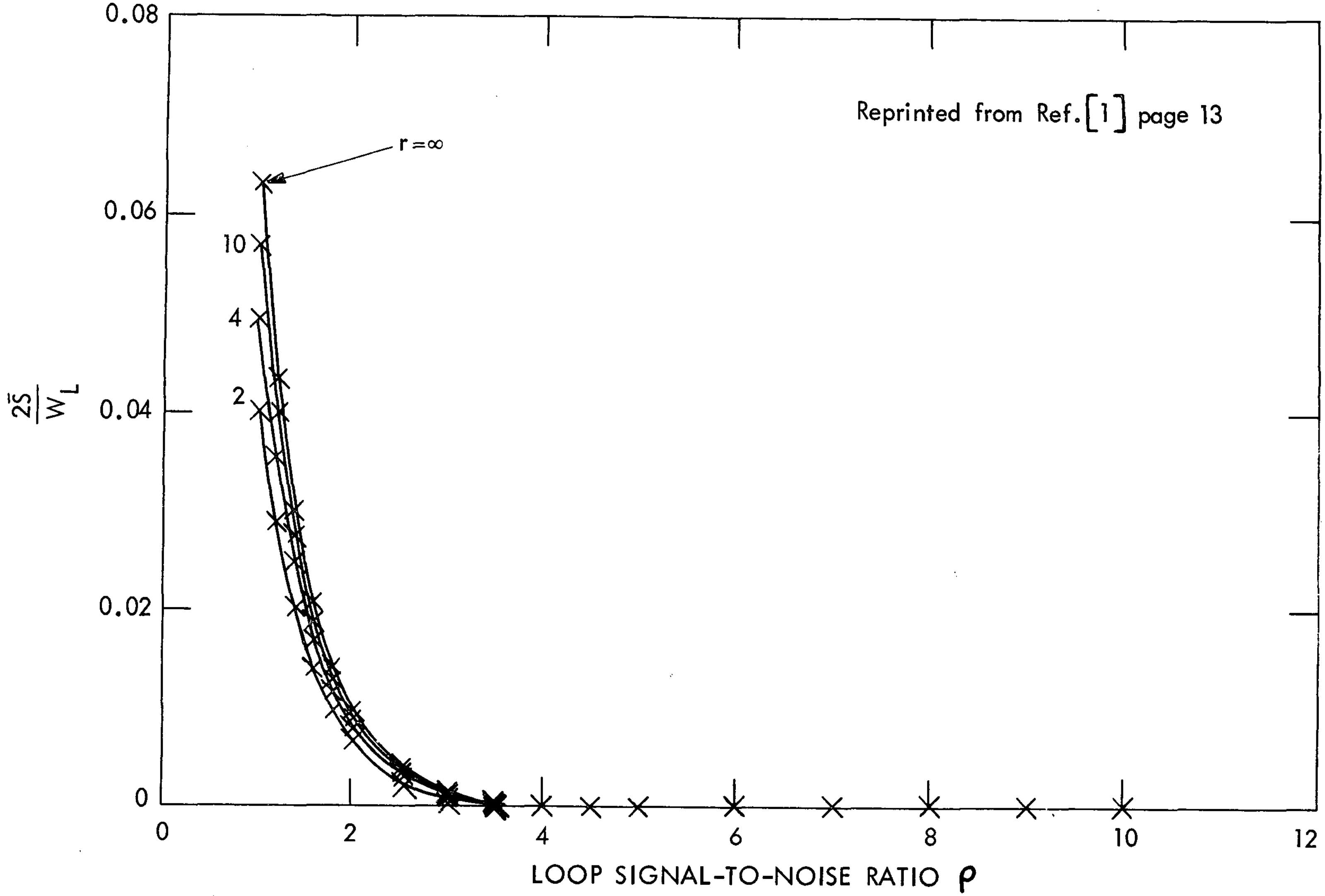


Fig. 8. Average number of cycle slips per unit time vs ρ .

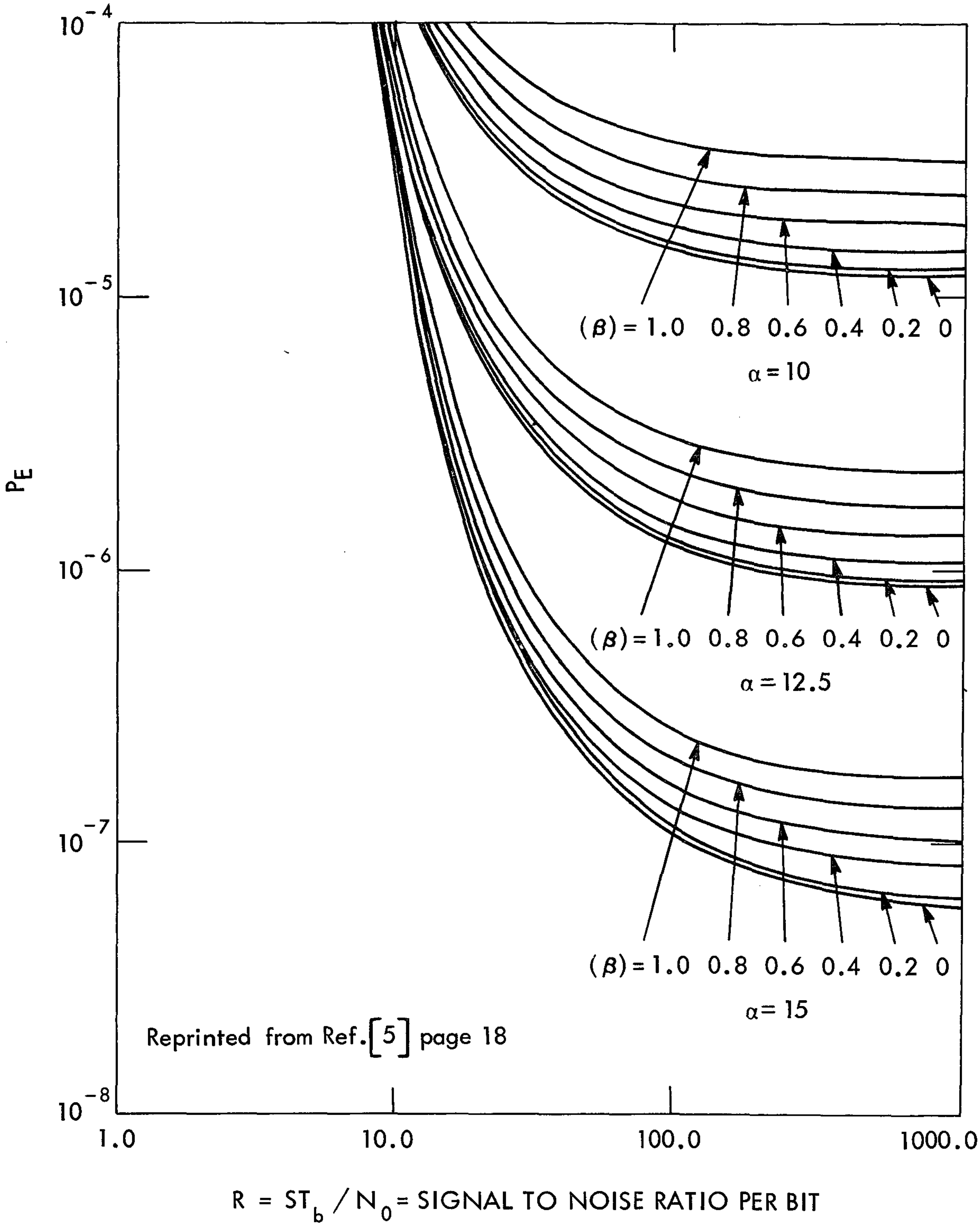


Fig. 9. BEP for slowly varying phase condition.

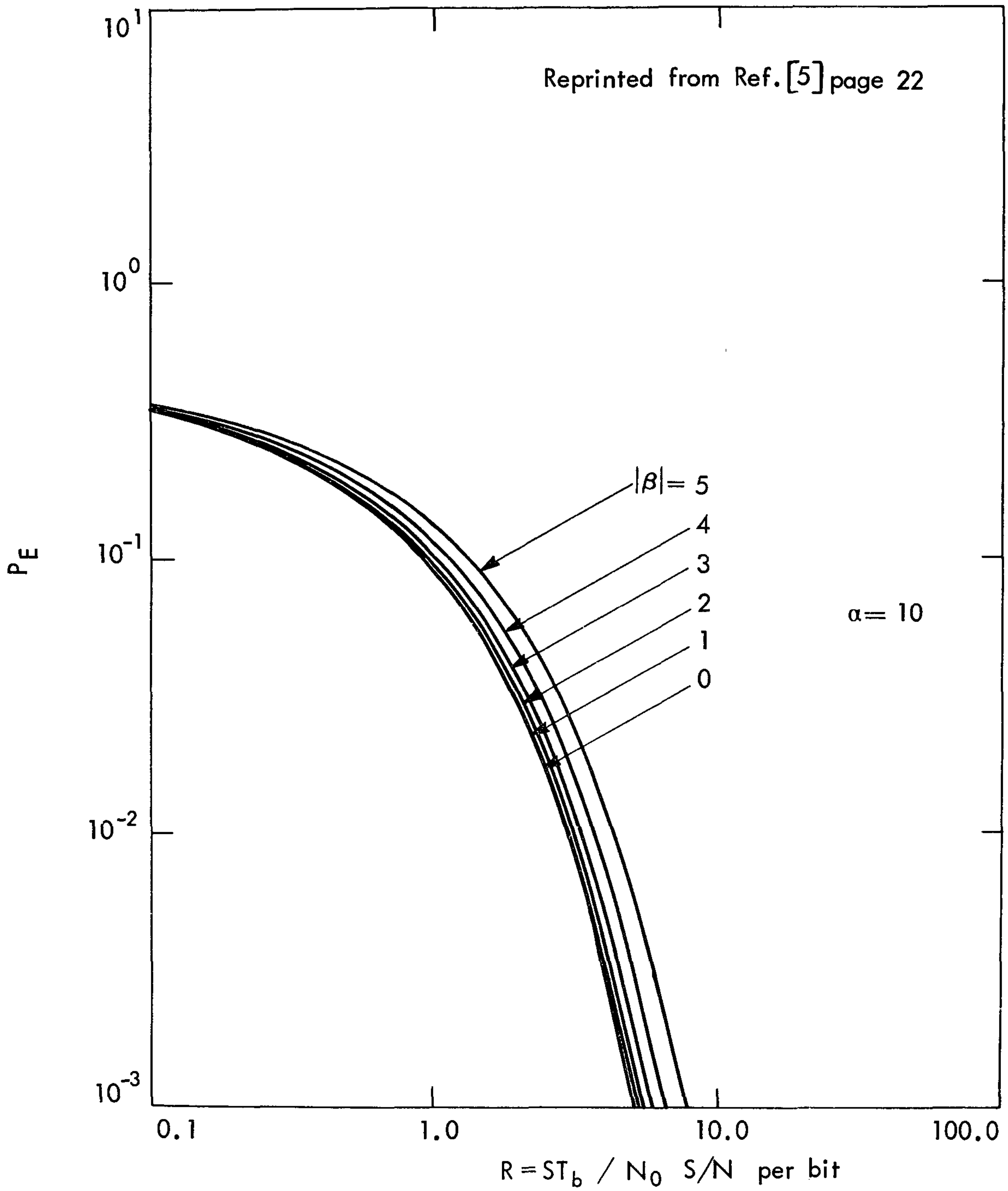


Fig. 10. BEP for rapidly varying phase condition.

Probability of losing lock
time t from Equation (35)

$$P_{s_t} \approx 1 - \exp(\bar{S}t) \approx 0.$$

Note: The above (7-8) convey the notion that tracking preformance of the specified design is satisfactory. More precise values of σ_ϕ^2, $\bar{S}$ and Pst in conjunction with Pdf's for the reduced phase error process are available via Equations (30), (31), (35), and (22, 25) respectively.

9. If the tracking performance was not satisfactory; for example, suppose the probability of losing lock over some period (Td) was too large.

   The procedure to now follow would be to specify an acceptable Pst from which one could work backwards to obtain the required loop bandwidth (necessarily less than before). The new loop bandwidth would in turn dictate a longer acquisition time. Such an increase in acquisition time would of course force a compromise in data transmit time which in turn would direct the designer to seek higher data rates or reduce the quantity of data to be transmitted.

VII. FUTURE WORK

Based on the above review of work generally available throughout the literature, the following future efforts seem warrented in order to improve current understanding and performance capability of phase locked loops.

A. Obtain more definitive and complete information pertaining to:

1. Acquisition (phase and frequency) of second and third order loops in noise subject to a range of system parameters including: elements of the difference phase process ($\theta^{(0)}$, $\theta^{(1)}$, $\theta^{(2)}$), loop parameters (ζ, W_n), signal to noise ratio and VCO sweep rate. Of particular value would be to concentrate on a time domain analysis in which the dynamics of the acquisition problem can be clearly understood and modelled at least in a probabilistic setting.

2. Effect of appreciable doppler rates $\theta^{(2)}$ on the tracking and acquisition behavior of the P. L. L. in noise.

B. A significant improvement in acquisition performance could come from a fast, reliable means of estimating the location of the carrier frequency prior to injecting the received signal into the P.L.L.

When operating within the pull in range of the P.L.L. the desired properties of such an estimator/P.L.L. combination would be:

1. To acquire the carrier frequency in a time less than $t_{a\dot{\phi}}$.

2. To perform the above acquisition with a probability comparable to the P.L.L. operating alone.

Such an estimator is of course essential should the carrier initially lie outside the pull in range of the P.L.L., at the time the loop seeks to acquire it.

VIII. ACKNOWLEDGEMENT

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* Derived herein or self explanatory from earlier result.