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ELECTROTHERMAL OSCILLATIONS AND THE QUASI-LINEAR THEORY OF ELECTRON ENTHALPY FLUCTUATIONS IN MAGNETOHYDRODYNAMIC GENERATORS AND MAGNETOPLASMADYNAMIC ARC THRUSTERS

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16. Abstract

Fluctuations in electron density and temperature coupled through Ohm's Law are studied for MHD power generator and MPD arc thruster applications. The dispersion relation based on linear theory is derived, and the two limiting cases of infinite ionization rate and frozen flow are examined. The nonlinear effects of the frozen flow case are then studied in the quasi-linear limit. Equations are derived for the amplitude of the fluctuation and its effect upon Ohm's Law and the electron temperature equation. Conditions under which a steady state can exist in the presence of the fluctuation are examined, and effective transport properties are determined.

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ELECTROTHERMAL OSCILLATIONS AND THE QUASI-LINEAR THEORY OF ELECTRON ENTHALPY FLUCTUATIONS IN MAGNETOHYDRODYNAMIC GENERATORS AND MAGNETOPLASMADYNAMIC ARC THRUSTERS

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SUMMARY

Electrothermal oscillations arising due to fluctuations in electron density and temperature coupled through a generalized Ohm's Law are studied with regard to magnetohydrodynamic (MHD) power generator and magnetoplasmadynamic (MPD) arc thruster applications. The dispersion relation is derived. In the limit of infinite ionization rate the dispersion relation reduces to that case which has been extensively analyzed for MHD generators. In the MPD thruster configuration this limit results in both a radial and a rotational oscillation. For the opposite limit of zero ionization rate, or frozen flow, it is shown that an unstable mode can exist which, while in general not critical to MHD generator operation, can be responsible for the rotational oscillation observed in MPD thrusters. The nonlinear effect of this fluctuation upon the electron properties is studied in the quasi-linear limit. Equations are derived for the amplitude of the fluctuation and its effect upon Ohm's Law and the electron temperature equation. Conditions under which a steady state is reached in the presence of the fluctuation are determined.

The consequences of a purely rotational fluctuation were also investigated. Based on the assumption of zero gas temperature in the onset region, which was previously used to reduce the existing experimental data for comparison with linear theory, the quasi-linear theory predicts unstable growth of fluctuations. Therefore, for the zero gas temperature case a "steady state" fluctuation of finite amplitude is not predicted by quasi-linear theory.

For nonzero values of the gas temperature the quasi-linear theory leads to four regions in the $\left(T_0/\langle T_e\rangle\right)$-against-$\beta_0^2$ plane. These regions correspond to absolute instability (both linear and nonlinear terms are growth terms), absolute damping (both linear and nonlinear terms are damping), stability at finite amplitude (linear term is growth term while nonlinear term is damping), and instability at finite amplitude (linear term is damping while nonlinear term is a growth term).

INTRODUCTION

Electrothermal oscillations can arise in gases due to fluctuations in the electron temperature. These temperature fluctuations can be amplified through increased ohmic heating arising as a result of increased ionization and/or decreased collision frequency. These changes in frequency occur as a consequence of the initial electron temperature perturbation. The dispersion relation depicting this phenomenon is derived in the first section of this report by considering the time-dependent electron density and temperature equations coupled through a generalized Ohm's Law. The neutral particles and ions are assumed to be incapable of responding at the frequencies of oscillation of interest since they are much heavier and much less mobile than the electrons. These heavy particles only form a time-independent, uniform background gas in which the electrons fluctuate.

The dispersion relation is then examined in two limits. In the first limit, the ionization rate is taken to be infinitely large compared to the frequency of the disturbance. This limiting solution has been widely investigated with regard to MHD power generators (refs. 1 to 4). In the geometry of the MPD arc thruster this mode produces both a radial and a rotational oscillation. Since this radial component has not been observed experimentally in high-pressure MPD arc thrusters, and due to the extensive investigation of this mode in the MHD generator case, it is not considered further in this report.

In the second limit, the ionization is assumed to be frozen. In this limit it is shown that the system can become unstable for electron collision frequencies which decrease sufficiently rapidly with electron temperature. For the MHD generator this mode does not appear to be important since operating conditions and working fluids of interest are such that the instability criterion is seldom satisfied. In the MPD arc thruster the general operating range is such that the collision frequency is Coulomb dominated. The electron temperature dependence of the collision frequency is such that this mode is unstable and can occur with a purely rotational component, as is observed experimentally.

In a recent paper (ref. 5) this rotational component was studied in order to make a comparison between theoretical and experimental determinations of the rotational frequency as a function of electric current, magnetic field strength, mass flow rate, and atomic weight of the working fluid. Reasonable agreement was obtained even through many of the values were far from the stability threshold, where linear results would be expected to apply.

Therefore, in the last section of this report the nonlinear effects of the electron enthalpy mode, that is, electrothermal oscillation in the limit of frozen ionization, are considered on the basis of a quasi-linear theory. In this theory the governing equations are expanded as in the linear case; however, nonlinear terms representing first-order corrections are retained. Conditions under which a steady state is reached in the presence of the fluctuation are determined.

ASSUMPTIONS AND GOVERNING EQUATIONS

Electrothermal instabilities can arise due to fluctuations in the electron temperature. These fluctuations increase the ionization and/or decrease the collision frequency, resulting in increased electrical conductivity and hence increased ohmic heating of the plasma. Under circumstances where this increased ohmic heating occurs, the temperature fluctuations are amplified and instability results.

Assumptions

In those regimes where electrothermal disturbances are predominant, the following assumptions, simplifications, and restrictions are applied:

(1) The analysis is restricted to sufficiently short wavelengths such that the steady-state properties do not vary appreciably over this distance.

(2) Only fluctuations in space and time of the electron density and temperature are considered. This is due to the fact that the relatively lighter and more mobile electrons respond to disturbances of greatly different frequencies than do the heavier and less mobile neutral atoms and ions.

(3) Only the propagation of magnetohydrodynamic disturbances is considered. This restriction implies neglecting the displacement current density in Maxwell's equations, ignoring induced magnetic fields relative to the applied constant magnetic field, and assuming the plasma to be quasi-neutral.

(4) Ion and neutral particle flow velocities are equal; that is, ion slip is ignored. Furthermore, all heavy particle flow velocities are assumed equal to the gas flow velocity $\underline{v}_0$.

(5) Only propagation in planes perpendicular to the applied magnetic field is considered.

(6) Ion and neutral particle temperatures are assumed to be equal and equal to the gas temperature T_0.

For these limitations, the problem is completely determined in terms of the fluctuations of electron density and temperature and their coupling through Ohm's Law and Maxwell's equations.

Maxwell's Equations

From Maxwell's equations, under the assumption of quasi-charge neutrality and neglecting the induced magnetic field, the following equations result:

$$\overline{\nabla} \times \underline{E} = 0 \tag{1}$$

$$\overline{\nabla} \cdot \underline{j} = 0 \tag{2}$$

All symbols are defined in the appendix.

Electron Continuity Equation

In the region of interest, the ionization is assumed to be dominated by electron-neutral atom ionizing collisions and by three-body recombination. The electron density equation then takes the form (see eq. 5.188 and discussion in chapter 6 of ref. 6.)

$$\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \overline{\nabla} n_e = n_e n_N \nu_i - n_e^3 \nu_r \tag{3}$$

This equation depends upon the steady-state gas flow velocity $\underline{v}_0$, rather than upon the electron flow velocity $\underline{v}_e$. This dependence arises as a result of assumption 4 and the assumption of quasi-charge neutrality. The current density is then given by $\underline{j} = n_e e(\underline{v}_0 - \underline{v}_e)$ and $\overline{\nabla} \cdot \underline{j} = 0$. Therefore,

$$\overline{\nabla} \cdot n_e \underline{v}_e = \overline{\nabla} \cdot n_e \left(\underline{v}_0 - \frac{1}{n_e e} \underline{j}\right) = \overline{\nabla} \cdot n_e \underline{v}_0 = \underline{v}_0 \cdot \overline{\nabla} n_e \tag{4}$$

since the gas flow velocity is constant to within the limits of assumption 1.

The electron number density equation is further simplified by assuming that the static solution is that of Saha equilibrium so that we are only concerned with small fluctuations from equilibrium. Under this condition the principle of detail balance (ref. 7) can be invoked to relate the ionization coefficient ν_i to the recombination coefficient ν_r through the equilibrium constant K. Then

$$\nu_i = K\nu_r \tag{5}$$

where

$$K = \left(\frac{n_e^2}{n_N}\right)_{\text{equilibrium}} \sim T_e^{3/2} e^{-T_i/T_e} \tag{6}$$

and T_i is the temperature equivalent of the ionization potential (see eq. 6.83 ref. 6). The electron number density equation then has the form

$$\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \overline{\nabla} n_e = -n_e^3 \nu_r \left(1 - \frac{n_N}{n_e^2} K\right) \tag{7}$$

Generalized Ohm's Law

The generalized Ohm's Law is (see ref. 6, eq. 5.168 neglecting ion slip)

$$\underline{j} = \sigma \underline{E}^* - \frac{e}{m_e \nu} \underline{j} \times \underline{B}_0 + \frac{e}{m_e \nu} \overline{\nabla} P_e \tag{8}$$

where $\underline{E}^* = \underline{E} + \underline{v}_0 \times \underline{B}_0$ is the electric field in the frame of reference moving with gas flow velocity $\underline{v}_0$ in the applied magnetic field $\underline{B}_0$. The magnetic field induced by the current $\underline{j}$ has been assumed negligible. The time-dependent and electron inertia terms are neglected since the current density relaxes in a mean collision time which is several orders of magnitude smaller than the period of the fluctuations of interest.

The electrical conductivity σ is given by

$$\sigma = \frac{n_e e^2}{m_e \nu} \tag{9}$$

where the total electron momentum collision frequency is

$$\nu = \nu_{eN} + \nu_{ei} \tag{10}$$

The species collision frequencies are allowed the following general temperature dependence:

$$\nu_{e\alpha} = A_{e\alpha} n_\alpha T_e^{a_{e\alpha}} \tag{11}$$

where $\alpha = i$ or N, and $A_{e\alpha}$ and $a_{e\alpha}$ are arbitrary constants chosen to fit average collision frequency data over the temperature range of interest.

Electron Energy Equation

The electron energy equation is

$$\frac{3}{2}k\left(n_e\frac{\partial}{\partial t}T_e + n_e\underline{v}_0\cdot\overline{\nabla}T_e\right) - \frac{5}{2}\underline{j}\cdot\overline{\nabla}\frac{kT_e}{e}$$

$$= -2\frac{m_e}{m_a}\nu n_e\frac{3}{2}k(T_e - T) + \underline{j}\cdot\underline{E}^* - \left(kT_i + \frac{3}{2}kT_e\right)\left(\frac{\partial}{\partial t}n_e + \underline{v}_0\cdot\overline{\nabla}n_e\right) \qquad (12)$$

where the $\overline{\nabla}T_e$ contribution to the heat conduction term and radiation losses are neglected. These terms contribute wavelength-dependent damping terms to the dispersion relation relative to the wavelength-independent elastic collision damping. The analysis is therefore restricted to wavelengths of sufficient magnitude so as to make these terms ignorable. The question of the relative importance of these terms has been extensively investigated for MHD-generator-type plasmas in references 8 and 9. Equation (12) is equation 5.196 of reference 6. However, $\underline{q}'_e$ in equation 5.196, as defined by equation 5.198, is in error. Equation 5.198 is the expression for $\underline{q}_e$, not $\underline{q}'_e$, as can be seen from equation 5.151 and the derivation thereof. Then $\underline{q}'_e$ is related to $\underline{q}_e$ by equation 4.175 so that the correct value is, neglecting terms the order of the electron-to-heavy-particle mass ratio,

$$\underline{q}'_e = \underline{q}_e - \frac{5}{2}p_e\underline{V}_e = -\underline{\underline{K}}^*_e\cdot\overline{\nabla}T_e$$

This is just the portion of the heat conduction vector which by prior argument we ignored. Therefore, in equation 5.196 of reference 6, $\overline{\nabla}\cdot\underline{q}'_e = 0$; and our equation (12) follows directly.

LINEAR STEADY-STATE EQUATION

Under the previously discussed assumptions, the zeroth order (or steady state) solution is depicted by a region of constant fluid dynamic and electromagnetic properties. Then equations (7), (8), and (12) reduce, respectively, to

$$\frac{n_{e0}^2}{n_{N0}} = K(T_{e0}) \equiv K_0 \tag{13}$$

$$\underline{j}_0 = \sigma_0 \underline{E}_0^* - \frac{e}{m_e \nu_0} \underline{j}_0 \times \underline{B}_0 \tag{14}$$

$$\underline{j}_0 \cdot \underline{E}_0^* = 2 \frac{m_e}{m_a} \nu_0 n_{e0} \frac{3}{2} k(T_{e0} - T_0) \tag{15}$$

where the subscript 0 refers to the zeroth order (or steady state) solution.

The steady state is therefore described by an electron density in Saha equilibrium at an electron temperature given by equating the Joule heating to the elastic collisional energy losses.

LINEAR STABILITY ANALYSIS

The essence of this analysis is to determine whether an infinitesimal perturbation to the steady-state solution will initially grow or decay. To this end we therefore consider solutions of the governing equations of the form

$$f = f_0 + f'(\underline{r}, t) \tag{16}$$

where f_0 is the steady-state solution which is constant in space and time and

$$f' = \text{real}\left[\tilde{f} e^{i(\underline{l} \cdot \underline{r} + \omega t)}\right] \tag{17}$$

is an infinitesimal perturbation to this solution such that the products of the f''s can be neglected. The $\tilde{f}$'s are independent of space and time. We furthermore consider only propagation in the plane perpendicular to $\underline{B}_0$, which is taken to be in the y-direction, so that

$$(l_x, l_y, l_z) = (l_x, 0, l_z) \tag{18}$$

By substituting functions of the preceding form into the governing equations, neglecting perturbations in the gas dynamic quantities n_a, $\underline{v}$, T, and utilizing the steady-state solutions, the equations for the perturbations reduce to the following:

Maxwell's equations:

$$\underline{l} \times \underline{\tilde{E}}^* = 0 \tag{19}$$

$$\underline{l} \cdot \underline{\tilde{j}} = 0 \tag{20}$$

Electron continuity equation:

$$\omega^* \frac{\tilde{n}_e}{n_{e0}} = -in_{N0}\nu_{i0}\left[\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\frac{\tilde{T}_e}{T_{e0}} - 2\frac{\tilde{n}_e}{n_{e0}}\right] \tag{21}$$

where $\omega^* = \omega + \underline{l} \cdot \underline{v}_0$ is the frequency in a coordinate system moving with the fluid.

Generalized Ohm's law:

$$\underline{\tilde{j}} = \sigma_0\underline{\tilde{E}}^* + \tilde{\sigma}\underline{E}_0^* - \frac{e}{m_e\nu_0}\underline{\tilde{j}}\times\underline{B}_0 + \frac{\tilde{\nu}}{\nu_0}\frac{e}{m_e\nu_0}\underline{j}_0\times\underline{B}_0 + i\underline{l}\frac{e}{m_e\nu_0}P_{e0}\left(\frac{\tilde{n}_e}{n_{e0}} + \frac{\tilde{T}_e}{T_{e0}}\right) \tag{22}$$

where

$$\frac{\tilde{\sigma}}{\sigma_0} = \frac{\tilde{n}_e}{n_{e0}} - \frac{\tilde{\nu}}{\nu_0} \tag{23}$$

$$\frac{\tilde{\nu}}{\nu_0} = \left(a_{eN}\frac{\nu_{eN0}}{\nu_0} - \frac{3}{2}\frac{\nu_{ei0}}{\nu_0}\right)\frac{\tilde{T}_e}{T_{e0}} + \frac{\nu_{ei0}}{\nu_0}\frac{\tilde{n}_e}{n_{e0}} \tag{24}$$

and the classical Coulomb potential has been assumed.

Electron energy equation:

$$\frac{3}{2}kn_{e0}\omega^*\tilde{T}_e - \frac{5}{2}\underline{l} \cdot \underline{j}_0\frac{k\tilde{T}_e}{e} = i\underline{j}_0 \cdot \underline{E}_0^*\left(\frac{\tilde{\nu}}{\nu_0} + \frac{\tilde{n}_e}{n_{e0}} + \frac{\tilde{T}_e}{T_{e0} - T_0}\right)$$

$$- i\left(\underline{j}_0 \cdot \underline{\tilde{E}}^* + \underline{\tilde{j}} \cdot \underline{E}_0^*\right) - \left(\frac{3}{2}kT_{e0} + kT_i\right)\omega^*\tilde{n}_e \tag{25}$$

Dispersion Relation

The dispersion relation can be obtained formally from the condition for the existence of a solution to the homogeneous linear equations (19) to (25), namely, that the determinant of coefficients be zero. However, since this method cannot be applied to the quasi-linear analysis, it is instructive to obtain the dispersion relation by direct substitution. This approach is taken in the quasi-linear analysis.

We start by observing that the perturbed ohmic heating term in equation (25) can, with the use of equation (8), be written as

$$\widetilde{\underline{j} \cdot \underline{E}^*} = \underline{j}_0 \cdot \underline{\tilde{E}}^* + \underline{\tilde{j}} \cdot \underline{E}_0^* = \widetilde{\frac{1}{\sigma} j^2} - \widetilde{\frac{e}{n_e \nu} \frac{1}{\sigma} \underline{j} \cdot \nabla P_e}$$

$$= -\frac{\tilde{\sigma}}{\sigma_0} \frac{1}{\sigma_0} j_0^2 + 2 \frac{1}{\sigma_0} \underline{j}_0 \cdot \underline{\tilde{j}} - i \frac{kT_{e0}}{e} \underline{l} \cdot \underline{j}_0 \left(\frac{\tilde{T}_e}{T_{e0}} + \frac{\tilde{n}_e}{n_{e0}} \right) \tag{26}$$

The variable $\underline{\tilde{E}}$ is eliminated from $\underline{\tilde{j}}$ in equation (22) by forming the vector product $\underline{l} \times (\underline{l} \times \underline{\tilde{j}})$ which, by use of equations (19) and (20), results in

$$\underline{l} \times \left(\underline{l} \times \underline{\tilde{j}} \right) = -l^2 \underline{\tilde{j}} = \tilde{\sigma} \underline{l} \times \left(\underline{l} \times \underline{E}_0^* \right) + \frac{\tilde{\nu}}{\nu_0} \frac{e}{m_e \nu_0} \underline{l} \times \left[\underline{l} \times \left(\underline{j}_0 \times \underline{B}_0 \right) \right] \tag{27}$$

Elimination of $\underline{E}_0^*$ by using equations (14) and (23) yields

$$-l^2 \underline{\tilde{j}} = \frac{\tilde{\sigma}}{\sigma_0} \left[\underline{l} \left(\underline{l} \cdot \underline{j}_0 \right) - l^2 \underline{j}_0 \right] + \frac{\tilde{n}_e}{n_{e0}} \frac{e}{m_e \nu_0} \left\{ \underline{l} \left[\underline{l} \cdot \left(\underline{j}_0 \times \underline{B}_0 \right) \right] - l^2 \left(\underline{j}_0 \times \underline{B}_0 \right) \right\} \tag{28}$$

which upon substitution into equation (26) gives

$$\underline{j}_0 \cdot \underline{\tilde{E}}^* + \underline{\tilde{j}} \cdot \underline{E}_0^* = \frac{\tilde{\sigma}}{\sigma_0} \left[1 - 2 \frac{\left(\underline{l} \cdot \underline{j}_0 \right)^2}{l^2 j_0^2} \right] \frac{1}{\sigma_0} j_0^2 - \frac{\tilde{n}_e}{n_{e0}} 2 \frac{e}{m_e \nu_0} \frac{\left(\underline{l} \cdot \underline{j}_0 \right) \left[\underline{l} \cdot \left(\underline{j}_0 \times \underline{B}_0 \right) \right]}{l^2 j_0^2}$$

$$\times \frac{1}{\sigma_0} j_0^2 - i \frac{kT_{e0}}{e} \underline{l} \cdot \underline{j}_0 \left(\frac{\tilde{T}_e}{T_{e0}} + \frac{\tilde{n}_e}{n_{e0}} \right) \tag{29}$$

Substituting equation (29) into (25) and using (23) and (24) to eliminate $\tilde{\sigma}/\sigma_0$ and $\tilde{\nu}/\nu_0$ in terms of $\tilde{n}_e/n_{e0}$ and $\tilde{T}_e/T_{e0}$ yields, after some rearranging,

$$
\begin{aligned}
&\left\{\frac{3}{2} P_{e0}\omega^* - \frac{3}{2}\frac{kT_{e0}}{e}\underline{l}\cdot\underline{j}_0 - i\frac{T_{e0}}{T_{e0} - T_0}\frac{1}{\sigma_0} j_0^2\right. \\
&\qquad \left. - i\,2\left(a_{eN}\frac{\nu_{eN0}}{\nu_0} - \frac{3}{2}\frac{\nu_{ei0}}{\nu_0}\right)\left[1 - \frac{\left(\underline{l}\cdot\underline{j}_0\right)^2}{l^2 j_0^2}\right]\frac{1}{\sigma_0} j_0^2\right\}\frac{\tilde{T}_e}{T_{e0}} \\
&\qquad = \left\{i\,2\left[\frac{\nu_{ei0}}{\nu_0} + \left(1 - \frac{\nu_{ei0}}{\nu_0}\right)\frac{\left(\underline{l}\cdot\underline{j}_0\right)^2}{l^2 j_0^2}\right]\frac{1}{\sigma_0} j_0^2 + i\,2\frac{e}{m_e\nu_0}\frac{\left(\underline{l}\cdot\underline{j}_0\right)\left[\underline{l}\cdot\left(\underline{j}_0\times\underline{B}_0\right)\right]}{l^2 j_0^2}\right. \\
&\qquad \left.\times\frac{1}{\sigma_0} j_0^2 - \frac{kT_{e0}}{e}\underline{l}\cdot\underline{j}_0 - n_{e0}\left(\frac{3}{2}kT_{e0} + kT_i\right)\omega^*\right\}\frac{\tilde{n}_e}{n_{e0}}
\end{aligned}
\tag{30}
$$

When equations (14) and (15) are used to eliminate $(1/\sigma_0)\, j_0^2$ and $\underline{l}\cdot\left(\underline{j}_0\times\underline{B}_0\right)$, the preceding equation reduces to

$$
\begin{aligned}
&\left(\omega^* - \frac{\underline{l}\cdot\underline{j}_0}{n_{e0}e} - i\,2\frac{m_e}{m_a}\nu_0\left\{1 + 2\left(a_{eN}\frac{\nu_{eN0}}{\nu_0} - \frac{3}{2}\frac{\nu_{ei0}}{\nu_0}\right)\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{\left(\underline{l}\cdot\underline{j}_0\right)^2}{l^2 j_0^2}\right]\right\}\right)\frac{\tilde{T}_e}{T_{e0}} \\
&\qquad = -\left(\frac{2}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\omega^* + \frac{2}{3}\frac{\underline{l}\cdot\underline{j}_0}{n_{e0}e} - i\,2\frac{m_e}{m_a}\nu_0\left(1 - \frac{T_0}{T_{e0}}\right)\right. \\
&\qquad \left.\times\left\{2\frac{\nu_{ei0}}{\nu_0}\left[1 - \frac{\left(\underline{l}\cdot\underline{j}_0\right)^2}{l^2 j_0^2}\right] + 2\sigma_0\frac{\left(\underline{l}\cdot\underline{j}_0\right)\left(\underline{l}\cdot\underline{E}_0^*\right)}{l^2 j_0^2}\right\}\right)\frac{\tilde{n}_e}{n_{e0}}
\end{aligned}
\tag{31}
$$

The dispersion relation is then obtained by using equation (21) to eliminate $\tilde{n}_e/n_{e0}$ from equation (31) so that

$$\omega^{*2} - \left(\frac{\underline{l} \cdot \underline{j}_0}{n_{e0}e} + i\,2\frac{m_e}{m_a}\nu_0\left\{1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{(\underline{l} \cdot \underline{j}_0)^2}{l^2 j_0^2}\right]\left(a_{eN}\frac{\nu_{eN0}}{\nu_0} - \frac{3}{2}\frac{\nu_{ei0}}{\nu_0}\right)\right\}\right.$$

$$\left. + i\,2\,n_{N0}\nu_{i0}\left[1 + \frac{1}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)^2\right]\right)\omega^* - (2n_{N0}\nu_{i0})\left(2\frac{m_e}{m_a}\nu_0\right)\left\{1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\right.$$

$$\times\left[1 - \frac{(\underline{l} \cdot \underline{j}_0)^2}{l^2 j_0^2}\right]\left[a_{eN}\frac{\nu_{eN0}}{\nu_0} + \frac{1}{2}\left(\frac{T_i}{T_{e0}} - \frac{3}{2}\right)\frac{\nu_{ei0}}{\nu_0}\right] + \left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)$$

$$\left.\times\left(1 - \frac{T_0}{T_{e0}}\right)\frac{(\underline{l} \cdot \underline{j}_0)(\underline{l} \cdot \underline{E}_0^*)}{l^2(\underline{j}_0 \cdot \underline{E}_0^*)}\right\} - i(2n_{N0}\nu_{i0})\frac{1}{3}\left(\frac{T_i}{T_{e0}} - \frac{3}{2}\right)\frac{\underline{l} \cdot \underline{j}_0}{n_{e0}e} = 0 \qquad (32)$$

Infinite Ionization Rate Limit

We consider two limiting cases. The first is that in which the ionization rate $n_{N0}\nu_{i0}$ is much greater than the frequency of the oscillation. In this case the electron number density is exactly in phase with the electron temperature fluctuation. This is seen from equation (21) in the limit $(n_{N0}\nu_{i0}/\omega^*) \rightarrow \infty$; that is,

$$2\frac{\tilde{n}_e}{n_{e0}} = \left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\frac{\tilde{T}_e}{T_{e0}} \qquad (33)$$

This limit is equivalent to assuming that the electrons remain in Saha equilibrium. Equation (33) can also be obtained directly from Saha's equation by finding its perturbed form. This limit is the one considered by Kerrebrock in reference 1 and has been successfully used in describing the fluctuations occurring in MHD generators (see refs. 2 and 3).

With regard to MPD arc thrusters, this mode was shown in reference 5 to result in radial as well as rotational oscillations. Since only rotational disturbances have so far been detected in these devices, it is not known how significant a role radial oscillations play or even if they exist in these devices. Due to the already extensive investigation of this mode, both on the basis of linear analysis and quasi-linear analysis (see review of present status in ref. 4), it will not be considered further in this report.

Frozen Flow Limit

The other limiting case is the opposite of the case just considered, that is, the limit in which the ionization rate $n_{N0}\nu_{i0}$ is small compared to the frequency of oscillation. This limit arises either as a result of the ionization coefficient ν_{i0} being small or when the gas becomes fully ionized, $n_{N0} \rightarrow 0$. In this case, the electron density is frozen as can be seen from equation (21) in the limit $(n_{N0}\nu_{i0}/\omega^*) \rightarrow 0$; that is,

$$\frac{\tilde{n}_e}{n_{e0}} = 0 \tag{34}$$

This mode will be called the electron enthalpy mode since only the electron temperature fluctuates. Taking the limit $(n_{N0}\nu_{i0}/\omega^*) \rightarrow 0$ in the general dispersion relation yields for $\omega^* = \omega_r^* - i\omega_i^*$

$$\omega_r^* = \frac{\underline{l} \cdot \underline{j}_0}{n_{e0}e} \tag{35}$$

$$\omega_i^* = -2 \frac{m_e}{m_a} \nu_0 \left\{1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{\left(\underline{l} \cdot \underline{j}_0\right)^2}{l^2 j_0^2}\right]\left(a_{eN} \frac{\nu_{eN0}}{\nu_0} - \frac{3}{2} \frac{\nu_{ei0}}{\nu_0}\right)\right\} \tag{36}$$

The condition that the system be unstable is then

$$1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{\left(\underline{l} \cdot \underline{j}_0\right)^2}{l^2 j_0^2}\right]\left(a_{eN} \frac{\nu_{eN0}}{\nu_0} - \frac{3}{2} \frac{\nu_{ei0}}{\nu_0}\right) \leq 0 \tag{37}$$

Since in MHD power generators or MPD arc thrusters $T_e \geq T_0$, the instability criterion requires that the total collision frequency be a decreasing function of the electron temperature (see eqs. (10) and (11) with $a_{ei} = 3/2$). It is further obvious from the inequality (37) that the unstable disturbance propagates at an angle relative to the steady-state current density $\underline{j}_0$, and that the growth rate ω_i^* is a maximum for disturbances propagating at right angles to $\underline{j}_0$.

Physical mechanism for instability of electron enthalpy mode. - The physical mechanism for this instability can be understood by considering a local temperature fluctuation propagating in the direction as depicted in figure 1 where, for simplicity of argument, we shall neglect the Hall effect and the electron pressure gradient in equation (8). Consider the region of incremental temperature increase δT_e. The total collision frequency depends upon the electron temperature $\nu \sim T_e^a$ so that $\delta\nu \sim a(\delta T_e)$. The electrical conductivity is inversely proportional to the total collision frequency $\sigma \sim 1/\nu$ so that $\delta\sigma \sim -a(\delta T_e)$. In the presence of the electric field, this incremental change in conductivity will result in an incremental change in the electric current density. Due to the condition $\overline{\nabla} \cdot \underline{j} = 0$ (eq. (2)), this incremental current density is forced to flow perpendicular to $\underline{l}$ rather than in the direction of the unperturbed current density $\underline{j}_0$. To see how this comes about, we consider the charge continuity equation. The condition $\overline{\nabla} \cdot \underline{j} = 0$ arises as a result of the fact that the time for significant space charge buildup is inversely proportional to the plasma frequency, which for typical MHD generator or MPD arc thruster conditions is many orders of magnitude greater than the 0- to 10^6-hertz range of interest. Hence, the space charge, for all practical purposes, instantaneously readjusts over the time scale of the wave phenomenon of interest. Thus, over the time scale of interest the time rate of change of the charge density term $\partial q/\partial t$ in the charge continuity equation $(\partial q/\partial t) + \overline{\nabla} \cdot j = 0$ can be ignored. Therefore, there is an instantaneous buildup of an electric field δE along $\underline{l}$ (see eq. (19)) cancelling out that component of current density which would otherwise flow across the inhomogenity in the $\underline{l}$ direction. To obtain the magnitude of this field, consider the component of change in current density along $\underline{l}$ due to $(\delta\sigma)$, $(\delta\sigma)(1/l)\underline{l} \cdot \underline{E}_0$. The incremental electric field in the direction of $\underline{l}$ needed to cancel this change in current density is then

$$(\delta\underline{E}) = -\frac{\delta\sigma}{\sigma_0}\frac{1}{l^2}\underline{l}(\underline{l} \cdot \underline{E}_0)$$

so that the current which actually flows, $\delta\underline{j}$, is given by

$$(\delta\underline{j}) = -\frac{1}{l^2}\underline{l} \times \left[\underline{l} \times (\delta\underline{j})\right] = -(\delta\sigma)\frac{1}{l^2}\underline{l} \times (\underline{l} \times \underline{E}_0)$$

The resultant change in the local ohmic heating is then given by

$$\underline{j}_0 \cdot (\delta\underline{E}) + (\delta\underline{j}) \cdot \underline{E}_0 = -\frac{(\delta\sigma)}{\sigma_0}\frac{1}{l^2}(\underline{l} \cdot \underline{j}_0)(\underline{l} \cdot \underline{E}_0) - (\delta\sigma)\frac{1}{l^2}\left[(\underline{l} \cdot \underline{E}_0)^2 - l^2\underline{E}_0^2\right]$$

$$= -a\left[1 - 2\frac{\left(\underline{l} \cdot \underline{j}_0\right)^2}{l^2 j_0^2}\right]\left(\frac{1}{\sigma_0} j_0^2\right)\frac{(\delta T_e)}{T_{e0}} \tag{38}$$

where we have used the relations $(\delta\sigma) \sim -a(\delta T_e)$ and $\underline{j}_0 = \sigma_0 \underline{E}_0$.

For equilibrium to be maintained, the change in ohmic heating must be balanced by the change in the collisional energy loss which is given by

$$\delta\left[2\frac{m_e}{m_a}\nu n_e \frac{3}{2}k(T_e - T)\right] = (a + 1)\left(\frac{1}{\sigma_0} j_0^2\right)\frac{(\delta T_e)}{T_{e0}} \tag{39}$$

We have used the steady-state relation

$$\frac{1}{\sigma_0} j_0^2 = 2\frac{m_e}{m_a}\nu_0 n_{e0}\frac{3}{2}k(T_{e0} - T_0)$$

and for simplicity have set $T_0 = 0$. The time rate of change in electron temperature is then proportioned to the difference between equations (38) and (39)

$$\frac{\partial}{\partial t}(\delta T_e) \sim -a\left[1 - 2\frac{\left(\underline{l} \cdot \underline{j}_0\right)^2}{l^2 j_0^2}\right]\left(\frac{1}{\sigma_0} j_0^2\right)\frac{(\delta T_e)}{T_{e0}} - (a + 1)\left(\frac{1}{\sigma_0} j_0^2\right)\frac{(\delta T_e)}{T_{e0}}$$

$$= -\left(\frac{1}{\sigma_0} j_0^2\right)\left\{1 + 2\left[1 - \frac{\left(\underline{l} \cdot \underline{j}_0\right)^2}{l^2 j_0^2}\right]a\right\}\frac{(\delta T_e)}{T_{e0}}$$

which obviously, within the argumentative simplifications made, is fully equivalent to the result (eq. (37)) obtained in a formal mathematical way.

Having developed this physical understanding of the electron enthalpy mode of instability we conclude this section by considering how devices of interest, that is, MHD generators or MPD arc thrusters, may be affected by this instability.

MHD generators. - Two types of MHD generators must be considered. In the combustion-type generator, inelastic collisions with the numerous molecular species in the combustion gas working fluid make it impossible to raise the electron temperature much above the gas temperature. In this case then the factor $(1 - (T_0/T_{e0}))$ in the inequality (37) is in general too small for the instability criterion to be satisfied.

In the nonequilibrium-type generator operating on inert gases seeded with alkali metals, the goal is to obtain an elevated electron temperature. An electron temperature approximately twice the gas temperature is typical. Therefore, these devices could exhibit the electron enthalpy mode of instability for values of

$$\left(a_{eN} \frac{\nu_{eN0}}{\nu_0} - \frac{3}{2} \frac{\nu_{ei0}}{\nu_0}\right) \lessapprox -1$$

In general, over the temperature range and for neutral gases of interest, the temperature characteristic a_{eN} is positive or at best only slightly negative; see Maxwell-averaged cross sections for electron temperatures near 3000 K presented in reference 10. Therefore, it appears that, at least with the proper choice of neutral gas, this mode can only become important when the operation is in the upper range of electron temperature, where the collision frequency is dominated by electron-ion collisions. Furthermore, conditions such as optimal operation tend to require concentration ratios of seed (ionizable component) to inert carrier gas (nonionized component) of the order of 0.1 percent or less (refs. 10 and 11). Since at electron temperatures generally obtained ($\simeq$3000 K) the degree of ionization is of the order of 10 percent or less, the free electron concentration is such that collisions with the neutral component are significant if indeed not dominant. Therefore, while the electron enthalpy mode could occur in nonequilibrium MHD generators, it appears that its range of importance is small and can be avoided without great difficulty.

MPD arc thrusters. - In order to simplify the discussion of the MPD arc thruster, the geometric complexity of the problem is reduced by considering the interelectrode gap width to be small compared to the inner electrode radius so that radial variations can be neglected. For this situation the geometry can be approximated by two-plane electrodes parallel to the x-y plane separated by the gap width. The fluid flow velocity and applied magnetic field are taken to be in the y-direction. The applied electric field is in the z-direction; and the solution is periodic in the x-direction with period $2\pi r$, where r is the mean radius of the fluid flow gap (see fig. 2). For this geometry the steady-state Ohm's Law (eq. (14)) has the components

$$j_{0x} = \beta_0 j_{0z} \tag{40}$$

$$j_{0y} = 0 \tag{41}$$

$$j_{0z} = \frac{\sigma_0}{1 + \beta_0^2} E_0 \tag{42}$$

where

$$\beta_0 = \frac{eB_0}{m_e \nu_0} \tag{43}$$

is the Hall parameter. Substituting equations (40) to (42) into the inequality (37), the instability criterion becomes

$$1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{\left(l_x\beta_0 + l_z\right)^2}{l^2\left(1 + \beta_0^2\right)}\right]\left(a_{eN}\frac{\nu_{eN0}}{\nu_0} - \frac{3}{2}\frac{\nu_{ei0}}{\nu_0}\right) \leq 0 \tag{44}$$

Furthermore, the operating regime of the MPD arc thruster is generally such that the electron temperature is much greater than the gas temperature. Also, the degree of ionization of the working fluid is generally in the range of 10 percent or better, so that the collision processes are dominated by electron-ion collision (refs. 12 and 13), that is, $(\nu_{ei0}/\nu_0) \approx 1$. Therefore, the instability condition for the thruster can be written approximately as

$$1 - 3\left[1 - \frac{\left(l_x\beta_0 + l_z\right)^2}{l^2\left(1 + \beta_0^2\right)}\right] \leq 0 \tag{45}$$

Obviously, if the disturbance can propagate in any direction relative to $\underline{E}_0$, the mode is always unstable. The maximum growth rate (magnitude of destabilizing or negative term is maximum) occurs at the angle given by

$$\tan\theta \equiv \frac{l_x}{l_z} = \frac{-1}{\beta_0} \tag{46}$$

where θ is the acute angle between the direction of propagation $\underline{l}$ and the applied electric field $\underline{E}_0$.

If the mode is restricted by some means to a purely rotational characteristic, that is, $l_z = 0$, then the instability condition reduces to

$$\beta_0 \leq \sqrt{2} \tag{47}$$

This was the case which was investigated in reference 5, where the theory was applied to the onset of the electric-current-formation region near the throat of the device, where the pressure is sufficient for condition (47) to be satisfied.

QUASI-LINEAR THEORY

We now consider the effect of the nonlinear terms upon the growth of the electron enthalpy mode. In doing so, we consider perturbed quantities of the form

$$f = \langle f \rangle + \tilde{f}(\underline{r}, t) \tag{48}$$

The quantity $\tilde{f}$ is the oscillator function for which the time- and/or space-averaged value defined by the brackets $\langle \tilde{f} \rangle$ is zero. In the zeroth order, that is, $\tilde{f} = 0$, the quantity $\langle f \rangle$ satisfies the steady-state equations (13) to (15). The value of $\tilde{f}$ is taken to be sufficiently small so that only first-order corrections to the linear equations need be considered. Furthermore, since we are considering small departures, only the fundamental mode is considered; that is, mode coupling between harmonics or other fundamental modes is ignored. Then

$$\tilde{f} = f'(t)e^{i\underline{l}\cdot\underline{r}} \tag{49}$$

Before proceeding to the development of the quasi-linear equations it is appropriate to consider some of the average quantities which will appear in the theory. First of all, since quantities of interest are all real, definition (49) can be written as

$$\tilde{f} = f_s(t)\sin \underline{l} \cdot \underline{r} + f_c(t)\cos \underline{l} \cdot \underline{r} \equiv f_s(t)\sin \varphi + f_c(t)\cos \varphi$$

The following pertinent averages over all φ are then obvious:

$$\left\langle \tilde{f} \right\rangle = 0$$

$$\left\langle \tilde{f}^2 \right\rangle = \frac{f_s^2 + f_c^2}{2}$$

$$\left\langle \tilde{f}^3 \right\rangle = 0$$

$$\left\langle \tilde{f}^4 \right\rangle = \frac{3}{2} \left\langle \tilde{f}^2 \right\rangle^2 \qquad (50)$$

The quasi-linear equations are now obtained by utilizing the preceding perturbation form and retaining the dominate order in nonlinear terms.

Maxwell's Equations

Since equations (1) and (2) are linear in the perturbed quantities, the results are of the same form as previously obtained, that is, equations (19) and (20).

Electron Continuity Equation

The electron enthalpy mode as previously defined (see eq. (34)) requires that

$$\tilde{n}_e = 0 \qquad (51)$$

Generalized Ohm's Law

Substituting quantities of the form (48) into equation (8), we obtain

$$\langle \underline{j} \rangle + \underline{\tilde{j}} = \left(\langle \sigma \rangle + \tilde{\sigma}\right)\left(\left\langle \underline{E}^* \right\rangle + \underline{E}^*\right) - \left(\langle \beta \rangle + \tilde{\beta}\right)\left(\langle \underline{j} \rangle + \underline{\tilde{j}}\right) \times \underline{\hat{b}} + \left(\langle \sigma \rangle + \tilde{\sigma}\right)\left(i\underline{l} \frac{k\tilde{T}_e}{e}\right) \qquad (52)$$

where $\underline{\hat{b}}$ is a unit vector in the direction of $\underline{B}_0$ and

$$\beta \equiv \frac{eB_0}{m_e \nu} \qquad (53)$$

The average current density is obtained by averaging equation (52)

$$\langle \underline{j} \rangle = \langle \sigma \rangle \left\langle \underline{E}^* \right\rangle + \left\langle \tilde{\sigma} \underline{\tilde{E}}^* \right\rangle - \langle \beta \rangle \langle \underline{j} \rangle \times \hat{\underline{b}} - \left\langle \tilde{\beta} \tilde{\underline{j}} \right\rangle \times \hat{\underline{b}} + i \underline{l} \left\langle \tilde{\sigma} \frac{k\tilde{T}_e}{e} \right\rangle \tag{54}$$

which we note is just the previous steady-state equation (14) with the addition of the non-linear correlation terms $\left\langle \tilde{\sigma} \underline{\tilde{E}}^* \right\rangle$, $\left\langle \tilde{\beta} \tilde{\underline{j}} \right\rangle \times \hat{\underline{b}}$, and $i\underline{l}\left\langle \tilde{\sigma}\, k\tilde{T}_e/e \right\rangle$.

We shall now determine these correlation terms as functions of the mean square value of the amplitude of the electron temperature fluctuation $\left\langle \tilde{T}_e^2 \right\rangle$. To do this, we proceed to determine $\tilde{\underline{j}}$ and then $(\underline{\tilde{E}}^* + i\underline{l}\, k\tilde{T}_e/e)$ as functions of $\tilde{\sigma}$, $\tilde{\beta}$, and the average quantities $\langle j \rangle$ and $\left\langle \underline{E}^* \right\rangle$. From this, $\langle \underline{j} \rangle$ is determined as a function of $\tilde{\sigma}$, $\tilde{\beta}$, and $\left\langle \underline{E}^* \right\rangle$. In this form the expression is general and applies even in the presence of electron density fluctuations. The results are then specialized to the electron enthalpy mode by determining $\tilde{\sigma}$ and $\tilde{\beta}$ in terms of $\tilde{T}_e$.

The fluctuating current density $\tilde{\underline{j}}$ is determined by subtracting equation (54) from equation (52).

$$\tilde{\underline{j}} = \tilde{\sigma}\left\langle \underline{E}^* \right\rangle + \langle \sigma \rangle \underline{\tilde{E}}^* + \left(\tilde{\sigma}\underline{\tilde{E}}^* - \left\langle \tilde{\sigma}\underline{\tilde{E}}^* \right\rangle\right) - \tilde{\beta}\langle \underline{j} \rangle \times \hat{\underline{b}} - \langle \beta \rangle \tilde{\underline{j}} \times \hat{\underline{b}} - \left(\tilde{\beta}\tilde{\underline{j}} - \left\langle \tilde{\beta}\tilde{\underline{j}} \right\rangle\right) \times \hat{\underline{b}}$$
$$+ i\underline{l}\langle \sigma \rangle \frac{k\tilde{T}_e}{e} + i\underline{l}\left(\tilde{\sigma}\frac{k\tilde{T}_e}{e} - \left\langle \tilde{\sigma}\frac{k\tilde{T}_e}{e} \right\rangle\right) \tag{55}$$

This equation can be simplified by taking the triple cross product $\underline{l} \times (\underline{l} \times \tilde{\underline{j}})$, utilizing equations (19) and (20), and using the fact that we are only considering propagation in the plane perpendicular to $\underline{B}_0$.

$$\underline{l} \times (\underline{l} \times \tilde{\underline{j}}) = -l^2 \tilde{\underline{j}} = \tilde{\sigma}\underline{l} \times \left(\underline{l} \times \left\langle \underline{E}^* \right\rangle\right) - \tilde{\beta}\underline{l} \times \left[\underline{l} \times \left(\langle \underline{j} \rangle \times \hat{\underline{b}}\right)\right] \tag{56}$$

The fluctuating electric field $\underline{\tilde{E}}^*$ is determined from equation (55) by taking the dot product $\underline{l} \cdot \tilde{\underline{j}}$ and utilizing equations (19) and (20) to obtain

$$\langle \sigma \rangle l^2 \left(\underline{\tilde{E}}^* + i\underline{l}\frac{k\tilde{T}_e}{e}\right) = -\tilde{\sigma}\underline{l}\left(\underline{l} \cdot \left\langle \underline{E}^* \right\rangle\right) + \underline{l}\tilde{\beta}\underline{l} \cdot \left(\langle \underline{j} \rangle \times \hat{\underline{b}}\right) + \underline{l}\langle \beta \rangle \underline{l} \cdot \left(\tilde{\underline{j}} \times \hat{\underline{b}}\right)$$
$$+ \text{Higher order terms} \tag{57}$$

The quantity $\left(\underline{\tilde{E}}^* + i\underline{l}\, k\tilde{T}_e/e\right)$ will only be used in this report to eliminate the term

$\left\langle \tilde{\sigma} \left(\underline{\tilde{E}}^* + i\underline{l}\, k\tilde{T}_e/e \right) \right\rangle$ in equation (54), which is a second-order term. Therefore, the terms of second degree and higher in equation (57) are ignored. Substituting equation (56) into equation (57) to eliminate $\underline{\tilde{j}}$ gives

$$\langle \sigma \rangle l^2 \left(\underline{\tilde{E}}^* + i\underline{l} \frac{k\tilde{T}_e}{e} \right) = -\tilde{\sigma} \underline{l} \left(\underline{l} \cdot \left\langle \underline{E}^* \right\rangle \right) + \underline{l} \tilde{\beta} \underline{l} \cdot \left(\langle \underline{j} \rangle \times \underline{\hat{b}} \right)$$

$$+ \underline{l} \langle \beta \rangle \left[\widetilde{\sigma \underline{l}} \cdot \left(\left\langle \underline{E}^* \right\rangle \times \underline{\hat{b}} \right) + \widetilde{\beta \underline{l}} \cdot \langle \underline{j} \rangle \right] \tag{58}$$

The average current density $\langle j \rangle$ can now be written as a function of $\tilde{\sigma}$ and $\tilde{\beta}$ by substituting $\underline{\tilde{j}}$ and $\underline{\tilde{E}}^* + i\underline{l}\, k\tilde{T}_e/e$ from equations (56) and (58), respectively, into equation (54). After some amount of algebraic manipulation $\langle \underline{j} \rangle$ can be written in the form

$$\langle \underline{j} \rangle = \langle \sigma \rangle \left\langle \underline{E}^* \right\rangle - \langle \beta \rangle \langle \underline{j} \rangle \times \underline{\hat{b}} - \frac{1}{l^2} \underline{l} \left[\frac{\left\langle \tilde{\sigma}^2 \right\rangle}{\langle \sigma \rangle^2} + \left\langle \left(\frac{\tilde{\sigma}}{\langle \sigma \rangle} - \frac{\tilde{\beta}}{\langle \beta \rangle} \right)^2 \right\rangle \langle \beta \rangle^2 \right] \underline{l} \cdot \langle \underline{j} \rangle \tag{59}$$

in which only the dominant order of the nonlinear terms is retained. In this form, the equation is quite general and applies to arbitrary electron number density and/or temperature fluctuations.

These results are now specialized to the case of the electron enthalpy mode by determining $\tilde{\sigma}$ and $\tilde{\beta}$ in terms of $\tilde{T}_e$ when $\tilde{n}_e = 0$. Furthermore, since as previously discussed, the instability is in general only important in the Coulomb-collision-dominant regime, the subsequent work will be restricted to this limit; that is, $\nu_{ei0} \simeq \nu_0$. This approximation results in a great simplification of the algebra, without which the subsequent equations become unwieldy. With these simplifications

$$\sigma \equiv \frac{n_{e0} e^2}{m_e \nu} = \frac{n_{e0} e^2}{m_e \nu_0} \left(1 + \frac{\tilde{T}_e}{\langle T_e \rangle} \right)^{3/2} = \sigma_0 \left(1 + \frac{3}{2} \frac{\tilde{T}_e}{\langle T_e \rangle} + \frac{3}{8} \frac{\tilde{T}_e^2}{\langle T_e \rangle^2} - \frac{1}{16} \frac{\tilde{T}_e^3}{\langle T_e \rangle^3} + \ldots \right) \tag{60}$$

where the subscript 0 in this and subsequent equations signifies evaluation at the average electron temperature $\langle T_e \rangle$. These quantities will be referred to as zeroth order quantities. The average value of the conductivity is then

$$\langle \sigma \rangle = \sigma_0 \left(1 + \frac{3}{8} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) \tag{61}$$

and the fluctuating part

$$\tilde{\sigma} \equiv \sigma - \langle \sigma \rangle$$

$$= \sigma_0 \left[\frac{3}{2} \frac{\tilde{T}_e}{\langle T_e \rangle} + \frac{3}{8} \left(\frac{\tilde{T}_e^2}{\langle T_e \rangle^2} - \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) - \frac{1}{16} \frac{\tilde{T}_e^3}{\langle T_e \rangle^3} + \dots \right] \tag{62}$$

The quantity of importance throughout the subsequent analysis is the ratio

$$\frac{\tilde{\sigma}}{\langle \sigma \rangle} = \left[\frac{3}{2} \frac{\tilde{T}_e}{\langle T_e \rangle} + \frac{3}{8} \left(\frac{\tilde{T}_e^2}{\langle T_e \rangle^2} - \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) - \frac{1}{16} \frac{\tilde{T}_e^3}{\langle T_e \rangle^3} - \frac{9}{16} \frac{\tilde{T}_e}{\langle T_e \rangle} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} + \dots \right] \tag{63}$$

It is furthermore obvious from definitions (53) and (60) that

$$\left. \begin{aligned} \frac{\tilde{\beta}}{\langle \beta \rangle} &= \frac{\tilde{\sigma}}{\langle \sigma \rangle} \\ \frac{\tilde{\beta}}{\beta_0} &= \frac{\tilde{\sigma}}{\sigma_0} \end{aligned} \right\} \tag{64}$$

The average current density as defined by equation (59) can now be written solely as a function of the applied electric field $\langle \underline{E}^* \rangle$, the average electron temperature $\langle T_e \rangle$, and the electron temperature fluctuation amplitude function $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ through the use of equations (61), (62), and (64):

$$\langle \underline{j} \rangle = \left(1 + \frac{3}{8} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) \left(\sigma_0 \left\langle \underline{E}^* \right\rangle - \beta_0 \langle \underline{j} \rangle \times \hat{\underline{b}} \right) - \frac{1}{l^2} \underline{l} \frac{9}{4} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \underline{l} \cdot \langle \underline{j} \rangle \tag{65}$$

Therefore $\langle \underline{j} \rangle$ is completely specified once $\langle T_e \rangle$ and $\left\langle \tilde{T}_e^2 \right\rangle / \langle \tilde{T}_e \rangle^2$ are established from the electron heating equation. However, before proceeding to that task, we place the fluctuating current density expression (56) in a simpler and a more convenient form for later calculations by noting from equation (59) that

$$\underline{l} \times \left(\underline{l} \times \langle \underline{j} \rangle\right) = \langle \sigma \rangle \underline{l} \times \left(\underline{l} \times \left\langle \underline{E}^* \right\rangle\right) - \langle \beta \rangle \underline{l} \times \left[\underline{l} \times \left(\langle \underline{j} \rangle \times \hat{\underline{b}}\right)\right]$$

and using equation (64) to yield

$$\tilde{\underline{j}} = -\frac{1}{l^2} \frac{\tilde{\sigma}}{\langle \sigma \rangle} \underline{l} \times \left(\underline{l} \times \langle \underline{j} \rangle\right) \tag{66}$$

Electron Energy Equation

The electron energy equation (12) can be expanded to yield

$$\frac{3}{2} k n_{e0} \frac{\partial}{\partial t} \left(\langle T_e \rangle + \tilde{T}_e\right) = -2 \frac{m_e}{m_a} n_{e0} \nu_0 \left(1 - \frac{3}{2} \frac{\tilde{T}_e}{\langle T_e \rangle} + \frac{15}{8} \frac{\tilde{T}_e^2}{\langle T_e \rangle^2} - \frac{35}{16} \frac{\tilde{T}_e^3}{\langle T_e \rangle^3} + \dots\right)$$

$$\times \frac{3}{2} k \left(\langle T_e \rangle - T_0\right) \left(1 + \frac{\tilde{T}_e}{\langle T_e \rangle - T_0}\right) + \frac{1}{\langle \sigma \rangle} \left(1 - \frac{\tilde{\sigma}}{\langle \sigma \rangle} + \frac{\tilde{\sigma}^2}{\langle \sigma \rangle^2}\right.$$

$$\left. - \frac{\tilde{\sigma}^3}{\langle \sigma \rangle^3} + \dots\right) \left(\langle j \rangle^2 + 2 \langle \underline{j} \rangle \cdot \tilde{\underline{j}} + \tilde{j}^2\right) \tag{67}$$

Since in the work to follow we shall only be interested in the growth or damping of the wave, only terms contributing to this effect have been retained in the preceding equation. We also have used equation (8) to replace $\underline{j} \cdot \underline{E}^*$ by $(1/\sigma) j^2$. The advantage of writing the fluctuating current density in the form given by equation (66) is now apparent since in this form the ohmic heating term in equation (67) reduces to a power series in $\tilde{\sigma}/\langle \sigma \rangle$.

The average electron temperature equation is obtained by averaging equation (67) which results in

$$\frac{3}{2} k n_{e0} \frac{\partial}{\partial t} \langle T_e \rangle \simeq -2 \frac{m_e}{m_a} n_{e0} \nu_0 \frac{3}{2} k \Big(\langle T_e \rangle - T_0\Big) \left[\left(1 + \frac{15}{8} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) - \frac{\langle T_e \rangle}{\langle T_e \rangle - T_0} \frac{3}{2} \frac{\tilde{T}_e^2}{\langle T_e \rangle^2} \right]$$

$$+ \frac{1}{\langle \sigma \rangle} \left[\langle j \rangle^2 + \frac{1}{l^2} \Big(\underline{l} \cdot \langle \underline{j} \rangle\Big)^2 \frac{9}{4} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right] \tag{68}$$

The equation for the fluctuating part is then obtained by subtracting equation (68) from equation (67)

$$\frac{3}{2} k n_{e0} \frac{\partial}{\partial t} \tilde{T}_e = -2 \frac{m_e}{m_a} n_{e0} \nu_0 \frac{3}{2} k \Big(\langle T_e \rangle - T_0\Big) \left[-\frac{3}{2} \frac{\tilde{T}_e}{\langle T_e \rangle} + \frac{\tilde{T}_e}{\langle T_e \rangle - T_0} + \frac{15}{8} \left(\frac{\tilde{T}_e^2}{\langle T_e \rangle^2} \right. \right.$$

$$\left. - \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) - \frac{3}{2} \frac{\langle T_e \rangle}{\langle T_e \rangle - T_0} \left(\frac{\tilde{T}_e^2}{\langle T_e \rangle^2} - \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) + \left(\frac{15}{8} \frac{\langle T_e \rangle}{\langle T_e \rangle - T_0} - \frac{35}{16} \right)$$

$$\left. \times \frac{\tilde{T}_e^3}{\langle T_e \rangle^3} + \ldots \right] + \frac{\tilde{\sigma}}{\langle \sigma \rangle^2} \langle j \rangle^2 - \frac{1}{\langle \sigma \rangle} \left[2 \frac{\tilde{\sigma}}{\langle \sigma \rangle} - \left(\frac{\tilde{\sigma}^2}{\langle \sigma \rangle^2} - \frac{\left\langle \tilde{\sigma}^2 \right\rangle}{\langle \sigma \rangle^2} \right) \right.$$

$$\left. + \frac{\tilde{\sigma}^3}{\langle \sigma \rangle^3} + \ldots \right] \frac{1}{l^2} \Big(\underline{l} \cdot \langle \underline{j} \rangle\Big)^2 \tag{69}$$

The quantity $\left\langle \tilde{T}_e^2 \right\rangle$ is determined by multiplying the preceding equation by $\tilde{T}_e \langle T_e \rangle$, utilizing equations (50) and (63), and averaging to obtain

$$\frac{3}{2} k n_{e0} \langle T_e \rangle \frac{1}{2} \frac{\partial}{\partial t} \left\langle \widetilde{T}_e^2 \right\rangle = -2 \frac{m_e}{m_a} n_{e0} \nu_0 \frac{3}{2} k \left(\langle T_e \rangle - T_0 \right) \left[-\frac{3}{2} - \frac{35}{16} \frac{3}{2} \frac{\left\langle \widetilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} + \frac{\langle T_e \rangle}{\langle T_e \rangle - T_0} \right.$$

$$\left. \times \left(1 + \frac{15}{8} \frac{3}{2} \frac{\left\langle \widetilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) \right] \left\langle \widetilde{T}_e^2 \right\rangle + \frac{1}{\langle \sigma \rangle} \left[\left(\frac{3}{2} - \frac{21}{32} \frac{\left\langle \widetilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) \langle j \rangle^2 \right.$$

$$\left. - 2 \left(\frac{3}{2} + \frac{51}{32} \frac{\left\langle \widetilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right) \frac{1}{l^2} \left(\underline{l} \cdot \langle \underline{j} \rangle \right)^2 \right] \left\langle \widetilde{T}_e^2 \right\rangle \tag{70}$$

Discussion and Results of Quasi-Linear Theory

The effect of fluctuations on Ohm's Law. - Of particular interest here is the determination of the effective values of the transport properties, that is, the electrical conductivity and Hall parameter in the presence of the fluctuation as given by equation (59). The effective electrical conductivity σ_{eff} and the effective Hall parameter β_{eff} are defined by the equation

$$\langle \underline{j} \rangle \equiv \sigma_{eff} \langle \underline{E}^* \rangle - \beta_{eff} \langle \underline{j} \rangle \times \hat{\underline{b}} \tag{71}$$

Reducing equation (59) to this form yields

$$\sigma_{eff} \equiv \frac{\langle \sigma \rangle}{1 + \dfrac{\dfrac{\left\langle \widetilde{\sigma}^2 \right\rangle}{\langle \sigma \rangle^2} + \left\langle \left(\dfrac{\widetilde{\sigma}}{\langle \sigma \rangle} - \dfrac{\widetilde{\beta}}{\langle \beta \rangle} \right)^2 \right\rangle \langle \beta \rangle^2}{1 + \tan^2 \varphi}} \tag{72}$$

where φ is defined as the angle between $\underline{l}$ and $\langle \underline{j} \rangle$ $(0 \leq \varphi \leq 2\pi)$. By using equations (61), (63), and (64), the effective conductivity becomes, in terms of the zeroth order conductivity σ_0,

$$\sigma_{eff} = \sigma_0 \frac{1 + \frac{3}{8} \frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}}{1 + \frac{9}{4} \frac{1}{1+\tan^2 \varphi} \frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}} \tag{73}$$

In figure 3, the ratio of the effective to the zeroth order conductivity is plotted as a function of the amplitude function for various angles of propagation. The interesting point to note is that the effective conductivity is enhanced by the presence of the fluctuation for values of $|\tan \varphi|$ greater than $\sqrt{5}$ but is reduced for values less than $\sqrt{5}$.

The effective Hall parameter is determined by comparing the defining equation (71) to equation (59). The effective Hall parameter is then found to be

$$\beta_{eff} = \frac{\langle \beta \rangle + \frac{\tan \varphi}{1 + \tan^2 \varphi} \left[\frac{\langle \tilde{\sigma}^2 \rangle}{\langle \sigma \rangle^2} + \left\langle \left(\frac{\tilde{\sigma}}{\langle \sigma \rangle} - \frac{\tilde{\beta}}{\langle \beta \rangle} \right)^2 \right\rangle \langle \beta \rangle^2 \right]}{1 + \frac{1}{1 + \tan^2 \varphi} \left[\frac{\langle \tilde{\sigma}^2 \rangle}{\langle \sigma \rangle^2} + \left\langle \left(\frac{\tilde{\sigma}}{\langle \sigma \rangle} - \frac{\tilde{\beta}}{\langle \beta \rangle} \right)^2 \right\rangle \langle \beta \rangle^2 \right]} \tag{74}$$

Using equations (61), (63), and (64), the effective Hall parameter becomes, in terms of the zeroth order Hall parameter β_0,

$$\beta_{eff} = \frac{\beta_0 \left(1 + \frac{3}{8} \frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2} \right) + \frac{9}{4} \frac{\tan \varphi}{1 + \tan^2 \varphi} \frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}}{1 + \frac{9}{4} \frac{1}{1 + \tan^2 \varphi} \frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}} \tag{75}$$

Equations (73) and (75) show that in the limiting cases $\tan \varphi = 0$ and $|\tan \varphi| = \infty$.

$$\frac{\beta_{eff}}{\beta_0} = \frac{\sigma_{eff}}{\sigma_0} \tag{76}$$

At intermediate values of $|\tan\varphi|$, β_{eff}/β_0 depends upon the value of β_0. Therefore, the dividing point between enhancement and degradation of the Hall parameter is not simply a function of the angle of propagation as it was for the conductivity ratio. The plot of β_{eff}/β_0 for the limiting cases is therefore identical to the σ_{eff}/σ_0 plot in figure 3, except that the neutral point $\beta_{eff}/\beta_0 = 1$ occurs for values of $\tan\varphi$ given by

$$\tan\varphi = -\frac{3}{\beta_0}\left(1 \pm \sqrt{1 + \frac{5}{9}\beta_0^2}\right) \tag{77}$$

The values of $\tan\varphi$ are plotted in figure 4. The angles of propagation for which $\beta_{eff} = \beta_0$ are restricted to $0 \le \tan\varphi \le \sqrt{5}$ and $-\infty \le \tan\varphi \le -\sqrt{5}$. The asymptotic values $\tan\varphi = \pm\sqrt{5}$ occur in the limit $\beta_0 \to \infty$. At the limiting point, equality (76) again holds, as can be seen by taking the limit $\beta_0 \to \infty$ in equation (75) and comparing the results to equation (73).

The effect of fluctuations for the case of maximum growth rate. - We are interested in determining whether a steady state can exist in the presence of an infinitesimal oscillation, and if so, under what circumstance the oscillation occurs and what its amplitude is as a function of experimentally measured parameters. If the system is steady, then $\partial/\partial t\langle T_e\rangle = 0$ and the balance between Joule heating and electron collision losses as given by equation (68) takes the form

$$2\frac{m_e}{m_a}n_{e0}\nu_0\frac{3}{2}k\left(\langle T_e\rangle - T_0\right) = \frac{1 + \dfrac{\left(\underline{l}\cdot\langle \underline{j}\rangle\right)^2}{l^2\langle j\rangle^2}\dfrac{9}{4}\dfrac{\left\langle \tilde{T}_e^2\right\rangle}{\langle T_e\rangle^2}}{1 + \left(\dfrac{15}{8} - \dfrac{3}{2}\dfrac{\langle T_e\rangle}{\langle T_e\rangle - T_0}\right)\dfrac{\left\langle \tilde{T}_e^2\right\rangle}{\langle T_e\rangle^2}}\frac{1}{\langle\sigma\rangle}\langle j\rangle^2 \tag{78}$$

This equation reduces to the previously obtained linear steady-state solution (eq. (15)) in the limit $\left\langle \tilde{T}_e^2\right\rangle/\langle T_e\rangle^2 \to 0$.

We first consider the amplitude equation (70) for the direction of propagation which results in the maximum growth rate of the disturbance. Since in this direction, the fluctuation is first unstable and grows most rapidly, it is assumed that for small displacements from the stability threshold this is the observed disturbance. This direction, as discussed following the development of the instability criterion (eq. (37)), occurs for propagation perpendicular to the average current density; that is, $\underline{l}\cdot\langle \underline{j}\rangle = 0$.

For this case, the amplitude function is given by

$$\frac{1}{2}\frac{\partial}{\partial t}\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2} = 2\frac{m_e}{m_a}\nu_0\left[\left(2 - 3\frac{T_0}{\langle T_e\rangle}\right) + \frac{3}{2}\frac{29}{8}\left(\frac{2}{29} - \frac{T_0}{T_e}\right)\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2}\right]\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2} \tag{79}$$

where we have used equation (78) to eliminate $(1/\langle\sigma\rangle)\langle j\rangle^2$ and have neglected terms greater than second order in $\left\langle \tilde{T}_e^2 \right\rangle/\langle T_e\rangle^2$. We note that the linear term is just the stability condition obtained from the linear analysis and obviously is damping until the instability point, $T_0/\langle T_e\rangle \leq 2/3$, is reached. Beyond this point the linear term causes a growth in the amplitude of the oscillation until the nonlinear term, which is damping for $T_0/\langle T_e\rangle$ is greater than 2/29, offsets the linear growth term. A steady state is reached when

$$\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2} = -\frac{2 - 3\dfrac{T_0}{\langle T_e\rangle}}{\dfrac{3}{2}\dfrac{29}{8}\left(\dfrac{2}{29} - \dfrac{T_0}{\langle T_e\rangle}\right)} \tag{80}$$

Obviously, in the case in which $T_0/\langle T_e\rangle$ is less than 2/29 the nonlinear term contributes to the growth, and the thermal perturbation is unconditionally unstable. However, it is obvious from equation (80) that as $T_0/\langle T_e\rangle$ approaches 2/29, $\left\langle \tilde{T}_e^2 \right\rangle/\langle T_e\rangle^2$ approaches infinity, which violates the quasi-linear condition of small amplitude oscillations; that is $\left\langle \tilde{T}_e^2 \right\rangle/\langle T_e\rangle^2 \ll 1$. Therefore, this point is not considered further.

The amplitude function $\left\langle \tilde{T}_e^2 \right\rangle/\langle T_e\rangle^2$, as given by equation (80) can be calculated by determining the temperature ratio $T_0/\langle T_e\rangle$ from equation (68). For $\underline{l} \cdot \langle \underline{j}\rangle = 0$, the steady-state condition $\partial/\partial t\, \langle T_e\rangle = 0$, and the geometry shown in figure 2, this equation reduces to

$$\left(1 - \frac{T_0}{\langle T_e\rangle}\right)\left[1 + \left(\frac{15}{8} - \frac{3}{2}\frac{1}{1 - \dfrac{T_0}{\langle T_e\rangle}}\right)\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2}\right] = \frac{T_0}{\langle T_e\rangle}\left(1 - \frac{3}{8}\frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e\rangle^2}\right)\frac{\langle\beta\rangle^2}{1 + \langle\beta\rangle^2}R \tag{81}$$

where

$$
R \equiv \frac{\frac{1}{2} m_a \left(\frac{E}{B_0}\right)^2}{\frac{3}{2} kT_0} \tag{82}
$$

is the ratio of the ion drift energy to its random kinetic energy and $E = \langle E^* \rangle$ and B_0 are the magnitude of the applied electron and magnetic fields, respectively. In deriving equation (81) from equation (68), we have also used equations (59), (61), and (64).

Equations (80) and (81) can be solved simultaneously for given values of the Hall parameter $\langle \beta \rangle$ and the parameter R to determine the amplitude function $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$. (Note that a physically acceptable solution requires that $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2 \geq 0$.) These values are shown in figure 5, where the amplitude function has been plotted as a function of the Hall parameter for various values of the parameter R. The first thing to be noted from this figure is that for a given value of R, no physically acceptable solutions exist, that is, there is no instability, until a critical value of $\langle \beta \rangle$ is reached corresponding to the onset of instability. This critical value of $\langle \beta \rangle$ increases with decreasing values of R, as shown in figure 6. Also to be noted from figure 5 is that the amplitude function reaches an asymptotic value as $\langle \beta \rangle \to \infty$, the magnitude of which depends upon the parameter R. These asymptotic values are plotted as a function of R in figure 7.

In figure 8, the ratio of the effective values of the electrical conductivity and Hall parameter to their steady-state values are plotted as a function of the zeroth order Hall parameter β_0 for various values of the parameter R. These ratios are defined by equations (73) and (75), respectively, for $\varphi = \pi/2$; that is, $\underline{l} \cdot \langle \underline{j} \rangle = 0$. It is seen that the effect of the fluctuation is to enhance the conductivity and Hall parameter, as was pointed out previously in figure 3.

<u>The effect of fluctuations for the case of a purely rotational disturbance</u>. - In the case considered in reference 5 where the propagation was taken to be purely rotational, the amplitude equation to second order in $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ reduces, after considerable algebraic manipulation, to

$$
\frac{1}{2} \frac{\partial}{\partial t} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} = 2 \frac{m_e}{m_a} \nu_0 \frac{1}{1 + \beta_0^2} \left[\left(2 - 3 \frac{T_0}{\langle T_e \rangle} - \beta_0^2 \right) + \frac{3}{2} \frac{29}{8} \left(\frac{2}{29} - \frac{T_0}{\langle T_e \rangle} + \frac{23}{29} \beta_0^2 - \frac{53}{29} \beta^2 \frac{T_0}{\langle T_e \rangle} - \frac{3}{29} \beta_0^4 \right) \frac{1}{1 + \beta_0^2} \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \right] \frac{\left\langle \tilde{T}_e^2 \right\rangle}{\langle T_e \rangle^2} \tag{83}
$$

Unfortunately, the dependence of equation (83) upon $T_0/\langle T_e \rangle$ and β_0^2 is such that it is not possible to incorporate the above quasi-linear results in the framework of the linearized theory of reference 5. The difficulty is that, in order to obtain from the available experimental data the values of the local gas dynamic properties required in the theory, a number of assumptions had to be made in the analysis of reference 5. One of these was to take the local gas temperature T_0 to be sufficiently small so that it could be ignored in the onset region where the frequency of the oscillation was evaluated. If this is done here, the nonlinear term of equation (83) becomes

$$\frac{3}{2}\frac{29}{8}\left(\frac{2}{29} + \frac{23}{29}\beta_0^2 - \frac{3}{29}\beta_0^4\right)\frac{1}{1 + \beta_0^2}\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}$$

which obviously is positive and hence destabilizing even for the maximum value of β_0^2 for which the linear term in equation (83) is destabilizing; that is, $\beta_0^2 = 2$. Therefore, a more complete steady-state solution is required and/or local gas properties must be measured experimentally if this assumption is the cause of the non-steady-state condition.

However, a second interpretation is possible. Since, in reference 5, the rotational frequency was calculated at what was assumed to be the upstream onset point, it could well be that the amplitude of the oscillation is growing at this point and reaches its quasi-steady-state value downstream of the point in question. In this case, the amplitude may indeed be small at the point at which the frequency was calculated, so that nonlinear effects can be ignored. The effect of nonlinearity then manifests itself downstream in the growth of the oscillation.

Of further interest is the fact that the linear and nonlinear coefficients in equation (83),

$$C_0 \equiv 2 - 3\frac{T_0}{\langle T_e \rangle} - \beta_0^2 \tag{84}$$

$$C_1 \equiv \frac{3}{2}\frac{29}{8}\left(\frac{2}{29} - \frac{T_0}{\langle T_e \rangle} + \frac{23}{29}\beta_0^2 - \frac{53}{29}\beta_0^2\frac{T_0}{\langle T_e \rangle} - \frac{3}{29}\beta_0^4\right)\frac{1}{1 + \beta_0^2} \tag{85}$$

respectively, can be positive or negative, depending upon the magnitude of $T_0/\langle T_e \rangle$ and β_0. In figure 9, the curves of $C_0 = 0$ and $C_1 = 0$ are plotted as functions of $T_0/\langle T_e \rangle$ and β_0^2. For both coefficients, points $(\beta_0^2, T_0/\langle T_e \rangle)$ above their respective curves result in negative values of the coefficient and hence damp the fluctuation. However, for points (β_0^2, $T_0\langle T_e\rangle$) below their respective curves, the coefficients are positive so that the terms contribute to the growth of the fluctuation. As a result, the $T_0/\langle T_e\rangle$-against-β_0^2 plane is divided into four regions having the following characteristics:

(1) Region I: Both coefficients are damping, and hence the fluctuation is unconditionally damped.

(2) Region II: The linear term is damping; the nonlinear term is growing. In this case the usual linear stability theory would predict a damped disturbance. However, the growth contribution of the nonlinear term indicates that, for disturbances with initial magnitude of the amplitude function $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ greater than $|C_0/C_1|$, the disturbances grow unstably. Therefore, in this region, fluctuations are damped for $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ less than $|C_0/C_1|$ and grow unstably for $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ greater than $|C_0/C_1|$.

(3) Region III: Both coefficients are positive, and hence any disturbances grows unstably.

(4) Region IV: The linear term is growing; the nonlinear term is damping. Disturbance grows or damps until a stable equilibrium point is reached with amplitude function $\left\langle \tilde{T}_e^2 \right\rangle / \langle T_e \rangle^2$ equal to $|C_0/C_1|$.

Whether these regions exist under actual experimental conditions only further experimentation can answer. Furthermore, the quasi-linear theory is probably not applicable to those regions in which it predicts the unstable growth of the disturbance due to its requirement of infinitesimal disturbances. In these regions, therefore, a fully nonlinear theory must be developed.

CONCLUDING REMARKS

Electrothermal oscillations due to fluctuations in the electron properties of a partially ionized gas have been considered. Two limiting cases have been analyzed. In the first case, the limit of infinite ionization rate is considered. This limit reduces the dispersion relation to the form which has been extensively analyzed for MHD generators (refs. 3 and 4). For the MPD arc thruster configuration this limit results in a rotational as well as a radially pulsating component. The second limiting case is that of zero ionization rate, or frozen flow. For the MHD generator this mode does not appear to be important since operating conditions and working fluids of interest are or can be chosen such that the instability criterion is not satisfied. However, MPD arc thrusters operating at high degrees of ionization are unstable in this mode, and the oscillation can occur with a purely rotational component as is observed experimentally.

A quasi-linear analysis for the frozen flow limit was also derived. The consequences of a purely rotational fluctuation for the MPD arc thruster configuration were considered. Based on the assumption of zero gas temperature in the onset region, which was used in reference 5 to reduce the existing experimental data for comparison with linear theory, the quasi-linear theory predicts unstable growth of the fluctuation. Therefore, as a consequence of this assumption, a "steady state" fluctuation of finite amplitude is not predicted by quasi-linear theory. The validity of the zero gas temperature assumption requires further experimental information or a more complete steady-state theory.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, December 2, 1971,
112-02.

APPENDIX - SYMBOLS

a, A	defined by eq. (11)
$\underline{B}$	magnetic field vector
$\hat{\underline{b}}$	unit vector in magnetic field direction
C_0	defined by eq. (84)
C_1	defined by eq. (85)
$\underline{E}$	electric field vector
e	magnitude of charge of an electron
$\underline{j}$	electric current density vector
K	Saha equilibrium constant
k	Boltzmann constant
$\underline{l}$	wave vector
m	particle mass
n	particle number density
P	pressure
q	electric charge density
R	defined by eq. (82)
r	mean radius of fluid flow gap
$\underline{r}$	position vector
T	temperature of neutrals and ions
T_e	temperature of electrons
T_i	temperature equivalent of ionization potential
t	time coordinate
$\underline{v}$	gas flow velocity vector
β	Hall parameter
σ	electrical conductivity
ν	total electron momentum collision frequency
ν_{eN}	electron-neutral momentum collision frequency
ν_{ei}	electron-ion momentum collision frequency

ν_i	ionization coefficient
ν_r	recombination coefficient
φ	angle between $\underline{l}$ and $\underline{j}$
θ	angle defined by eq. (46)
ω	wave frequency
$-\omega_i$	imaginary part of ω
ω_r	real part of ω

Subscripts:

a	heavy particles (neutral atoms and ions)
e	electron
eff	effective value
N	neutral atom
$\underline{l}$	component parallel to the vector $\underline{l}$
x, y, z	components of x, y, z coordinate system
0	zeroth order value

Superscripts:

'	perturbed quantities
~	defined by eq. (17)
*	value in coordinate system moving with gas

Brackets:

$\langle \ldots \rangle$	average value of quantity within brackets

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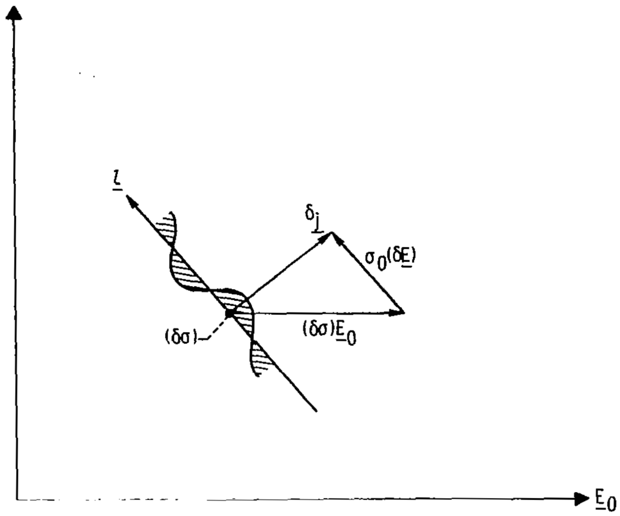
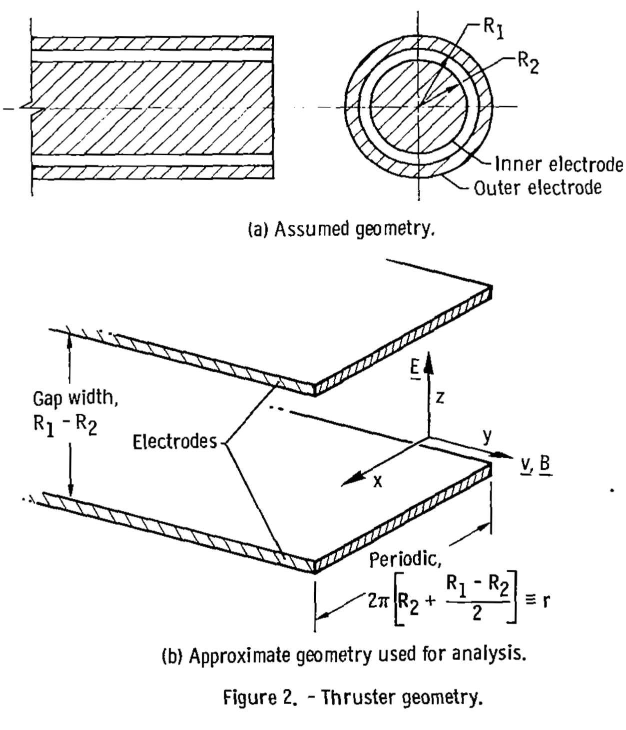


Figure 1. - Depiction of electron enthalpy fluctuation.



(a) Assumed geometry.

(b) Approximate geometry used for analysis.

Figure 2. - Thruster geometry.

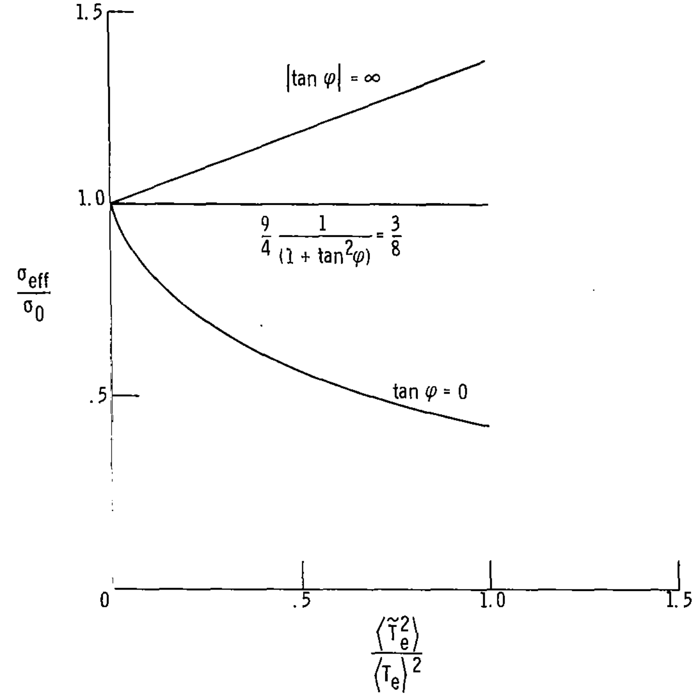


Figure 3. - Ratio of effective to zeroth order electrical conductivity as function of amplitude function for various angles of propagation.

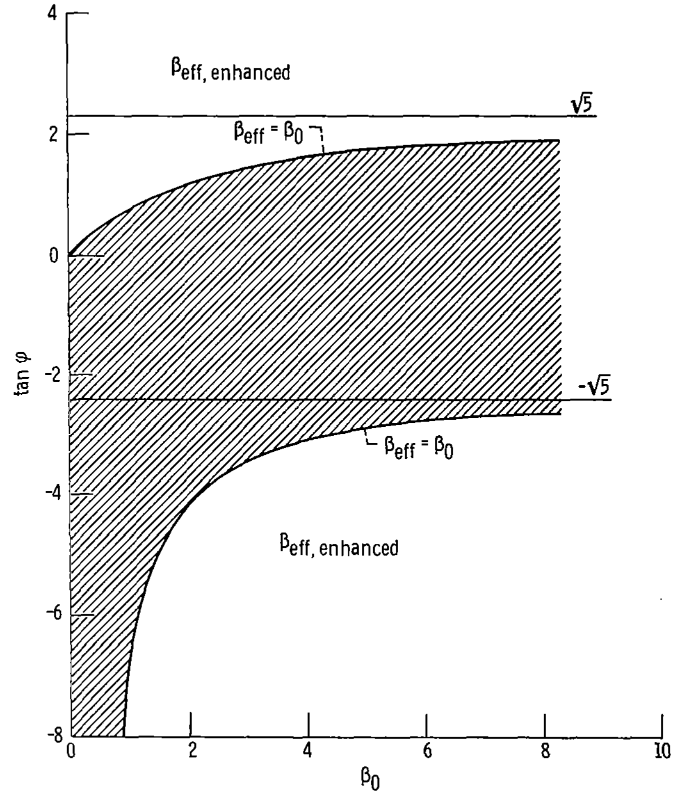


Figure 4. - Value of $\tan\varphi$ at which effective Hall parameter β_{eff} equals zeroth order Hall parameter β_0 as function of β_0.

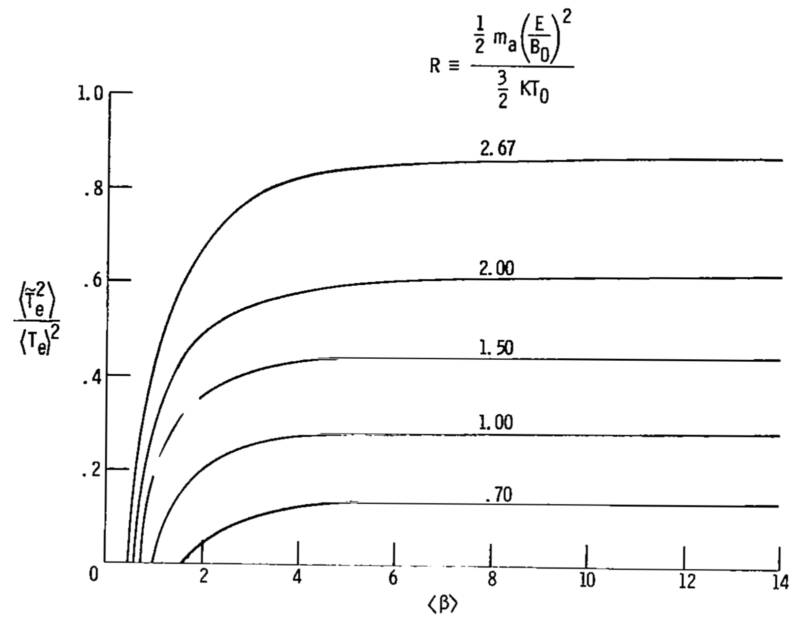


Figure 5. - Amplitude function as function of Hall parameter.

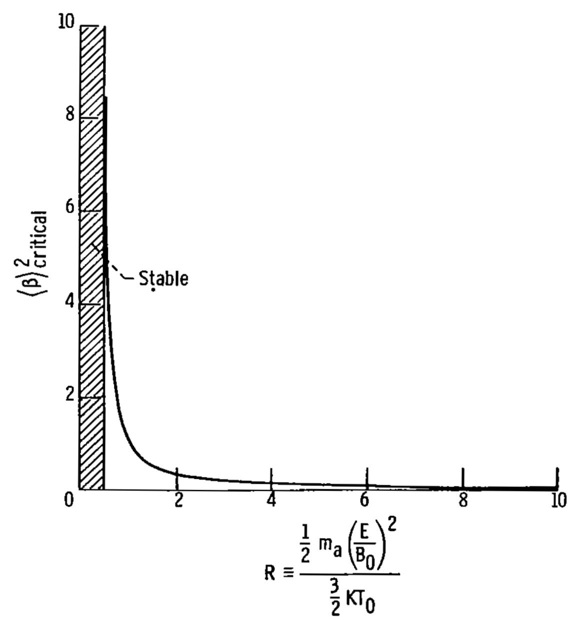


Figure 6. - Critical Hall parameter as function of parameter R.

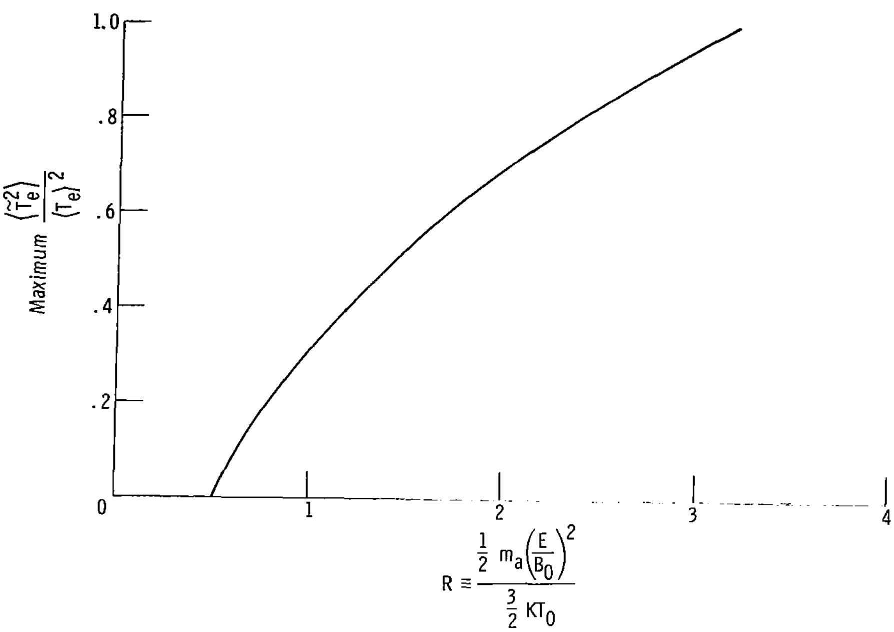


Figure 7. - Asymptotic value of amplitude function as function of parameter R.

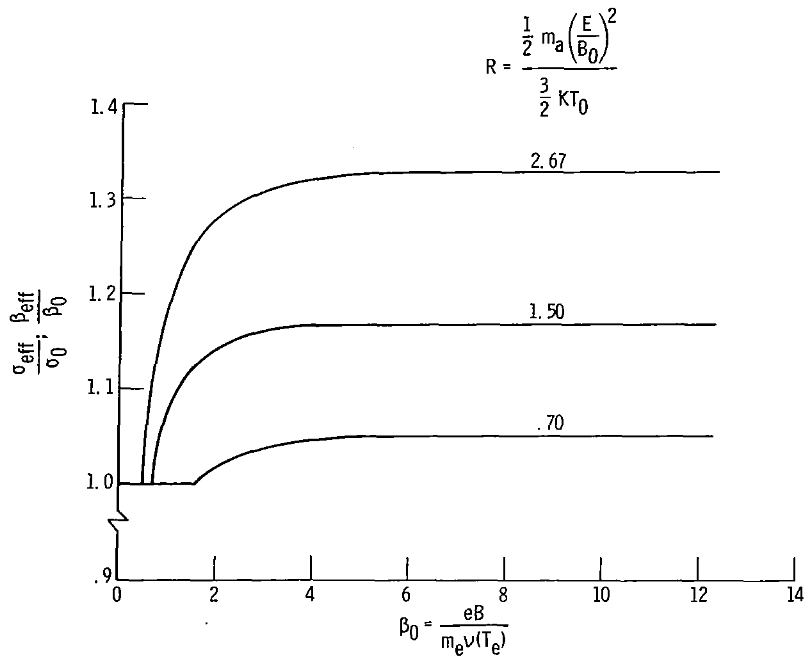


Figure 8. - Ratio of effective to zeroth order conductivity and ratio of effective to zeroth order Hall parameter as a function of zeroth order Hall parameter.

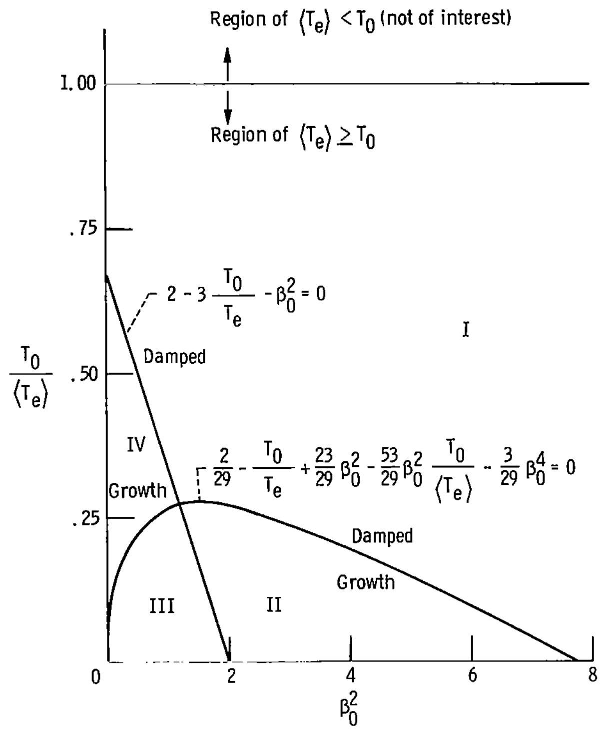


Figure 9. - Damping and growth regions for rotational disturbances.

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