# DESIGN OF RECURSIVE DIGITAL FILTERS HAVING SPECIFIED PHASE 

## AND MAGNITUDE CHARACTERISTICS

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# DESIGN OF RECURSIVE DIGITAL FILTERS HAVING SPECIFIED PHASE AND MAGNITUDE CHARACTERISTICS 

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## SUMMARY

A method for a computer-aided design of a class of optimum filters, having specifications in the frequency domain of both magnitude and phase, is described. The method, an extension to the work of Steiglitz, uses the Fletcher-Powell algorithm to minimize a weighted squared magnitude and phase criterion. Results using the algorithm for the design of filters having specified phase as well as specified magnitude and phase compromise are presented.

## INTRODUCTION

Recursive filters, wherein the output sequence is both a function of the input as well as past output samples, are commonly used in digital signal processing and analysis. Such digital filters in many applications offer distinct advantages of precision and versatility over their continuous or analog counterparts. There exist a number of design procedures for implementing digital filters (see ref. 1) each one of which strives to attain some analogy between discrete and continuous systems. Transform methods such as the matched-z, bilinear $-z$, and standard-z which lead to specific property invariances are available (see ref. 2) to the designer familiar with continuous filter design.

For frequency-domain synthesis (see refs. 3 and 4), realization is normally by means of cascade or parallel combinations of pole and zero pairs in the complex plane. The synthesis problem is, in fact, reduced to one of approximation since the filter topology is generally specified. In none of the available design procedures, which can yield filters having excellent magnitude-frequency characteristics, however, do the resultant filters, in themselves, have particularly useful phase characteristics. Indeed, in striving for particular magnitude characteristics by using any of the available design methods, there is no control over the filter phase properties.

In practice, it is often desirable to specify a digital filter in the frequency domain by its phase (see ref. 5) or even a compromise between magnitude and phase. The procedure in this paper meets these requirements through the use of an iterative computeraided design leading to an optimum set of parameters for a specified filter topology and is an extension of the technique described by Steiglitz (see ref. 6) for determining the optimum coefficients of a cascade filter having magnitude specifications alone. The extension makes possible the design of a new class of digital filters having the prescribed phase characteristics.

## SYMBOLS

| A | filter multiplier |
| :---: | :---: |
| $D_{k}^{\text {i }}$ | denominator of ith stage of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ |
| $\mathrm{E}_{\mathrm{k}}^{\mathrm{M}}$ | magnitude error at $\Omega_{\mathrm{k}}$ |
| $\mathrm{E}_{\mathrm{k}}^{\phi}$ | phase error at $\Omega_{\mathrm{k}}$ |
| $\overrightarrow{\mathrm{e}}_{\mathrm{k}}$ | error vector at $\Omega_{\mathrm{k}}$ |
| $\partial \vec{e}_{\mathrm{k}} / \partial \mathrm{A}$ | derivative of error vector at $\Omega_{\mathrm{k}}$ with respect to zero frequency gain |
| $\mathrm{f}_{\mathrm{k}}$ | frequency at kth specification point, Hz |
| $\mathrm{f}_{\mathrm{S}}$ | sampling frequency, Hz |
| $\mathrm{H}(\mathrm{z})$ | unity gain discrete transfer function |
| $\left\|\mathrm{H}_{\mathrm{k}}\right\|$ | magnitude of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ |
| $\bar{H}_{k}$ | conjugate of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ |
| $\partial\left\|\mathrm{H}_{\mathrm{k}}\right\| / \partial \overrightarrow{\mathrm{p}}$ | gradient vector of magnitude of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ with respect to parameter vector |
| I( ) | imaginary part of quantity |
| i, . . .,N | denotes filter stage |


| $\vec{J}_{k}$ | Jacobian at $\Omega_{\mathrm{k}},\left[\mathrm{A}^{*} \frac{\partial\left\|\mathrm{H}_{\mathrm{k}}\right\|}{} \mathrm{l}\right.$ |
| :---: | :---: |
| k | sample point |
| $\mathrm{M}_{\mathrm{k}}$ | specification magnitude at $\Omega_{\mathrm{k}}$ |
| $\mathrm{N}_{\mathrm{k}}^{\mathrm{i}}$ | numerator of ith stage of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ |
| - | parameter vector |
| $\overrightarrow{\mathrm{p}}_{\mathrm{i}}$ | set of filter parameters for the ith stage, $a_{i}, b_{i}, c_{i}$, and $d_{i}$ |
| $\mathrm{q}_{1}^{\mathrm{i}}(\mathrm{k})$ | first system state of ith stage at kth sample point |
| $q_{2}^{i}(\mathrm{k})$ | second system state of ith stage at kth sample point |
| R( ) | real part of quantity |
| $\mathrm{u}_{\mathrm{i}}(\mathrm{k})$ | input to ith stage at kth sample point |
| V | criterion functional, that is, $\mathrm{V}(\mathrm{A}, \overrightarrow{\mathrm{p}})$ |
| $\mathrm{V}_{\mathrm{k}}$ | criterion functional at $\Omega_{k}$, that is, $V_{k}(\mathrm{~A}, \overrightarrow{\mathrm{p}})$ |
| $\hat{\mathrm{V}}$ | reduced criterion functional, that is, $\mathrm{V}\left(\mathrm{A}^{*}, \overrightarrow{\mathrm{p}}\right)$ |
| $\partial \mathrm{V} / \partial \mathrm{A}$ | slope of criterion functional with respect to zero frequency gain |
| $\partial \mathrm{V}_{\mathrm{k}} / \partial \overrightarrow{\mathrm{e}}_{\mathrm{k}}$ | gradient vector of criterion functional at $\Omega_{\mathrm{k}}$ with respect to error vector at $\Omega_{\mathrm{k}}$ |
| $\vec{W}_{\text {k }}$ | weighting matrix at $\Omega_{\mathrm{k}}$ |
| $\mathrm{W}_{\mathrm{k}}^{\mathrm{M}}$ | magnitude weighting at $\Omega_{\mathrm{k}}$ |
| $\mathrm{W}_{\mathrm{k}}{ }^{\text {¢ }}$ | phase weighting at $\Omega_{\mathrm{k}}$ |
| $\mathrm{w}^{\mathrm{i}}$ (k) | dummy variable of ith stage at kth sample point |


| $\mathrm{Y}(\mathrm{z})$ | digital filter discrete transfer function |
| :--- | :--- |
| $\mathrm{y}_{\mathrm{i}}(\mathrm{k})$ | output of ith stage at kth sample point |
| z | transform variable <br> $\mathrm{z}_{\mathrm{k}}$ |
| discrete transform variable at $\Omega_{\mathrm{k}}, \quad \mathrm{e}^{\mathrm{j} \pi \Omega_{\mathrm{k}}}$  <br> $\theta_{\mathrm{k}}$ specification phase at $\Omega_{\mathrm{k}}$, radians <br> $\lambda$ collective phase weight <br> $\phi_{\mathrm{k}}$ phase of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$, radians <br> $\partial \phi_{\mathrm{k}} / \partial \overrightarrow{\mathrm{p}}$ gradient vector of phase of $\mathrm{H}(\mathrm{z})$ at $\Omega_{\mathrm{k}}$ with respect to parameter vector <br> $\Omega_{\mathrm{k}}$ fractional frequency at kth specification point |  |

An asterisk on a symbol denotes an optimum value. A circumflex denotes optimization with respect to $A$. A superscript $T$ denotes the transpose.

## DISCUSSION

## The Filter Form

The fundamental advantages of the N -stage cascade canonical form of recursive digital filter whose signal flow graph is shown in figure 1 and which is described by the product operator

$$
\left.\begin{array}{l}
Y(z)=A \prod_{i=1}^{N} \frac{1+a_{i} z^{-1}+b_{i} z^{-2}}{1+c_{i} z^{-1}+d_{i} z^{-2}}  \tag{1}\\
Y(z)=A H(z)
\end{array}\right\}
$$

are (1) its relative insensitivity to perturbations in the denominator coefficients, an important consideration in digital filters, especially of high order and particularly where finite register lengths (see ref. 1) are involved; (2) its simplicity of implementation; and (3) the simplicity of factoring the filter operator to determine its roots. This form has found extensive application in practical filters for signal processing, and a version employing serial arithmetic (ref. 7) is commercially available.

For completeness, an alternative description of the filter is given in terms of the system states $q_{1}^{i}$ and $q_{2}^{i}$ and clearly demonstrates the recursive nature of the filter. The set of difference equations describing the filter and required in developing a computer algorithm is presented. Thus, for the ith stage in figure 1 at the kth sample point

$$
\begin{aligned}
& w^{i}(k)=A_{i} u_{i}(k)-c_{i} q_{1}^{i}(k)-d_{i} q_{2}^{i}(k) \\
& q_{1}^{i}(k+1)=w^{i}(k) \\
& q_{2}^{i}(k+1)=q_{1}^{i}(k) \\
& y_{i}(k)=w^{i}(k)+a_{i} q_{1}^{i}(k)+b_{i} q_{2}^{i}(k)
\end{aligned}
$$

where

$$
u_{i}(k)=y_{i-1}(k)
$$

is the input to the ith stage and is identical to the output of the (i -1 ) stage and

$$
A_{i}= \begin{cases}A & (i=1) \\ 1 & (i \neq 1)\end{cases}
$$

## The Synthesis Problem

The design problem considered in this paper can be stated as follows: When the magnitude and phase specifications ( $\mathrm{M}_{\mathrm{k}}$ and $\theta_{\mathrm{k}}$, respectively) at the kth fractional Nyquist frequencies $\Omega_{k}=2 f_{k} / f_{S}$ (where $f_{S}$ is the sampling frequency in Hz ) are known, determine the set of optimum parameters $\overrightarrow{\mathrm{p}}^{*}$ of an N -stage cascade filter having the form of equation (1) so that the resultant digital filter will have a minimum sum squared magnitude and phase error for all specified frequencies.

By constraining the filter topology, the optimum synthesis problem becomes one of parametric optimization with respect to a given criterion of fit. The composite criterion which can weight the magnitude and phase requirements independently and as functions of frequency is chosen as the inner product

$$
\begin{equation*}
\mathrm{V}(\mathrm{~A}, \overrightarrow{\mathrm{p}})=\sum_{\mathrm{k}}\left\langle\overrightarrow{\mathrm{e}}_{\mathrm{k}}, \overrightarrow{\mathrm{~W}}_{\mathrm{k}} \overrightarrow{\mathrm{e}}_{\mathrm{k}}\right\rangle=\sum_{\mathrm{k}} \mathrm{~V}_{\mathrm{k}} \tag{2}
\end{equation*}
$$

where

$$
\overrightarrow{\mathrm{e}}_{\mathrm{k}}=\left[\begin{array}{c}
\mathrm{A}\left|\mathrm{H}_{\mathrm{k}}\right|-\mathrm{M}_{\mathrm{k}} \\
\phi_{\mathrm{k}}-\theta_{\mathrm{k}}
\end{array}\right]=\left[\begin{array}{c}
\mathrm{E}_{\mathrm{k}}^{\mathrm{M}} \\
\mathrm{E}_{\mathrm{k}}^{\phi}
\end{array}\right]
$$

is the error vector and

$$
\overrightarrow{\mathrm{W}}_{\mathrm{k}}=\left[\begin{array}{cc}
\mathrm{W}_{\mathrm{k}}^{\mathrm{M}} & 0 \\
0 & \lambda \mathrm{~W}_{\mathrm{k}}^{\phi}
\end{array}\right]
$$

is the diagonal weighting matrix. Clearly, $V(A, \vec{p})$ is a nonlinear function of the parameter vector $\vec{p}=\left(a_{1}, b_{1}, c_{1}, d_{1}, \ldots, a_{N}, b_{N}, c_{N}, d_{N}\right)^{T}$, which involves the $4 N$ filter coefficients, and of the filter multiplier $A$.

The Minimization Algorithm
Through formal differentiation of the criterion function (eq. (2)) with respect to the multiplier $A$, the minimization procedure can be slightly simplified to that of finding the minimum of a reduced functional $\hat{\mathrm{V}}(\overrightarrow{\mathrm{p}})=\mathrm{V}\left(\mathrm{A}^{*}, \overrightarrow{\mathrm{p}}\right)$ involving only 4 N parameters. Thus

$$
\frac{\partial \mathrm{V}}{\partial \mathrm{~A}}=\sum_{\mathrm{k}}\left\langle\frac{\partial \overrightarrow{\mathrm{e}}_{\mathrm{k}}}{\partial \mathrm{~A}}, \frac{\partial \mathrm{~V}_{\mathrm{k}}}{\partial \overrightarrow{\mathrm{e}}_{\mathrm{k}}}\right\rangle=2 \sum_{\mathrm{k}}\left[\left|\mathrm{H}_{\mathrm{k}}\right| \mathrm{w}_{\mathrm{k}}^{\mathrm{M}}: 0\right] \overrightarrow{\mathrm{e}}_{\mathrm{k}}
$$

and $\partial \mathrm{V} / \partial \mathrm{A}=0$ implies

$$
2 \sum_{k}\left|H_{k}\right| W_{k}^{M}\left(A^{*}\left|H_{k}\right|-M_{k}\right)=0
$$

or

$$
\begin{equation*}
A^{*}=\frac{\sum_{\mathrm{k}}\left|\mathrm{H}_{\mathrm{k}}\right| \mathrm{W}_{\mathrm{k}}^{\mathrm{M}_{\mathrm{M}_{\mathrm{k}}}}}{\sum_{\mathrm{k}}\left|\mathrm{H}_{\mathrm{k}}\right|^{2} \mathrm{~W}_{\mathrm{k}}^{\mathrm{M}}} \tag{3}
\end{equation*}
$$

An additional necessary condition for existence of an extremum is that the gradient vector be zero; thereby, the optimum parameter vector $\overrightarrow{\mathrm{p}}^{*}$ is obtained. From equation (2)

$$
\begin{equation*}
\frac{\partial \hat{\mathbf{V}}}{\partial \overrightarrow{\mathrm{p}}}=2 \sum_{\mathrm{k}}\left\langle\overrightarrow{\mathrm{~J}}_{\mathrm{k}}, \overrightarrow{\mathrm{w}}_{\mathrm{k}} \overrightarrow{\mathrm{e}}_{\mathrm{k}}\right\rangle \tag{4}
\end{equation*}
$$

where the $(4 N \times 2)$ Jacobian $\vec{J}_{k}$ is

$$
\begin{equation*}
\overrightarrow{\mathrm{J}}_{\mathrm{k}}^{\mathrm{T}}=\nabla_{\overrightarrow{\mathrm{p}}} \overrightarrow{\mathrm{e}}_{\mathrm{k}}=\left[\mathrm{A}^{*} \frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial \overrightarrow{\mathrm{p}}}: \frac{\partial \phi_{\mathrm{k}}}{\partial \overrightarrow{\mathrm{p}}}\right]^{\mathrm{T}} \tag{5}
\end{equation*}
$$

Clearly, each element of the gradient vector is the sum of two weighted functions of the magnitude and phase error. By writing

$$
\left|\mathrm{H}_{\mathrm{k}}\right|^{2}=\mathrm{H}_{\mathrm{k}} \overline{\mathrm{H}}_{\mathrm{k}}
$$

where $\bar{H}_{k}$ is the conjugate of $H_{k}$ evaluated at the fractional frequency $\Omega_{k}$, it is readily shown (see ref. 6), where $\overrightarrow{\mathrm{p}}_{\mathrm{i}}$ is the set of filter parameters for the ith stage, that

$$
\frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial \overrightarrow{\mathrm{p}}_{\mathrm{i}}}=\frac{1}{\left|\mathrm{H}_{\mathrm{k}}\right|} \mathrm{R}\left(\overline{\mathrm{H}}_{\mathrm{k}} \frac{\partial \mathrm{H}_{\mathrm{k}}}{\partial \overrightarrow{\mathrm{p}}_{\mathrm{i}}}\right)
$$

For the cascaded filter topology in terms of the elements of $\vec{p}_{i}$,

$$
\begin{aligned}
& \frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial \mathrm{a}_{\mathrm{i}}}=\left|\mathrm{H}_{\mathrm{k}}\right| \mathrm{R}\left(\frac{\mathrm{z}_{\mathrm{k}}^{-1}}{\mathrm{~N}_{\mathrm{k}}^{\mathrm{i}}}\right) \\
& \frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial \mathrm{b}_{\mathrm{i}}}=\left|\mathrm{H}_{\mathrm{k}}\right| \mathrm{R}\left(\frac{\mathrm{z}_{\mathrm{k}}^{-2}}{\mathrm{~N}_{\mathrm{k}}^{\mathrm{i}}}\right) \\
& \frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial c_{\mathrm{i}}}=-\left|\mathrm{H}_{\mathrm{k}}\right| \mathrm{R}\left(\frac{z_{\mathrm{k}}^{-1}}{D_{\mathrm{k}}^{\mathrm{i}}}\right)
\end{aligned}
$$

and

$$
\frac{\partial\left|H_{k}\right|}{\partial d_{i}}=-\left|H_{k}\right| R\left(\frac{z_{k}^{-2}}{D_{k}^{i}}\right)
$$

where, with $z_{k}=e^{j \pi \Omega_{k}}$,

$$
N_{k}^{i}=N^{i}\left(z_{k}\right)=1+a_{i} z_{k}^{-1}+b_{i} z_{k}^{-2}
$$

and

$$
D_{k}^{\mathrm{i}}=D^{\mathrm{i}}\left(\mathrm{z}_{\mathrm{k}}\right)=1+\mathrm{c}_{\mathrm{i}} \mathrm{z}_{\mathrm{k}}^{-1}+\mathrm{d}_{\mathrm{i}} \mathrm{z}_{\mathrm{k}}^{-2}
$$

By letting

$$
H_{\mathrm{k}}=\left|\mathrm{H}_{\mathrm{k}}\right| e^{j \phi_{\mathrm{k}}}
$$

it follows that

$$
\phi_{\mathbf{k}}=\mathrm{I}\left(\log _{\mathrm{e}} \mathrm{H}_{\mathrm{k}}\right)
$$

whence

$$
\frac{\partial \phi_{\mathrm{k}}}{\partial \overrightarrow{\mathrm{p}}}=\mathrm{I}\left(\frac{\partial}{\partial \overrightarrow{\mathrm{p}}} \log _{\mathrm{e}} \mathrm{H}_{\mathrm{k}}\right)=\mathrm{I}\left(\frac{1}{\mathrm{H}_{\mathrm{k}}} \frac{\partial \mathrm{H}_{\mathrm{k}}}{\partial \overrightarrow{\mathrm{p}}}\right)
$$

which takes on a particularly simple form for the cascade topology. For the ith stage parameters, in fact,

$$
\begin{aligned}
& \frac{\partial \phi_{k}}{\partial a_{i}}=I\left(\frac{z_{k}^{-1}}{N_{k}^{i}}\right) \\
& \frac{\partial \phi_{k}}{\partial b_{i}}=I\left(\frac{z_{k}^{-2}}{N_{k}^{i}}\right) \\
& \frac{\partial \phi_{k}}{\partial c_{i}}=-I\left(\frac{z_{k}^{-1}}{D_{k}^{i}}\right)
\end{aligned}
$$

and

$$
\frac{\partial \phi_{\mathrm{k}}}{\partial \mathrm{~d}_{\mathrm{i}}}=-\mathrm{I}\left(\frac{\mathrm{z}_{\mathrm{k}}^{-2}}{\mathrm{D}_{\mathrm{k}}^{\mathrm{i}}}\right)
$$

The special case of a one-stage ( $\mathrm{N}=1$ ) filter is illustrated. Here

$$
\begin{aligned}
& \mathrm{H}_{\mathrm{k}}=\mathrm{A} \frac{1+\mathrm{az}}{\mathrm{k}} \mathrm{~B}^{-1}+\mathrm{bz}_{k}^{-2} \\
& 1+\mathrm{cz}_{\mathrm{k}}^{-1}+\mathrm{dz}_{k}^{-2} \\
& \hat{\mathrm{~V}}=\sum_{\mathrm{k}}\left(\mathrm{~A}^{*}\left|\mathrm{H}_{\mathrm{k}}\right|-\mathrm{M}_{\mathrm{k}}\right)^{2} \mathrm{~W}_{\mathrm{k}}^{\mathrm{M}}+\lambda \sum_{\mathrm{k}}\left(\phi_{\mathrm{k}}-\theta_{\mathrm{k}}\right)^{2} \mathrm{~W}_{\mathrm{k}}^{\phi}
\end{aligned}
$$

and

$$
\frac{\partial \hat{\mathrm{V}}}{\partial \mathrm{a}}=2 \sum_{\mathrm{k}}\left(\mathrm{E}_{\mathrm{k}}^{\mathrm{M}_{\mathrm{W}}} \mathrm{M}_{\mathrm{k}} \frac{\partial\left|\mathrm{H}_{\mathrm{k}}\right|}{\partial \mathrm{a}}+\lambda \mathrm{E}_{\mathrm{k}}^{\phi} \mathrm{W}_{\mathrm{k}}^{\phi} \frac{\partial \phi_{\mathrm{k}}}{\partial \mathrm{a}}\right)=\sum_{\mathrm{k}}\left[\mathrm{Q}_{\mathrm{k}}^{\mathrm{M}_{\mathrm{R}}}\left(\frac{\mathrm{z}_{\mathrm{k}}^{-1}}{\mathrm{~N}_{\mathrm{k}}^{\mathrm{i}}}\right)+\lambda \mathrm{R}_{\mathrm{k}}^{\phi} \mathrm{I}\left(\frac{\mathrm{z}_{\mathrm{k}}^{-1}}{\mathrm{~N}_{\mathrm{k}}^{\mathrm{i}}}\right)\right]
$$

Similarly,

$$
\begin{aligned}
& \frac{\partial \hat{V}}{\partial b}=\sum_{k}\left[Q_{k}^{M} R\left(\frac{z_{k}^{-2}}{N_{k}^{i}}\right)+\lambda R_{k}^{\phi}\left(\frac{z_{k}^{-2}}{N_{k}^{i}}\right)\right] \\
& \frac{\partial \hat{V}}{\partial c}=\sum_{k}\left[Q_{k}^{M_{R}}\left(\frac{z_{k}^{-1}}{D_{k}^{i}}\right)+\lambda R_{k}^{\phi}\left(\frac{z_{k}^{-1}}{D_{k}^{i}}\right)\right] \\
& \frac{\partial \hat{V}}{\partial d}=\sum_{k}\left[Q_{k}^{M} R\left(\frac{z_{k}^{-2}}{D_{k}^{i}}\right)+\lambda R_{k}^{\phi} I\left(\frac{z_{k}^{-2}}{D_{k}^{i}}\right)\right]
\end{aligned}
$$

where

$$
\mathrm{Q}_{\mathrm{k}}^{\mathrm{M}}=2 \mathrm{E}_{\mathrm{k}}^{\mathrm{M}_{\mathrm{k}}} \mathrm{~W}_{\mathrm{k}}^{\mathrm{M}}\left|\mathrm{H}_{\mathrm{k}}\right|
$$

and

$$
\mathrm{R}_{\mathrm{k}}^{\phi}=2 \mathrm{E}_{\mathrm{k}}^{\phi} \mathrm{W}_{\mathrm{k}}^{\phi}
$$

are the weighted errors. It is obvious that the frequency intervals of the input data (specifications) need not be uniform and may, in fact, be intentionally unequal to allow for nonuniform frequency weighting.

## Complementary Root Reflection and Stability

In deriving the frequency response of a discrete operator by letting $\mathrm{z}_{\mathrm{k}}$ lie on the unit circle $\Gamma$, it is possible to take advantage of a unique property of the discrete transform pertaining to its magnitude when a root lying outside the unit circle is imaged or reflected into the unit circle. It is easy to show that the magnitude of a phasor $z-z_{0}$, where $z_{0}$ is a root of the discrete transform lying outside the unit circle, is equal to

$$
\left|z-z_{0}\right|=\left|z_{0}\right|\left|z-\frac{1}{z_{0}}\right| ; z \in \Gamma
$$

Since $z_{0}$ has been assumed to be outside the unit circle, $1 / z_{0}$ must be inside, the term $\left|z_{0}\right|$ correcting for magnitude changes. Thus, if in the optimization procedure a pole should stray outside the unit circle and thereby lead to an unstable filter, root reflection guarantees stability with no magnitude change. There is no analogous simple identity for the phase of a reflected root. Experience with the procedure has shown that provided the design requirements can be met by means of a stable filter, that is, that a feasible solution exists, an optimum will indeed be found through repeated application of root reflection.

## The Computer Algorithm

A complete listing of the filter design algorithm, which is an adaptation of the program written by Steiglitz, is given in the appendix. The main program is termed STGZ 3 which calls four principal subroutines: (1) FUNCT performs the functional and gradient computation for each iteration as well as putting out the final optimum parameters and plots, (2) FLPWL is a Fletcher-Powell conjugate gradient routine, (3) INSIDE computes root reflection, and (4) ROOTS determines the poles and zeros of the filter. Singleprecision arithmetic has been employed.

When minimization of the functional has been attained in the first pass or the minimization algorithm has iterated 300 times, a test is made to ascertain that all the roots are within the unit circle, a necessary requirement for the poles for stability reasons and for the zeros to insure minimum phase. If the design should result in an unstable
configuration, the roots are reflected about the unit circle and minimization is resumed in a second pass. If a minimum does indeed exist and all the roots then lie within the unit circle, the program computes and prints out the frequency response and commences plotting.

Minimization is deemed to be achieved when the absolute difference in functionals between successive iterations $\epsilon=\left|\hat{\mathrm{V}}_{\text {new }}-\hat{\mathrm{V}}_{\text {old }}\right|$ or the norm of the gradient vector falls below preassigned limits. Convergence is generally fast for magnitude or phase filters but can be very slow for the case of compromise filters.

When the design specifications cannot be met after LIM iterations (see appendix), the program will stop; this situation indicates that the optimum could not be found and the resultant characteristic which may be unusable is plotted. Generally, feasible designs have been determined in less than 2000 iterations.

Minimization of the criterion function does not guarantee determination of a global minimum but rather determination of a local minimum. Depending upon the parameter vector utilized for initialization of the algorithm computation, different minima may be achieved. Experience has shown that stage-by-stage optimization, that is, utilization of the ith-stage optimum parameter vector as the initial parameter vector for the $(i+1)$ stage of an N -stage filter, yields lower minimum values of the criterion function than does single-pass optimization.

## APPLICATIONS

## Linear-Phase Filter

This example considers a digital filter having application as a phase discriminator with a linear phase characteristic and arbitrary magnitude characteristic and is shown in figure 2. In this example all magnitude weights were set to zero and all phase weights to unity, the multiplier A being arbitrarily made unity since it has no effect on the phase characteristic.

The phase requirements were $\theta_{\mathrm{k}}=1-2 \Omega_{\mathrm{k}}\left(0 \leqq \Omega_{\mathrm{k}} \leqq 1\right)$, and a two-stage filter was specified. When an initial parameter vector $\overrightarrow{\mathrm{p}}=(0,0,0,0.25,0,0,0,0) \mathrm{T}$ was used, the algorithm converged to the optimum, with $\epsilon=10^{-4}$, in 52 iterations and a Control Data 6600 computer time of 14 seconds. The optimum parameter values computed were to four places

$$
\begin{array}{llll}
A=1.0 & & \\
\mathrm{a}_{1}=0 & \mathrm{~b}_{1}=-0.9871 & c_{1}=0 & d_{1}=0.0395 \\
\mathrm{a}_{2}=0 & \mathrm{~b}_{2}=-0.9871 & c_{2}=0 & \mathrm{~d}_{2}=-0.0127
\end{array}
$$

It is interesting to note that the phase requirements were met to within $0.008 \pi$ radian for approximately 95 percent of the frequency range.

## Constant-Phase Filters

Two cases were considered to obtain filters having constant phases of $-\pi / 2$ and $\pi / 2$ radians over a frequency range $0.3 \leqq \Omega_{\mathrm{k}} \leqq 0.7$. As in the previous case, the form of the magnitude characteristic was of no concern; hence, zero magnitude weighting was specified. With the same initial parameter state used in the previous example, the first case (lag network) optimized in 1673 iterations and 42 seconds to yield a hyperbolic magnitude characteristic and phase errors of less than $0.0003 \pi$ radian throughout the specified band.

The computed parameters for the lag case were

$$
\begin{array}{llll}
\mathrm{A}=1.0 & \\
\mathrm{a}_{1}=0.5580 & \mathrm{~b}_{1}=-0.1857 & c_{1}=-0.4752 & d_{1}=0.0363 \\
\mathrm{a}_{2}=0.5580 & \mathrm{~b}_{2}=-0.1857 & c_{2}=-0.3712 & d_{2}=-0.5686
\end{array}
$$

The positive phase filter (lead network), however, took only 165 iterations and 17 seconds to yield the desired phase characteristic with errors nowhere exceeding $0.001 \pi$ radian in the specified band.

The optimum filter parameters for this second case were determined to be

$$
\begin{array}{lll}
\mathrm{A}=1.0 & \\
\mathrm{a}_{1}=-0.4768 & \mathrm{~b}_{1}=-0.1548 & \mathrm{c}_{1}=0.5022
\end{array} \mathrm{~d}_{1}=-0.1082
$$

It is noted that for both cases, the phase weights outside the specified band were set to zero, and thereby allowed for arbitrary phase in these regions. Figures 3(a) and 3(b) show the resultant frequency characteristics for the lag and lead cases, respectively, of two-stage filters. The combination of the two filters, although they have antagonistic magnitude characteristics, suggests the possibility of a phase-splitting digital network.

## Limited-Band Constant-Gain Linear-Phase Filter

The third example demonstrates a compromise design of a digital filter having constant-magnitude and linear-phase characteristics, over a limited frequency band, typical of phase discriminators. Here, except for $\lambda=0$, the specifications were stated as

$$
\begin{aligned}
& \mathrm{M}_{\mathrm{k}}=\left\{\begin{array}{lr}
1 & \left(0.3 \leqq \Omega_{\mathrm{k}} \leqq 0.7\right) \\
0 & (\text { Elsewhere })
\end{array}\right. \\
& \theta_{\mathrm{k}}=\left\{\begin{array}{lr}
1-2 \Omega_{\mathrm{k}} & \left(0.3 \leqq \Omega_{\mathrm{k}} \leqq 0.7\right) \\
0 & (\text { Elsewhere })
\end{array}\right.
\end{aligned}
$$

Equal error and frequency weights were employed and the effects of changes in $\lambda$ are shown in figure 4 for a two-stage design. Figure $4(a)$ shows the case of $\lambda=0$, that is, a magnitude-only filter being specified, and coincidentally yields the linear-phase-filter characteristic derived in the first example. (See fig. 2.) Figures 4(b) and 4(c) show the magnitude and phase characteristics for the cases of $\lambda=10$ and $\lambda=1000$, respectiveiy. The increasing weight on phase and resultant degradation in the magnitude characteristic are shown. The optimum parameters were
$\lambda=0:$

$$
\begin{array}{llll}
\mathrm{A}=0.2063 & & \\
\mathrm{a}_{1}=0.0000 & \mathrm{~b}_{1}=-1.0000 & \mathrm{c}_{1}=0.0000 & \mathrm{~d}_{1}=0.1539 \\
\mathrm{a}_{2}=0.0000 & \mathrm{~b}_{2}=-1.0000 & \mathrm{c}_{2}=0.0000 & \mathrm{~d}_{2}=0.1539
\end{array}
$$

$\lambda=10:$

$$
\begin{array}{llll}
\mathrm{A}=0.3658 & \\
\mathrm{a}_{1}=-0.9754 & \mathrm{~b}_{1}=0.7300 & \mathrm{c}_{1}=0.4529 & \mathrm{~d}_{1}=0.7211 \\
\mathrm{a}_{2}=0.8632 & \mathrm{~b}_{2}=0.5632 & \mathrm{c}_{2}=-0.6119 & \mathrm{~d}_{2}=0.7443
\end{array}
$$

$\lambda=1000:$

$$
\begin{array}{llll}
A=0.4232 & & \\
a_{1}=-1.1739 & \mathrm{~b}_{1}=0.8489 & \mathrm{c}_{1}=0.7596 & \mathrm{~d}_{1}=0.6691 \\
\mathrm{a}_{2}=1.1739 & \mathrm{~b}_{2}=0.8489 & \mathrm{c}_{2}=-0.7596 & d_{2}=0.6691
\end{array}
$$

## Low-Pass Zero-Phase Filter

The fourth example considers a compromise filter, having two and three stages, with specifications that are intentionally conflicting. A filter described by

$$
\begin{aligned}
& \mathrm{M}_{\mathrm{k}}=\left\{\begin{array}{lr}
1.0 & \left(0.0 \leqq \Omega_{\mathrm{k}}<0.5\right) \\
0.5 & \left(\Omega_{\mathrm{k}}=0.5\right) \\
0.0 & \left(0.5<\Omega_{\mathrm{k}} \leqq 1.0\right)
\end{array}\right. \\
& \theta_{\mathrm{k}}=\left\{\begin{array}{lr}
0 & \left(0.0 \leqq \Omega_{\mathrm{k}} \leqq 0.5\right) \\
\text { Unspecified } & \text { (Elsewhere) }
\end{array}\right.
\end{aligned}
$$

is specified.
Figures 5 and 6 show the results for the two- and three-stage designs, respectively, with figures 5 (a) and 6(a) showing the magnitude-only ( $\lambda=0$ ) case. The degradation in the magnitude characteristics when greater emphasis is placed on the phase specifications is evident in figures 5(b) and 6(b) for $\lambda=10$ and in figures 5(c) and 6(c) for $\lambda=1000$. Comparison of figure 6 with figure 5 demonstrates the improvement brought about by increasing the number of stages. The optimum parameters for the two-stage filter were $\lambda=0:$

$$
\begin{array}{llll}
A=0.1196 & & \\
a_{1}=1.0240 & b_{1}=1.0000 & c_{1}=-0.1713 & d_{1}=0.7676 \\
a_{2}=1.0240 & b_{2}=1.0000 & c_{2}=-0.5324 & d_{2}=0.2286
\end{array}
$$

$\lambda=10:$
$\mathrm{A}=0.4879$
$a_{1}=0.2018$
$b_{1}=0.6684$
$c_{1}=0.3560$
$\mathrm{d}_{1}=0.4612$
$a_{2}=0.6597$
$\mathrm{b}_{2}=0.4335$
$c_{2}=0.0806$
$\mathrm{d}_{2}=0.7671$
$\lambda=1000:$

$$
\mathrm{A}=0.5343
$$

$\mathrm{a}_{1}=0.0205$
$b_{1}=0.7169$
$c_{1}=-0.0836$
$\mathrm{d}_{1}=0.6255$
$a_{2}=0.6286$
$\mathrm{b}_{2}=0.7905$
$c_{2}=0.2123$
$d_{2}=0.6681$

The optimum parameters for the three-stage filter were $\lambda=0$ :

$$
\mathrm{A}=0.0510
$$

$a_{1}=0.8537$
$b_{1}=1.0000$
$c_{1}=-0.1068$
$d_{1}=1.0000$
$\mathrm{a}_{2}=0.8537$
$\mathrm{b}_{2}=1.0000$
$c_{2}=-0.4046$
$\mathrm{d}_{2}=0.5990$
$\mathrm{a}_{3}=0.8537$
$b_{3}=1.0000$
$c_{3}=-0.6799$
$\mathrm{d}_{3}=0.2069$
$\lambda=10:$

$$
\mathrm{A}=0.5109
$$

| $\mathrm{a}_{1}=1.3302$ | $\mathrm{~b}_{1}=0.5515$ | $\mathrm{c}_{1}=-0.1731$ | $\mathrm{~d}_{1}=0.8097$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{a}_{2}=0.6844$ | $\mathrm{~b}_{2}=0.7157$ | $\mathrm{c}_{2}=1.1880$ | $\mathrm{~d}_{2}=0.5850$ |
| $\mathrm{a}_{3}=-0.0373$ | $\mathrm{~b}_{3}=0.7012$ | $\mathrm{c}_{3}=0.3825$ | $\mathrm{~d}_{3}=0.5262$ |

$\lambda=1000:$

$$
\begin{array}{llll}
\mathrm{A}=0.4515 & & \\
\mathrm{a}_{1}=1.5107 & \mathrm{~b}_{1}=0.5286 & c_{1}=-0.1771 & \mathrm{~d}_{1}=0.8972 \\
\mathrm{a}_{2}=0.5825 & \mathrm{~b}_{2}=0.7490 & c_{2}=1.3094 & \mathrm{~d}_{2}=0.4191 \\
\mathrm{a}_{3}=-0.1663 & \mathrm{~b}_{3}=0.7485 & c_{3}=0.2002 & d_{3}=0.6393
\end{array}
$$

A three-stage design of this example is used to demonstrate the existence of two distinct local minima, dependent upon the initial parameter vector. In the first case, a singlepass optimization was accomplished with $\vec{p}=(0,0,0,0.25,0,0,0,0)^{T}$ for the initial parameter vector and resulted in the optimum filter shown in figure 6(a). In the second case, a stage-by-stage optimization was accomplished by utilizing the optimum parameter vector from a two-stage design for the initial parameter vector of a three-stage design and resulted in the optimum filter shown in figure 7. Comparison of these results demonstrates the existence of two distinct local minima, the stage-by-stage minimization yielding superior results.

## CONCLUDING REMARKS

A method has been developed for a computer-aided design of cascade canonical digital filters having prescribed magnitude or phase characteristics or a compromise between the two. The method, which uses an unconstrained minimization algorithm, allows for arbitrary error and frequency weighting. Representative designs of phase and compromise filters have demonstrated the utility of the technique. Although convergence is generally fast for magnitude phase filters, it may be slow for the case of compromise filters.

Langley Research Center,
National Aeronautics and Space Administration, Hampton, Va., February 17, 1972.

## APPENDIX

## PROGRAM LISTING

This appendix contains a program listing written for the Control Data 6600 computer at the Langley Research Center，Hampton，Virginia，and is an adaptation of that written by Kenneth Steiglitz at Princeton University for the design of specified magnitude－only filters．

009003 $00000^{3}$ $00000^{2}$ 000003 000003 000004 000005 000011 000011 000012 000014 000 O31 000031 00005

003051
000054 000058 003051 $0) 00 \in 2$ 00010K 000106 000117 000112 000117 000117 000141

000141 00014 ？ 007147 000155 000157 000157 000176 000174 000つの？ 000213 000715 000214 0002 二2 000223 000226 000230

```
            PRCGRAM STG73IINPUT, RUTFUT,TAPE5=INPIJT,TAPFE=[UTPLT,PUNCHI
            FXTFRNAL FUNCT OC14
            IINFNSISN H(IRG),X(1t),G(1+) CO15
            CTNNON/RAW/W(1(O).Y(100).N.PHASF!)1OC),ALAMCA,FR,WTM(LCO). OOIS
            C.WTOILOJI.KTYD OC17
            CCNNCN/RAWl/ICALL.KCALL.LIF
            CALL CALCCMP
                0C19
            CALL LFRJY 0020
            WRITEIG.NII OC21
            51 FRFNAT(* INPUT DATA*) 00フ2
            M=0
            aC N}=M+
            PFAC,(=, \geq1)N(M),Y(N),PHASEC(N),WTY{N),WTR(M)
                                    21 FRPNAT(5F10.5)
            WPITF(E.2Z)M,W(M),Y(N),FHASEC(N),WTN(M),WTP(N)
```



```
            C. * W「M=*.F7.4.* WTP=*.F7.41 0029
            IF(n\M).LT.1.CCIGCTOミ0 COZO
            Dil 15 J=1.16 0031
    1F X(J)=0.00
            x(4)=.?5
                            003?
                            CC3?
            GC RFACIJ,SOIL,LIN,FST,FPS.HNAX,AIAMFA,FR,KTYP
60 FMRNAT(2I5.5FIC.5.IIC)
                                    Cc35
            IF(FQ.IT..OOI) FK=1. OC35
            N=4휴
            IF(FAF,5) CGG.FAQ
&gR CONTIM:UF
                            0%27
        WPITFIG,SIIL,LIN,FST,FPS.FNAX,FK,ALAMCA
    61 FIJKNAT(* L=*.I2.* LIN=*,I5,* ECT=*,FIN.5.* FFS=*.FlO.5.
            C* hNAX=*,F1O.5.* FRFCRANCE=*.FIO.5.* LANB)A=*,FIC.ES OC42
            IC.ALI =0
                                    CC43
    GF KCALL =O
            CALL FLPNLIFUNCT,N,X,F,G,FST,FPS,FC,IFP,H)
            CALL RCOTS(N,x)
                                    0045
            CALL INSTDEIN.X,KFLACI
                                    0045
            WHITF(6.26)IEK.KFLAG.ICGLL.KCALL
    >E FCPNAT!* IFR=*.I5.* KFLAG=*,I5.* ICALL=%,IF,* KCALL=*,I5)
            IF(KCALL.GT. 2CC) Gi) TO C&
            IF(IKFLAG.AF.C.CD.IER.NE.O).AND.I(ALL.LF.LIN) GCTCGQ
            CALLKCITS(N.X) CO50
            ICALL=-10 CC51
            CAIL FLVCTIN,X,F,G,HNAX) CC52
            guTCCa
ЭCO CALL (ALPLT(Co,^.,Oつ?)
    STOP
    EvD
CC=
O!ge
CC-4
Core
```


## APPENDIX－Continued

angolo ononlo onjoln coonln nnoola nocolo

0 CJO10 conol nnOOL2 nnon 14 on） 015 00）O1； 0nocion 07040 neoras nonr 4 ？ Onnn $4=$ $00004+$ 000のラ： のクロクラィ Oの） 057 nnol．t． noo：のミ 00）！？r 007147 00015．7 00ク： 50 U001 5 0n2167 $00077^{1}$ 000＞0？ cno 04 $003>0=$ ヘNOPO7 ヘOこ？ $0^{7}$
00（ ）11 0กつつ14 00：）？ 71 0クロフフ4 กロコプラ ロッロアクロ 000ファา COOP4 ${ }^{\circ}$ のヘロテニッ 0ヘプら $0007+5$ 000，72 OのO2：$=$ 0クへ3！ 7 000277 OกO322 กのワマム？ nのO35 000352 のOO： 0003ヶi 00）スか4 の吹ヶя
「ついるが 0กころ7つ 0.30377 $0004 \mathrm{C}, 7$ $0004 \%$ ？ －03＋19 On．）414 $003 \div 10$

```
            ClABR`!TIVF FIJNCT(A,X.F.G.HNAX) 0057
```



```
            IIMFNSIBN XPL(ZCO),YPL(ZOC),CX(2`C),CY(2O))
            1)INGASTO\because H{!&4),X(1&), (11t),VHT(10O),F(100)
            C.C.MFIEX 7OUAR.7K.77RUH.77CUF2
            CLMDIFX 7(ICRI.TUM{ICO.4).JFNIIOC.4).O.GEAR,ZCUF,ZCURZ.ONFC
            CCMN(N/2JW/W(ICC),Y(1)C),N,FHASET(ICO).AINNDA,FR,HTM(IOO).
            C KTPI:Oへ).KTYF
            C[NN]N/DANI/ICALI,KCALL.LIN
            1NF心=C.NPLX(1.00.).ON)
            PI=-3.141572;5 35ac70
            K=N/4
            IF(ICAIL.NF.OIGCTCIRI
            nr 10> I=1.*
```



```
    10: 1I=0.Cい
                                0072
            A 2=C.00
            On 4) T=:.N
            7Cしぐ=711)
                    0074
            7CHRT=7C10*7CHF
```




```
            OC コ2 J=1.K 0C78
            J4=(J-1)*4 0079
            TUM(I.J)=1.70+X(J4+1)* 年UF+X(J4+2)*7CIJF? 0080
```



```
    z: O=0*TUN(I.J)/DFAII,J) 0082
    CFAR= CU*jG(D)
    YHT(I)= =*QHAP
    \thereforeZ=Aつ+YHT(I)*hTM(1)
0084
    YHT(I)= 50RT(YHT(I))
    4) A!=\Delta!+YHT(I)*Y(I)*WTN(I)
    IF(KTYD.^!E.N) R.C TA the
    A=A1/A?
    fir Tr GST
AEh A=1.
6&7 CPNTIVGF
    or <!? I
    7CL<=ノ!!) 0094
```



```
    7\Omega=CMOLX(1..n.) 0096
    OH\DeltaS=?.
    0)[ E.1] j=1,K
    |<l=(, J-1)*4
    0078
```



```
    0100
    |A1=7!J\Mk
    7Aつ=CNP\X(0..-ミ.)*70ト\DeltaF
    PHAS= DHAS + ATANZ1l,A2.7A1)
    7?=7\cap*7 \n\triangleN
```



```
    7\Delta1=7つ2\DeltaU
```



```
    PHAS=PHAS-ATAN=(/\DeltaZ.IAI)
    7心=20/7.2HAF
&ll CRNTTNUE
    DHAII)=-'HHAS/FI
    }12 PHF(I)=OHA {I\-PHASFC(I)
        )「 巨.J J=1.!^
    =7r,(J)=).Jn
        VI=0.
        v2=?.
        「い<< 1=1,.M
        7(い,R=7!!)
        7(11こ)=7C10*2C.1F
        YHT\I!=A*YHT| I)
        c(I)=v&T(I)-v(I)
        VI=VI + E(I)*E{ I|**TM{||
        Vて=V r的保(I)*PFS(I)*NTF(I)}012
0124
    v=F(I)*mTM(I)
```


## APPENDIX－Continued

007427 $001+7$ ？ $0034>5$ 00044 1 000451 0034\％： $00347=$ onjeñ DOOF1／ 07351～ 000 5？ cojar？ 0กグックフ 00リの5 oncs 7 ？
 000613 onohla 0nos？n 0006？！ 000634 000657 000657 OODRAR onORbs 000ヶ力6 000ヶ73

000673 000701 $00970!$ 00）707 $00) 707$ 000731 000731 000753 000753 009750 001711

001011 001016 001016 001017 001070 001071 001073 $0010>7$ 001032 001047 001045 00105 ） 001051 001055 001057 001176 $00110 n$ 001117 001114 001127 001144 001146 001156 001152 001177 001207 001212 001214 001716 001217

```
    O\Gamma 4>つJ=1.K 0120
    J4=(.J-\)告 0:27
```



```
    ;(Jム+l)=`(Jム+?)+J*2ClR 0120
    G(J~+?)=0(J4+2)+O*7CLQ? 0130
    i.)=-`.*!, *YHT|\1/EN(1,J) 0131
    C:(Jち+3)=r(J4+2)+)*7CUR 0:22
    (i(J4+4)=R(J++6)+1)*7CUF? 0133
4วつ r(NTIN:JE
            ')行 }J={.
            14=(1- !) & 4
            (ild4+l)=r,(J4+i)+T*NIMA(,| ZCLF/TlN(1,J))
```




```
            ;(J<+4)=T(.)<+4)+T*A[**(-7ClF?/つ巨N(I.J))
+> C.NTIN:JF
            F=V1+\DeltaLAMDA*V2
            ICALI =ICALL+1
            KCALL=KCALL+1
            IF(KRALL.GT. 'CCIOFTURN
            IFI(IC,ALL/IO)*1O.EO.ICALL-I)WRITF(A.2=)ICALL.F.(;(J),J=1,N)
                0144
    วE FCRMATI* CALL NC.*.l4.* F=*.FI5.9/( 25x.4F!5.R)| 0145
            IFIICALL.GT.?\CO) TO) TC 45C
            IFIICALL.GT.OJRETURN
            CO Tn 449
    4FC IF(ICALL.GT.LIN+1) GC TO 44C
            RETURN
C......PFINT TUT
0148
    44% WDITEI%.50)F
    EC FORMATI* FINAL FUNCTISN VAILF=%.FIF.Q1 O150
            NRITF(f.51)A
        51 FOFMAT(* A=#,F1E.8)
            WPITF(6.52)(X(J),J=1,N)
        52 FORNAT(* FINAL x =*/(***4FlF.bl)
            WRITF(f,54)(G(J),J=1,N)
        54 FIIRMAT(* FINAL GRADIENT =*/(***.4FIF.&))
            \Piत 55 I=1.M
        55 WRITE(6,55)I,W(I),Y(I),YHT(I),E(I), PHASFFIII), PHA(I),PHE(I)
    56 FITRNATI*I=*.I?.* W=*.FE.5,* Y=*,FR.E.* YHT=*,FR.5,* E=*.FR.5,
            r. # PHASFD=*,F8.5,* PHA=*,FF.5.* DFE=&,FQ.jl
            WHITF(5.59)K
    5¢ F[RNATI* FINAL TABLE FOR At,I 2.* STAGE FILTFR*)
            YMIN:3=0.
            YMAX3?=0.
            YNAX11=0.
            S=F之/200.
            \capO t0 I=1.201
            FFFO=S*FLOATII-1)
            ZCUR= CFXPI CNPIX(O.CO.FRFG#PI)I
            ZCUR? = ICUR*7C.UR
            D= C.MPLX(1.OO.C.OC)
            PHASE=0.00
            DC G1 J=1.K
            j4=(J-2)*4
            OHAR=L'NEO+X(J4+1)*2CLR+X(J4+?)* ZCUR?
            AI=ORAR
            AR=C4PLX(0.00,-1.00)*QEAR
            PHASF=PHASE+ ATANZ(A2. AI)
            0=L&OPAR
            OBAR=[BNF)+X(J4+2)*ZC(UK+X(J4+4)*Z「LF?
            Al=CA\DeltaR
            A? =CMP1\times(O.OO.-1.CO)*MBAR
            PHASF=PHASF- ATANO(A2.A1)
    G1 D=O/LRAO
            A1=0^ CONJG(0)
            \Delta1=\Delta* SORT(\Delta1)
            PHASF=-PHASF/P!
            RMFCAIII=FPEO
            AMAGII)=AI OL89
0149
                0151
                                    0:5?
                01う3
            MASF=0.oO
                            C154
0155
0155
0:57
0158
C157
C157
015?
01*1
0102
0:42
0164
0165
c16t
0167
C16%
-167
0169
C17C
0171
0:7?
0173
C:74
0175
0175
017%
0177
0177
0178
0179
0179
C1R1
ClR1
0183
0193
0195
0186
0198
            PH.ACFXIII=PHASF 
```


## APPENDIX－Continued

| $001>21$ |  | IF（FHASE－YMAX33） $201.3 C 1.4 C 1$ | 0191 |
| :---: | :---: | :---: | :---: |
| $0017 ? 6$ | 4 Cl | YMAX ${ }^{\text {P }}$ 3＝PHASF | 0192 |
| 001230 | 3 C ？ | r．ontinus | 0193 |
| 001230 |  | IF（DHASE－YMIN33）402．3C2．3C2 | 0194 |
| 001233 | 4 C 2 | YM1N3 3＝－HASF | 0195 |
| 001235 | $30 ?$ | CCATINUE | 0195 |
| 001735 |  | IF（AT－YMAX12） $2 \mathrm{SC} .3 \mathrm{CC,4CO}$ | 0197 |
| 001240 | 4 CO | YMAXII $=A 1$ | 0198 |
| 001242 | 3 CO | COn TI NUF | 0199 |
| 001247 | 6C | WFITF1S．S2IFREG．PHASE．A1 | 0200 |
| 001751 | 62 |  | 0201 |
|  | C SCAI | E COMPUTATIUNS | 0202 |
| 001261 |  | IYMAXIL＝YMAX11＋．9c90 | 0203 |
| 001764 |  | YN $\Delta \times 1=F L T A T(I Y N A X I I)$ | 0204 |
| 00：256 |  | IFIHMAX．FQ．O．）COCI TR 3 3 | 0205 |
| 001767 |  | $Y M A X I=H M A X$ | 0205 |
| 001270 | $33^{2}$ | CONTINUE | 0207 |
| 001770 |  | I YNAXミ3＝YMAX3？ Y ．C¢9 | 0209 |
| 001773 |  | YMAX2 $=$ FLOAT（IYNAX23） | 0209 |
| 001275 |  | IYMTNここ Y YMINE 2 －．9¢99 | 0212 |
| 001300 |  | YMIN＝FL．JAT（IYNIN 3 ） | $0<11$ |
| 001301 |  | $\Delta X N=1$ Or F／IFS／2） | 0212 |
| 001302 |  | $A Y N=1$ OHMAGNITUCES | C213 |
| OC1304 |  | $A F R=730 . * F R$ | 0214 |
| 001706 |  | $N F R=70$ | 02：3 |
|  | C NAGA | itude（crimputed－fkegleacy plot | 0216 |
| 001307 |  |  | 0217 |
|  |  | AYN，OJ | 0218 |
|  | C NAGN | ITUCF（DESIRFII－FPFOUENCY plot | $021 \%$ |
| 001377 |  |  | 0220 |
| 001347 |  | AYM＝1 OH PHASE／FI | 0221 |
|  | C PHAS | E－FRFGUENCY PICT | 0222 |
| Oロ1231 |  |  | 0223 |
|  | （ |  | 0224 |
|  | C PliAs | ¢（Ciくfofil－F2FOUJFNCY DLCT | 0225 |
| orla |  |  | 0226 |
|  | C |  | 0227 |
| Cos 41. |  | $\therefore$ FTH：$V$ | 0228 |
| nn：411 |  | 「AD | 0229 |
|  |  |  | 0230 |
| 003915 |  |  | 0231 |
| 00071： |  | （GMN $\operatorname{CN/2AW1/!~CAIL.KCALI,LIN~}$ |  |
| 0000.9 |  |  | 0233 |
| an＠rop |  | frikrall．gT．̇rri rin Til 7 ？ |  |
| nounz？ |  | IFIIGALEST．LIN）（\％Tr フフa |  |
| 90） 0 ミニ |  | C－T $\mathrm{Cl}^{\text {¢ }}$ |  |
| の○プニ5 | ？ 3 j | IFD＝？ |  |
| ，（0） 37 |  | 「FTUM |  |
|  | G：7 | 15R＝？ |  |
| OnOn¢ |  | Krunt＝0） | 0235 |
| 00J041 |  | $v ?=\+V$ | 0236 |
| 0（0）${ }^{\text {a }}$ |  | $\backslash 2=V\rangle+v$ | 0237 |
| 000042 |  | ，21－A $2+2$ | C238 |
| OnOn4 5 | $\therefore \quad$ | $\kappa=\mathrm{v}=1$ | 0239 |
| Cnonci |  |  | 0240 |
| の（T） 51 |  | $\mathrm{H}(\mathrm{K})=1.00$ | 0241 |
| 037） 5 ＝ |  | A．$J=\wedge-1$ | 0242 |
| 000054 |  |  | 0243 |
| （n）ran | $?$ | $)^{\prime \prime}$ j L＝？ 0 ， | 0244 |
| ornos） |  | $k \mathrm{l}=\mathrm{k}+\mathrm{l}$ | 0245 |

## APPENDIX－Continued

| oconat | 3 | $H(K L)=0.00$ | 0246 |
| :---: | :---: | :---: | :---: |
| conrisa | 4 | $K=K L+i$ | 0247 |
| onon7？ | う | KruNT＝KCUVT + ？ | 0248 |
| ¢0 or 7： |  | Weritit -.5011 KCOL T | 0249 |
| 00017 ： | 50.1 |  | 0250 |
| ก0ロ10！ |  | － $\mathrm{LD} \mathrm{D}=\mathrm{F}$ | 0251 |
| nnolio |  |  | 0252 |
| －00？ 07 |  | $\mathrm{K}=\mathrm{N}+\mathrm{J}$ | 0253 |
| OnJ110 |  | $H(K)=G(1)$ | 0254 |
| ncoll4 |  | $k=K+N$ ． | 0255 |
| OC7：15 |  | H（K）＝X（J） | 0255 |
| （0）1 ${ }^{\text {Pr }}$ |  | $k=1+y^{\prime}$ ？ | 0257 |
| $00.21 ?$ ？ |  | $T=0.00$ | 0258 |
| nool 2 － |  | ¢0 2 $1=1 . v$ | 0259 |
| Co） 124 |  | $T=T-r .1 \mid)=4(k)$ | 0260 |
| nool 3 ！ |  | 1F（L－J） 5 ， 7.7 | 0261 |
| nod： 34 | 1. | $K=\langle+1-1$ | 0262 |
| 0nn：37 |  | ； 0 T．7 $\mu$ | 0263 |
| Q0．7137 | 7 | $k=k+1$ | 0264 |
| 00314 | $\cdots$ | こ「NTIV | 0265 |
| 0n 2144 | $\rightarrow$ | $\mathrm{H}(\mathrm{J})=\mathrm{T}$ | 0266 |
| 000150 |  | DY $=\mathrm{C} .00$ | 0267 |
| 000150 |  | HNRM $=0.0 .0$ | 0268 |
| 000151 |  | GNRN＝0．0） | 0269 |
| 000153 |  | On $10 \mathrm{~J}=1 . \mathrm{N}$ | 0270 |
| 000154 |  |  | 0271 |
| 000150 |  | GNRN＝GNRM＋ARS（C）${ }_{\text {G }}$ ） | 0272 |
| 000163 | 10 | DY＝DY＋H（J）＊G（J） | 0273 |
| 000173 |  | IFIDY 1 1，51．51 | 0274 |
| 000174 | 11 | IF（HNRN／GNRM－EPS）51．51．12 | 0275 |
| 000200 | 12 | $\mathrm{F} Y=\mathrm{F}$ | 0276 |
| 000201 |  | ALFA $=2.00 *$（FST－F）／OY | 0277 |
| 000204 |  | $\triangle M B \Gamma A=1.00$ | 0278 |
| 000705 |  | IFIALFAI 15．15．13 | 0279 |
| $0007 \mathrm{C7}$ | 13 | IF（ALFA－AMBDA）14．15．15 | C260 |
| 000） 12 | 14 | $A M B C A=\triangle L F A$ | 0281 |
| 000214 | 15 | $A L F A=0.30$ | 0282 |
| 000215 | $1 \epsilon$ | $F X=F Y$ | 0283 |
| 000216 |  | DX $=\mathrm{CY}$ | 0284 |
| 000227 |  | DC $17 \mathrm{I}=1, \mathrm{~N}$ | 0285 |
| $0007 ? 2$ | 17 | X（I）＝X（I）＋AMACA＊H（I） | 0296 |
| 000231 |  | CALL FUNCTIN，X，F，G，HMAX） | C289 |
| 000243 |  | IFIKCALL．GT．3OC）GO TO 724 |  |
| 000746 |  | IF（ICALL．GT．LIN）GO TO 724 |  |
| 000251 |  | GП TO 918 |  |
| 000251 | 724 | IFR＝3 |  |
| 000253 |  | RFTURN |  |
| 000253 | ¢18 | $F Y=F$ |  |
| 000254 |  | OY $=$ C． 00 | 0292 |
| 000755 |  | DO 19［＝1．N | 0293 |
| 000757 | 18 | $D Y=C Y+G(1) * H I I)$ | 0294 |
| 000766 |  | IF（OY）19．26． 22 | 0297 |
| 000267 | 19 | IF（FY－FX）20．22．22 | 0298 |
| 000277 | 20 | $\triangle M B D A=\triangle M B C A+\triangle L F A$ | C299 |
| 000774 |  | $A L F A=\triangle M Q D A$ | 0300 |
| 070275 |  | $F R R C R=1 . E 10$ | 0301 |
| 000276 |  | IF（HNRM＊AMRDA－ERRDR） 16.15 .21 | 0302 |
| 000307 | 21 | I $E$ R＝？ | 0303 |
| 000304 |  | RETURN | 0304 |
| 000304 | 22 | $T=0.00$ | 0305 |
| 000305 | 23 | IF（ $\triangle M B \cap A) 24,36,24$ | 0306 |
| 000306 | 24 | $Z=3.00 *(F X-F Y) / A M B D A+D X+D Y$ | 0307 |
| 000314 |  | ALFA＝AMAXI（ AES（Z），$A B S(0 X)$ ，ABS（DY） | 0308 |
| 000326 |  | CALFA $=$ Z／ALFA | 0309 |

## APPENDIX－Continued

| 000327 |  | DALFA $=$ CALFA＊DALFA－DX／ALFA＊［Y／ALFA | 0310 |
| :---: | :---: | :---: | :---: |
| 000337 |  | IF（DALFA）51．25．25 | 0311 |
| 000335 | 25 | W＝ALFA＊SORTICALFA） | 0312 |
| 000340 |  | ALFA $=(D Y+W-Z) * A M B D A /(D Y+2 \cdot C O * W-D X)$ | 0313 |
| 000351 |  | Dก 26 I $=1 . N$ | 0314 |
| 000356 | 26 | X（I）$=$ X（I）＋（T－ALFA） $\mathrm{H}_{(1)}^{(I)}$ | 0315 |
| 000366 |  | CALL FUNCTIN，X，F，G，HNAXI | 0320 |
| 000400 |  | IFIKCALL．GT． 3001 GO IO 725 |  |
| 000403 |  | IFIICALL．GT．LIN）GO TO 725 |  |
| 000406 |  | GO TO 919 |  |
| 000406 | 725 | $I E R=3$ |  |
| 000410 |  | RFTLRN |  |
| 0004！ 9 | 919 | IF（F－FX）27．27．28 |  |
| 000413 | 27 | IF（F－FY）36．36．29 | 0323 |
| 000416 | 28 | DALFA $=0.00$ | 0324 |
| 000417 |  | DG ？$\quad 1=1, N$ | 0325 |
| 000421 | 29 | CALFA＝DALFA＋G（I）＊H（I） | 0326 |
| 000431 |  | IF（CALFA） $30,33.33$ | 0329 |
| 000431 | 3 C | IF（F－FX）32．31．23 | 0330 |
| $000+34$ | 31 | IF（EX－［ALFA） 22.36 .32 | 0321 |
| 000436 | 32 | $F X=F$ | 0332 |
| 000437 |  | DX＝CALFA | 0333 |
| 000440 |  | $T=A L F A$ | 0334 |
| 00044 ？ |  | $\triangle M B C A=A L F A$ | 0335 |
| 000443 |  | GT TH ？ 3 | 033 5 |
| 000443 | 33 | IFIFY－F）35．34．35 | 0337 |
| 000445 | 34 | IFICY－C．ALFA） 35.36 .35 | 0338 |
| 000447 | 35 | $F Y=F$ | 0339 |
| 000450 |  | $T Y=$ CALFA | 0340 |
| 000451 |  | $\triangle M B C A=\triangle M B D A-\triangle L F A$ | 0341 |
| 000454 |  | GO In 22 | 0342 |
| 000454 | 36 | Dก $37 \mathrm{~J}=1 \cdot \mathrm{~N}$ | 0343 |
| 000456 |  | $\mathrm{K}=\mathrm{N}+\mathrm{J}$ | 0344 |
| 000457 |  | $H(K)=G(J)-H(K)$ | 0345 |
| 000463 |  | $K=K+N$ | 0346 |
| 000464 | 37 | H（K）$=X(1)-H(K)$ | 0347 |
| 000477 |  | IF（OLDF－F＋FPS）52．38．39 | 0348 |
| 000476 | 38 | IFR＝0 | 0349 |
| 000477 |  | IF（KOUNT－N 42.39 .39 | 0350 |
| 000501 | 39 | $T=0.00$ | 0351 |
| $00050 ?$ |  | $Z=0.00$ | 0352 |
| 000502 |  | DC $40 \mathrm{~J}=1 . \mathrm{N}$ | 0353 |
| 000504 |  | $\mathrm{K}=\mathrm{N}+\mathrm{J}$ | 0354 |
| coosem |  | $w=H(k)$ | 0355 |
| 00cs10 |  | $K=K+\lambda$, | 0356 |
| 00051 ？ |  | $T=T+A P S(H\{K))$ | 0357 |
| 000514 | 40 |  | 0358 |
| Cos52： |  | IFIHVRN－FPS）41．41．42 | 0359 |
| 000ヶ3 | 41 | IFIT－FPS）5t．5t．4？ | 0360 |
| $000=31$ | 47 | IFIKCUNT－LIMIT；42．5）．5C | 0361 |
| nove 34 | 43 | $\triangle L F \Delta=C .(T)$ | 0362 |
| 000535 |  | Mn $47 \mathrm{~J}=1$ ， N | 0363 |
| 000527 |  | $k=J+N$ ？ | 0364 |
| 000540 |  | $\cdots=C .00$ | 0365 |
| 00.544 |  | OC 4＊L $=1, N^{\prime}$ | 0366 |
| 00.7542 |  | $k I=N+1$. | 0367 |
| 000544 |  | $N=W+H(K L) * H(K)$ | 0368 |
| 00n55 |  | IF（L－J）44．45．4E | 0369 |
| 00フラ54 | 44 | $K=K+N-L$ | 0370 |
| COO5 57 |  | GC TM 4 ¢ | 0371 |
| 000557 | 45 | $k=k+1$ | 0372 |
| 00056］ | 46 | rentivuf | 0373 |
| 000564 |  | $K=N+J$ | 0374 |
| 000565 |  | $A L F A=A L F A+W * H(K)$ | 0375 |

## APPENDIX－Continued

| 000577 | 47 | H（J）$=W$ | 0376 |
| :---: | :---: | :---: | :---: |
| 000576 |  |  | 0377 |
| coos00 | 48 | $K=N ? 1$ | 0378 |
| $00060 ?$ |  |  | 0379 |
| c00503 |  | $K \mathrm{~L}=\mathrm{N} 2+\mathrm{L}$ | 0380 |
| 000605 |  | Oก $49 \mathrm{~J}=\mathrm{L}$ ． N | 0381 |
| OOODOF |  | $\mathrm{A}_{\mathrm{I}} \mathrm{J}=\mathrm{N} \boldsymbol{\lambda}+\mathrm{J}$ | 0382 |
| 0006C7 |  | $H(K)=H(K)+H(K L) * H(N J) / Z-H(L) * H(J) / \Delta L F A$ | 0383 |
| 000く？ | 49 | $K=K+1$ | 0384 |
| 00063？ |  | TO T） 5 | 0385 |
| 000634 | 5 G | $I F P=1$ | 0385 |
| 000636 |  | PFTURN | 0387 |
| OnO636 | 51 | クU $5>\mathrm{J}=1, \mathrm{~N}$ | 0388 |
| 000647 |  | $k=N \geqslant+J$ | 0389 |
| 000044 | 52 | $X(J)=H(K)$ | 0390 |
| 000647 |  | C．ALL FUNCT（N，X．F．G．HNAX） | 0353 |
| 000661 |  |  |  |
| 000664 |  | IFIITALL．gT．LIN）GO in 726 |  |
| 000ちら7 |  | GOT TH G 20 |  |
| 0006ヵ7 | $77 t$ | IFR＝3 |  |
| 000671 |  | FFTUFN |  |
| 000671 | $5 ? \mathrm{C}$ | It（GVRM－5PS）¢5．5E．5？ |  |
| 000674 | うミ | IF（IFR）55．54．54 | 0396 |
| 000475 | 54 | IFR $=-1$ | 0397 |
| COフ700 |  | GC Tl 1 | 0398 |
| 00.9700 | 55 | 1FR＝〕 | 0399 |
| 000701 | 56 | RETURN | 0400 |
| 000707 |  | ¢ ND | 0401 |

000006 000006 000006 000007 000011 000014 000016 000020 000022

000024 000025 000027 000031 000033 000035 000051 000051 000054 000060 000064 000066

000066 000071 000071 000072 000074 000076 000077

SUBROUTINE INSIDE（N，X，XFLAG）
DIMENSION X（16）
$J=-1$
$K F L A G=0$
$10 \mathrm{~J}=\mathrm{J}+$ ？
IFIJ．GT．NIRETURN
e＝－．500＊X（J）
$C=x(1)+1)$
DISC＝e＊e－C
IF（OISC．LE．O．00）GOTO2O
C．．．．．．REAL ROOTS
DISC＝SORT（DISC）
R1 $=8+$ CISC
R2＝B－DISC
CRI $=A B S(R 1)$
DR2 $=A B S(R 2)$
IFIOR1．LE．1．00．AND．DR2．LE．1．00）GOTO10
$K F L A G=1$
IF（DR1．GT．1．001R1 $=1.00 / R 1$
IF（DR2．GT．1．00）R2 $=1.00 / R 2$
$x(J)=-1.00 *(R 1+R 2)$
X（J＋1）＝R1＊R2
GCTOIO
C．．．．．．CEMPLEX ROOTS
20 IFIC．LE． 1.00 ）GOTO10
KFLAG＝1
$\mathrm{C}=1.00 / \mathrm{C}$
$x(J+1)=C$
$x(J)=x(J) * C$
GOTOLO
END

APPENDIX - Concluded

000005 000005 000010 000010 000011 000013 000017 000021 000023 000025

000027 000030 000032 000035 000044 000044

000046
000052 000055 000070 000070 000072
SUBROUTINE ROOTS(N,X) 0433
DIMENSICN X(16) ..... 0434
WRITE(6,40) ..... C435
40 FORMAT(* ROOTS*/6X,*REAL*,11X,*1MAG*,11X,*REAL*,11X**IMAG*) ..... 0436
$J=-1$ ..... 04370438
IF(J.GT.NIRETURN ..... C4 39
$B=-500 * X(J)$ ..... C440
$C=X(J+1)$C441
DISC=B*B-C 0442
IF (DISC.LE. O.00) GOTO20 ..... 0443
C.......REAL RCOTS ..... 0444
DISC = SORT(DISC) ..... C445R1 $=\mathrm{B}+\mathrm{DISC}$
$R 2=B-D I S C$
C446WRITE(E, 30)R1,R20447
30 FORMAT (* \&,F15.8.15X,F15.\&)
0448
GOTOLO
Complex roots ..... 04510449
20 DISC $=$ SORT $(-1.00 * D I S C I$ ..... 0452
DI SCM $=-1.00 * D I S C$ ..... 0453
WRITE(6.50)B,DISC,B,DISCM ..... 0454
50 FORMAT (* *.4F15.8) ..... 0455
GOTO10 ..... 0456
END ..... 0457

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Figure 1.- Signal flow graph of cascaded digital filter.

$$
\theta_{k}=1-2 \Omega_{k} \quad\left(0 \leqq \Omega_{k} \leqq 1\right)
$$

$$
M_{k}=\text { Unspecified } \quad\left(0 \leqq \Omega_{k} \leqq 1\right)
$$



Figure 2.- Two-stage linear-phase filter.

$$
\theta_{k}= \begin{cases}-\pi / 2 & \left(0.3 \leqq \Omega_{k} \leqq 0.7\right) \\ \text { Unspecified } & \text { (elsewhere) }\end{cases}
$$

$$
M_{k}=\text { Unspecified } \quad\left(0 \leqq \Omega_{k} \leqq 1\right)
$$


(a) Lag filter.

Figure 3.- Two-stage constant-phase filters.

$$
\theta_{\mathrm{k}}= \begin{cases}\pi / 2 & \left(0.3 \leqq \Omega_{\mathrm{k}} \leqq 0.7\right) \\ \text { Unspecified } & \text { (elsewhere) }\end{cases}
$$



(b) Lead filter.

Figure 3.- Concluded.

(a) Unspecified phase filter. $\lambda=0$.

Figure 4.- Two-stage limited-band constant-gain filters.

$$
M_{k}= \begin{cases}1 & \left(0.3 \leqq \Omega_{k} \leqq 0.7\right) \\ 0 & \text { (elsewhere) }\end{cases}
$$


$\theta_{k}=\left\{\begin{array}{cl}1-2 \Omega_{k} & \left(0.3 \leqq \Omega_{k} \leqq 0.7\right) \\ 0 & \text { (elsewhere) }\end{array}\right.$

(b) Linear-phase filter. $\lambda=10$.

Figure 4.- Continued.

(c) Linear-phase filter. $\lambda=1000$.

Figure 4.- Concluded.
$\theta_{k}=$ Unspecified $\quad\left(0 \leqq \Omega_{k} \leqq 1\right)$


$$
M_{k}= \begin{cases}1.0 & \left(0.0 \leqq \Omega_{k}<0.5\right) \\ 0.5 & \left(\Omega_{k}=0.5\right) \\ 0.0 & \left(0.5<\Omega_{k} \leqq 1.0\right)\end{cases}
$$


(a) Unspecified-phase filter. $\lambda=0$.

Figure 5.- Two-stage low-pass filters.

$$
\theta_{k}=\left\{\begin{array}{ll}
0 & \left(0.0 \leqq \Omega_{k} \leqq 0.5\right) \\
\text { Unspecified } & \text { (eIsewhere) }
\end{array} \quad M_{k}= \begin{cases}1.0 & \left(0.0 \leqq \Omega_{k}<0.5\right) \\
0.5 & \left(\Omega_{k}=0.5\right) \\
0.0 & \left(0.5<\Omega_{k} \leqq 1.0\right)\end{cases}\right.
$$



(b) Zero-phase filter. $\lambda=10$.

Figure 5.- Continued.

(c) Zero-phase filter. $\lambda=1000$.

Figure 5.- Concluded.

$$
\theta_{k}=\text { Unspecified } \quad\left(0 \leqq \Omega_{k} \leqq 1\right)
$$



$$
M_{k}= \begin{cases}1.0 & \left(0.0 \leqq \Omega_{\mathrm{k}}<0.5\right) \\ 0.5 & \left(\Omega_{\mathrm{k}}=0.5\right) \\ 0.0 & \left(0.5<\Omega_{\mathrm{k}} \leqq 1.0\right)\end{cases}
$$


(a) Unspecified-phase filter. $\lambda=0$.

Figure 6.- Three-stage low-pass filters.

(b) Zero-phase filter. $\lambda=10$.

Figure 6.- Continued.

$$
\theta_{k}=\left\{\begin{array}{ll}
0 & \left(0.0 \leqq \Omega_{k} \leqq 0.5\right) \\
\text { Unspecified } & \text { (el sewhere) }
\end{array} \quad M_{k}= \begin{cases}1.0 & \left(0.0 \leqq \Omega_{k}<0.5\right) \\
0.5 & \left(\Omega_{k}=0.5\right) \\
0.0 & \left(0.5<\Omega_{k} \leqq 1.0\right)\end{cases}\right.
$$



(c) Zero-phase filter. $\lambda=1000$.

Figure 6.- Concluded.

$$
\theta_{k}=\text { Unspecified } \quad\left(0 \leqq \Omega_{k} \leqq 1\right)
$$



$$
M_{k}= \begin{cases}1.0 & \left(0.0 \leqq \Omega_{k}<0.5\right) \\ 0.5 & \left(\Omega_{k}=0.5\right) \\ 0.0 & \left(0.5<\Omega_{k} \leqq 10\right)\end{cases}
$$



Figure 7.- Three-stage low-pass filter.

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