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FOUR DIMENSIONAL STUDIES IN EARTH SPACE

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R. S. Mather
Geodynamics Branch

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CONTENTS

Page

SUMMARY v

1. INTRODUCTION 1

1.1 Preamble 1
1.2 A Guide to Notation 1

2. THE INSTANTANEOUS SYSTEM OF REFERENCE 3

2.1 Preamble 3
2.2 The Instantaneous Axis of Rotation 4
2.3 The Axis of Reference 5

3. A SYSTEM OF REFERENCE FOR FOUR DIMENSIONAL STUDIES IN EARTH SPACE 10

3.1 Description 10
3.2 The Euler Angles and Observations 10
3.3 Definition of the Motion of the Geocenter 13
3.4 A Complete Definition of Earth Space 17

4. DISCUSSION 19

4.1 The Scale for Earth Space 19
4.2 Gravitation and Earth Space 21
4.3 A Note on "Motion" in the Four Dimensional Representation of Earth Space 21

5. CONCLUSIONS 22

6. ACKNOWLEDGMENTS 23

REFERENCES 24

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FOUR DIMENSIONAL STUDIES IN EARTH SPACE

R. S. Mather*
Geodynamics Branch

SUMMARY

A system of reference which is directly related to observations, is proposed for four dimensional studies in Earth space. The requisite data is used to define both global control network and also polar wandering. The determination of variations of the Earth's gravitational field with time also forms part of such a system. Techniques are outlined for the unique definition of the motion of the geocenter, and the changes in the location of the axis of rotation of an instantaneous Earth model, in relation to values at some epoch of reference. The instantaneous system referred to is directly related to a fundamental equation in geodynamics. The reference system defined would provide an unambiguous frame for long period studies in Earth space, provided the scale of the space were specified.

* On leave of absence from the University of New South Wales, Sydney, Australia.

FOUR DIMENSIONAL STUDIES IN EARTH SPACE

1. INTRODUCTION

1.1 Preamble

Earth space is defined as the Euclidian space which has the same galactic and rotational motion as the Earth. A detailed review of the background is given in (Mather 1972). The definition of this space is a concept which is fundamental to all geodetic reference systems which provide a framework for the conversion of observations to notions of position. The definition of Earth space poses no difficulties for the postulation of reference frames whose utility extends over short periods of time, and in the context of observational accuracies which were not better than 1 ppm. It is however becoming increasingly evident that control networks, based on observations whose precision could very well be at least one, and possibly two orders better, will become commonplace in the foreseeable future (NASA-MIT Report 1969). It is also probable that portable devices for measuring absolute gravity with equivalent precision will become available (Section IV, IAG Proceedings 1971). It is not unreasonable to envisage measuring accuracies of $\pm 1 \mu$gal in the context of geodetic studies in the future, in view of the accuracies being achieved with fixed apparatus at the present time (Sakuma 1971, p. 159), and it follows that these instruments will become invaluable for the study of secular changes in gravity over periods of time which are short enough to be commensurate with those required to obtain results of significance from laser ranging techniques (e.g., Smith & Kolenciewicsz 1971; Escobal & Muller 1972; Faller, Bender et al. 1971) and very-Long-Baseline Interferometry (VLBI) (Shapiro & Knight 1970).

The contentions underlying the instantaneous reference system for the definition of Earth space are outlined in section 2, while the techniques for the continuous definition of Earth space over long periods of time are presented in section 3. This includes the recovery of the changes in position of the Earth's center of mass or geocenter. The space so defined and the changes in its scale and physical characteristics are discussed, along with other problems, in section 4.

1.2 A Guide to Notation

Symbols

c = velocity of light

G = gravitational constant

H = angular momentum of the Earth

I = i^{th} order inertia tensor of the Earth

$\imath$ = the set of unit vectors 1, 2, & 3 along the axis x_1, x_2 and x_3 respectively

L = torque acting on the Earth

ℓ_i = the set of direction cosines defining the position vector R with respect to the x_i axis system

M = mass of the Earth

R = the position vector, given by $R = x_i \imath$ when referred to the instantaneous system

t = time

X_i = the co-ordinates X_1, X_2, X_3; the X_1 X_2 X_3 rectangular Cartesian axis system, used as an inertial system of reference, being coincident with the x_i system at some epoch of reference ($\tau = t_0$).

x_i = the instantaneous three dimensional Cartesian co-ordinate system $x_1 x_2 x_3$, with origin at the geocenter, the x_3 axis being normal to the $x_1 x_2$ plane and coincident with the rotation axis of the Earth, the x_1 axis lying in the plane of the reference meridian while the x_2 axis completes the rectangular triad; also the co-ordinates x_1, x_2 x_3

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1}$$

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if } i = j,\ i = k,\ j = k \\ 1 & \text{if subscripts occur in the order 12312} \\ -1 & \text{if subscripts occur in the order 13213} \end{cases} \tag{2}$$

θ_i = the Euler angles relating the axes χ_i to the inertial axes X_i, all rotations being anti-clockwise

λ = longitude, positive east

τ = epoch of observation

ϕ = latitude, positive north

χ_i = the x_i axis system used as a local inertial frame of reference for the study of elemental rotations

ω = the angular velocity of rotation of the Earth about an instantaneous axis of rotation

ω_i = the components of ω about the instantaneous reference axes χ_i. Thus

$$\omega = \left(\sum_{i=1}^{3} \omega_i^2 \right)^{1/2} ;$$

also could have the diurnal component of the rotation removed

Subscripts

g = referring to the instantaneous geocenter

Conventions adopted

$x_i \, y_i \equiv x_1 y_1 + x_2 y_2 + x_3 y_3$

$x_i = a_{ij} \, y_j \equiv$ three equations of the form $x_1 = a_{i1} y_1 + a_{i2} y_2 + a_{i3} y_3$ obtained on replacing i by 1, 2 and 3 respectively.

2. THE INSTANTANEOUS SYSTEM OF REFERENCE

2.1 Preamble

The relevance of a system of reference for any space is governed by the ease with which observations can be referred to it. A macro characteristic which dominates all observations made in relation to the Earth as an observing platform, is its rotation which is essentially stable in the present epoch. The rotation axis of the Earth is therefore one of symmetry and is conventionally used as a direction of reference for positioning ellipsoids when defining geodetic datums. Ambiguities which may occur due to variations in the rate of rotation of the Earth and other factors with those physical changes producing polar motion, are not considered when defining the geodetic datum itself, but rather as corrections to observations in relation to some epoch of reference. This would apply to the latest system defined by the International Association of Geodesy (Reference System 1967). It is also common practice to refer three dimensional Cartesian systems of reference used in satellite geodesy to the gross characteristics of the Earth's rotation.

The systems defined above are intended to provide a frame of reference for the definition of position with an accuracy of 1 part per million (ppm) and deal with limited intervals of time. The center of these reference systems is located at either the Earth's center of mass (geocenter) or at a location whose extent of approximation to the geocenter depends on the degree of sophistication adopted in defining some datum point at the surface of the Earth from geometrical considerations.

The Earth itself is non-rigid. This implies a constant re-distribution of mass in Earth space on a variety of time scales. A related consequence is a time dependent gravity field. Many considerations in geodynamics are based on the Liouville equation (Routh 1905, p. 20) and it is therefore desirable that any reference frame proposed is capable of direct relation to this equation.

2.2 The Instantaneous Axis of Rotation

The problems arising from the departures of the rotation of the Earth from that of a rigid body model, are definitively dealt with by Munk & MacDonald (1960). They are essentially of two types.

1. Departures which are a consequence of variations in the rate of rotation; and

2. Those due to changes in the location of the rotation axis with respect to the Earth's crust.

There is no reason to reject the possibility that these changes have a frequency spectrum which ranges from at least semi-diurnal to secular terms. The contributions of terms corresponding to seasonal and Chandlerian periods continues to be the subject of considerable research. The original investigations were based on astronomical observations, but the tracking of both artificial Earth satellites and the lunar reflectors afford alternate techniques which not only promise greater accuracy but also the possibility of resolving the contributions with higher frequency.

The origin of the system of reference could be either a monumented geodetic station at the surface of the Earth or a point implied by the physical characteristics of the space, e.g., the instantaneous geocenter.

It would be preferable to adopt the latter for both geopolitical and aesthetic reasons, provided it were capable of definition with adequate precision. Further, in view of the fluidity of the inner core and its relatively high density, changes in the location of the geocenter are likely to be significantly smaller than those at a point on the Earth's surface.

The instantaneous axis of rotation will therefore be a line through the instantaneous geocenter about which the Earth, treated as a rigid body would rotate. There appears to be no objection to the use of differential calculus in modelling the changes in the direction cosines of this axis in relation to an inertial frame of reference X_i, which is defined in section 3. The instantaneous axis of rotation is therefore uniquely defined if the Earth were modeled by a rigid body equivalent in much the same way that the motion of an artificial Earth satellite is by the concept of a Keplerian ellipse.

The variations in the position of the pole can be deduced from astronomical considerations of the variations in latitude $\Delta\phi_i$ at selected observatories P_i, using observation equations of the form

$$\Delta\phi_i = y_1 \cos\lambda_i + y_2 \sin\lambda_i + y_3 \qquad (3)$$

where the $y_1 y_2$ plane is tangential to the Earth at the pole of the reference axis of rotation for the epoch considered, the y_1 axis being oriented in the plane of the meridian of reference, and the y_2 axis being perpendicular to it, the y_3 axis being coincident with the reference axis of rotation. λ_i is the longitude of the meridian passing through the observatory P_i. The values of y_1 and y_2 can be used to determine the direction cosines of the instantaneous rotation aix, as described in section 3.2, while the term y_3, which is the z-term of Kimura, absorbs any tendency for the observatories to move en-masse in relation to the pole.

A similar determination of polar motion could be made by studying the orbits of artificial Earth satellites. On holding the tracking station positions fixed, the along-track residuals obtained on fitting the orbital solution to the tracking stations could be analyzed to determine y_1 and y_2 (Anderle & Beuglass 1970).

2.3 The Axes of Reference

The identification of the axes comprising the reference system at any given epoch requires some reflection. It is well established that the instantaneous pole is in motion relative to a network of observing stations whose positions are considered to be fixed. This latter assumption is acceptable for short period studies as the magnitudes of the changes in the position of the pole are two orders of magnitude greater than any likely changes in station positions. On the other hand, estimates of secular variations in the position of the pole (e.g., Markowitz 1968, p. 26) have a magnitude which is not essentially different from those of movements of continental plates.

The nature of this motion of the pole due to departures of the Earth from a rigid body require a consideration of the Earth's inertia tensors. It is straightforward to show (e.g., Mather 1972, p. 18) that the geocenter should lie on the instantaneous axis of rotation of a rigid body undergoing free rotation, the location of the geocenter being related to the first order inertia tensor. The second order inertia tensor is defined by the equation

$$I_{2ij} = \int\int\int_v \rho \, x_i \, x_j \, dV \tag{4}$$

where the x_i axis system, assumed rectangular Cartesian, is centered at the geocenter G in figure 1, ρ being the density of matter at the element of volume dV whose co-ordinates are x_i. The moment of inertia I_{GP} about any axis GP through G, is defined by the Equation (ibid, p. 16)

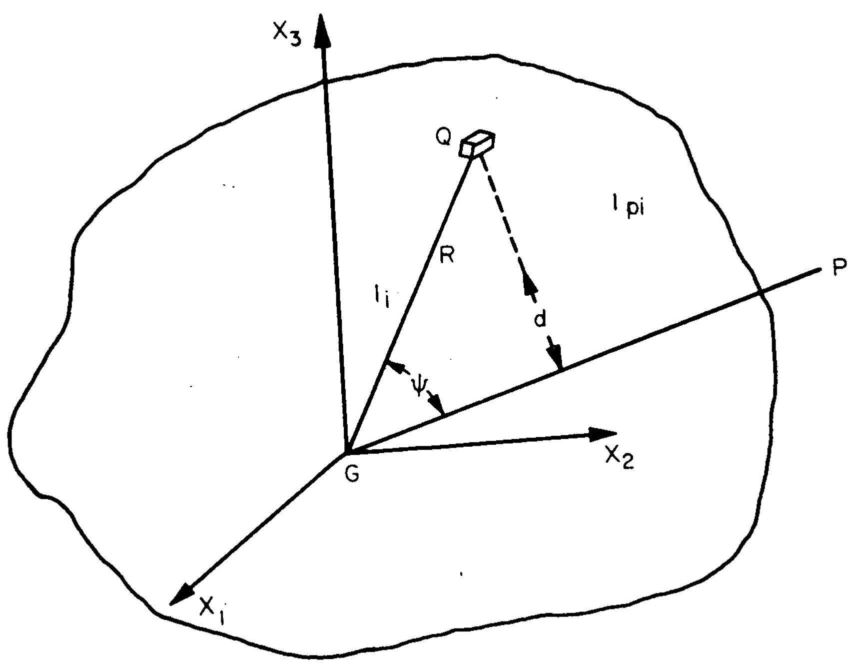


Figure 1. The Second Order Inertia Tensor.

$$I_{GP} = L^T \, I_2 \, L \tag{5}$$

where

$$L^T = |\ell_{p1} \, \ell_{p2} \, \ell_{p3}| \tag{6}$$

and the elements I_{2ij} of I_2 are given by

$$I_{2ij} = \int\int\int_v \rho \, (\delta_{ij} \, R^2 - x_i \, x_j) \, dV \tag{7}$$

the quantities ℓ_{pi} being the direction cosines of GP, R being the distance of the element of volume dV from G, and δ_{ij} being defined by equation 1.

The axes of greatest and least moments of inertia are obtained by differentiating equation 5 with respect to ℓ_{pi} when

$$I_{2ij} \, \ell_{pj} = 0 \tag{8}$$

On defining the principal axes of inertia as those which diagonalize the array I_2, it follows that

$$I_{2ij} = 0 \quad \text{if} \quad i \neq j \tag{9}$$

when non-trivial solutions are obtained if

$$\ell_{pi} = 0.$$

Thus the principal axes of inertia are also the axes of greatest and least moments of inertia when they coincide with the co-ordinate axes.

It has to be decided whether the instantaneous axes of reference should be related to the axis of rotation or to those of greatest and least moments of inertia, knowing full well that the amplitude of the Chandler wobble indicates that any discrepancy is unlikely to exceed 0."5.

The relation between these two sets of axes is commonly investigated by recourse to the techniques of Poinsot (Routh 1905, p. 86 et. seq.). If the principal axes of inertia were adopted as a set of body axes y_i for the instantaneous rigid body model of the Earth, as shown in figure 2, the moment of inertia in any general direction GR can be represented by the equation

$$I_{GR} = I_{2ii}\, \ell^2_{gi} \tag{10}$$

from a consideration of equations 5, 7 and 8. On assuming GR = 1, as a scale for the y_i co-ordinate system, this equation can be written as

$$I_{2ii}\, y_i^2 = I_{GR} \tag{11}$$

which is the equation of an ellipsoid, called the momental ellipsoid.

If the Earth were to rotate about the instantaneous axis of rotation GR, the Poinsot treatment calls for the momental ellipsoid to roll on a plane tangential to it at R, in such a manner that the distance GP from G to the tangent plane along the y_3 axis remains fixed in relation to the latter plane. As P and R do not coincide and GP remains fixed for a rigid body model, the momental ellipsoid

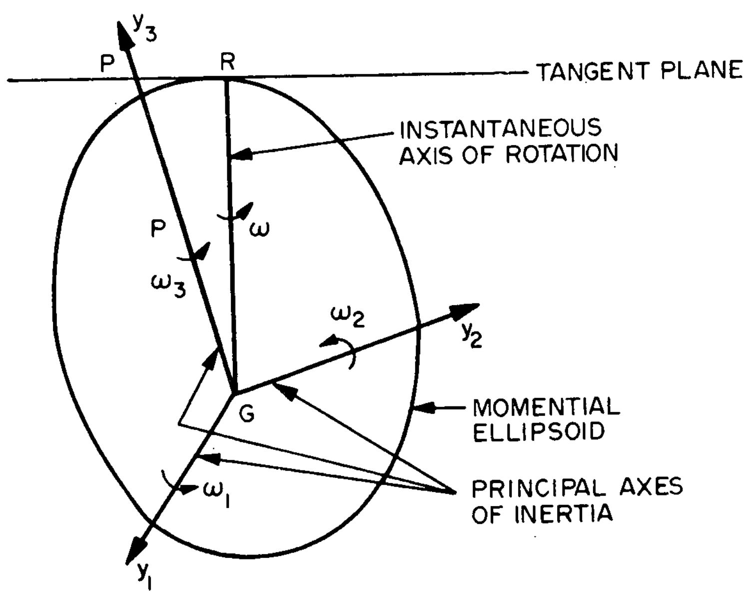


Figure 2. The Instantaneous Axis of Rotation and the Principal Axes of Inertia

rolls on the tangent plane with GP fixed. The locus of points of contact between the plane and the momental ellipsoid on the latter is a conic section, called the polhode, being elliptical about the axes of greatest and least moments of inertia and hyperbolic about any other axis. The smaller the eccentricity of the polhode, the stabler the rotation.

It can be shown that a change in the rate of rotation of the Earth can be represented by a change in the location of the instantaneous axis of rotation on the momental ellipsoid. The new path traced out on the momental ellipsoid is a neighboring polhode, whose stability will depend on the relation between the polhodes concerned and the principal axes of inertia.

Any change in the mass distribution of the Earth will cause a change in the locations of both the principal axes of inertia and the geocenter. The polhode manifests itself, from the point of view of observations, as polar motion, as do changes in the location of the principal axes. As the Earth has stable rotational characteristics, the instantaneous polhode should be nearly circular, changing to a neighboring polhode when either the rate of rotation changes or a mass redistribution were to occur. The correlations pointed out between breaks in the pole path and earthquakes by Smylie & Mansinha (1971) are in accordance with the above concepts.

The principal axes of inertia are of interest in geodesy even though observations are never directly related to them, as they provide an interpretation of the harmonics C_{21} and S_{21} in the spherical harmonic analysis of the external gravitational field of the Earth.

It is therefore possible to draw the following conclusions

(i) The motion of the Earth as detected by observations at any instant of time, can be represented as that of a rigid body freely rotating about an axis passing through the geocenter.

(ii) The location of the principal axes of inertia cannot be accurately related to observations and are unsuited for the purpose of defining an instantaneous reference frame, even though changes in the location of these axes in Earth space due to large scale mass re-distributions will cause the instantaneous axis to seek out the axis of greatest moment of inertia to maintain the stability of the rotational motion. This is a dynamic consequence without direct geodetic interest.

Instantaneous Earth space is therefore best defined in relation to a three dimensional Cartesian system x_i which is centered on the geocenter with the x_3 axis coincident with the instantaneous rotation axis of the rigid body model of the Earth at epoch ($\tau = t$). The instantaneous reference system is completed by defining the x_1 axis in the plane of the meridian of reference, with the x_2 axis completing the rectangular triad.

3. A SYSTEM OF REFERENCE FOR FOUR DIMENSIONAL STUDIES IN EARTH SPACE

3.1 Description

The development in section 2 has indicated that the Earth at any given epoch, can be modeled by an equivalent rigid body which is rotating freely about an instantaneous axis passing through the geocenter. Earth space itself has the same galactic and rotational motion as the Earth. The likely consequences of galactic motion are discussed in (Mather 1972). On taking this into account the desired Euclidian space is obtained by reversing the effect of rotation.

Each of the instantaneous systems so defined in terms of the last paragraph of section 2.3, can be related to the system at some epoch of reference ($\tau = t_0$) which is shown as the inertial axes X_i in figure 3, centered on the location G_0 of the geocenter during this epoch. The instantaneous axes χ_i at some epoch ($\tau = t$) are related to the inertial reference system X_i by

(i) the co-ordinates X_{gi} of the geocenter G at epoch ($\tau = t$); and

(ii) the Euler angles θ_i, as shown in figure 3, which define the rotation of the χ_i axes with respect to the inertial axes X_i.

While the concept itself is straightforward, certain problems of interpretation arise. They are

1. How are observations related to the increments in the Euler angles?

2. How is the motion of the geocenter determined?

These are dealt with in the next sub-sections.

3.2 The Euler Angles and Observations

Changes in the locations of the reference axes over small intervals of time can be represented by the axis systems χ_i which is the instantaneous system at epoch ($\tau = t$) and x_i which is that at epoch ($\tau = t + dt$). If the former is chosen as an inertial frame of reference for the purpose of studying positional changes, and if the geocenter has moved from G to G' whose co-ordinates referred to the

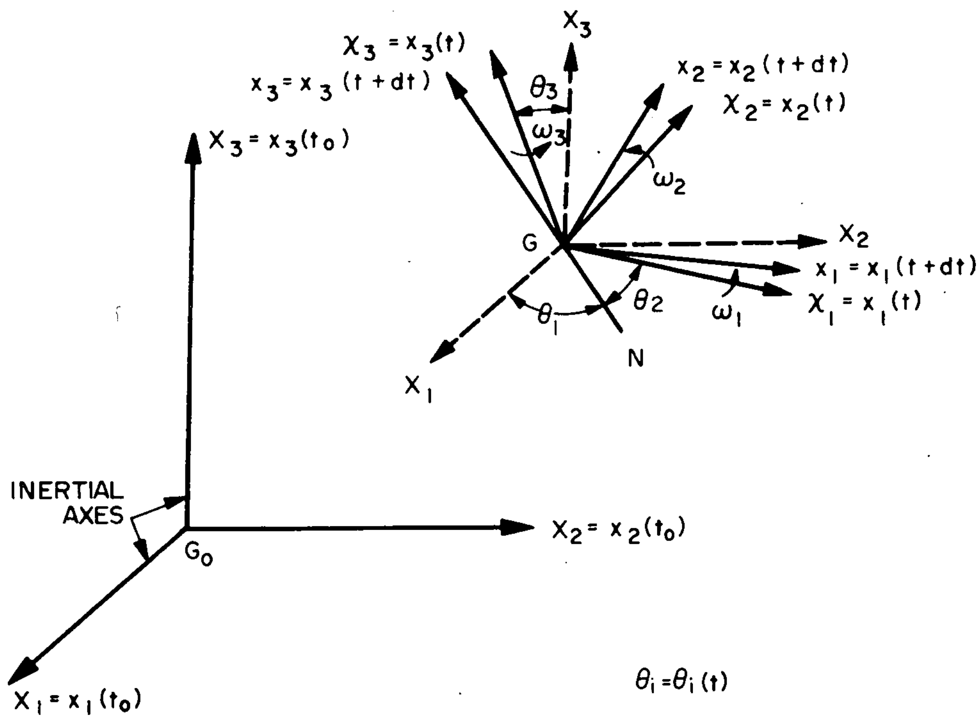


Figure 3. Inertial and Rotating Axes in Euclidian Space

χ_i system is χ_{gi}, the co-ordinates of any point P referred to the local inertial system χ_i are given by (e.g., Mather 1972, p. 23)

$$\chi_i = \chi_{gi} + x_i - \epsilon_{ijk}\, x_j\, \omega_k\, dt \tag{12}$$

where x_i are the co-ordinates of P on the x_i system, ω_i being the components of the rotation of the axes about the χ_i axis system, as shown in figure 3. ϵ_{ijk} is defined by equation 2.

If the diurnal rotational motion of the Earth has been reversed, equation 12 refers to changes in Earth space, ω_i being the rotations about the axes χ_i due to polar motion, the attendant mass re-distributions and continental plate movements.

The purely rotational effects are best considered by ignoring the motion of the geocenter for the time being. These can be related to the changes $d\theta_i$ in the Euler angles θ_i on direct resolution in figure 3. If the angles θ_i were treated as vectors acting along the directions GX_3, $G\chi_3$ and GN respectively, it follows that

$$\Omega \, d\, t = R \, d\, \theta \tag{13}$$

where

$$\Omega^T = |\omega_1 \, \omega_2 \, \omega_3|$$

$$d\,\theta^T = |d\,\theta_1 \; d\,\theta_2 \; d\,\theta_3| \tag{14}$$

and

$$R = \begin{vmatrix} \sin\theta \cos\theta & 0 & \cos\theta \\ \sin\theta \cos\theta & 0 & -\sin\theta \\ \cos\theta & 1 & 0 \end{vmatrix} \tag{16}$$

Equations 13 to 16 define changes $d\theta_i$ in the Euler angles θ_i, in terms of the quantities ω_i dt. These, in turn, are the angles which define the rotation of the instantaneous axes which were specified in section 3.1. This purely rotational effect is a consequence of the change in the orientation of the instantaneous axis of rotation in Earth space, and that of the reference meridian which defines the instantaneous x_1x_3 plane. At any instant, the former is defined by the locations of the geocenter, the pole and the "reference meridian." The latter should be a plane containing both the rotation axis and the geocenter. Thus it is necessary to define just a single point to complete the description of the reference meridian and hence the instantaneous system of reference.

This third point could be either an identifiable location of the surface of the Earth, when the quantity ω_3 dt is the correctly interpreted component of any continental movement of the plate containing the point of reference in a direction parallel to the equator; or the representation of this variation could be purely conceptual, being based on the global analysis of changes in position in this direction, as obtained from a network adjustment on a world-wide basis.

The quantities ω_1 and ω_2 are evaluated from equation 12, on neglecting the motion of the geocenter using the observational data for polar motion which is conventionally expressed in the form given in equation 1.

3.3 Definition of the Motion of the Geocenter

The term quasi-geocentric will be used in the present development to describe those systems whose center is near the geocenter but not made to coincide with it by design. A quasi-geocentric system does not satisfy equation 12 as its origin is not at the goecenter. This would apply to all systems of geodetic control which are established by purely geometric means, a differentiation existing from those whose origin is based on a consideration of the gravitational field on the assumption that this could be achieved with sufficient precision.

The geocenter can be related to a geodetic system by comparing gravimetric determinations of the separation vector with those determined by astro-geodetic methods, at a network of geodetic stations which provide a well spaced control at the surface of the Earth. The existing regional geodetic networks could be linked to form such a scheme with an accuracy estimated in ±6 m in each co-ordinate at the present time (Mather 1971).

Another system affording similar accuracy at the present time is one based on the fact that the orbit of both an artificial Earth satellite and the moon are directly related to the geocenter. If the co-ordinates of two or three events per orbital arc are determinable with respect to the tracking stations, whose quasi-geocentric co-ordinates are known, this information could be used to determine the relation between the center of the quasi-geocentric system and the geocenter (Koch and Schmid 1970). The accuracy estimate quoted is based on data obtained by the use of optical techniques. Laser ranging techniques should result in a significant reduction in this error estimate as it is likely that the location of the geocenter can be recovered with a precision equivalent to that in defining the tracking station co-ordinates.

It can be concluded with confidence that the accuracies from both techniques can be improved. In the case of the gravimetric method, the improvements would be as a consequence of a better definition of the Earth's gravitational field due to an increased density of determinations at the surface of the Earth. It is important that these two methods be used in a complementary manner in defining the geocentric orientation vector for any quasi-geocentric system. This will not be possible unless techniques are developed for the transfer of the direction of the vertical onto an extra-terrestrial reference frame with a resolution in the vicinity of 0."001. As this may not be possible in the foreseeable future, it would appear that the best procedure is to define the geocentric orientation vector for a quasi-geocentric system of laser ranging stations evenly distributed over the Earth and use the gravimetric determination whose accuracy is restricted due to limits on the accuracy of separation vectors determined astro-geodetically, as an independent check.

The precision so obtained would be at least an order better than that achieved at present and hopefully two orders better. This would provide the definition of the geocenter at the epoch of reference ($\tau = t_0$).

Techniques for defining the motion of the geocenter can also be defined by both the laser tracking of satellites and gravimetric means. In the former case, the repetition of the determinations at a later epoch and analysis on lines similar to principle to those developed below, would define the motion of the geocenter. The use of gravimetric methods is more complex.

Re-determinations of the Earth's gravitational field to provide the requisite order of accuracy are a fearsome task if only direct extrapolation of present day technology set the limits of portable techniques, and if solutions of the boundary value problem were resorted to. A much simpler system is based on the contention that a portable device for measuring absolute gravity with a precision of ±1 μgal is well within the scope of present day meterology. It is also assumed that a year's field program would provide sufficient data for meaningful studies over an area the size of the United States.

The system calls for the establishment of a global net of geodetic observatories whose Earth space positions are established and maintained using the techniques described above and with connections to global networks established by VLBI. A regional network of ancillary stations consisting of stable observing platforms is established to provide a uniform coverage of the local area (e.g., the United States). This is vital if it is hoped to meaningfully interpret changes of a few cm per decade on a global basis. The present development will only deal with the recovery of the motion of the geocenter as a result of changes in the value of observed gravity at these specially constructed ancillary observing platforms, assuming that compatibility of gravity units existed globally. The time spans involved are decades, not centuries.

The gravity anomaly Δg at epoch ($\tau = t$) is obtained by referring the value of observed gravity to a reference ellipsoid which is geocentric. The geocentricity is implicit in the conventional solution of the boundary value problem. The relation of the global geodetic control network to this reference figure determines the value of normal gravity γ_i which is differenced from observed gravity g_i at the point P_i to give the gravity anomaly Δg_i, on allowing for the appropriate reduction c_i. Thus

$$(\Delta g_i)_t = g_i + c_i - \gamma_i \tag{17}$$

If the value of observed gravity at this same platform at epoch (τ = t = dt) were re-observed as ($g_i + \delta g_i$), the gravity anomaly at the same point referred to the same ellipsoid, on allowing for rotational variations between epochs, and retaining its center at the location of the geocenter for epoch (τ = t) is given by

$$(\Delta g_i)_{t+dt} = g_i + \delta g_i + c_i - \gamma_i = (\Delta g_i)_t + \delta g_i \tag{18}$$

The change in the value of observed gravity and hence in the gravity anomaly, as referred to the geocentric reference ellipsoid for the epoch (τ = t) is due to the following causes.

1. Changes in g due to lateral motion. This is affected only by motions involving changes in latitude, when a change of 1 m will give rise to a change of 1 μ gal in g. As both the ranging systems and VLBI are likely to be at least an order more sensitive to such changes, it is preferable to remove such effects before the analysis of residual variations.

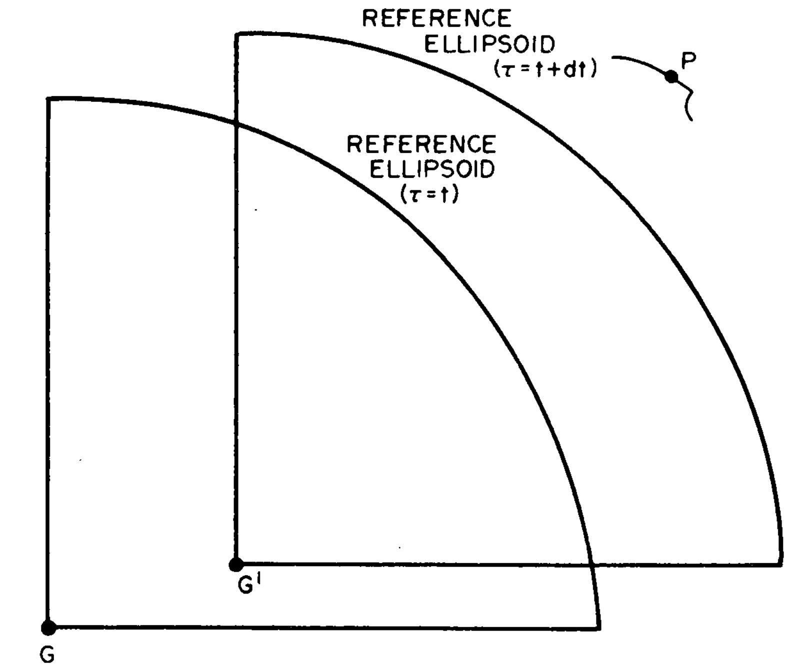
2. Other changes of significance can be interpreted on the global harmonic analysis of δg_i.

   (a) The term of zero degree can be interpreted as either a decrease in the gravitational constant as postulated by the scalar-tensor theory of gravitation (Brans & Dicke 1961) or an increase in the size of the Earth (Jordan 1969).

   (b) The terms of first degree could be eliminated by moving the center of the reference ellipsoid from the Earth space locations of the geocenter G at epoch (τ = t) to that G' at epoch (τ = t = dt), as illustrated in figure 4. If the instantaneous frame of reference at epoch (τ = t) is χ_i as specified in section 3.1, let the co-ordinates of G' be χ_{gi} with respect to these axes.

   The resulting change $\delta \gamma_i$ in normal gravity on shifting this position of the reference ellipsoid without change of basic mass characteristics, at any point on the surface of the Earth, can be related to the change δh_{si} in the ellipsoidal elevation h_{si} as a consequence of the shift χ_{gi} in the location of the center. δh_{si} is given by

$$\delta h_{si} = \ell_{ij} \chi_{gj} \tag{19}$$



G = LOCATION OF GEOCENTER AT EPOCH (τ = t)
G' = LOCATION OF GEOCENTER AT EPOCH (τ = t + dt)

Figure 4. The Motion of the Geocenter

where ℓ_{ij} are the direction cosines of the ellipsoid normal at the observing platform P_i at the surface of the Earth, as defined by its geodetic co-ordinates. The equivalent effect $\delta\gamma_i$ on normal gravity is due to

(i) the change in position of the center of the reference ellipsoid and hence its center of mass with respect to P_i and

(ii) the gravitational effects of the difference in the distribution of matter in relation to the observing platform P_i in Earth space.

It therefore follows that the required observation equation at each observing platform is of the form

$$v_i = \delta g_{1i} - K (\ell_{ij} \chi_{gj}) \tag{20}$$

where δg_{1i} is the contribution of first degree to δg_i at the observing platform P_i , and K is the conversion factor for taking (i) and (ii) above into account. This would be a global constant for all practical purposes. The residuals v_i would allow for ambiguities in the model adopted for the Earth's crust, and observational errors in δg_i.

The solution of equation (20) in the normal manner will give the values of χ_{gi} and hence define the motion of the geocenter on a relative basis:

(c) Similar interpretations could be given to the existence of magnitudes of the various coefficients of degree two, in terms of changes in the second order inertia tensor of the Earth. This is not considered in the present development as it is not of direct relevance in the definition of Earth space.

(d) The existence of harmonic terms of other degrees are due to variations in g due to mass re-distributions and once again, of no direct consequence in the definition of Earth space.

The main concern, from a practical standpoint, is the accuracy with which χ_{gi} can be determined when observations are restricted to continental areas due to the requirements governing the stability of the observing platforms. As the solution proposed is solely concerned with the determination of χ_{gi} through the terms comprising the first degree harmonic as obtained on the global analysis of changes in observed gravity, no serious concern is caused by the bias of stations toward land areas except from the standpoint of data distribution for the purpose of analysis. It is still desirable that regions like the Pacific be represented by a well spaced net of ancillary stations for the purpose of this analysis as in the first instance, it could provide contrasting spectra of variations in g for analytical comparisons with results obtained for continental regions.

It is envisaged that absolute gravity will be re-determined at points comprising a network which would afford a uniform coverage of the globe. Those stations which are not permanent observatories will be occupied intermittently in conjunction with mobile ranging units to effect the densification necessary for extracting reliable results. It can be anticipated that $\delta g_{1i} \not> 10$ μgal over a decade, calling for an analytical accuracy of 1 part in 10^2. There should be no objections therefore to using spherical harmonic methods in carrying out this proposed analysis, provided an acceptable distribution of data of adequate accuracy were available.

3.4 A Complete Definition of Earth Space

Space considerations of the Euclidian domain which has the same galactic and rotational motion as the Earth, are uniquely defined in four dimensions by the concept of an instantaneous Earth space which is the Euclidian space having the same rotational and galactic motion as the Earth at the epoch considered. Observations in such a space can be referred without ambiguity to a three dimensional Cartesian system centered on the geocenter G such that the x_3 axis coincides with the axis of rotation, the x_1 and x_2 axes which complete the frame, lying in the plane of the reference meridian and at right angles to it. The meridian of reference could either pass through some point on the Earth's surface which is a station in the global control net or it could be a location not included in the latter. In either case, the meridian of reference is defined in terms of a specific point.

All physical concepts are instantaneous in the above definition. Such a procedure is warranted as the Earth can be represented at any instant of time by a rigid body model, freely rotating about an instantaneous axis of rotation passing through the geocenter.

The variations in position of the instantaneous reference frame with time can be represented by reference to an inertial system afforded by the position occupied by the frame at some epoch of reference ($\tau = t_0$) and designated as the inertial reference axes X_i. The definition is effected by means of

(i) the translational motion of the geocenter, represented by X_{gi}; and

(ii) the Euler angles θ_i defining the rotational displacement of the instantaneous axes x_i with respect to the reference axes X_i.

The inertial co-ordinates X_i of a point P at epoch ($\tau = t$) are related to the values x_i, obtained from observations, and referred to the instantaneous axes x_i, as illustrated in figure 3, by the relations

$$X_i\ (t) = X_{gi}\ (t) + \alpha_{ij}\ x_i\ (t) \tag{21}$$

where the elements α_{ij} are obtained from figure 3 by direct resolution as those of the array

$$\begin{vmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\cos\theta_3 & \sin\theta\sin\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_3 \\ \sin\theta_2\sin\theta_3 & \cos\theta_2\sin\theta_3 & \cos\theta_3 \end{vmatrix} \tag{22}$$

Notes:

1. The motion due to mass re-distributions is initially determined from observations in the instantaneous frame x_i, thus providing a complete system of reference for the variations in position of points at the surface of the Earth.

2. Geodetic observations should be planned so that the motion of the geocenter and that of the reference point defining the meridian of reference are capable of determination, along with polar motion.

3. The metric of an Euclidian space is not complex, but the scale could be a problem in long period studies. This is discussed further in section 4.

4. The system can be shown (Mather 1972, p. 25) to be capable of direct relation to the Louiville equation which is of fundamental importance in geodynamic studies (Munk and MacDonald 1960, p. 9). The components L_i of the torque for the system described in section 3.1, can be expressed in terms of the rate of change of angular momentum H_i by the relation

$$L_i = \frac{d H_i}{dt} + \epsilon_{ijk}\, \omega_j\, H_k + \frac{d}{dt}\,(M\, \epsilon_{ijk}\, x_{gj}\, v_k) \tag{23}$$

where

$$H_i = I_{2ij}\, \omega_j + \iiint_v \rho\, \epsilon_{ijk}\, x_j\, u_k\, dV \tag{24}$$

all quantities on the right being evaluated with respect to the instantaneous axis system x_i; u_i being the components of the velocity of the element of volume dV with respect to the origin, v_i being the velocity of the geocenter G' with respect to the origin, its co-ordinates being x_{gi}. The last term vanishes if the geocenter coincides with the origin of the instantaneous system. The other terms have been defined earlier and have been summarized in section 1.2.

4. DISCUSSION

4.1 The Scale for Earth Space

Whilst the development in the previous section has provided a consistent basis for the definition of the axes of reference for geodetic observations, and which can be related to physical characteristics of the observation space, it has not looked into the problem of the space metric and its definition over long periods of time. The metric tensor for Euclidian space is the simplest in concept. A more complex problem arises in defining the scale of Earth space.

No complication arises when defining the scale for the instantaneous system of reference described in section 2. All distances are deduced from techniques which come under the classification of "Electronic Distance Measurement," unless the classical technique of baseline measurement is resorted to. The former techniques are based on deducing the distance from the measurement of the time lapse for a frequency controlled signal to cover the return distance over the range concerned. The great accuracy of microwave frequency standards, estimated at 1 part in 10^{12} for cesium standards and 1 part in 10^{14} for that provided by a hydrogen maser, makes the measurement intervals of time the most significant contributor to enhanced geodetic accuracy.

The distance itself is deduced from the value of the velocity of light in vacuo whose relative uncertainty, estimated till recently as 1 part in 600,000, was adjudged to be a scale factor of the space, and therefore posed no limitations on the accuracy of ranging techniques. Any determination of the velocity of light will have to be in terms of an independently measured distance which, in turn, is related to the velocity of light through the adoption of the Krypton-86 standard for length. It therefore follows that the scale of the instantaneous space is defined by the velocity of light.

The extension of these concepts to four dimensions poses certain problems. This is not as a consequence of any correlation that may exist between scale and the determination of the Earth's gravitational field as the values of observed gravity will themselves be scaled by the velocity of light, as a consequence of the meteorological techniques involved. The problem is essentially the consequence of galactic motion and effects of Mach's principle. It is illustrated by Shapiro's experiment which involved the reflection of radar transmissions off the planet Mercury when solar occultation studies indicated delays of travel time of up to 125 μsec (Shapiro 1969). In other words, if it were possible to produce some absolute invariant standard of length, would the velocity of light be invariant with time? If this were not so, it would follow that X_{gi} in equation (21) would be made up of several increments each of which had a different "unit of length." The angles θ_i would themselves be fictitious in any absolute sense, being computed as increments each of which is based on this variable scale.

It is much simpler from a geodetic point of view to adopt the relativist's contention that the velocity of light is invariant, in preference to some physically impossible "absolute" standard of length.

4.2 Gravitation and Earth Space

Solutions of the boundary value problem are based on Newtonian gravitation. As pointed out in the previous sub-section, the most accurate determination of g is based on the acceleration/deceleration of falling/rising bodies, the system being scaled by length and time standards implied from the velocity of light. Thus the frame of reference for measuring gravity is consistent with that defining the scale of Earth space.

The linearization of the basic boundary relation in physical geodesy is achieved by introducing the concept of a reference system with gravitational characteristics approximating those of the Earth. The convention adopted at present calls for the use of a reference ellipsoid with a bounding equipotential, whose rotational characteristics are equivalent to those of the Earth. It must be assumed that this model will be revised as and when necessary to take into account any changes of significance in the Earth's rotation rate. The nature of the mass distribution is not specified except that of symmetry about the rotation axis and the equator. The effect of the shortening of the length of day by 1 msec affects the model by approximately 1 part in 10^{10} per century. This is of the same order of magnitude as that due to the secular decrease of the gravitational constant G as predicted by a scalar-tensor theory of gravitation (Dicke 1967, p. 68). The effect manifests itself as part of the term of zero degree obtained on the global analysis of the changes δg in observed gravity as explained in section 3.3. The Earth space effect of this zero degree term will therefore have to be analyzed in conjunction with the results of ranges used to define the global system and filtered out of space considerations.

4.3 A Note on "Motion" in the Four Dimensional Representation of Earth Space

The system of reference elaborated on in section 3 is a four dimensional one which has been represented in three dimensions, in terms of position on an inertial frame. The instantaneous axes of reference are based on physical characteristics which may be identified with different mass elements in Earth space at different epochs. The exception is the point at the Earth's surface which defines the reference meridian. To labor the point, none of the points concerned, with this exception, can be "monumented" as a means of reference, their positions being implied for observations.

The studies of polar motion have already familiarized geodesists with this concept. Thus, any "motion" of the geocenter is not that of a particle of matter but of physical characteristic. In addition, the definition of $\dot{X}_{gi}$ and $\dot{\theta}_i$ cause a problem as they are based on the scale defined by the velocity of light in the epoch of observation. This in turn, implies the possibility of a system of units which is time dependent and thereby destroys the conventional Euclidian character of the inertial space to which the instantaneous frame is referred.

Earth space, in the strictest sense, is therefore an instantaneous concept. Its representation in four dimensions as described in section 3, departs from these Euclidian concepts due to the possibility of a variable and indeterminate scale which has a variation which cannot be quantified. This problem is circumvented by defining the scale in terms of the velocity of light and treating the resulting inertial representation as Euclidian. Thus X_{gi} and θ_i are apparent quantities, any variations in the velocity of light from some "absolute" standard being absorbed in their magnitudes.

5. CONCLUSIONS

Earth space, which is described as the Euclidian space having the same rotational and galactic motion as the Earth, can be uniquely defined at any epoch ($\tau = t$) with reference to a three dimensional Cartesian co-ordinate system x_i, centered on the Earth's center of mass or geocenter, with the x_3 axis coincident with the axis of rotation, the x_1 axis lying in the plane of the meridian of reference, noting that all physical characteristics are based on an <u>instantaneous rigid body model of the Earth.</u>

The space is scaled by the velocity of light which is used for the definition of both time and distance and therefore, for the framework in which absolute gravity is defined.

The instantaneous system has the advantage of being directly related to observations and easy definition. The fourth dimension is introduced by relating these frames of reference x_i at epoch ($\tau = t$) to that (X_i) at some epoch of reference ($\tau = t_0$), on disregarding galactic motion, which is essentially indeterminate in the current frame of reference for observations.

The use of such a system of reference is essential for the unambiguous interpretation of the results of studies into the nature of secular variations in position. These investigations should be based on

(a) A definitive determination of the global reference network in relation to the geocenter at some epoch ($\tau = t_0$); and

(b) the determination of differential changes in both the reference networks and the frames of reference at subsequent epochs ($\tau = t$) in terms of the instantaneous frame of reference defined above.

Possible changes in the velocity of light in relation to some "absolute" space metric, and in the gravitational constant will manifest themselves as apparent changes in the co-ordinates of the geocenter and the Euler angles on the one hand,

and in the zero degree term in the global harmonic analysis of changes in absolute gravity, on the other.

Systematic differences between the values obtained for $\dot{X}_{gi}$ and $\dot{\theta}_i$ from the different methods must be anticipated in the first instance. The writer feels that the resolution of these differences will form a major scientific endeavour in the future. Obviously, confidence in the interpretations will be directly proportional to the number of independent solutions available with the requisite precision. It is therefore important to incorporate techniques involving the establishment of the direction of the vertical in Earth space into the overall scheme. This appears to be ineligible at the present time as it is not possible to transfer this direction onto an extra-terrestrial frame of reference with an accuracy on par with that obtainable from laser ranging techniques, determinations of absolute g and VLBI, due to refraction problems. VLBI techniques appear to promise a framework of radio sources which are capable of affording resolution to 0.″001. The development of a technique for transferring changes in the direction of the vertical onto such a system of reference while retaining the accuracy inherent in the framework of reference, is a matter requiring urgent attention.

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