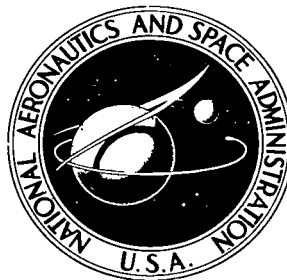


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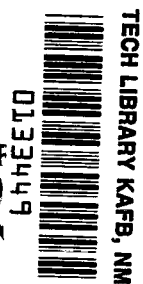


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## DECAY OF HOMOGENEOUS TURBULENCE FROM A SPECIFIED STATE

*by Robert G. Deissler  
Lewis Research Center  
Cleveland, Ohio 44135*





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# DECAY OF HOMOGENEOUS TURBULENCE FROM A SPECIFIED STATE

by Robert G. Deissler

Lewis Research Center

## SUMMARY

The homogeneous turbulence problem is formulated by first specifying the multi-point velocity correlations or their spectral equivalents at an initial time. Those quantities, together with the correlation or spectral equations, are then used to calculate initial time derivatives of correlations or spectra. The derivatives in turn are used in time series to calculate the evolution of turbulence quantities with time. When the problem is treated in this way, the correlation equations are closed by the initial specification of the turbulence and no closure assumption is necessary. An exponential series which is an iterative solution of the Navier-Stokes equations gave much better results than a Taylor power series when used with the limited available initial data. In general, the agreement between theory and experiment was good.

## INTRODUCTION

A basic difficulty in the usual analyses of homogeneous turbulence is the closure problem; that is, the set of correlation or moment equations contains more unknowns than equations. The problem occurs, of course, because of the nonlinearity of the Navier-Stokes equations from which the correlation equations are obtained (ref. 1).

Although many approximations have been introduced into the correlation equations (or equivalent spectral equations) in attempts to obtain closure, those suggestions have varying degrees of arbitrariness. The analyses in references 2 and 3, although based on definite physical ideas, contain dimensionless constants which must be determined by experiment. Those in references 4 to 7, although somewhat more deductive, have other difficulties. That in reference 4, at least for the restricted initial condition for which it has been worked out, sometimes gives negative energies (ref. 8). The analyses in references 5 and 6 give reasonable results for moderately weak turbulence but become unduly complex for high Reynolds numbers. That in reference 7, although it has yielded some realistic deductions, also has computational difficulties because of its complexity.

There is another way of looking at the problem of homogeneous turbulence. In order not to lose sight of our goal, we will first give a statement of that problem. The statement given by Batchelor is essentially the following: given the statistical state of a homogeneous turbulent field at an initial instant, the problem is to predict the evolution of the turbulence (in probability) as a function of time. Note that the initial development of turbulence from a nonturbulent state produced by, say, flow through a grid, is not considered in this report. Rather we are concerned with the evolution of turbulence after a time when the flow is already turbulent. In order to specify completely a turbulent field at an initial time, it is necessary to give all of the multipoint velocity correlations or their spectral equivalents at that time (ref. 1). It is not hard to show that, given these multipoint correlations and the correlation equations, all the time derivatives of the turbulent energy tensor and of other pertinent turbulence quantities can be calculated. These time derivatives can then be used in a series, for instance a Taylor series, to calculate the evolution of the turbulent energy tensor (or of the equivalent energy spectrum tensor) and of other turbulence quantities.

It is noted that when the turbulence is treated in this way, we no longer have the problem of closing the infinite set of correlation or spectral equations. The correlation equations are used only to relate the correlations at an initial time to their time derivatives, and those correlations must be given in order to have a complete specification of the turbulence at that time. Of course, in practice only a small number of the correlations, and thus of their time derivatives, will ordinarily be available, but a sufficient number may be known to give a reasonably good representation. It might be pointed out that even in those analyses which require a closure assumption, the turbulence should be specified initially by its correlations or spectra since the correlation equations require initial conditions.

Kraichnan (ref. 9) has very recently studied the convergence properties of series such as those considered in this report. As mentioned in another article by that author (ref. 10), it is not necessary that an expansion be convergent in order to be useful, since divergent series can provide excellent asymptotic approximations (ref. 11).

Although the present problem circumvents the closure problem in the usual sense, there is still the question of the legitimate truncation of the time series to obtain explicit results. This report is not concerned primarily with convergence questions but will use as a test the agreement of the results with experiment. Although a Taylor series might give good results if sufficient statistical information were available at the initial time, it will be seen that an exponential series which arises in a study of the nonlinear decay of a disturbance in a fluid (ref. 12) is much more satisfactory. This is not surprising since the exponential series is an iterative solution of the Navier-Stokes equations and thus contains information which is not contained in the Taylor series. The resulting formulation gives results which are in quite good agreement with the available experimental data.

## INITIAL TIME DERIVATIVES AND SIMPLE EXPANSIONS

As mentioned in the INTRODUCTION, if the multipoint correlations are known at an initial instant, as they must be for a complete specification of the turbulence at that instant, then the time derivatives of the correlations can be calculated from the correlation equations. For illustrative purposes we will consider the derivatives of the turbulent energy tensor  $\overline{u_i u_j}$ , where  $u_i$  and  $u_j$  are respectively velocity components at the points  $P$  and  $P'$  separated by the vector  $\vec{r}$ , and the overbar indicates an averaged value. Then the first time derivative of  $\overline{u_i u_j}$  at  $t = t_1$  is given directly by the two-point correlation equations (ref. 5) evaluated at  $t = t_1$ :

$$\begin{aligned} \left( \frac{\partial \overline{u_i u_j}}{\partial t} \right)_{t=t_1} = & - \frac{\partial}{\partial r_k} \left[ \left( \overline{u_i u_j u'_k} \right)_{t=t_1} - \left( \overline{u_i u'_j u_k} \right)_{t=t_1} \right] \\ & - \left[ \frac{1}{\rho} \frac{\partial}{\partial r_j} \left( \overline{u_i p'} \right)_{t=t_1} - \frac{\partial}{\partial r_i} \left( \overline{p u'_j} \right)_{t=t_1} \right] + 2\nu \frac{\partial^2 \left( \overline{u_i u'_j} \right)_{t=t_1}}{\partial r_k \partial r_k} \end{aligned} \quad (1)$$

where the pressure-velocity correlations are given by

$$\frac{1}{\rho} \frac{\partial^2 \left( \overline{p u'_j} \right)_{t=t_1}}{\partial r_k \partial r_k} = - \frac{\partial^2 \left( \overline{u_l u_k u'_j} \right)_{t=t_1}}{\partial r_k \partial r_l} \quad (2)$$

and a similar equation for  $\left( \overline{u_i p'} \right)_{t=t_1}$ . The pertinent solution of equation (2) is (ref. 1)

$$\frac{1}{\rho} \left( \overline{p u'_j} \right)_{t=t_1} = \frac{1}{4\pi} \int \frac{1}{|\vec{r} - \vec{s}|} \frac{\partial^2 \left( \overline{u'_i u'_k u'_j} \right)_{t=t_1}}{\partial s_i \partial s_k} d\vec{s}$$

where  $u'_j$  is the velocity at the point  $\vec{x}' = \vec{x} - \vec{s}$ , and the integration is over all  $\vec{s}$  space. This solution is for an infinite fluid, for which case the boundary conditions are that  $\overline{p u'_j}$  is bounded for  $\vec{r} = 0$  and zero for  $\vec{r} = \infty$ . The quantity  $\rho$  is the density,

$\nu$  is the kinematic viscosity, and  $p$  is the pressure. A repeated subscript in a term indicates a summation, with the subscript successively taking on the values 1, 2, and 3. The correlation equations are, of course, derived from the Navier-Stokes equations. The quantity  $\overline{\partial u_i u_j' / \partial t}$  at  $t = t_1$  can be calculated from equations (1) and (2) if  $\overline{u_i u_j'}$  and the two-point triple correlations are known at  $t = t_1$ .

The second time derivative of  $\overline{u_i u_j'}$  is obtained by differentiating the two-point correlation equations and evaluating the result at  $t_1$ . This gives

$$\begin{aligned} \left( \frac{\partial^2 \overline{u_i u_j'}}{\partial t^2} \right)_{t=t_1} = & - \frac{\partial}{\partial r_k} \left[ \left( \frac{\partial}{\partial t} \overline{u_i u_j' u_k'} \right)_{t=t_1} - \left( \frac{\partial}{\partial t} \overline{u_i u_j' u_k} \right)_{t=t_1} \right] \\ & - \frac{1}{\rho} \left[ \frac{\partial}{\partial r_j} \left( \frac{\partial \overline{u_i p'}}{\partial t} \right)_{t=t_1} - \frac{\partial}{\partial r_i} \left( \frac{\partial \overline{p u_j'}}{\partial t} \right)_{t=t_1} \right] + 2\nu \frac{\partial^2}{\partial r_k \partial r_k} \left( \frac{\partial \overline{u_i u_j'}}{\partial t} \right)_{t=t_1} \end{aligned} \quad (3)$$

and

$$\frac{1}{\rho} \frac{\partial^2}{\partial r_k \partial r_k} \left( \frac{\partial \overline{p u_j'}}{\partial t} \right)_{t=t_1} = - \frac{\partial^2}{\partial r_k \partial r_l} \left( \frac{\partial \overline{u_l u_k' u_j'}}{\partial t} \right)_{t=t_1} \quad (4)$$

The quantity  $\left[ \left( \partial / \partial t \right) \left( \overline{u_i u_j' u_k'} \right) \right]_{t=t_1}$  in equation (3) is obtained from the three-point correlation equations (ref. 5) written for  $t = t_1$  and  $\vec{r}' = \vec{r}$ . (The vector  $\vec{r}'$  separates the points P and P'.) Thus,

$$\begin{aligned} \left( \frac{\partial}{\partial t} \overline{u_i u_j' u_k'} \right)_{t=t_1} = & \left\{ \frac{\partial}{\partial r_l} \left( \overline{u_i u_j' u_k' u_l} \right)_{t=t_1} + \frac{\partial}{\partial r_l'} \left( \overline{u_i u_j' u_k' u_l} \right)_{t=t_1} - \frac{\partial}{\partial r_l} \left( \overline{u_i u_j' u_k' u_l'} \right)_{t=t_1} - \frac{\partial}{\partial r_l'} \left( \overline{u_i u_j' u_k' u_l'} \right)_{t=t_1} \right. \\ & - \frac{1}{\rho} \left[ - \frac{\partial}{\partial r_i} \left( \overline{p u_j' u_k'} \right)_{t=t_1} - \frac{\partial}{\partial r_i'} \left( \overline{p u_j' u_k'} \right)_{t=t_1} + \frac{\partial}{\partial r_j} \left( \overline{p' u_i u_k'} \right)_{t=t_1} + \frac{\partial}{\partial r_j'} \left( \overline{p' u_i u_k'} \right)_{t=t_1} \right] \\ & \left. + 2\nu \left[ \frac{\partial^2 \left( \overline{u_i u_j' u_k'} \right)_{t=t_1}}{\partial r_l \partial r_l} + \frac{\partial^2 \left( \overline{u_i u_j' u_k'} \right)_{t=t_1}}{\partial r_l \partial r_l'} + \frac{\partial^2 \left( \overline{u_i u_j' u_k'} \right)_{t=t_1}}{\partial r_l' \partial r_l'} \right] \right\}_{\vec{r}' = \vec{r}} \end{aligned} \quad (5)$$

where  $(\overline{pu_j^i u_k^{i'}})_{t=t_1}$  is given by

$$\begin{aligned} \frac{1}{\rho} \left[ \frac{\partial^2 (\overline{pu_j^i u_k^{i'}})_{t=t_1}}{\partial r_l \partial r_l} + 2 \frac{\partial^2 (\overline{pu_j^i u_k^{i'}})_{t=t_1}}{\partial r_l \partial r_l'} + \frac{\partial^2 (\overline{pu_j^i u_k^{i'}})_{t=t_1}}{\partial r_l' \partial r_l'} \right] = & - \frac{\partial^2 (\overline{u_l u_m u_j^i u_k^{i'}})_{t=t_1}}{\partial r_m \partial r_l} \\ & - \frac{\partial^2 (\overline{u_l u_m u_j^i u_k^{i'}})_{t=t_1}}{\partial r_m \partial r_l'} - \frac{\partial^2 (\overline{u_l u_m u_j^i u_k^{i'}})_{t=t_1}}{\partial r_m' \partial r_l} \\ & - \frac{\partial^2 (\overline{u_l u_m u_j^i u_k^{i'}})_{t=t_1}}{\partial r_m' \partial r_l'} \end{aligned} \quad (6)$$

Similar equations are obtained for the other pressure-velocity correlations. The boundary conditions for equation (6) are similar to those for equation (2); that is,  $\overline{pu_j^i u_k^{i'}}$  is bounded for  $\vec{r}$  or  $\vec{r}' = 0$  and zero for  $\vec{r}$  or  $\vec{r}' = \infty$ . Also, an expression for  $\left[ (\partial/\partial t) (\overline{u_i u_j^i u_k^i}) \right]_{t=t_1}$  in equation (3) is obtained by letting  $\vec{r}' = 0$  instead of  $\vec{r}' = \vec{r}$  in equation (5). Thus, if the turbulence is specified sufficiently well at  $t = t_1$  that the double, triple, and quadruple velocity correlations are known,  $(\partial^2 \overline{u_i u_j^i} / \partial t^2)_{t=t_1}$  can be calculated. Similarly, higher order derivatives are obtained by considering four or more point correlations in the turbulent field (ref. 6). With the time derivatives of  $\overline{u_i u_j^i}$  known at  $t = t_1$ , a Taylor series gives  $\overline{u_i u_j^i}$  as a function of time as

$$\overline{u_i u_j^i} = (\overline{u_i u_j^i})_{t=t_1} + \left( \frac{\partial \overline{u_i u_j^i}}{\partial t} \right)_{t=t_1} (t - t_1) + \frac{1}{2!} \left( \frac{\partial^2 \overline{u_i u_j^i}}{\partial t^2} \right)_{t=t_1} (t - t_1)^2 + \dots \quad (7)$$

A similar analysis can be carried out in wave number space. For instance, the energy spectrum function  $E$ , which shows the contributions at various wave numbers to  $\overline{u_i u_j^i}/2$ , can be written as

$$\mathbf{E} = (\mathbf{E})_{t=t_1} + \left(\frac{\partial \mathbf{E}}{\partial t}\right)_{t=t_1} (t - t_1) + \frac{1}{2!} \left(\frac{\partial^2 \mathbf{E}}{\partial t^2}\right)_{t=t_1} (t - t_1)^2 + \dots \quad (8)$$

where  $\partial \mathbf{E} / \partial t$  is obtained from the Fourier transform of the two-point correlation equation (eq. (9) of ref. 5) as

$$\frac{\partial \mathbf{E}(\kappa)}{\partial t} = \int_{\mathbf{A}} \frac{1}{2} \left\{ -2\nu\kappa^2 \varphi_{ii}(\vec{\kappa}) + i\kappa_k \left[ \varphi_{iki}(\vec{\kappa}) - \varphi_{iki}(-\vec{\kappa}) \right] \right\} d\mathbf{A}(\kappa) \quad (9)$$

where  $d\mathbf{A}$  is an element of surface area of a sphere of radius  $\kappa$ ,  $\vec{\kappa}$  is the wave number vector corresponding to the spatial vector  $\vec{r}$ , and  $\varphi_{ii}$  and  $\varphi_{iki}$  are respectively the Fourier transforms of  $\overline{u_i u_i'}$  and  $\overline{u_i u_k u_i'}$ . Extracting from the integral that portion which can be written in terms of  $\mathbf{E}$  and setting the rest of the integral equal to  $\mathbf{T}$  give

$$\frac{\partial \mathbf{E}}{\partial t} = \mathbf{T} - 2\nu\kappa^2 \mathbf{E} \quad (10)$$

Equation (10) is the well known scalar form of the two-point spectral equation. The transfer term  $\mathbf{T}$  produces energy transfer between wave numbers and arises from the triple correlation term in equation (1) (with  $i = j$  (ref. 1)). (Note that the pressure-velocity correlation terms in eq. (1) drop out for  $i = j$ .) The second time derivative of  $\mathbf{E}$  is

$$\begin{aligned} \left(\frac{\partial^2 \mathbf{E}}{\partial t^2}\right)_{t=t_1} &= \left(\frac{\partial \mathbf{T}}{\partial t}\right)_{t=t_1} - 2\nu\kappa^2 \left(\frac{\partial \mathbf{E}}{\partial t}\right)_{t=t_1} \\ &= \left(\frac{\partial \mathbf{T}}{\partial t}\right)_{t=t_1} - 2\nu\kappa^2 (\mathbf{T})_{t=t_1} + (2\nu\kappa^2)^2 (\mathbf{E})_{t=t_1} \end{aligned}$$

The quantity  $(\partial \mathbf{T} / \partial t)_{t=t_1}$  can be calculated from the two- and three-point spectral equations if the two- and three-point spectral quantities in those equations are known at  $t = t_1$ . From equations (20), (23), and (24) of reference 5 we obtain



$$\frac{\partial \mathbf{T}}{\partial t} = \int_A \int_{-\infty}^{\infty} \frac{1}{2} \left( -2\nu\kappa^2 \left\{ i\kappa_k [\beta_{iik}(\vec{\kappa}) - \beta_{iik}(-\vec{\kappa})] \right\} + f(\beta_{ijk}, \beta_{ijk\ell}) \right) d\vec{\kappa}' dA(\kappa)$$

where  $\vec{\kappa}'$  is the wave number vector corresponding to  $\vec{r}'$ ,  $d\vec{\kappa}' = d\kappa_1 d\kappa_2 d\kappa_3$ , and  $\beta_{ijk}$  and  $\beta_{ijk\ell}$  are respectively the Fourier transforms of  $\overline{u_i u_j u_k'}$  and  $\overline{u_i u_j u_k' u_\ell'}$ . If by analogy with the procedure used for obtaining equation (10), we extract from the integral that portion which can be written in terms of spectral quantities already defined (E and T), we have

$$\begin{aligned} \frac{\partial \mathbf{T}}{\partial t} + 2\nu\kappa^2 \mathbf{T} &= \int_A \int_{-\infty}^{\infty} \frac{1}{2} f(\beta_{ijk}, \beta_{ijk\ell}) d\vec{\kappa}' dA(\kappa) \\ &= V[\beta_{ijk}(\vec{\kappa}'), \beta_{ijk\ell}(\vec{\kappa}')] \end{aligned} \quad (11)$$

where  $V$  is a quantity related to the three-point spectral tensors  $\beta_{ijk}$  and  $\beta_{ijk\ell}$ . More precisely we can say that  $V$  is a functional of  $\beta_{ijk}$  and  $\beta_{ijk\ell}$ , since each value of  $V$  depends on values of  $\beta_{ijk}$  and  $\beta_{ijk\ell}$  at all points of  $\vec{\kappa}'$  space. With equation (11), the expression for  $(\partial^2 \mathbf{E} / \partial t^2)_{t=t_1}$  becomes

$$\left( \frac{\partial^2 \mathbf{E}}{\partial t^2} \right)_{t=t_1} = V_{t=t_1} - 4\nu\kappa^2 T_{t=t_1} + (2\nu\kappa^2)^2 E_{t=t_1} \quad (12)$$

The Taylor series for  $\mathbf{E}$  then becomes

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{t=t_1} + \left( \mathbf{T}_{t=t_1} - 2\nu\kappa^2 \mathbf{E}_{t=t_1} \right) (t - t_1) \\ &\quad + \frac{1}{2!} \left[ V_{t=t_1} - 4\nu\kappa^2 T_{t=t_1} + (2\nu\kappa^2)^2 E_{t=t_1} \right] (t - t_1)^2 + \dots \end{aligned} \quad (13)$$

Equation (13) was used in conjunction with available experimental data at an initial time (ref. 13) in an attempt to calculate the variation with time of  $\mathbf{E}$  and thus of  $\overline{u_i u_i}$ . However, with the available initial data  $(\mathbf{E}_{t=t_1}, \mathbf{T}_{t=t_1}, \text{ and } V_{t=t_1})$ , reasonable results were not obtained except at small times (fig. 2). It thus appears that, in order to obtain good

results by using a simple Taylor series, initial statistical information of much higher order than that which is available would have to be given. Thus, an alternative approach which makes more efficient use of the initial statistical information and also incorporates additional information from the equations of motion will be considered.

## A WORKABLE FORMULATION FOR THE DEVELOPMENT OF TURBULENCE FROM A GIVEN INITIAL STATE

In order to obtain a more efficient means for calculating the evolution of turbulence than by a Taylor series in time, we consider an iterative solution of the Navier-Stokes equations similar to that in reference 12. In addition to the initial statistical information and calculated time derivatives, we will then have information about the form of the decay law from the equations of motion.

Although attention was confined to determinate initial conditions in reference 12, for the present purposes we can just as well assume the initial velocity fluctuations to be random or turbulent. Thus, we consider a field of homogeneous turbulence to be made up of a very high density of eddies or harmonic disturbances in wave number space. For all practical purposes then, since the density of disturbances is very high, the spectrum of the turbulence can be considered continuous. The velocity and pressure at any point in the field are given by

$$\frac{\partial u_i}{\partial t} - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial (u_i u_k)}{\partial x_k} \quad (14)$$

and

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_k \partial x_k} = - \frac{\partial^2 (u_k u_l)}{\partial x_k \partial x_l} \quad (15)$$

The latter equation is obtained by taking the divergence of equation (14) and applying the continuity equation.

From the spectrum of harmonic disturbances we arbitrarily select two cosine terms with wave number vectors  $\vec{q}$  and  $\vec{r}$ . Then, the velocity associated with those disturbances will be

$$u_i^{cc} = a_i \cos \vec{q} \cdot \vec{x} + b_i \cos \vec{r} \cdot \vec{x} \quad (16)$$

where the superscript  $cc$  on the velocity indicates that it depends on two cosine terms. The results that follow would be the same if two sine terms or a sine and a cosine term were considered. If  $u_i^{cc}$  is substituted for  $u_i$  in the right sides of equations (14) and (15), the time variations of  $a_i$  and  $b_i$  plus additional harmonic terms are obtained. If we then substitute that new expression into equations (14) and (15), another expression containing still more harmonic terms is obtained. In each approximation, the linear terms of the Navier-Stokes equations are considered as unknown and the nonlinear terms as known from the preceding approximation. As shown in reference 12, continuation of this process leads to

$$u_i^{cc} = \sum_{\vec{k}} \left( A_{i, \vec{k}}^{c'} \cos \vec{k} \cdot \vec{x} + A_{i, \vec{k}}^{s'} \sin \vec{k} \cdot \vec{x} \right) \quad (17)$$

where

$$A_{i, \vec{k}}^{c'} = \sum_q a_{i, \vec{k}, q}^c \exp \left[ - b_{\vec{k}, q}^c (t - t_1) \right] \quad (18)$$

and

$$A_{i, \vec{k}}^{s'} = \sum_r a_{i, \vec{k}, r}^s \exp \left[ - b_{\vec{k}, r}^s (t - t_1) \right] \quad (19)$$

Comparison of equations (17) to (19) with the first and second approximations in reference 12 shows that  $b_{\vec{k}, 1}^c = b_{\vec{k}, 1}^s = \nu k^2$ . Also, we note that since the two harmonic components in equation (16) were selected arbitrarily, expressions similar to equations (17) to (19) will be obtained for any other two components. But the nonlinear interaction of any number of harmonic components can be expressed as the sum of the interactions of pairs of components (eqs. (37) and (38) of ref. 12). Thus,  $u_i$ , the velocity resulting from all the harmonic components, will be of the form of equations (17) to (19) and can be written as

$$u_i = \sum_{\vec{k}} \left( A_{i, \vec{k}}^c \cos \vec{k} \cdot \vec{x} + A_{i, \vec{k}}^s \sin \vec{k} \cdot \vec{x} \right) \quad (20)$$

where

$$A_{i, \vec{\kappa}}^{(\cdot)} = a_{i, \vec{\kappa}, 1}^{(\cdot)} \exp[-\nu \kappa^2 (t - t_1)] + \sum_{\substack{q \\ q \neq 1}} a_{i, \vec{\kappa}, q}^{(\cdot)} \exp[-b_{\vec{\kappa}, q}^{(\cdot)} (t - t_1)] \quad (21)$$

The summations in equations (20) and (21) will, of course, contain more terms by many orders of magnitude than those in equations (17) to (19). Since the initial conditions are random, the quantities  $A_{i, \vec{\kappa}}^{(\cdot)}$ ,  $a_{i, \vec{\kappa}, q}^{(\cdot)}$ , and  $b_{\vec{\kappa}, q}^{(\cdot)}$  are assumed to be random variables. The space-averaged value of  $u_i^2$  (no sum on  $i$ ) is obtained from equation (20) by squaring, integrating over a cycle, and using the orthogonality property of sines and cosines. This gives

$$\overline{u_i^2} = \sum_{\vec{\kappa}} \frac{1}{2} \left[ \left( A_{i, \vec{\kappa}}^c \right)^2 + \left( A_{i, \vec{\kappa}}^s \right)^2 \right] \quad (22)$$

where

$$\begin{aligned} \left[ A_{i, \vec{\kappa}}^{(\cdot)} \right]^2 &= \left[ a_{i, \vec{\kappa}, 1}^{(\cdot)} \right]^2 \exp[-2\nu \kappa^2 (t - t_1)] + \sum_{\substack{q \\ q \neq 1}} \left[ a_{i, \vec{\kappa}, q}^{(\cdot)} \right]^2 \exp[-2b_{\vec{\kappa}, q}^{(\cdot)} (t - t_1)] \\ &+ \sum_{\substack{q, r \\ q \neq r}} a_{i, \vec{\kappa}, q}^{(\cdot)} a_{i, \vec{\kappa}, r}^{(\cdot)} \exp \left\{ -2 \left[ b_{\vec{\kappa}, q}^{(\cdot)} + b_{\vec{\kappa}, r}^{(\cdot)} \right] (t - t_1) \right\} \end{aligned} \quad (23)$$

According to equation (23),  $\left( A_{i, \vec{\kappa}}^c \right)^2$  and  $\left( A_{i, \vec{\kappa}}^s \right)^2$  in equation (22) have the same form, so that we do not need to carry along the superscripts  $c$  and  $s$ .

We want to obtain an averaged form of equation (23) which is a smoothed function of the magnitude of the vector  $\vec{\kappa}$  (but not of its direction). In order to do that, we divide the interval of  $\kappa = (\kappa_1 \kappa_1)^{1/2}$  over which disturbances occur into a large number of small increments  $\Delta \kappa$ . The terms in  $\sum_{\vec{\kappa}}$  in equations (22) and (23) are divided into groups each of which corresponds to a particular  $\Delta \kappa$ . Note that, while the magnitudes of the various vectors lying in a particular  $\Delta \kappa$  are approximately equal, their directions can, of

course, vary. The group of terms corresponding to each  $\Delta\kappa$  is then subdivided into groups in each of which the values of the  $b_{i, \vec{\kappa}, q}$  in  $\sum_{q \neq 1}$  do not vary appreciably from a value of  $b_s(\kappa)$ . The index  $s$  designates a particular increment in the values of the  $b_{i, \vec{\kappa}, q}$ . Also, for each  $s$ ,  $a_{i, \vec{\kappa}, q}^2$  will have an average value which we designate by  $\langle a_{i, \vec{\kappa}}^2 \rangle_s$ . The summation  $\sum_{q \neq 1}$  in equation (23), which applies to a particular  $\vec{\kappa}$ , is then replaced by

$$\sum_s n_{s, (i)} \langle a_{i, \vec{\kappa}}^2 \rangle_s(\kappa) \exp[-2b_s(\kappa)(t - t_1)]$$

which applies to a particular  $\Delta\kappa$ , and where  $n_{s, (i)}$  is the number of terms in  $\sum_{q \neq 1}$  which are assigned to the group  $s$  for the component  $i$ . The parentheses around  $i$  indicate that there is no summation on that subscript. A similar regrouping can be carried out for the terms in  $\sum_{q, r \neq 1}$ . However, that summation turns out to be zero, if we assume that the random  $a_{i, \vec{\kappa}}$  are uncorrelated, since  $\langle a_{i, \vec{\kappa}, q} a_{i, \vec{\kappa}, r} \rangle_s$  will be zero for  $q \neq r$ . Then the average value of  $A_{i, \vec{\kappa}}^2$  in the increment  $\Delta\kappa$  becomes (see eq. (23))

$$\langle A_{i, \vec{\kappa}}^2 \rangle(\kappa) = \langle a_{i, \vec{\kappa}, 1}^2 \rangle(\kappa) \exp[-2\nu\kappa^2(t - t_1)] + \sum_s \left( \frac{n_s}{n_\kappa} \right)_{(i)} \langle a_{i, \vec{\kappa}}^2 \rangle_s(\kappa) \exp[-2b_s(\kappa)(t - t_1)] \quad (24)$$

where  $n_\kappa$  is the number of terms in  $\sum_{q \neq 1}$  that lie in  $\Delta\kappa$ . The expression for  $\overline{u_i^2}$  (eq. (22)) then becomes

$$\overline{u_i^2} = \sum_\kappa \left\{ \langle a_{i, \vec{\kappa}, 1}^2 \rangle(\kappa) \exp[-2\nu\kappa^2(t - t_1)] + \sum_s \left( \frac{n_s}{n_\kappa} \right)_{(i)} \langle a_{i, \vec{\kappa}}^2 \rangle_s(\kappa) \exp[-2b_s(\kappa)(t - t_1)] \right\} \quad (25)$$

To obtain an expression for the energy spectrum function  $E$ , we note that (ref. 1)

$$\frac{1}{2} \overline{u_i u_i} = \int_0^\infty E \, d\kappa \quad (26)$$

where  $\overline{u_i u_i} = \overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}$ . Equations (25) and (26) then give

$$\int_0^\infty E \, d\kappa = \sum_{\kappa} \frac{1}{2} \left\{ \frac{\langle a_{i, \vec{\kappa}, 1} a_{i, \vec{\kappa}, 1} \rangle}{\Delta \kappa} \exp[-2\nu \kappa^2 (t - t_1)] + \sum_s \left( \frac{n_s}{n_{\kappa, i}} \right) \frac{\langle a_{i, \vec{\kappa}}^2 \rangle_s}{\Delta \kappa} \exp[-2b_s (t - t_1)] \right\} \Delta \kappa \quad (27)$$

where there is now a summation on  $i$ . If  $\Delta \kappa$  is very small, we can write, to a very good approximation,

$$E(\kappa) = B^2(\kappa) \exp[-2\nu \kappa^2 (t - t_1)] + \sum_s B_s^2(\kappa) \exp[-2b_s(\kappa)(t - t_1)] \quad (28)$$

Equation (28) gives the evolution in time of the energy spectrum function from an initial state which is specified by the  $B$ 's and  $b$ 's in the equation.

As shown in the last section, if the turbulence is specified at an initial instant, the time derivatives of  $E$  can be calculated at that instant by using the Fourier transformed correlation equations. Thus, it is desirable to write the  $B$ 's and  $b$ 's in equation (28) in terms of  $E$  and its derivatives at the initial time. That can be done by evaluating equation (28) and its time derivatives at  $t = t_1$  and solving the resulting system of equations for the  $B$ 's and  $b$ 's.

In what follows, we will first retain only two terms of equation (28). Equation (28) can then be written conveniently as

$$E = E_{t=t_1} \left\{ C(\kappa) \exp[-2\nu \kappa^2 (t - t_1)] + (1 - C) \exp[-2b(\kappa)(t - t_1)] \right\} \quad (29)$$

where  $0 \leq C \leq 1$ .

For  $C = 1$  equation (29) reduces to the well-known expression for the final period of decay (ref. 1). For the general case ( $C \neq 1$ ) we could determine  $C$  and  $b$  in terms of the first and second derivatives of equation (29) for  $t = t_1$  and then evaluate those derivatives by using the two-point spectral equations (see eqs. (10) to (12)). The following procedure turns out to be simpler, however. By substituting equation (29) into the spectral equation (10) we get for the energy transfer term

$$T = 2(1 - C)(\nu\kappa^2 - b)E_{t=t_1} \exp[-2b(t - t_1)] = T_{t=t_1} \exp[-2b(t - t_1)] \quad (30)$$

Then

$$\frac{\partial T}{\partial t} = -2bT_{t=t_1} \exp[-2b(t - t_1)] = \left(\frac{\partial T}{\partial t}\right)_{t=t_1} \exp[-2b(t - t_1)] \quad (31)$$

Comparing the last two members of equation (31) and using equation (11) gives

$$b = \nu\kappa^2 - \frac{V_{t=t_1}}{2T_{t=t_1}} \quad (32)$$

From equations (30) and (32) we have

$$C = 1 - \frac{T_{t=t_1}^2}{V_{t=t_1}E_{t=t_1}} \quad (33)$$

Equations (29) and (30) then become

$$E = E_{t=t_1} \left\{ C \exp[-2\nu\kappa^2(t - t_1)] + (1 - C) \exp\left[-2\left(\nu\kappa^2 - \frac{V_{t=t_1}}{2T_{t=t_1}}\right)(t - t_1)\right] \right\} \quad (34)$$

and

$$T = T_{t=t_1} \exp \left[ - 2 \left( \nu \kappa^2 - \frac{V_{t=t_1}}{2T_{t=t_1}} \right) (t - t_1) \right] \quad (35)$$

From equation (11)

$$V = V_{t=t_1} \exp \left[ - 2 \left( \nu \kappa^2 - \frac{V_{t=t_1}}{2T_{t=t_1}} \right) (t - t_1) \right] \quad (36)$$

where  $C$  is given by equation (33).

Equations (34) and (35) were obtained by retaining two terms on the right side of equation (28). We consider next a higher order approximation in which three terms are retained in that equation. If equation (28), with three terms retained, is substituted into equation (10), we get for  $T$

$$T = 2B_1^2(\kappa^2 - b_1) \exp[-2b_1(t - t_1)] + 2B_2^2(\kappa^2 - b_2) \exp[-2b_2(t - t_1)] \quad (37)$$

Equation (37) contains four unknown functions which are to be determined by the initial conditions. For that purpose we use equation (37) and its first three derivatives evaluated at  $t = t_1$ . Thus, we obtain

$$b_1 = - \frac{T_1 T_2 - T_{t=t_1} T_3}{4(T_1^2 - T_{t=t_1} T_2)} + \left\{ \frac{\left[ \frac{T_1 T_2 - T_{t=t_1} T_3}{4(T_1^2 - T_{t=t_1} T_2)} \right]^2 - \frac{T_2^2 - T_1 T_3}{4(T_1^2 - T_{t=t_1} T_2)}}{4(T_1^2 - T_{t=t_1} T_2)} \right\}^{1/2} \quad (38)$$

$$b_2 = - \frac{T_1 T_2 - T_{t=t_1} T_3}{4(T_1^2 - T_{t=t_1} T_2)} - \left\{ \frac{\left[ \frac{T_1 T_2 - T_{t=t_1} T_3}{4(T_1^2 - T_{t=t_1} T_2)} \right]^2 - \frac{T_2^2 - T_1 T_3}{4(T_1^2 - T_{t=t_1} T_2)}}{4(T_1^2 - T_{t=t_1} T_2)} \right\}^{1/2} \quad (39)$$



$$B_1^2 = \frac{2b_2 T_2 + T_3}{16b_1^2(\kappa^2 - b_1)(b_2 - b_1)} \quad (40)$$

and

$$B_2^2 = \frac{2b_1 T_2 + T_3}{16b_2^2(\kappa^2 - b_2)(b_1 - b_2)} \quad (41)$$

where  $T_1$ ,  $T_2$ , and  $T_3$  are the first, second, and third time derivatives of  $T$  at  $t = t_1$ . The first derivative  $T_1$  can be written in terms of the functional  $V_{t=t_1}$ , which gives a representation of three-point spectral quantities (eq. (11)). Equations for higher order functionals can be obtained by the procedure used for obtaining equation (11) for  $V$ . Thus, by using the four-point spectral equations of reference 6 (eqs. (11) and (12)) we get

$$\frac{\partial V}{\partial t} = -2\nu\kappa^2 V + R \quad (42)$$

where  $R$  is a functional of three- and four-point spectral quantities. Similarly,

$$\frac{\partial R}{\partial t} = -2\nu\kappa^2 R + S \quad (43)$$

where  $S$  is a functional of three-, four-, and five-point spectral quantities. By using equations (11), (42), and (43), the first, second, and third time derivatives of  $T$  at  $t = t_1$  in equations (38) to (41) can be written in terms of higher order spectral quantities as

$$T_1 = -2\nu\kappa^2 T_{t=t_1} + V_{t=t_1} \quad (44)$$

$$T_2 = (2\nu\kappa^2)^2 T_{t=t_1} - 4\nu\kappa^2 V_{t=t_1} + R_{t=t_1} \quad (45)$$

and

$$T_3 = -(2\nu\kappa^2)^3 T_{t=t_1} + 3(2\nu\kappa^2)^2 V_{t=t_1} - 6\nu\kappa^2 R_{t=t_1} + S_{t=t_1} \quad (46)$$

## RESULTS AND DISCUSSION

A comparison between the experimental data of Uberoi (ref. 13) and the present theory (eqs. (34) to (36)) is given in figures 1 to 4. (Another pertinent experimental investigation is that of C. W. Van Atta of the University of California at San Diego. He recently measured directly the individual terms in the two-point spectral equation; however, his data are for only one time.) The comparison in figures 1 to 4 is made for an initial time corresponding to  $X/M = 48$  in the experiment ( $t_1^* = (\nu/M^2)t = 0.001818$ ,  $X$  is the distance downstream from the grid, and  $M$  is the mesh size of the grid). For the initial specification of the turbulence values of  $E$  and  $T$  were obtained from figures 5, 9, and 10 of reference 13. Initial values of  $V$  were not given directly in reference 13 but were estimated from the decay data for  $T$  and equation (11). Except for experimental error those values will be the same as those that might have been measured directly.

The agreement between the predicted and experimental energy spectra for the same initial conditions (fig. 1) appears to be quite good, considering the difficulty of the measurements. The calculation of the experimental values of  $E$  required the differentiation of measured one-dimensional spectra and an assumption of isotropy.

Predicted and experimental values for the decay of  $\overline{u_1 u_1}$  are plotted in figure 2. The agreement between theory and experiment is excellent for values of  $t^*$  up to about 0.006. (Note that spectra were measured only for values of  $t^*$  between 0.00182 and 0.00417.) Elimination of the moderate deviation for  $t^* > 0.006$  might require a higher

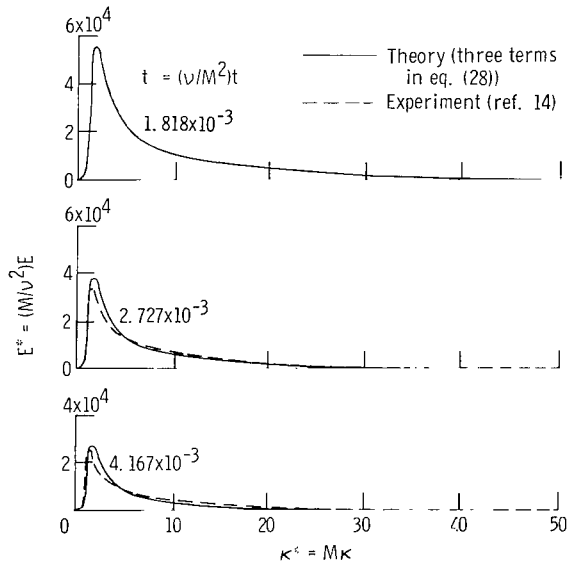


Figure 1. - Comparison of theory with experiment of reference 13 for decay of turbulent energy spectra.

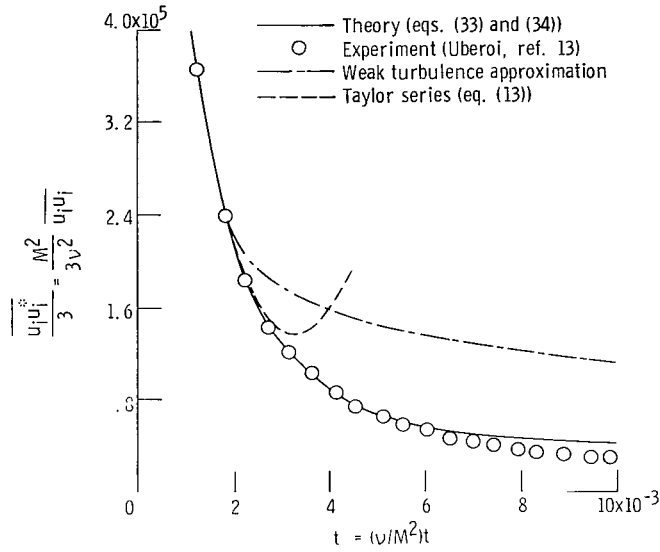


Figure 2. - Comparison of theory with experiment of reference 13 for decay of average component of velocity variance.

order theory (more terms in eq. (28)), together with additional initial statistical information. Alternatively, the deviation might be due to the amplification at large times of slight inaccuracies in the measured initial spectra. The theoretical values for  $t^*$  less than 0.00182 were calculated by working backwards from the measured initial spectra. Also included in figure 2 is a Taylor series solution which uses the same initial information as the exponential series and the curve for the weak turbulence approximation. It might be pointed out that the curve for the weak turbulence approximation is not the  $-5/2$  power decay law usually given for the final period (ref. 1) but is the curve obtained by using the measured initial energy spectrum and equation (29) with  $C = 1$ .

Spectra for the energy transfer term  $T$  are plotted in figure 3. The experimental and theoretical curves are in good agreement except near the value of  $\kappa$  where  $T_{t=t_1}$  changes sign. The deviation there results from a mathematical singularity in equation (35) when  $T_{t=t_1} = 0$ . However, that deviation does not seem to be serious, because the real physical curve in that region can easily be estimated graphically or by using an interpolation formula. This is particularly true since it is known that the total area enclosed by the  $T$  spectrum should be zero (ref. 1). It appears likely that the difficulty could be eliminated if another term were retained in equation (28). (More will be said about that possibility in the next paragraph.) The deviation also carries over to some extent into the results for  $E$  and  $\overline{u_i u_i}$ . However, if one does not use values of  $\kappa$  close to the point where  $T_{t=t_1}$  changes sign for calculating  $E$  and  $\overline{u_i u_i}$ , the inaccuracies in those quantities will be small. It appears that the overall agreement between theory and

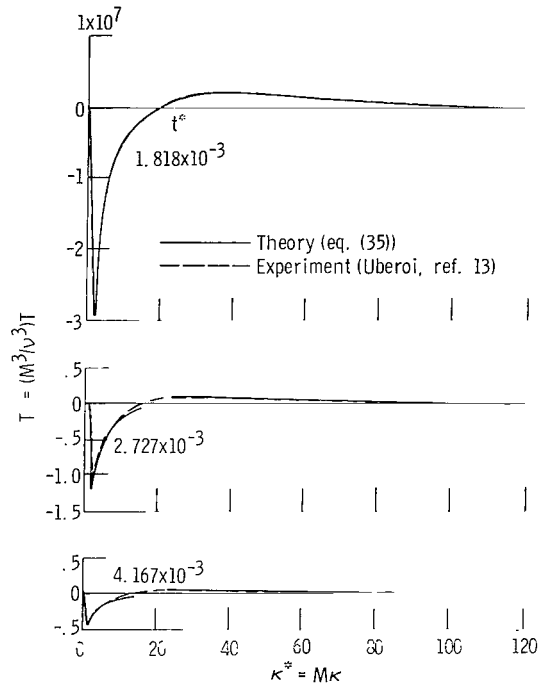


Figure 3. - Comparison of theory with experiment of reference 3 for decay of energy transfer spectra.

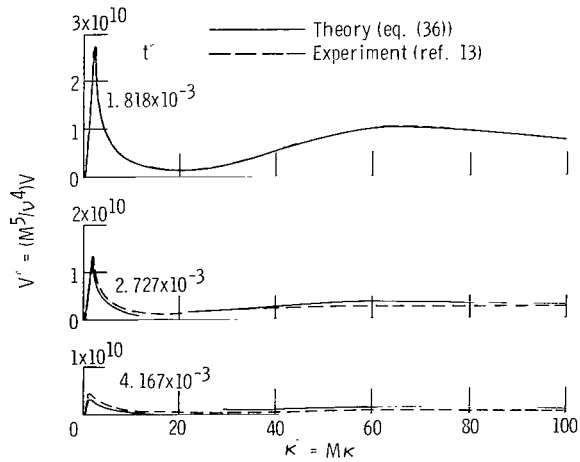


Figure 4. - Comparison of theory with experiment of reference 3 for decay of higher order spectral quantity  $V$  (eq. (11)).

experiment obtained by using equations (33) to (35) should be considered encouraging.

For the sake of completeness, spectra of the functional  $V$  (eqs. (11) and (36)), the third initial condition specified for the turbulence, are plotted in figure 4. The agreement between theory and experiment is probably within the uncertainty in estimating  $V$  from the decay data in reference 13, except in the vicinity of the point where  $T_{t=t_1}$  changes sign. Thus, the theory predicts the evolution in time of  $E$ ,  $T$ , and  $V$ , when those quantities are specified at an initial time.

We have not been able to apply a higher order theory to Uberoi's data, that is, to evaluate three instead of two terms in equation (28) by using the initial data given in his article. However, we can apply a higher order theory to an analysis in reference 5, since for that analysis we can, in effect, calculate as much initial information as is desired. That analysis neglects quadruple correlation terms in the three-point correlation equations and should apply, for a particular set of initial conditions, at times somewhat before the final period of decay. The initial conditions, as well as values at later times, are given by closed-form equations in that analysis and thus are better defined than may be possible in an experiment. For the present purposes, the analytical results from reference 5 might in fact be thought of as experimental results in which the initial conditions are specified exactly. This is true because the analysis of reference 5 is exact for the model chosen, and the initial conditions used in both that analysis and the present theory correspond to that model.

The case considered here corresponds to figure 6 of reference 5. Values of dimensionless  $E$ ,  $T$ , and time derivatives of  $T$  for the initial specification of the turbulence ( $t_1^* = 0.002$ ) are obtained from equations (40) and (39) in reference 5. We can eliminate the time derivatives of  $T$  by introducing  $V$  (eq. (11)) and the higher order functionals  $R$  and  $S$  (eqs. (42) and (43)). In the present case, those quantities will all be representations of correlations of order no higher than the third, since terms involving correlations of higher order than the third are assumed negligible in the analysis of reference 5.

Figure 5 gives a comparison between results for  $T$  calculated from the present analysis and those from reference 5. The quantity  $J_0$  is a constant related to conditions at  $t_0^* = -0.00633$  in the equations of reference 5. The starred quantities in figures 5 to 7 are the same as those in figures 1 to 4 if we let  $J_0 = M^3 \nu^2$ . As expected, when  $T$  is calculated from equation (35), the agreement with reference 5 is good except in the region where  $T_{t=t_1}$  changes sign. However, when a higher order theory is used by retaining three terms in the expression for  $E$  (two terms in the expression for  $T$  eq. (37)) the agreement is excellent at essentially all values of  $\kappa$ . It might be expected that a similar improvement would be obtained in figures 3 and 4 if a higher order theory could be used for comparison with the experimental data of Uberoi.

Because of the good agreement obtained for  $T$  in figure 5, one would expect the calculated energy spectra  $E$  to also be in good agreement with those from reference 5.

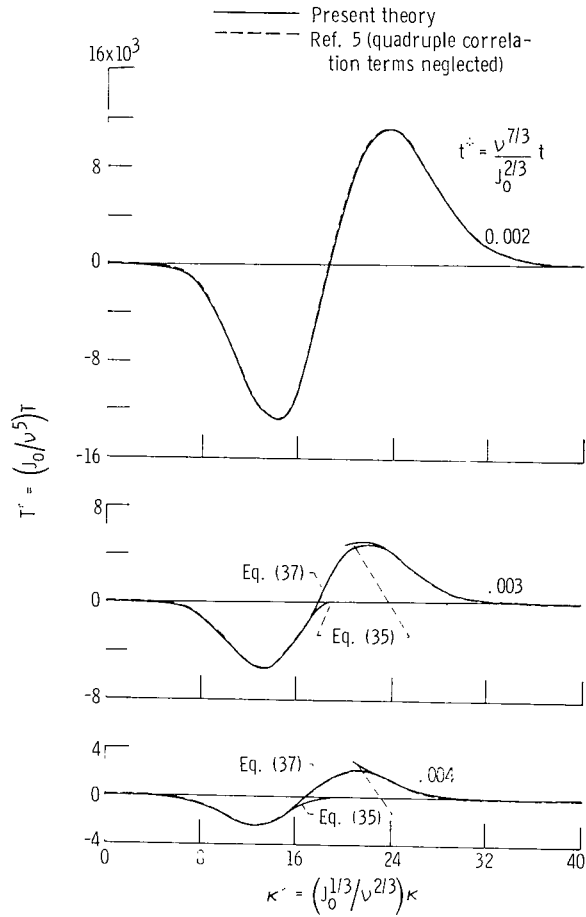


Figure 5. - Comparison of present theory with that of reference 5 for decay of energy transfer spectrum.

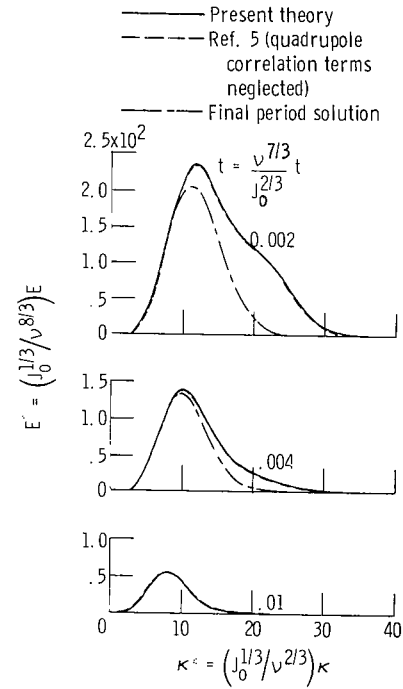


Figure 6. - Comparison of present theory with that of reference 5 for decay of turbulent energy spectrum.

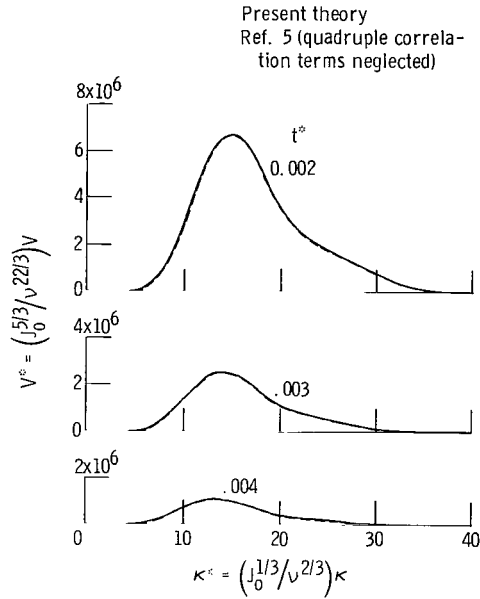


Figure 7. - Comparison of present (higher order) theory with that of reference 5 for decay of higher order spectral quantity  $V$  (eq. (11)).

Figure 6 shows that that is indeed the case. The energy spectrum, in this case, decays in a highly nonsimilar fashion. In order to show the effects of energy transfer between wave numbers, curves for the final period of decay (first term of eq. (40) of ref. 5) are also included in figure 6.

Figures 7 to 9 show plots for the decay of the higher order spectral quantities  $V$ ,  $R$ , and  $S$ . The agreement between the present higher order theory and the results of reference 5 is very good. Although the effects of the singularity at  $\kappa = 15.33$  are greater for these higher order quantities than for the lower order ones, they are still not apparent unless points close to the singularity are used in plotting the curves. For points close to the singularity, an interpolation formula can be used. Thus, by specifying the initial conditions for  $E$ ,  $T$ ,  $V$ ,  $R$ , and  $S$ , we can predict the evolution in time of those quantities by using the present higher order theory. That is, the required number of initial conditions is no greater than the number of quantities whose decay we can predict.

The higher order theory (three exponential terms retained in eq. (28)) can also be compared with some recent grid-turbulence data obtained in a water channel by Ling and Huang (ref. 14). For that comparison, the experimental input can be conveniently obtained from an empirical equation for  $E$  (eq. (22)) in their article. The higher order spectra were not measured directly in their experiment but could be calculated from their equation for  $E$  by using equations (10), (11), (42), and (43). Except for possible

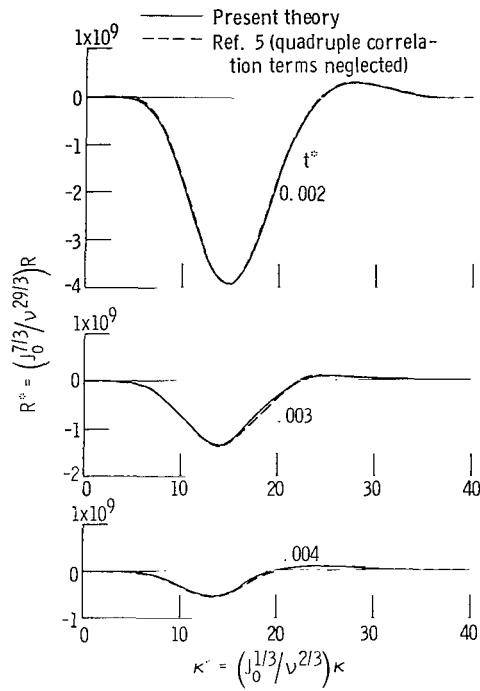


Figure 8. - Comparison of present (higher order) theory with that of reference 5 for decay of higher order spectral quantity  $R$  (eq. (42)).

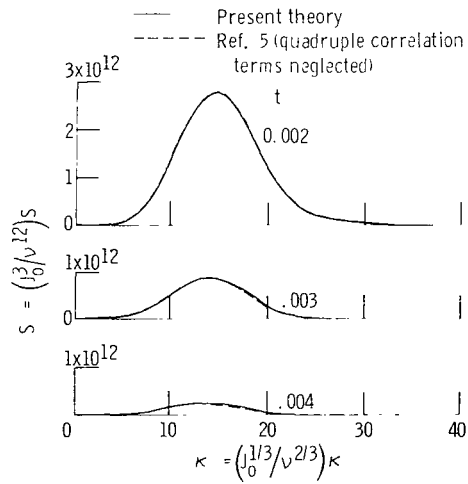


Figure 9. - Comparison of present (higher order) theory with that of reference 5 for decay of higher order spectral quantity  $S$  (eq. (43)).



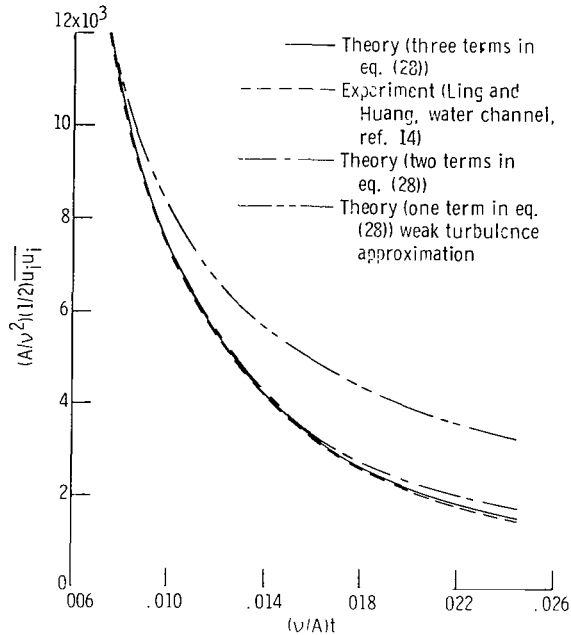


Figure 10. - Comparison of theory with experiment of reference 14 for decay of turbulent energy.

experimental error those values will be the same as those that might have been measured directly. The comparison is shown in dimensionless form in figures 10 to 15. The quantity  $A$  is a constant with the dimensions of a length squared and is related to conditions at a time  $t_0$  (ref. 14). As in the preceding comparisons, unphysical singularities occurred in the theoretical spectra at certain values of  $\kappa$ , particularly in the higher order spectra. Thus, in the vicinity of those points, four-point interpolation formulas were used.

Figure 10 compares theory and experiment for the decay of turbulent energy when the initial state is specified at  $(\nu/A)t_1 = 0.0075$ . Theoretical curves are shown for 1, 2, and 3 exponential terms retained in equation (28). The curve for three terms is in good agreement with the experiment for the whole decay period. The curve for two terms is in almost as good agreement. That is not the case for the spectra, where only the curves for three terms agree closely (see the curves for  $E$  in fig. 11). Comparison of the curve in figure 10 for one term retained (weak turbulence approximation) with the experimental curve shows the effect of inertia on the decay process. As in figure 2, the curve for the weak turbulence approximation in figure 10 is not the  $-5/2$ -power decay law usually given for the final period, since measured initial energy spectra were used in this report.

Figures 11 to 15 give a comparison of theory and experiment for the decay of the spectra used to specify the initial state of the turbulence at  $t_1$ . The curves indicate

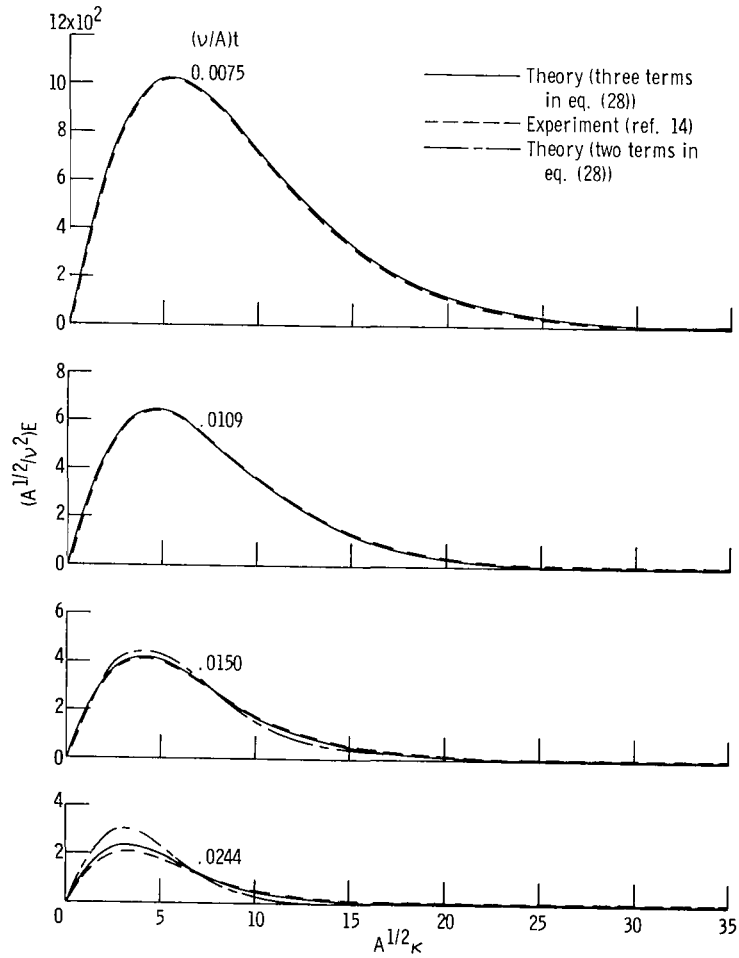


Figure 11. - Comparison of theory with experiment of reference 14 for decay of three-dimensional turbulent-energy spectra.

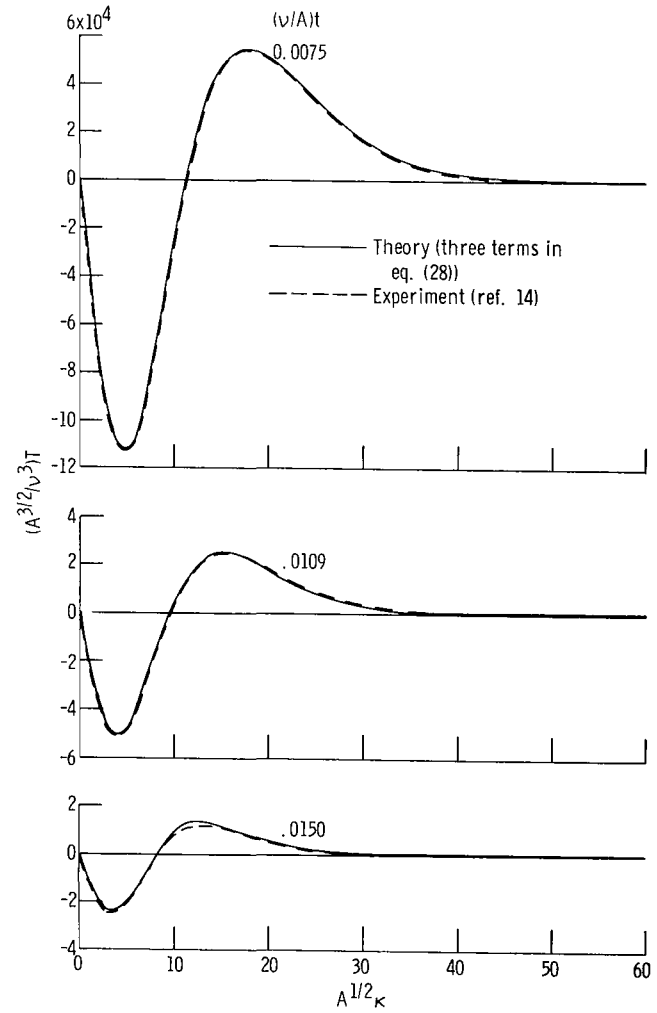


Figure 12. - Comparison of theory with experiment of reference 14 for decay of energy-transfer spectra.

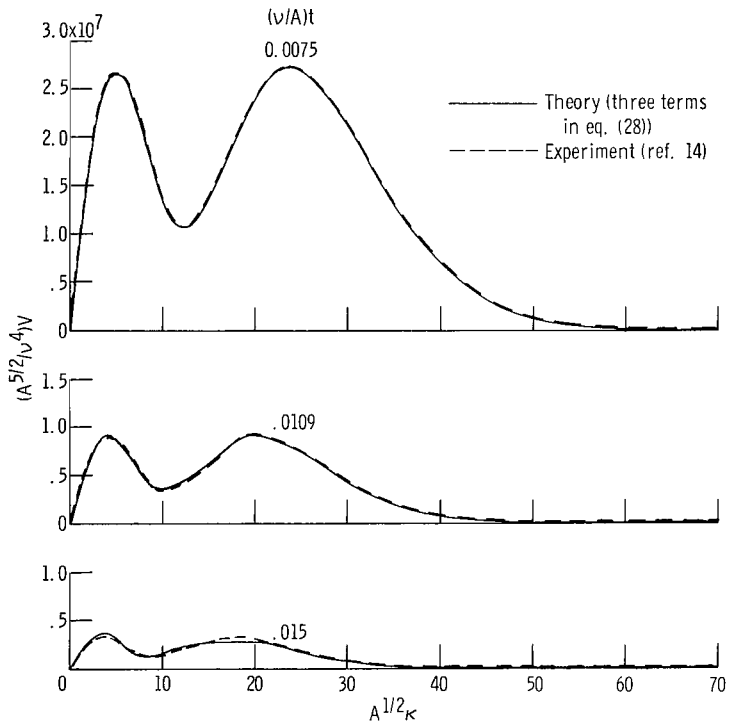


Figure 13. - Comparison of theory with experiment of reference 14 for decay of higher order spectral quantity  $V$  (eq. (11)).

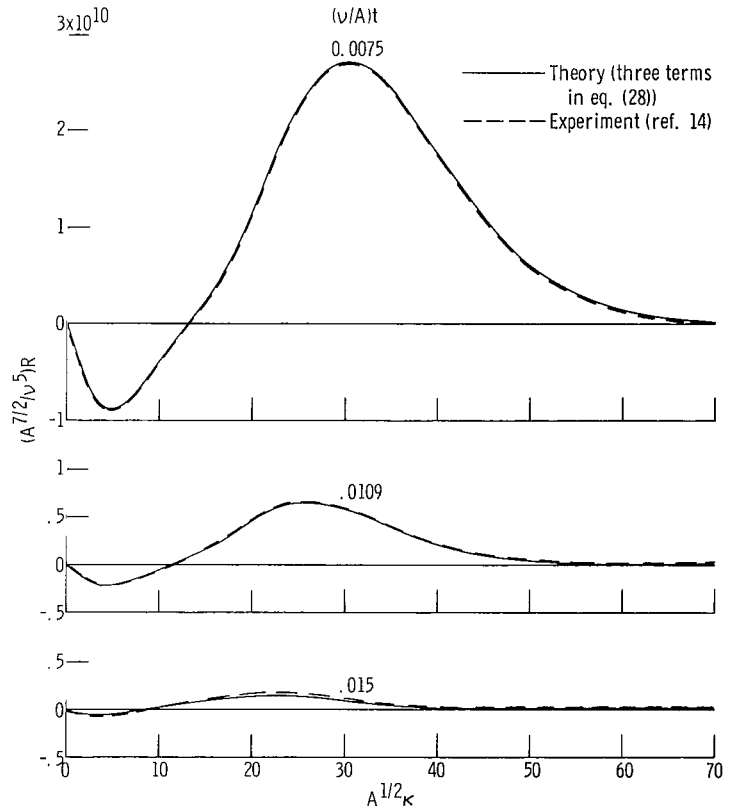


Figure 14. - Comparison of theory with experiment of reference 14 for decay of higher order spectral quantity  $R$  (eq. (42)).

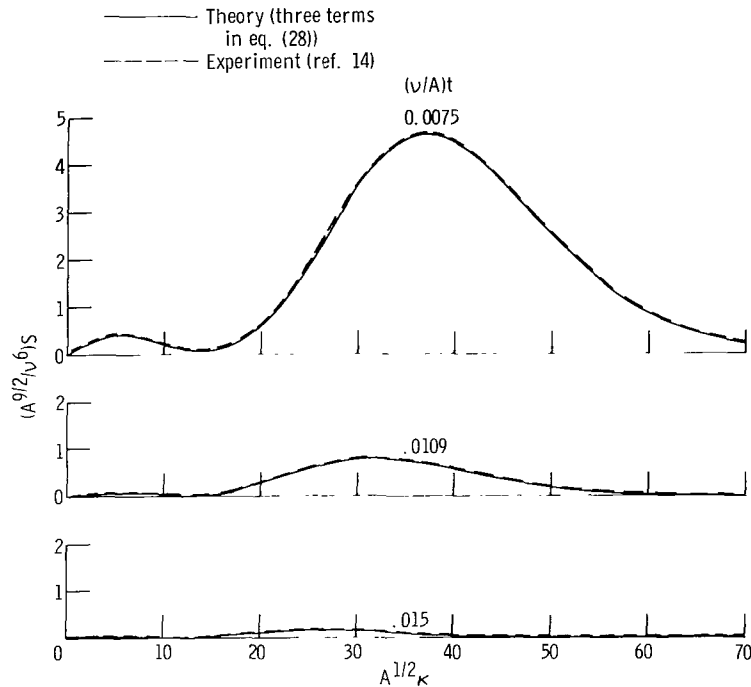


Figure 15. - Comparison of theory with experiment of reference 14 for decay of higher order spectral quantity  $S$  (eq. (43)).

good agreement with the higher order theory. That is, the theory is able to predict the decay of all of the spectra used to specify the initial turbulence, when three exponential terms are retained in equation (28).

## CONCLUDING REMARKS

If a homogeneous turbulent field is specified at an initial instant by its multipoint-velocity correlations (or their spectral equivalents), the initial time derivatives of those quantities can be calculated from the correlation or spectral equations. The development of the turbulence in time can then be obtained by using those derivatives in a series such as a Taylor power series. When the problem is formulated in this way, an assumption for closing the system of correlation equations is not required, since those equations are closed by the initially specified correlations or spectral quantities. A Taylor series expansion, however, did not give realistic results (except for small times) when the limited initial experimental data were used. An exponential series (eq. (28)) which was an iterative solution of the Navier-Stokes equations worked much better.

In general, when the energy and transfer spectra and a quantity related to three-

point spectra were specified at an initial time, the predicted changes with time of those spectra, as well as the turbulent energy, were in good agreement with the experimental wind-tunnel data of Uberoi. Since the prediction of the changes of those spectra with time is evidently an essential part of the homogeneous turbulence problem, the results are encouraging.

A higher order theory was given in which the three-point spectral quantities, as well as two additional higher order spectral quantities, were specified initially. The predicted decay of all of those quantities agreed very well with the water-channel data of Ling and Huang, as well as with the results for a previous analytical model. For the present purposes the results for the previous model might be thought of as experimental results in which the initial conditions are specified exactly. Thus, when the results from the present theory are compared with either experimental results or the results of an "analytical experiment," the agreement is good. The evolutions of the various spectra are interdependent on the initial specifications of those spectra.

By specifying  $n$  spectra at an initial time, where  $n$  is an odd integer greater than or equal to 3, we have been able to predict the evolution in time of those  $n$  spectra. We have not been able to obtain determinate results for  $n < 3$ , except for weak turbulence. But when one considers the fact that an infinite number of spectra (or correlations) would be required for a complete initial specification, there seems to be no obvious reason why we should be able to do so. In fact, if we were to claim that we should be able to predict the decay of the energy spectrum by specifying at an initial instant only that spectrum, we would in effect be saying that the Fourier components of the energy spectrum decay independently, as in the final period. If we want to include the effect of the interaction of those components, we will have to specify the initial energy transfer spectrum. The Fourier components of the transfer spectrum, and of higher order spectra, will also interact in this nonlinear problem, so it is not surprising that we have to give the initial specifications of at least three spectra in order to predict the decay of any of them.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, January 17, 1972,  
136-13.

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