NASA TECHNICAL MEMORANDUM

NASA TM X- 67952

NASA TM X- 67952

WIND-DRIVEN CURRENTS IN A SHALLOW LAKE OR SEA

by Marvin Goldstein and Richard T. Gedney
Lewis Research Center
Cleveland, Ohio

WIND-DRIVEN CURRENTS IN A SHALLOW LAKE OR SEA

by Marvin Goldstein and Richard T. Gedney

Lewis Research Center

ABSTRACT

For shallow lakes and seas such as the great lakes (especially Lake Erie) where the depth is not much greater than the Ekman depth the usual Ekman dynamics cannot be used to predict the wind driven currents. The necessary extension to include shallow bodies of water, given by Welander, leads to a partial differential equation for the surface displacement which in turn determines all other flow quantities. A technique for obtaining exact analytical solutions to Welander's equation for bodies of water with a large class of bottom topographies which may or may not contain islands is given. It involves applying conformal mapping methods to an extension of Welander's equation into the complex plane When the wind stress is constant (which is the usual assumption for lakes) the method leads to general solutions which hold for bodies of water of arbitrary shape (the shape appears in the solutions through a set of constants which are the coefficients in the Laurent expansion of a mapping of the particular lake geometry) The method is applied to an elliptically shaped lake and a circular lake containing an eccentrically located circular island

1. INTRODUCTION

A main source of motion in the oceans and lakes is the wind. When the wind acts on the surface of a deep body of water, it sets up a circulation pattern which consists of a top and bottom friction (boundary) layer separated by a geostrophic (inviscid) core. In Ekman dynamics, the bottom stress is assumed proportional to the geostrophic velocity. However, Ekman dynamics is only valid when the thickness of the friction (or Ekman) layer is negligible by comparison with the depth of the lake. The application of Ekman dynamics to any of the Great Lakes is questionable since they all have shallow regions of considerable extent. In Lake Erie, for moderate wind speeds, the thickness of the friction layer is comparable to the depth over much of the lake and Ekman dynamics can very definitely not be applied.

The necessary extension of the Ekman analysis to the case of a homogeneous shallow lake has been given by Welander (1957) and that theory is used here for the case of steady winds.

Gedney (1971) numerically calculated the wind-driven currents in Lake Erie using Welander's formulation. Comparison of these calculations with measurements was favorable and showed that the shallow sea formulation does indeed yield fairly accurate three-dimensional velocity profiles as a function of horizontal position and depth in the lake. The variation of bottom depth and boundary geometry must be included in any shallow lake analysis because the circulation depends very strongly on these factors.

Here a technique is given for obtaining exact analytical solutions to Welander's equation for bodies of water which have bottom topographies of a general type and which may or may not contain an island. The shallow lake model consists of a general second-order linear elliptic partial differential equation in two independent variables with coefficients which depend on the local bottom topography. This equation is transformed using conformal mapping methods into an equation whose coefficients are functions of only one of its two independent variables. By taking a finite Fourier transform the problem is reduced to the problem of solving a second-order ordinary differential equation. The solution is then represented by an infinite series with each term a product of a function of one of the independent variables and a function of the other. When the wind stress is constant the method leads to general solutions which hold for bodies of water with shore lines of arbitrary shape. The shape appears in the solution through a set of constants which are the coefficients in the Laurent expansion of a mapping of the particular lake geometry.

2. FORMULATION

In the present analysis the basic approximations are that the water density is constant, the vertical eddy viscosity is independent of depth but dependent on wind velocity, the pressure is hydrostatic, and the lateral friction and nonlinear acceleration terms can be neglected. The above assumptions reduce the momentum equations to two equations containing the horizontal velocities and the surface slope as unknowns. The effect of the Coriolis force is included but its variation with latitude is not taken into account. The appropriate boundary conditions for these

equations are a no-slip condition at the lake bottom and a specified shear stress (due to the wind) at the air-sea interface. These equations and boundary conditions can be solved analytically for the velocity

$$(U + iV)\frac{f_c}{g} = \left[\frac{(1 - i)\pi\tau^W}{gd}\right]\frac{\sinh\left[(1 + i)\pi\left(\frac{h + \xi}{d}\right)\right]}{\cosh\left[(1 + i)\pi\frac{h}{d}\right]}$$
$$- i\left\{\frac{\cosh\left[\pi(1 + i)\frac{\xi}{d}\right]}{\cosh\left[\pi(1 + i)\frac{h}{d}\right]} - 1\right\}\left(\frac{\partial\zeta}{\partial x} + i\frac{\partial\zeta}{\partial y}\right)$$

where ξ is the vertical coordinate measured upward from the lake surface, ζ is the surface displacement of the sea or lake above its zero-flow level, x and y are the horizontal, locally Cartesian coordinates, h is the depth of the body of water, $d = \pi\sqrt{2\nu/f_c}$ is the Ekman friction depth, ν is the coefficient of vertical eddy diffusivity and f_c is the Coriolis parameter which introduces the effect of the Earth's rotation. In addition U and V are the horizontal components of velocity in the x and y directions, respectively, g is the acceleration due to gravity and $\tau^W = \tau_1^W + i\tau_2^W$ where τ_1^W and τ_2^W are, respectively, the x and y components of the wind stress divided by the density of water.

By vertically integrating the horizontal velocity over the vertical direction and by using the continuity equation one can obtain a single equation for the surface displacement, which is

$$\zeta_{xx} + \zeta_{yy} + \left[A\left(2\pi\frac{h}{d}\right)_x - B\left(2\pi\frac{h}{d}\right)_y\right]\zeta_x + \left[A\left(2\pi\frac{h}{d}\right)_y + B\left(2\pi\frac{h}{d}\right)_x\right]\zeta_y$$
$$= \frac{1}{E}\left[\sigma_x^{(1)} + \sigma_y^{(2)}\right] \tag{1}$$

where the subscripts x and y denote partial derivatives with respect to these variables and

$$A + iB \equiv \frac{i \left[\sinh\left(2\pi \frac{h}{d}\right) - i \sin\left(2\pi \frac{h}{d}\right) \right]^2}{\left[\cosh\left(2\pi \frac{h}{d}\right) + \cos\left(2\pi \frac{h}{d}\right) \right] \left[\sinh\left(2\pi \frac{h}{d}\right) - \sin\left(2\pi \frac{h}{d}\right) \right]}$$

$$\sigma^{(1)} + i\sigma^{(2)} = \sigma \equiv \frac{2\pi i \ \tau^w}{gd} \left[\frac{2 \cosh \frac{\pi h}{d} (1 - i)}{\cosh\left(2\pi \frac{h}{d}\right) + \cos\left(2\pi \frac{h}{d}\right)} - 1 \right]$$

$$E \equiv \frac{\sinh\left(2\pi \frac{h}{d}\right) - \sin\left(2\pi \frac{h}{d}\right)}{\cosh\left(2\pi \frac{h}{d}\right) + \cos\left(2\pi \frac{h}{d}\right)}$$

Equation (1) is a slightly rearranged form of that given by Welander (1957). For details of the derivation of the above equation and a discussion of the approximations involved in the derivation, see Gedney (1971). For instance Gedney has shown for Lake Erie that the approximations used, except for the approximation of a constant eddy viscosity, induce at the most a small (of the order of 10^{-1}) error in the calculations.

The x and y components of the total volume flow Q_1 and Q_2 respectively are related to the surface displacement by the relation

$$Q \frac{f_c}{gd} = \frac{1}{2\pi} \left[\sigma - (E + iF)(\zeta_x + i\zeta_y) \right] \tag{2}$$

where $Q = Q_1 + iQ_2 = \int_{-h}^{0} (U + iV) \, d\xi$ and

$$F \equiv \frac{\sin\left(2\pi\frac{h}{d}\right) + \sinh\left(2\pi\frac{h}{d}\right)}{\cos\left(2\pi\frac{h}{d}\right) + \cosh\left(2\pi\frac{h}{d}\right)} - \frac{2\pi h}{d}$$

The boundary conditions for equation (1) are obtained by specifying the volume flow normal to the coasts. Thus if n_1 and n_2 are the x and y components, respectively, of the outward drawn unit normal $\hat{n}$ to the boundary, then for any closed body of water the boundary conditions for equation (1) are obtained by substituting equation (2) into the equation

$$Q_1 n_1 + Q_2 n_2 = 0 \tag{3}$$

3. GENERAL SOLUTION

We shall now suppose that there exists a nonconstant harmonic function u of x and y and an arbitrary function H of u only such that the depth distribution h can be expressed in the form

$$h(x,y)/d = \frac{1}{2\pi} H\left[e^{u(x,y)}\right] \tag{4}$$

This is a fairly general functional form and it will be possible, for any one of a large number of lakes and seas, to choose the functions u and H in equation (4) so that the depth distribution is approximated fairly closely by a relation of this type.

Now let v be the harmonic conjugate of u. Then the function

$$w = u + iv \tag{5}$$

is an analytic function of the complex variable

$$z = x + iy$$

We are interested in obtaining solutions to equation (1) for a shallow sea

or lake which is either bounded by a single closed curve C such as that shown in figure 1(a) or a lake which contains a single island as shown in figure 2(a). Thus in the latter case we consider the doubly-corrected region where the outer boundary (shore line) is C and the inner (island) boundary is S. In either case we shall suppose that the bottom topography of the basin can be approximately described by a function of the forms (4) and that the depths along the shore line C in figure 1(a) and C and S in figure 2(a) are constants (which may in general be different). Equation (4) therefore shows that the harmonic function u must also be constant along C in figure 1(a) or along C and S in figure 2(a). Without any loss of generality we can always redefine the functions H and u so that u = 0 on C. When an island is included in the lake we shall denote the constant value of u on S by u_S. Again without loss of generality, it is always possible to arrange matters so that $u_S < 0$. Now consider the analytic function w introduced in equation (5) and put

$$T = e^W$$

Then T is an analytic function either within the simply-connected region of figure 1(a) or the multiply-connected region of figure 2(a). Since

$$|T| = e^u$$

it follows that $|T| = 1$ for z on C and in the case where an island is present $|T| = e^{u_S} = $ constant for z on S. Hence, when no island is present the mapping

$$z \rightarrow T$$

transforms the simply connected region in figure 1(a) into the interior of the unit circle shown in figure 1(b) and in the case where a single island

is present the mapping

$$z \to T$$

transforms the multiply connected region of figure 2(a) into the interior of the concentric annular region in the T-plane shown in figure 2(b), with the outer boundary C mapping into the unit circle and the inner boundary S mapping into the inner circle which has radius $R_o = e^{u_S}$

As long as the boundaries of the regions shown in figures 1 and 2 are sufficiently smooth, the Reimann mapping theorems for simply and doubly connected regions guarantee that the mappings $z \to T$ described above will always exist.

We now introduce polar coordinates into the T-plane by the relation

$$T = re^{i\theta}$$

Since u is constant on the curve C and also when an island is present on the curve S, the boundary condition (3) becomes

$$\mathcal{Re} \, Q \left(\frac{dz}{dw} \right)^* = 0 \qquad \text{for } z \epsilon C \tag{6}$$

and when the island is present

$$\mathcal{Re} \, Q \left(\frac{dz}{dw} \right) = 0 \qquad \text{for } z \epsilon S \tag{7}$$

Upon substituting the relation (4) into equations (1) and (2), then substituting equation (2) into equations (6) and (7) and introducing r and θ as the new independent variables we get

$$r(r\zeta_r)_r + \zeta_{\theta\theta} + a(r)r\zeta_r + b(r)\zeta_\theta = \frac{2r^2}{e(r)} \mathcal{Re} \frac{\partial \sigma}{\partial T} \left(\frac{dz}{dT} \right)^* \tag{8}$$

$$r\zeta_r - \frac{f(r)}{e(r)}\,\zeta_\theta = \frac{1}{e(r)}\,\mathcal{Re}\,\sigma T^* \left(\frac{dz}{dT}\right)^* \left\{\begin{array}{l} \text{(for } r = 1)\\[4pt] \left(\begin{array}{l}\text{and for } r = R_o \text{ if}\\ \text{island is present}\end{array}\right)\end{array}\right. \quad -\pi \leq \theta \leq \pi \tag{9}$$

where

$$\left.\begin{array}{l} a + ib \equiv i\,\dfrac{\left[\sinh H(r) - i\sin H(r)\right]^2}{\left[\cosh H(r) + \cos H(r)\right]\left[\sinh H(r) - \sin H(r)\right]}\,r\,\dfrac{dH}{dr}\\[14pt] e \equiv \dfrac{\sinh H(r) - \sin H(r)}{\cosh H(r) + \cos H(r)}\\[14pt] f \equiv \dfrac{\sinh H(r) + \sin H(r)}{\cosh H(r) + \cos H(r)} - H(r)\end{array}\right\} \tag{10}$$

and the partial derivative with respect to T is taken with T^* held con-
stant. We have now shown that the surface displacement of the sea or
lake can be found by solving equation (8) either in the unit circle of fig-
ure 1(b) if no island is present or in the concentric annular region of
figure 2(b) if an island is present subject to the boundary condition (9).

It now follows from equations (8) and (9) that if the complex function
η is the solution to the boundary value problem

$$r\left(r\eta_r\right)_r + \eta_{\theta\theta} + a(r)r\eta_r + b(r)\eta_\theta = \frac{2r^2}{e(r)}\,\frac{\partial\sigma}{\partial T}\left(\frac{dz}{dT}\right)^* \tag{11}$$

$$r\eta_r - \frac{f(r)}{e(r)}\,\eta_\theta = \frac{\sigma T^*}{e(r)}\left(\frac{dz}{dT}\right)^* \left\{\begin{array}{l}\text{(for } r = 1)\\[4pt]\left(\begin{array}{l}\text{and for } r = R_o\\ \text{if island is present}\end{array}\right)\end{array}\right. \quad -\pi \leq \theta \leq \pi \tag{12}$$

Then the surface displacement ζ is determined by

$$\zeta = \mathcal{Re}\,\eta \tag{13}$$

The function η must certainly be periodic. Hence upon taking the
finite Fourier transforms of equations (11) and (12) and substituting into

equation (13) we find that

$$\zeta = \mathcal{R}e \sum_{n=-\infty}^{\infty} I_n^* \Omega_n(r) e^{in\theta} \tag{14}$$

$$r(r\Omega_n')' + ar\Omega_n' - (n^2 - inb)\Omega_n = \frac{1}{e} \Gamma_n \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{15}$$

$$r\Omega_n' - in\frac{f}{e}\Omega_n = \frac{1}{e}\gamma_n \left\{ \begin{array}{c} \text{(for } r = 1) \\ \left(\text{and for } r = R_o \right) \\ \text{if island is present} \end{array} \right\} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{16}$$

where

$$I_n \equiv \frac{1}{2\pi i} \oint_{C_o} T^n \frac{dz}{dT} dT \tag{17}$$

$$\Gamma_n \equiv \frac{r^2}{2\pi r^n I_n^*} \int_{-\pi}^{\pi} 2\frac{\partial\sigma}{\partial T}\left(T^n \frac{dz}{dT}\right)^* d\theta \tag{18}$$

$$\gamma_n \equiv \frac{1}{2\pi r^n I_n^*} \int_{-\pi}^{\pi} \sigma\left(T^{n+1} \frac{dz}{dT}\right)^* d\theta \tag{19}$$

And the contour C_o is to be taken as any contour within the unit circle enclosing the origin for the case where no island is present and any contour within the annular region of figure 2(b) which encloses the origin for the case where an island is present.

We have therefore reduced the problem of finding the surface displacement to that of solving the ordinary differential equation (15) subject to the boundary condition (or conditions) (16). In the case where no

island is present the second boundary condition is replaced by a bound-edness condition on Ω_n at $r = 0$.

4. CONSTANT WIND STRESS

For lakes and to a lesser extent for shallow seas the case of prin-cipal interest for steady state winds is that for which the wind stress is constant. The reason for this is that the uncertainties in the wind veloc-ity measurements are of the order of the variation of this velocity over the body of water. In comparing measurements and calculations Gedney (1971) found this to be true for Lake Erie. In this case σ is independent of θ and it follows from the Cauchy-Goursat theorem that equations (18) and (19) become

$$\Gamma_n = \frac{1}{r^n} r \frac{d\sigma}{dr} \tag{20}$$

$$\gamma_n = \frac{1}{r^n} \sigma \tag{21}$$

It is easy to see from equations (20) and (21) that the boundary value problems (15) and (16) for Ω_n is independent of the actual shape of the body of water and depends only on the bottom topography. Hence the Ω_n can be determined once and for all for an arbitrary lake. The effect of the lake shape enters the problem in two ways. First equation (14) determines the surface displacement in terms of the parametric vari-ables r and θ. The mapping

$$z \to T$$

which maps the particular shaped lake into the T-plane must then be used to relate r and θ to the physical coordinate and therefore allows ζ to be expressed as a function of x and y.

The effect of geometry also enters through sequences of constants $\{I_n\}$. In fact it is easy to see that I_n is simply the coefficient of $T^{-(n+1)}$ in the Laurent series expansion of dz/dT about $T = 0$. This shows that for the case where an island is not present no positive power of n appear in the series (14).

Thus, an exact solution for the sea-level elevation (and since all other physical quantities can be expressed in terms of this, an exact solution to the complete flow problem) can be obtained once the ordinary differential equation (14) with the boundary conditions (15) has been solved.

5. SOLUTIONS WHEN THE DEPTH IS GREATER THAN
ONE HALF THE EKMAN FRICTION THICKNESS

It is shown by Goldstein, et al (1970) that for $h/d \geq 1/2$ A, B, E, F, and σ can be replaced with only a small loss of accuracy by

$$A = 0$$

$$B = 1$$

$$E = 1 - \frac{2\pi h}{d}$$

$$F = 1$$

$$\sigma = \frac{2\pi i \tau^W}{gd} \left[2e^{-(1+i)H/2} - 1 \right] \tag{22}$$

Figure 3 shows how closely A, B, E, and F approximate these values for $h/d \geq 1/2$. This approximation holds, for example, over the major portion of the great lakes. With these approximations equations (15) and (16) become

$$r(r\Omega_n')' - (n^2 - inrH')\Omega_n = \frac{r}{r^n} \sigma' \tag{23}$$

$$r\Omega_n' + in(H - 1) \Omega_n = \frac{1}{r^n} \sigma \quad \left\{ \begin{array}{l} \text{(for } r = 1) \\ \left(\begin{array}{l} \text{and for } r = R_0 \text{ if} \\ \text{island is present} \end{array} \right) \end{array} \right. \tag{24}$$

In order to illustrate the method we shall work out specific examples using only the approximate equations (22) through (24).

(a) Elliptic Lake Without An Island

First consider the case of a lake with no island which has an approximately parabolic bottom topography given by

$$H(r) = H_0 + \delta(1 - r^2) \tag{25}$$

where $H_0 + \delta$ is the constant depth at the shore.

Since we are considering the case where no island is present, I_n is equal to zero for $n = 0, 1, 2, \cdots$. Hence only Ω_n with negative values of n occur in the solution (14). When equation (25) is substituted into equation (22) we obtain an equation which is easily transformed into Bessel's equation. The solution to this equation which satisfies the boundary conditions (24) and which is bounded at $r = 0$ is given by

$$\Omega_n = \frac{\pi}{2} \lambda_n \int_0^1 \frac{1}{\rho^n} \mathscr{K}_n(r;\rho)\sigma(\rho) \, d\rho \quad \text{for } n = -1, -2, \cdots$$

where

$$\mathscr{K}_n(r;\rho) \equiv \left\{ \begin{array}{ll} \left[Y_n(\lambda_n r) - \omega_n J_n(\lambda_n r) \right] J_{n+1}(\lambda_n \rho) & \text{for } \rho \leq r \\[2ex] \left[Y_{n+1}(\lambda_n \rho) - \omega_n J_{n+1}(\lambda_n \rho) \right] J_n(\lambda_n r) & \text{for } r \leq \rho \end{array} \right.$$

$$\omega_n \equiv \frac{n\left[1 + i(H_o - 1)\right] Y_n(\lambda_n) - \lambda_n Y_{n+1}(\lambda_n)}{n\left[1 + i(H_o - 1)\right] J_n(\lambda_n) - \lambda_n J_{n+1}(\lambda_n)}$$

$$\lambda_n \equiv \sqrt{-2in\delta}$$

and J_n and Y_n are Bessel functions.

This solution can be applied to various shaped bodies of water simply by finding the conformal mapping $z \to T$ of the region occupied by the body of water in the physical plane (see fig. 1(c)) into the unit circle. Thus for an elliptically shaped lake of semi-major axis L_1, and semi-minor axis L_2 the appropriate mapping which takes the center of the lake into the center of the unit circle (this causes the deepest portion of the lake to be located at its center) is

$$T = \sqrt{k} \, sn\left[\frac{2K}{\pi} \sin^{-1}\left(\frac{z}{I}\right), \, k\right]$$

where K is the complete elliptic integral of the first kind of modulus k, sn is the sine amplitude function

$$I \equiv \sqrt{L_1^2 - L_2^2}$$

and k is determined from

$$e^{-\pi K'/K} = \left(\frac{L_1 - L_2}{L_1 + L_2}\right)^2$$

The transformation $T(z)$ could have just as easily been constructed so the deepest portion of the lake occurs at any point other than at the center of the lake. This is accomplished by performing an additional transformation of the unit circle into itself with an arbitrary point mapping into the center and then forming the composite transformation.

(b) Circular Lake With An Eccentrically Located Circular Island

Next consider the case of a lake with a single island which has a bottom topography given by

$$H(r) = H_0 + \frac{\delta}{\ln R_0} \ln r \tag{26}$$

When this expression is substituted into equation (23) we obtain an equation with constant coefficients. The solution to this equation subject to the boundary conditions (24) is

$$\Omega_n = \frac{2\pi i \tau^W}{gd} \left\{ \frac{1}{K_n} \left[\omega_n(r; H) M_n(H_0 + \delta) R_0^{-n} - \omega_n\left(\frac{r}{R_0}; H_0 + \delta\right) M_n(H_0) \right] \right.$$
$$\left. - \frac{2e^{-[(1+i)/2]H(r)}}{r^n \left[2n + (1+i)\left(n + \frac{1}{2}\frac{\delta}{\ln R_0}\right)\right]} \right\} \tag{27}$$

$$\text{for } n = \pm 1, \pm 2, \pm 3, \cdots$$

where

$$\omega_n(r; \zeta) \equiv \left[\lambda_n - in(\zeta - 1)\right] r^{\lambda_n} + \left[\lambda_n + in(\zeta - 1)\right] r^{-\lambda_n}$$

$$K_n \equiv \left[\lambda_n - in(H_0 - 1)\right]\left[\lambda_n + in(H_0 + \delta - 1)\right] R_0^{\lambda_n}$$
$$- \left[\lambda_n + in(H_0 - 1)\right]\left[\lambda_n - in(H_0 + \delta - 1)\right] R_0^{-\lambda_n}$$

$$M_n(\alpha) = \frac{2n(2 + i\alpha)}{2n + (1+i)\left(n + \frac{1}{2}\frac{\delta}{\ln R_0}\right)} e^{-[(1+i)/2]\alpha} - 1$$

$$\lambda_n \equiv \sqrt{n^2 - \frac{in\delta}{\ln R_0}}$$

This solution can be applied to various shaped bodies of water and islands simply by finding the conformal mapping $z \to T$ of the doubly connected region occupied by the body of water in the physical plane into the concentric annular region in figure 2(b). For example, consider a circular body of water with a radius R containing a circular island of radius R_s whose center is located at some arbitrary point within the body of water. The configuration is shown in figure 2(c). In this case the appropriate mapping is

$$T = \frac{z - aR}{az - R} \tag{28}$$

where

$$a = \frac{\left(\frac{x_a}{R}\right) + \left(\frac{x_b}{R}\right)}{1 + \left(\frac{x_a}{R}\right)\left(\frac{x_b}{R}\right) + \sqrt{\left[1 - \left(\frac{x_a}{R}\right)^2\right]\left[1 - \left(\frac{x_b}{R}\right)^2\right]}}$$

and the radius R_0 of the inner circle in the T-plane is

$$R_0 = \frac{\left(\frac{x_a}{R}\right) - \left(\frac{x_b}{R}\right)}{1 - \left(\frac{x_a}{R}\right)\left(\frac{x_b}{R}\right) + \sqrt{\left[1 - \left(\frac{x_a}{R}\right)^2\right]\left[1 - \left(\frac{x_b}{R}\right)^2\right]}}$$

Upon substituting equation (28) into equation (17) and using the method of residues to evaluate the contour integral we find that

$$I_n = \begin{cases} 0 & \text{for } n = 0, 1, 2, \cdots \\ \\ R(a^2 - 1)a^{n-1} & \text{for } n = -1, -2, \cdots \end{cases} \tag{29}$$

Substituting equations (27) and (29) into equation (14) determines the surface displacement.

6. RESULTS

A method for obtaining the analytical solution to equation (1) for the surface displacement in a lake or shallow sea has been given in terms of the power series

$$\zeta = \mathcal{R}e \sum_{n=-\infty}^{+\infty} I_n^* \, \Omega_n(r) e^{in\theta} \tag{14}$$

The variables r and θ are related to physical coordinates x and y of the lake through a conformal mapping function which depends only on the shape of the lake. The coefficient I_n^* is determined solely by this mapping function. The bottom topography $h(x, y)$ has been chosen in such a way that it depends only on r in the transformed r, θ-plane. In general the effect of the lake bottom topography as well as the lake boundary shape enters through the functions $\Omega_n(r)$ which are determined from the ordinary differentiation equation (15). However, for a constant wind stress $\Omega_n(r)$ depends only on the bottom topography.

Results will be given here for the case when the bottom depth is greater than one half the Ekman friction thickness and the wind stress is a constant. Under these restrictions the functions $\Omega_n(r)$ are determined from the ordinary differential equation (23). Details of the determination of the mapping function $T(z)$, $\Omega_n(r)$, and I_n^* for each case are given in section 5. All solutions depend upon the wind shear stress, τ^W, the bottom depth, $h(x, y)$, the friction thickness, d, and the lake boundary geometry.

The first example concerns an elliptically shaped body of water without an island and with the approximately paraboloidal bottom topography given by equation (25). The surface displacement which results for a depth variation which increases from a maximum of $h/d = 9/2\pi$ at the center to a minimum of $h/d = 5/2\pi$ at the mainland shore is shown in figure 4. Here the wind direction is parallel to the x-axis and the ratio of semi minor to the semi major axis of the ellipse is 2/3. As is well known, the surface displacement for a constant depth basin with uniform wind stress is a plane inclined to the horizontal. The deviation of the surface displacement from a plane is then the effect of the variation in bottom depth. The s-shape of the surface across the wind direction in figure 4 will be found in most lakes because the dished-out type bottom topography used in this example occurs naturally.

The second example considered is a circular lake containing an eccentrically located circular island with the logarithmic bottom topography given by equation (26). The results shown in figure 5 are for an island diameter of $(x_a - x_b) = \frac{1}{2} R$ and an eccentricity of $(x_a + x_b)/2R = 0.5$. The bottom topography varies from a value of $h/d = 1.0$ at the mainland shore to a value of $h/d = 0.5$ at the island as shown in figure 5(a). In an actual basin without an island, the depth is smallest at the mainland shore and largest at some central interior point. Islands create local areas in the interior of the basin where the depth becomes shallow. This effect is modeled by the topography shown in figure 5(a).

Figure 5(b) shows the surface displacement for the circular lake to be more of a plane than that shown in the previous case. The reason

for this is that the circular lake bottom topography is much flatter than the one used in the elliptic lake case. Although the deviations of the surface displacement from an inclined plane are small, they have major effects on the local velocities.

The horizontal volume flow stream function is shown in figure 5(c). Here the stream function ψ is related to the volume flows Q_1 and Q_2 by

$$Q_1 = \frac{\partial \psi}{\partial y}, \quad Q_2 = -\frac{\partial \psi}{\partial x}$$

The stream function plot consists of two gyres; the gyre on the right is rotating clockwise and the one to the left of the wind is rotating counterclockwise. The two gyres are separated by a dividing streamline which in this case has a value of zero. The value of ψ on both the island and mainland shores is zero and the zero streamline in the interior has been twisted relative to the wind in a clockwise direction.

As shown by the ψ plots obtained by Gedney (1971), the dividing streamline for a dish shaped basin whose depth increases monotonically from the shore to some interior point always runs through the maximum depth point. If the depth decreases monotonically from the shore to an interior point, the dividing streamline would run through the minimum depth point. In the case of the island shown in figure 5(c), the zero streamline runs through the minimum depth point interior to the mainland shore which happens to be the island boundary. If the island was not placed at the minimum depth point the value of ψ on the island boundary would be different from zero since the path of the streamline dividing the

two gyres is determined primarily by the extremes in the bottom topography.

The horizontal velocities at $\xi/d = 0$, -0.125, -0.250, -0.375, and -0.500 are shown in figures 5(d) through 5(h). The magnitude of each velocity vector plotted can be determined from the scale included on the plot. The origin of the velocity vector is the position at which velocity is actually occurring. The velocities at the surface are skewed to the right of the wind due mainly to the Coriolis force. Local perturbations in the surface velocities due to the volume flow gyres are evident. The acceleration of the flow around the island and the deceleration near $x = -1$ and $+1$ can be directly attributed to the volume flow gyres shown in figure 5(c). At $\xi/d = -0.125$ mass is still being transported in the direction of the wind but an increasing amount is also being transported to the right of the wind. The flow acceleration around the island is still evident. At $\xi/d = -0.25$ the flow pattern is very similar to that of the integrated volume flow shown in figure 5(c). Return flow opposite to the wind direction is now occurring. The flow near the island is predominately tangent to it. At $\xi/d = -0.375$ and -0.50 there is much return flow opposite in direction to the top surface layer flow. The flow below $\xi/d = -0.50$ is very similar in direction to but smaller in magnitude than that shown at $\xi/d = -0.50$. This general flow pattern where mass in the top surface layer is transported along and to the right of the wind direction and returned in the opposite direction in the bottom layer is the dominant pattern in shallow lakes. The variations in this general pattern are due to the particular bottom topography. For this particular case where the bottom slopes upwards at the island an acceleration of flow occurs around the island.

REFERENCES

Gedney, R. T. 1971 Numerical Calculations of the Wind-Driven Currents in Lake Erie. Ph. D. Thesis, Case Western Reserve University.

Goldstein, M. E., Braun, W. H. & Gedney, R. T. 1970 A Method for Obtaining Analytical Solutions to the Equation for Wind-Driven Circulation in A Shallow Sea or Lake. NASA Tech. Note D-5989.

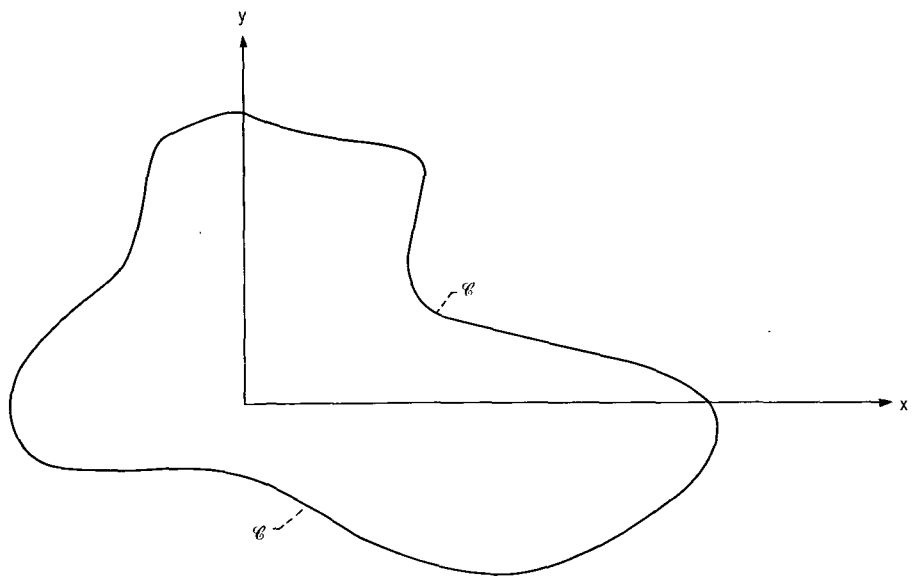Welander, P. 1957 Wind Action on a Shallow Sea: Some generalization of Ekman's theory. Tellus 9, 47-52.
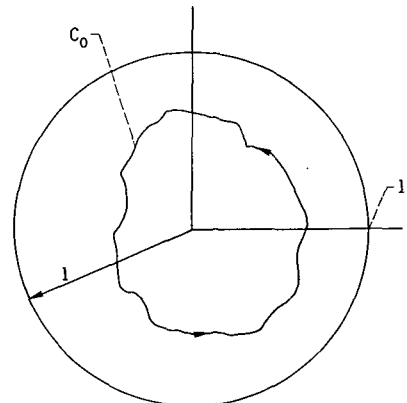
Figure 1(a). - Lake or sea configuration.

Figure 1(b). - Circular region in T-plane.
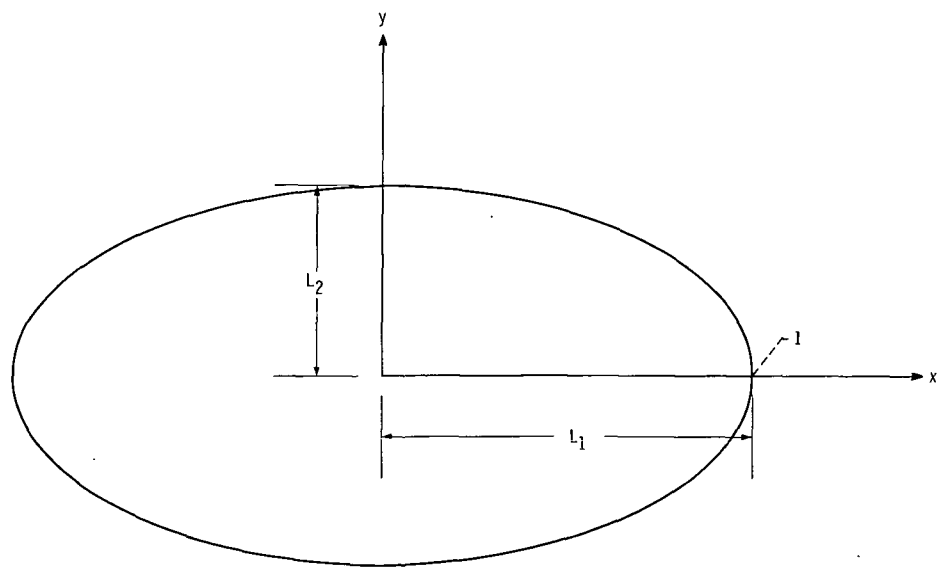
Figure 1(c). - Elliptic lake.
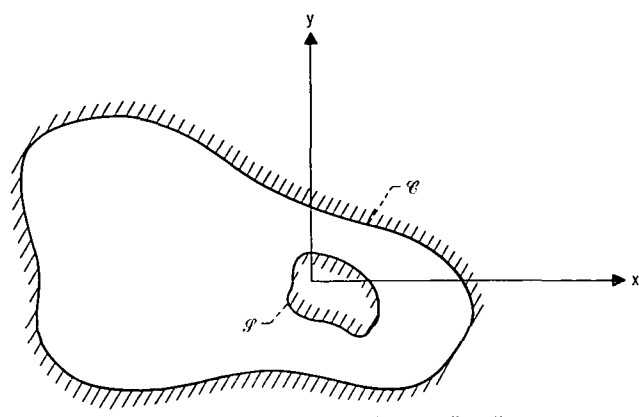
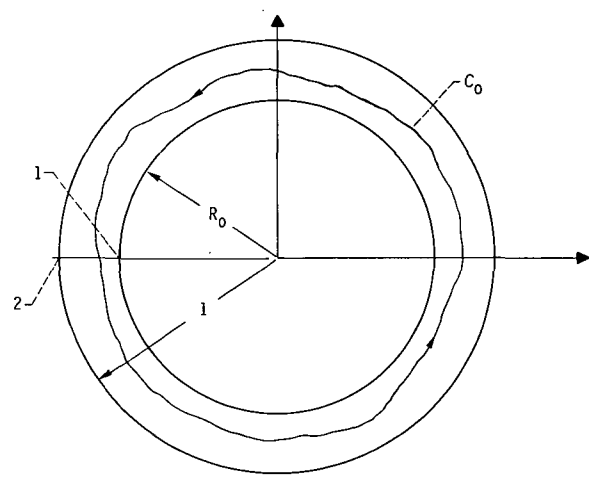Figure 2(a). - Lake (sea) - island configuration.
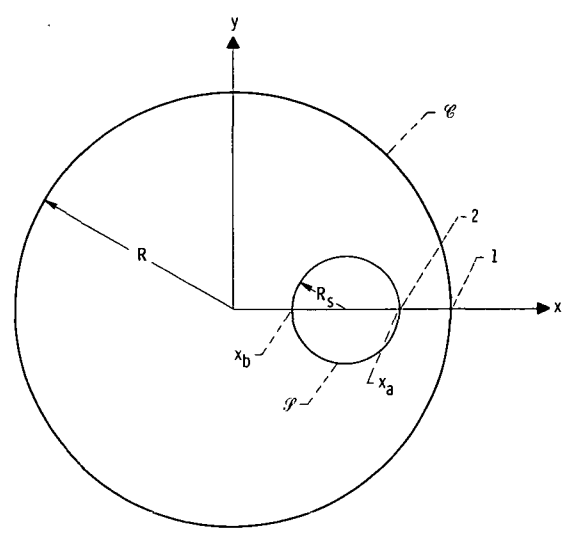

Figure 2(b). - Annular region in T-plane.
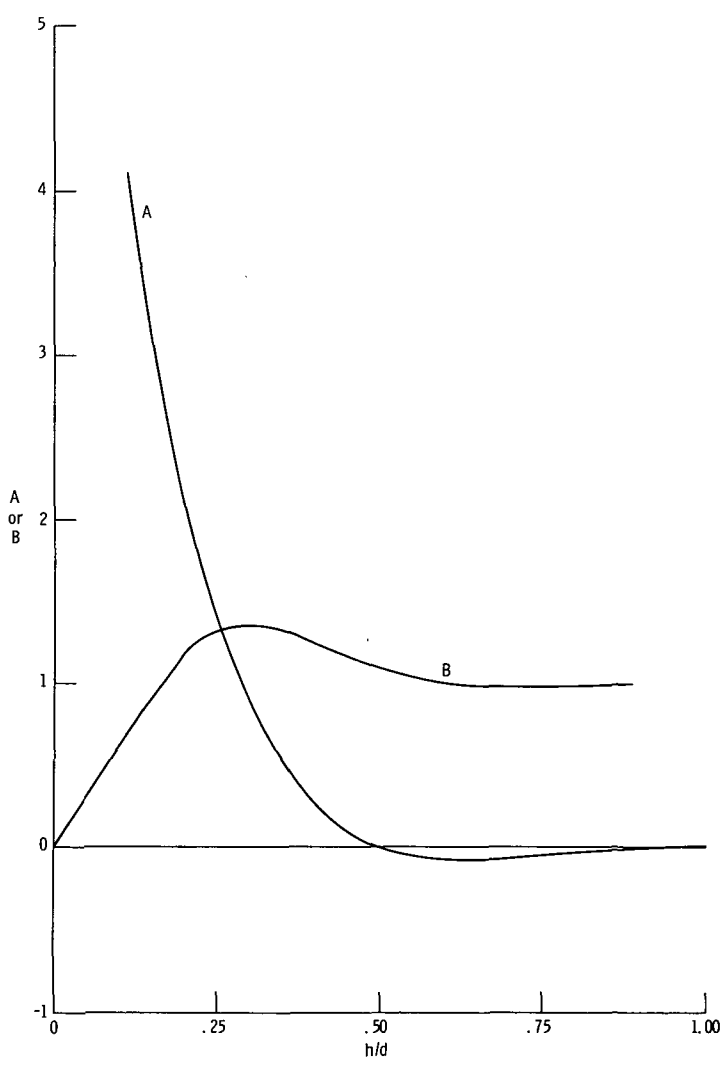

Figure 2(c). - Circular lake and island configuration.
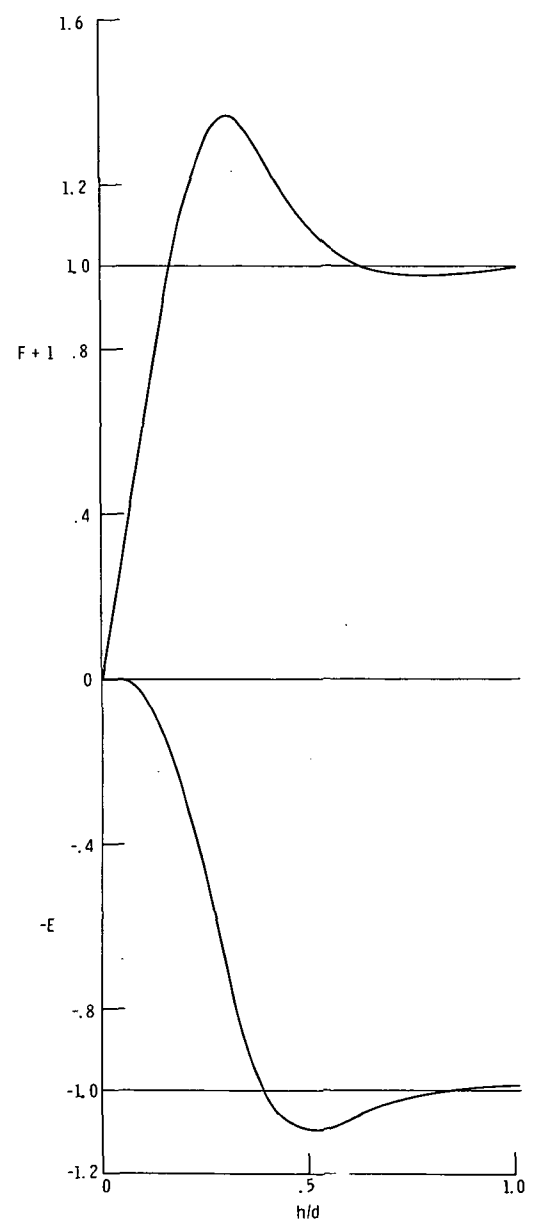
Figure 3(a). – Coefficients A and B.
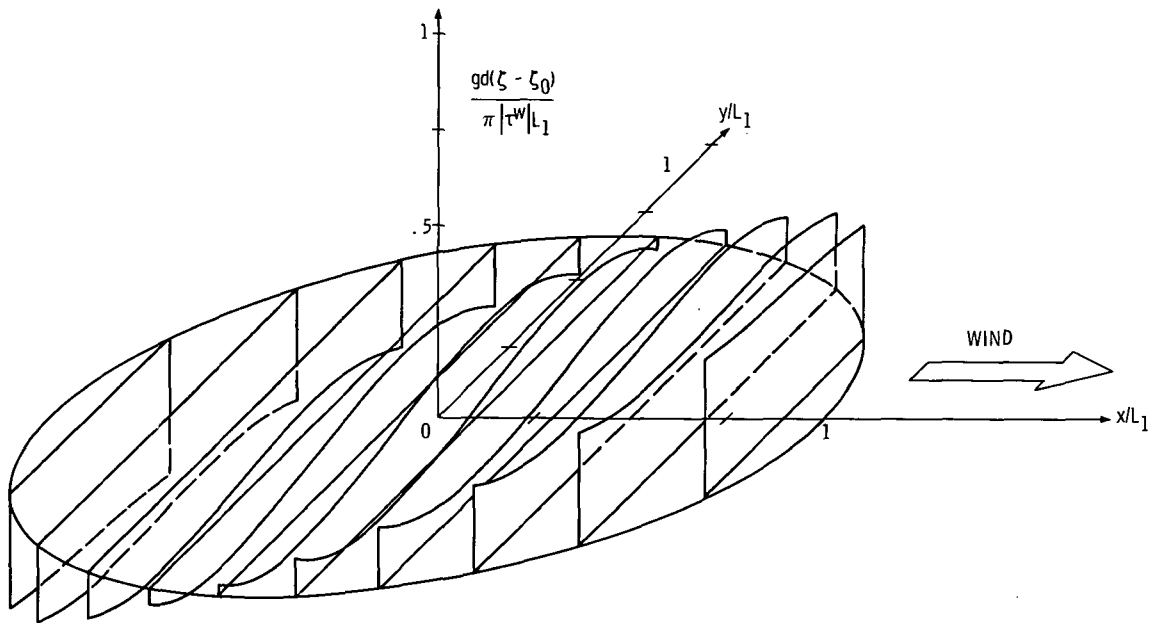

Figure 3(b). – Coefficients E and F + 1.

Figure 4. - Surface displacement of an elliptic lake. Ratio of minor to major axis. $L_2/L_1 = 2/3$; wind parallel to major axis $\delta = 4$; $H_s = 5$.
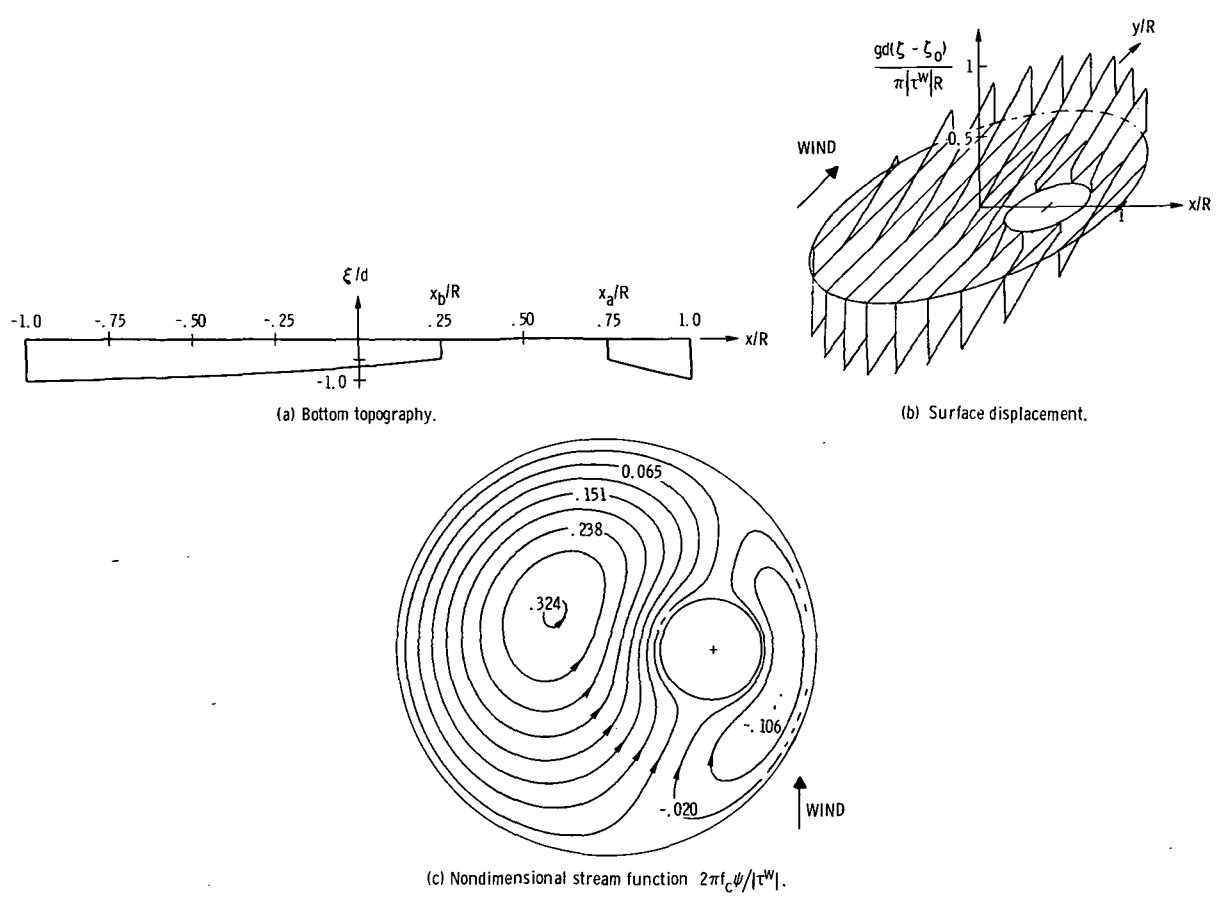
ξ/d

x_b/R x_a/R

-1.0   -.75   -.50   -.25   .25   .50   .75   1.0 x/R

-1.0

(a) Bottom topography.

$\frac{gd(\zeta - \zeta_0)}{\pi|\tau^W|R}$ y/R

1

WIND   0.5

x/R

1

(b) Surface displacement.

0.065

.151

.238

.324

-.106

-.020

WIND

(c) Nondimensional stream function $2\pi f_c \psi/|\tau^W|$.

Figure 5. – Circular basin circulation. Island eccentricity = 0.5 island diameter = 0.5; h/d = 1.0 at mainland shore; h/d = 0.5 at island; $\tau^W/|\tau^W|$ = i.
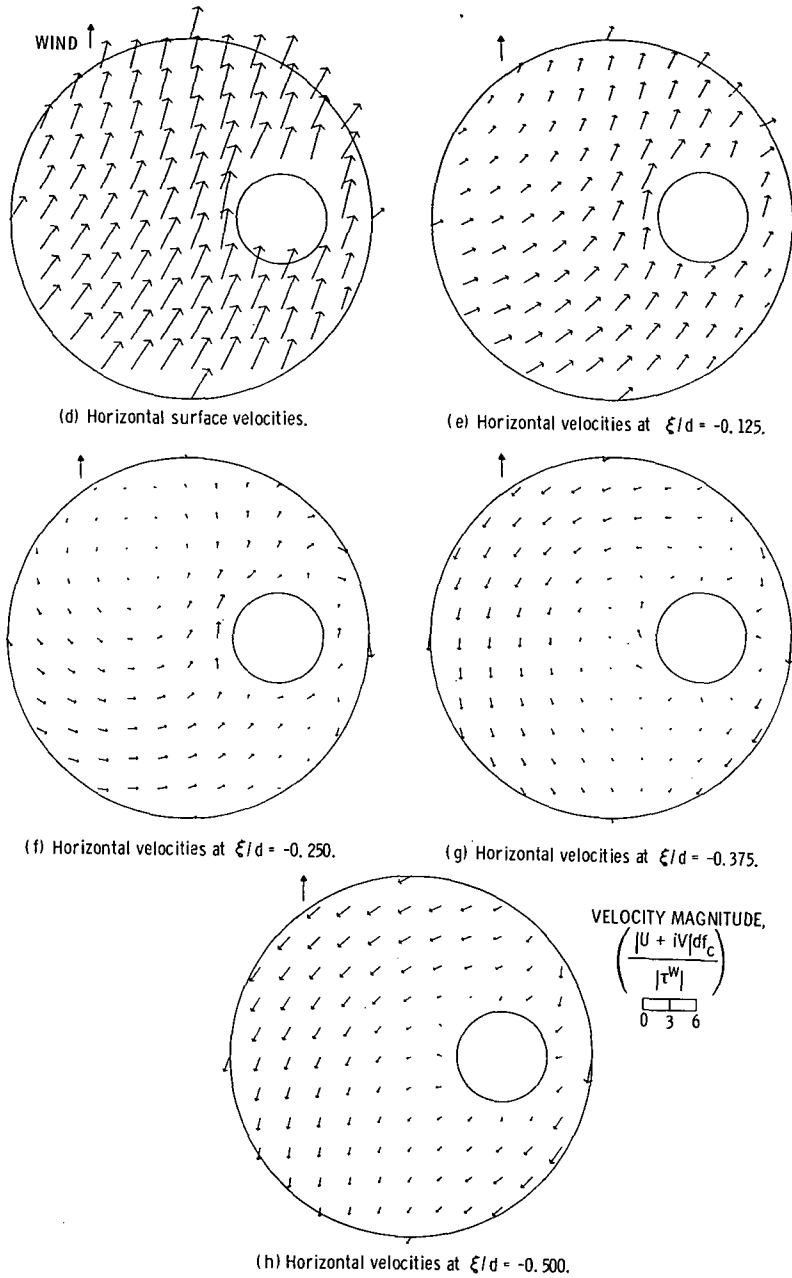
WIND

(d) Horizontal surface velocities.

(e) Horizontal velocities at $\xi/d = -0.125$.

(f) Horizontal velocities at $\xi/d = -0.250$.

(g) Horizontal velocities at $\xi/d = -0.375$.

VELOCITY MAGNITUDE,

$$\left(\frac{|U + iV|df_c}{|\tau^w|} \right)$$

0 3 6

(h) Horizontal velocities at $\xi/d = -0.500$.

Figure 5. - Concluded.