

Lagrangian Description of Warm Plasmas

by

Hongjin Kim

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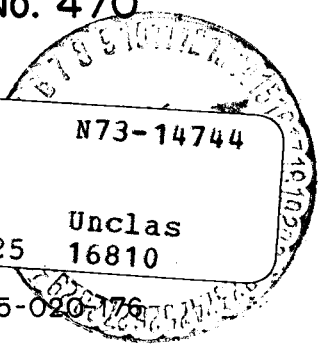
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Institute for Plasma Research
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ABSTRACT

The main aims of this research are to extend the averaged Lagrangian method of describing small signal wave propagation and nonlinear wave interaction, developed by earlier workers for cold plasmas, to the more general conditions of warm collisionless plasmas, and to demonstrate particularly the effectiveness of the method in analyzing wave-wave interactions. The theory is developed for both the microscopic description and the hydrodynamic approximation to plasma behavior. First, a microscopic Lagrangian is formulated rigorously, and expanded in terms of perturbations about equilibrium. Two methods are then described for deriving a hydrodynamic Lagrangian. In the first of these, the Lagrangian is obtained by velocity integration of the exact microscopic Lagrangian. In the second, the expanded hydrodynamic Lagrangian is obtained directly from the expanded microscopic Lagrangian. As applications of the microscopic Lagrangian, the small-signal dispersion relations and the coupled mode equations are derived for all possible waves in a warm, infinite, weakly inhomogeneous magnetoplasma, and their interactions are examined for propagation nearly parallel to the static magnetic field, and exactly perpendicular. As examples of the use of the hydrodynamic Lagrangian, the coupled mode equations are derived for interactions among two electron plasma waves and one ion acoustic wave; among one electron plasma wave, one ion acoustic wave and one ordinary wave, and among two electron plasma waves and one Alfvén wave. The work concludes with a brief discussion of specific cases of wave-wave interaction most likely to be worth subjecting to detailed numerical analysis, and some suggestions for extension of the Lagrangian method to bounded, inhomogeneous, weakly turbulent plasmas of practical interest.

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LIST OF SYMBOLS

<u>Symbol</u>	<u>Page where defined on first use</u>
(1) <u>Latin Alphabet</u>	
\underline{A}	vector potential 18
\underline{A}	vector potential at equilibrium 25
\underline{A}_0	vector potential at equilibrium 24
\underline{A}_k	slowly varying amplitude of \underline{a} of the \underline{k} -wave, related to \underline{a}_k by Equation (4.2) 55
\vec{A}_k	a matrix for wave equation, defined by Equation (3.7) 35
\underline{a}	vector potential of perturbations 24
\underline{a}_k	vector potential amplitude of the \underline{k} -wave, defined by Equation (3.1) 34
\underline{B}	magnetic flux density defined by Equation (2.10) 21
\underline{B}	static magnetic flux density 26
\underline{B}_0	static magnetic flux density 79
\underline{b}	magnetic flux density of perturbations, defined by Equation (2.25) 26
\underline{b}_k	magnetic flux density amplitude of the \underline{k} -wave, defined by Equation (3.16) 37
C_{UVW}	coupling coefficient of interaction among the U, V and W-modes, defined by Equation (4.14) 60
c	speed of light 39
\underline{E}	electric field intensity defined by Equation (2.10) 21
\underline{e}_k^U	slowly varying amplitude of \underline{e} of the U-mode, related to \underline{e}_k by Equation (4.15) 60
\underline{e}	electric field intensity of perturbations, defined by Equation (2.25) 26

LIST OF SYMBOLS (Contd.)

<u>Symbol</u>	<u>Page where defined on first use</u>
\underline{e}_k	electric field intensity amplitude of the \underline{k} -wave, defined by Equation (3.16) 37
\underline{e}_k^U	the \underline{e}_k of the U-mode, defined by Equation (4.15) 60
F	electron velocity distribution function in the Lagrangian coordinate system 20
\underline{F}	force exerted on an electron 20
\underline{F}	force exerted on an electron by static fields . 26
f	electron velocity distribution in the Eulerian coordinate system 21
f_0	electron velocity distribution at equilibrium . 22
f_1	first order electron velocity distribution function 29
\underline{f}	force exerted on an electron by time varying fields 26
\overline{G}_2	averaged generalized momentum density of second order in perturbation 40
$\underline{G}_k^{(2)}$	generalized momentum density amplitude of the \underline{k} -wave 40
\overleftrightarrow{G}_k	a matrix of a function of ω_k and \underline{k} , defined by Equation (3.37) 42
\overline{H}_2	averaged Hamiltonian density of second order in perturbation 39
$H_k^{(2)}$	Hamiltonian density amplitude of the \underline{k} -wave . . 39
I_n	modified Bessel's function 53
I_k^U	a function of ω_k and \underline{k} , defined by Equation (4.15) 60
i	$(-1)^{1/2}$ 34
J_n	Bessel's function 43
J	Jacobian 27
\underline{j}	first order current density 29

LIST OF SYMBOLS (Contd.)

Symbol	Page where defined on first use
\underline{j}_k	first order current density amplitude of the \underline{k} -wave 39
k_D	(Debye length) ⁻¹ 94
\underline{k}	wave vector 34
L	total Lagrangian of the system 20
L_n	total Lagrangian of the system of the n th order in perturbation
\mathcal{L}	Lagrangian density 77
\mathcal{L}_n	Lagrangian density of n th order in perturbation 80
\mathcal{L}^F	contribution of the fields to the Lagrangian density 20
\mathcal{L}^P	contribution of an electron to the Lagrangian density 20
\mathcal{L}^{Pe}	contribution of an electron to the Lagrangian density 88
\mathcal{L}^{Pi}	contribution of an ion to the Lagrangian density 88
\mathcal{L}_n^F	the \mathcal{L}^F of n th order in perturbation 25
\mathcal{L}_n^P	the \mathcal{L}^P of n th order in perturbation 25
\mathcal{L}_σ^P	contribution of an electron in an infinitesi- mal cell σ to the Lagrangian density 18
$\mathcal{L}_k^{(2)}$	contribution of the \underline{k} -wave to $\overline{\mathcal{L}}_2$ 35
$\mathcal{L}_k^{P(2)}$	contribution of the \underline{k} -wave to $\overline{\mathcal{L}}_2^P$ 35
\overleftrightarrow{M}_k	a matrix of a function of ω_k and \underline{k} , defined by Equation (3.35) 41

LIST OF SYMBOLS (Contd.)

Symbol	Page where defined on first use
m	mass of an electron 18
N_k	action of the k-wave defined by Equation (3.14) 36
\underline{n}	refractive index vector 39
n	electron density 76
n_0	electron density at equilibrium 53
$\underline{\underline{P}}$	pressure tensor defined by Equation (5.4) . . . 77
$P_{\mu, \nu}^n$	a polynomial defined by Equation (3.38) . . . 43
$P_{0, \mu, \nu}^n$	a polynomial defined by Equation (3.50) . . . 49
$Q_{kk'k''}^L, Q_{kk'k''}^R, Q_{kk'k''}^Z$	
$Q_{kk'k''}^O, Q_{kk'k''}^{\parallel}, Q_{kk'k''}^{(\alpha)}$	
$Q_{kk'k''}^{LL}, Q_{kk'k''}^{RR}, Q_{kk'k''}^{LZ}$	
$Q_{kk'k''}^{RZ}, Q_{kk'}^{x(\alpha)}, Q_{kk'}^{z(\alpha)}$	
	Coefficient in the averaged third order Lagrangians, defined by Eqs. (4.11), (4.27), (5.48) and (5.81) 58, 64, 93 and 108
\underline{r}	Eulerian position vector 20
$\underline{\underline{r}}$	position vector in equilibrium 25
\underline{r}_0	position vector in equilibrium 22
\underline{r}'	Lagrangian position vector 20
s	a diagonal element of the matrix $\underline{\underline{s}}$ of an isotropic plasma, defined by Equation (5.22). . 83
s_{\perp}, s_{\parallel}	diagonal elements of the matrix $\underline{\underline{s}}$, defined by Equation (5.11) 79

LIST OF SYMBOLS (Contd.)

<u>Symbol</u>	<u>Page where defined on first use</u>
\underline{s}	a matrix defined by Equation (5.5) 76
\underline{s}_n	the \underline{s} of n th order in perturbation 78
$T, T_{\perp}, T_{\parallel}$	electron temperature 80
U	drift velocity in equilibrium 78
\underline{U}	a unitary matrix 41
u	drift velocity 76
v	electron velocity, or Eulerian velocity coordinates 20
\underline{v}	electron velocity in equilibrium 25
v'	Lagrangian velocity coordinates 20
v_0	electron velocity in equilibrium 22
v_{σ}	velocity of an infinitesimal cell σ 18
v_k	group velocity defined by Equation (3.13) 37
v_k^U	group velocity of the U-mode defined by Equation (4.14) 60
\underline{w}	random velocity in equilibrium defined by Equation (5.27) 85
\underline{w}	random velocity defined by Equation (5.5) 77

(2) Greek Alphabet

α_k, β_k	correction factor accounting for the nonlinear effect 55
Γ	$\gamma-1$ 126
γ	specific heat ratio 110
$\hat{\gamma}$	nonlinear growth (or decay) rate defined by Equation (4.35) 67

LIST OF SYMBOLS (Contd.)

Symbol	Page where defined on first use
ϵ	a small parameter 54
ϵ_0	permittivity in free space 20
$\epsilon_{\alpha\beta\gamma}$	skew symmetric tensor in Cartesian-coordinate 87
ζ	a function of $\underline{\xi}$ defined by Equation (C.5) 127
θ	phase angle 34
$\theta_{\underline{k}}$	phase angle of the \underline{k} -wave 34
κ	Boltzmann's constant 53
$\vec{\kappa}_{\underline{k}}$	equivalent plasma permittivity tensor 39
$\Lambda_{\underline{k}}^{(2)}$	a component of $\mathfrak{L}_{\underline{k}}^{(2)}$, defined by Equation (4.6) 56
$\Lambda_{\underline{k}}^{F(2)}$,	
$\Lambda_{\underline{k}}^{P(2)}$	components of $\mathfrak{L}_{\underline{k}}^{F(2)}$ and $\mathfrak{L}_{\underline{k}}^{P(2)}$, respectively, defined by Equation (4.8) 57
$\Lambda_{\underline{k}\underline{k}'\underline{k}''}^{(3)}$,	
$\Lambda_{\underline{k}\underline{k}'\underline{k}''}^{P(3)}$	components of $\overline{\mathfrak{L}}_3$, defined by Equation (4.8) 57
$\lambda_{\underline{k}}$	$k_{\perp} v_{\perp} / \omega_c$ 43
$\lambda_0^{(n)}$,	
$\lambda_{\ell,m}^{(n)}$	components of $\Lambda_{\underline{k}}^{(2)}$ and $\Lambda_{\underline{k}\underline{k}'\underline{k}''}^{(3)}$, respectively, defined by Equations (5.33) and (5.34) 86
μ_0	permeability of free space 20
$\overline{H}_{\underline{k}}$	slowly varying amplitude of $\underline{\xi}$ of the \underline{k} -wave, related to $\underline{\xi}_{\underline{k}}$ by Equation (4.2) 55
$\underline{\xi}, \dot{\underline{\xi}}$	spatial and velocity displacement vectors from equilibrium, respectively 22
$\underline{\xi}_{\underline{k}}$	the $\underline{\xi}$ due to the \underline{k} -wave 34

LIST OF SYMBOLS (Contd.)

<u>Symbols</u>	<u>Page where defined on first use</u>
π_μ canonical momentum conjugate to Ψ_μ , defined by Equation (3.21)	39
ρ first order charge density	27
Φ scalar potential function in equilibrium . . .	25
Φ_0 scalar potential function in equilibrium . . .	24
φ scalar potential function	18
φ scalar potential function of perturbations . .	25
φ_1 scalar potential function of perturbations . .	24
ϕ azimuthal angle of particle velocity	37
ϕ_k azimuthal angle of the wave vector	37
Ψ dynamical variable	21
Ψ_μ dynamical variable	39
ω_c electron cyclotron frequency	37
ω_k angular-frequency of the k-wave	34
ω_N electron plasma frequency	46
$\omega_{k,\mu}$ $\omega_k - k \frac{v_z}{z} + \mu \omega_c$ in Chapters 3 and 4.	37
$\omega_{k,\mu}$ $\omega_k - k \frac{U_z}{z} + \mu \omega_c$ in Chapter 5	80

(3) Operators

D total time derivative along the equilibrium particle trajectory, defined by Equation (2.19)	24
$D_{k,\mu}$ particle differential operator in ϕ (Fourier transform of D), expressed by Equation (3.17)	37
\bar{D}_k differential operator in velocity space, defined by Equation (3.33)	42
∇_0 $\partial/\partial \underline{r}_0$	24

LIST OF SYMBOLS (Contd.)

<u>Symbols</u>	<u>Page where defined on first use</u>
∇_v $\partial/\partial \underline{v}$	21
∇_{v0} $\partial/\partial \underline{v}_0$	24
∇_W $\partial/\partial \underline{W}$	86
$\underline{\partial}_k$ a matrix of partial differential operators in the \underline{W} -space, defined by Equation (5.62)	99
$\bar{\Psi}$ average of Ψ over time and space	35
$\langle \Psi \rangle$ average of Ψ over velocity space	46

(3) Subscripts

k, k' or k'' quantities of the k, k' or k'' -wave	56
p, s or i quantities of the pump, signal or idler wave	60
L a component of a vector in the coordinate rotating in a left-handed sense about the z-axis	97
R a component of a vector in the coordinate rotating in a right-handed sense about the z-axis	97
Z, z a component of a vector in the z-axis	97
α, β, γ components of a vector in Cartesian coordinates	88
\perp a component of a vector perpendicular to the z-axis	37

(4) Superscripts

A Alfvén wave	107
I ion acoustic wave	95
L a component of a vector in the coordinate rotating in a left-handed sense about the z-axis; also left-hand polarized wave	41

LIST OF SYMBOLS (Contd.)

<u>Symbol</u>	<u>Page where defined on first use</u>
O	ordinary cyclotron harmonic wave or ordinary wave 64
P	electron plasma wave 60
R	a component of a vector in the coordinate rotating in a right-handed sense about the z-axis; also right-hand polarized wave 60
Z, z	z-component of a vector 41
(e), (i)	quantities for electrons and ions, respectively: 89
α, β, γ	components of a vector in Cartesian coordinates 88
(α)	to represent (e) or (i) 92
\parallel, \perp	components parallel and perpendicular to the wave vector, respectively 49
\parallel	longitudinal cyclotron harmonic wave 52

ACKNOWLEDGMENTS

It is a great pleasure to express my gratitude to my research adviser, Professor Frederick W. Crawford, for introducing me to plasma physics, and for stimulating my interest in this research problem and guiding it through to completion. I would especially like to thank him for his encouragement and enthusiastic support which has helped me through many difficult periods of this research.

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Chapter I

INTRODUCTION

1.1 Motivation of the Research

During the last forty years, the theory of plasma waves has developed from the simple analysis of Tonks and Langmuir¹ of electron plasma oscillations to very comprehensive treatments based on kinetic theory. A wide variety of modes have been predicted by linear theory for hot and cold magnetoplasmas, and have been studied experimentally, particularly during the last decade. The stage has now been reached where it may reasonably be stated that, at least for stable modes in homogeneous plasmas, the relevant dispersion relations have been very well verified.²

The linear approximation is, of course, inadequate to describe unstable waves: the predicted exponential growth of small perturbations must ultimately be limited by nonlinear effects. As a result, nonlinear and turbulent processes are found to play an important role in many laboratory and space plasma situations of considerable interest. For example, turbulent heating and anomalous diffusion are vital processes in the heating and confinement of plasmas, and the possible achievement of controlled thermonuclear fusion conditions. These, together with such phenomena as charged particle diffusion out of the van Allen belt, and the dynamics of the earth's bow shock, are highly dependent on nonlinear effects and illustrate the importance of understanding them. A further example is the need to understand propagation characteristics of electromagnetic waves through turbulent plasmas, either from the point

of view of communications or diagnostics. The description of nonlinear effects is therefore one of the central problems in modern plasma physics.

Unfortunately, nonlinear phenomena lead very quickly to formidable theoretical and algebraic complexities.³⁻⁵ Refuge may be sought in computer simulation of the problems, but this can be extremely expensive if complicated situations are to be modeled realistically, and it is easy for the physics to be obscured. There is consequently very strong motivation to elucidate the mechanisms involved with improved analytical tools. What is required is a formalism which will simplify the analysis to the greatest possible degree. Such a formalism is provided by the Lagrangian methods which form the subject of the present work.

1.2 Review of Nonlinear Wave-Wave Interaction

If two waves are propagating as $\exp[i(\omega_1 t - \underline{k}_1 \cdot \underline{r})]$ and $\exp[i(\omega_2 t - \underline{k}_2 \cdot \underline{r})]$, nonlinearity of the plasma will manifest itself in the generation of beat waves propagating as $\exp[i(\omega_3 t - \underline{k}_3 \cdot \underline{r})]$, where $\omega_3 = \omega_1 + \omega_2$ and $\underline{k}_3 = \underline{k}_1 + \underline{k}_2$. If a beat wave happens to be a normal mode of the plasma, synchronism is said to occur, and an exchange of energy and momentum will take place among the three waves. The present work is concentrated on this nonlinear wave-wave interaction process. It is not the only significant nonlinear process; the beat wave can also interact with charged particles moving at its phase velocity ω_3 / k_3 . Nonlinear wave-particle interaction then occurs.⁶

A simple degenerate form of nonlinear wave-wave interaction was explored by Pierce⁷ in connection with the traveling wave tube. In 1956, Manley and Rowe⁸ established an important relationship between the powers at different frequencies flowing into a nonlinear capacitor

which is now known as the Manley-Rowe relation. Its generalization to waves is extremely useful in the analysis of wave-wave interaction because it implies certain relations between the coupling coefficients in the equations describing the wave coupling.⁹ By 1960, amplification due to wave-wave interaction, so-called parametric amplification, had already been studied extensively in the context of microwave devices.¹⁰ Scarcely a paper on parametric amplifiers appeared without mentioning the relevant Manley-Rowe relations. Since then, parametric amplification has been studied in other fields which involve wave interactions in nonlinear media, e.g., nonlinear optics¹¹ and microwave acoustics.¹²

Nonlinear wave-wave interaction in plasmas seems to have been considered first by Sturrock¹³, in 1957. He obtained Manley-Rowe type energy transfer relations among four Langmuir waves, and calculated nonlinear corrections to the frequencies of the waves. Subsequently, Kino¹⁴ analyzed the nonlinear coupling between three waves propagating on a cold plasma column, with and without an infinite static axial magnetic field. Sturrock¹⁵ later showed that the Manley-Rowe relations represent a special case of the action-transfer relations applicable to any system describable by a Hamiltonian. Petschek¹⁶ pointed out the possibility of having instabilities due to nonlinear interactions in systems stable to small perturbations. The nonlinear interaction of longitudinal electron and ion oscillations was studied by Oraevskii and Sagdeev.¹⁷ Following these pioneer works, a large number of papers have appeared. Many of them will be mentioned in succeeding sections.

Depending on the situation of interest, two theoretical approximations have been made when analyzing wave-wave interaction: the coherent

phase approximation and the random phase approximation. The former is based on the assumption that each of the interacting waves is coherent. In plasmas, many waves are available to satisfy the synchronism conditions, and the phases of the waves of interest lose coherence due to wave-wave interaction with other existing waves, wave-particle interaction or other higher order processes. However, if significant energy transfer among the three waves can occur in a time small compared with the time required for changes in phases, the waves may be considered as coherent, and the coherent phase approximation is adequate. On the other hand, if the correlation among the phases of the waves vanishes due to the random phase change before appreciable transfer of energy from one wave to another, then the modulus of the wave amplitudes averaged over the phases is of interest, and the random phase approximation should be used. This situation was first considered by Galeev and Karpman,¹⁸ and has been used extensively in turbulence theory.^{5,19} The present work is treated by the coherent phase approximation.

The most important equations to be obtained in the analysis of wave-wave interaction are the coupled mode equations, from which spatial and temporal variations of amplitudes of the interacting waves can be calculated. The Manley-Rowe relations follow from them, and are useful in understanding certain features of the parametric interaction. It also provide a check on the tedious algebraic manipulations usually required in the derivation of the coupled mode equations. We shall now discuss several of the mathematical methods that have been proposed and used to obtain them.

1.2.1 The Iterative Method

The most direct line of attack on wave-wave interactions is the iterative method. The procedure is to expand up to second order in the perturbation amplitudes the current equation and the equations describing the particle dynamics, i.e., the single particle equation of motion for cold plasmas, the moment equations for the hydrodynamic approximation, or the Vlasov equation for the microscopic treatment of warm plasmas, and to simplify second order terms by the use of small signal relations and the synchronism conditions. Combination of the resulting equations with Maxwell's equations gives the coupled mode equations and the Manley-Rowe relations.

Up to now, the majority of analyses of wave-wave interaction have been done by this method, partly because it is sure and well known from the fields of microwaves and optics, and partly because other methods have not been well developed. Specific cases treated by either the cold plasma or the hydrodynamic approximation are: interaction among three longitudinal waves,^{17,20} among two ordinary waves and one Langmuir wave,²¹ among two ordinary waves and one extraordinary wave,²² among three modified ordinary waves,^{23,24} among three extraordinary waves,²⁵ among two circularly polarized waves and one Langmuir wave,²⁶ and among three right-hand polarized waves.²⁷ The theory has been extended to the microscopic treatment for the specific cases of: interaction among three longitudinal waves,²⁸ among two transverse waves and one ion acoustic wave,²⁹ among two ordinary waves and an extraordinary wave,³⁰ and among three cyclotron harmonic waves.^{31,32} Stenflo³³ has considered generally the interactions of waves propagating parallel to the static

magnetic field, and the extension to waves with their wave vectors slightly oblique to the magnetic field have been investigated by Kim, Harker and Crawford.³⁴

The iterative method has the advantage of being straightforward, and can be applied to systems for which the Lagrangian approach to be described below does not apply, i.e., a Lagrangian function is not derivable. In practice, however, it is extremely tedious. For warm plasmas, especially, the charged particle velocity distribution function has to be expanded to second order in perturbation by solving the differential equations derived from the Vlasov equation, with a driving term containing the first order distribution function. Since the driving term is already complicated, the entire process requires very laborious algebraic manipulation.

1.2.2 The Averaged Lagrangian Method

In this subsection, we shall discuss the averaged Lagrangian method and its application to cold plasmas. The warm plasma case has formed the main topic of our work, and will be discussed separately in Section 1.3. The procedure is as follows.³⁵ First, the plasma perturbation parameters, i.e., the position vectors of particles and the fields, are expanded in terms of a sum of sinusoidal perturbations from equilibrium whose amplitudes, frequencies, and wave vectors are assumed to vary slowly in space and time due to the nonlinearity. The Lagrangian is then expanded in terms of these perturbations, and averaged over space and time so as to remove rapidly varying terms. The Euler-Lagrange equations derived from the zeroth order Lagrangian yield the dynamical equations for the equilibrium state. The first order Lagrangian vanishes.

Variations of the second order Lagrangian with respect to the amplitudes and phases give the small signal equations and the equation of action conservation in each mode. Similar variations of the third order Lagrangian give the coupled mode equations and the Manley-Rowe relations.

For the method to be applicable, it must be possible to set up a Lagrangian density for a given system, i.e., such that application of Hamilton's principle leads to a set of Euler-Lagrange equations corresponding to those chosen to describe the plasma behavior. It is well known that the description of charged particles in an electromagnetic field satisfies this requirement.³⁶ Sturrock³⁷ used Hamilton's principle for electron beams and for cold plasmas to show that the first order Lagrangian vanished to within a total divergence, and that the second order Lagrangian gave the required small signal equations.

The averaging process has been discussed by several authors. Sturrock³⁸ introduced a parameter in which the Lagrangian and the dynamical variables are cyclic, and showed that the time averaged Euler-Lagrange equation obtained from variation of this parameter yielded the action transfer equations. Galloway³⁹ considered the equation for energy conservation averaged in time and space in order to get the wave coupling coefficient. Whitham⁴⁰ studied a Lagrangian averaged over the fast varying waves in a slowly varying medium. The dynamical equations were obtained by variation of the averaged Lagrangian with respect to the wave amplitudes. From the variation in the phase of waves he obtained an adiabatically conserved quantity. Bretherton and Garrett⁴¹ showed that this adiabatically conserved quantity was the wave energy density divided by frequency. This conservation equation is precisely the

action-transfer relation previously obtained by Sturrock³⁸, who had concluded that the Manley-Rowe relations are a special case of the action-transfer relation. Dewar⁴² has since shown rigorously that this conservation relation holds to all orders in ϵ , where the small parameter ϵ is the ratio of the period (or wavelength) of the fast varying wave to that of the slow variation.

The averaged Lagrangian method has been applied by Dougherty⁴³ to obtain equations for ray tracing, to demonstrate conservation of wave action, and to derive nonlinear coupled mode equations in a relativistically covariant form. While Dougherty presented the application of the method in general terms, specific cases have been studied by various authors. Galloway and Crawford⁴⁴ have applied it to the nonlinear interaction of waves at arbitrary propagation angles in an infinite magneto-plasma, and illustrated its effectiveness in obtaining explicit expressions for the coupling coefficients. Dysthe³⁵ not only gave a clear presentation on how the averaged Lagrangian method is applied to nonlinear wave interactions, but also studied the self-action effects and the decay problem of a finite amplitude wave into sidebands by this method. The coupled mode equations for Langmuir and ion acoustic waves have been derived by Vedenov and Rudakov.⁴⁵ Dougherty considered the waves in a cold unmagnetized plasma⁴⁶ as an example of his earlier study.⁴³

It is clear from the work cited so far that the Lagrangian method is very efficient for treating nonlinear wave-wave interaction. Its advantages over the iterative method derive mainly from the following features. First, in the iterative method, the coupled mode equations are obtained as the end-product of tedious algebraic manipulation.

Furthermore, the Manley-Rowe relations are obtained only from the coupled mode equations. In the averaged Lagrangian method, however, general expressions for the coupled mode equations and Manley-Rowe relations appear naturally before specifying an explicit expression for the Lagrangian function for a particular system. Further, the third order Lagrangian itself gives the coupling coefficient, and the expansion of the Lagrangian only needs relatively simple and straightforward algebra.

1.2.3 The Hamiltonian Method

As the Manley-Rowe relations indicate, wave energy appears as an important quantity in nonlinear wave interaction. Therefore, the development of a Hamiltonian method is quite natural. This approach was initiated by Sturrock^{15,47} in his study of the nature of the Manley-Rowe relation attributable to any system which may be described by a Hamiltonian. The application to wave-wave interaction is similar to that of the Lagrangian method. The Hamiltonian generates higher order perturbation terms which are averaged over the fast varying temporal and spatial variations. The canonical equations from the second order Hamiltonian give the small signal equations, and the equations describing the nonlinear effect are obtained from the third or higher order Hamiltonian.

In this method, there are two approaches dependent on the choice of canonical variables: the first uses action-angle variables,⁴⁸ and was initiated by Sturrock.^{15,17} The Euler-Lagrange equation obtained by variation of the Lagrangian with respect to angle is the action transfer relation. The other canonical equation gives the frequency-shift relation. Even though this frequency-shift relation is obtainable by the Lagrangian method,³⁵ it emerges more naturally in the Hamiltonian

method. This approach is analogous to the averaged Lagrangian method, but does not provide the coupled mode equation obtainable by the Lagrangian method. In order to overcome this disadvantage, Harker⁴⁹ has recently developed another approach more closely parallel to the second quantization method (see Section 1.2.4.) than the Lagrangian method. This second approach uses for canonical variables the classical parameters corresponding to the probability amplitudes in quantum mechanics. These parameters and their complex conjugates are then canonical conjugates, and the canonical equations yield the coupled mode equations. He has applied this approach to several illustrative cases: the interactions of two longitudinal waves and one transverse wave, of three longitudinal waves on a positive column, and of three longitudinal waves.

1.2.4 The Second Quantization Approach

Although the concepts of quantum mechanics, in particular the idea of quasiparticles, have been used by Soviet scientists,^{18,45,50-52} it was not until recently that a formal quantum mechanical method was developed for wave-wave or wave-particle interaction problems.⁵³ The procedure consists of expanding the Hamiltonian about equilibrium, and then quantizing the fields⁵⁴ in order to express the Hamiltonian in quantum mechanical form. The transition probability from one state to another can then be obtained by the Fermi golden rule in the time dependent perturbation technique. For a classical plasma, the transition probability is taken to the classical limit.

The method was originally developed by Pines and Schrieffer.⁵⁵ Walters and Harris⁵⁶ used the method to study the nonlinear interactions of longitudinal waves in a magnetized plasma. The static magnetic field

was assumed to be so strong that the electron motion was essentially one-dimensional. It was demonstrated that the classical limit of three-wave interaction among plasma waves yielded an earlier result of Aamodt and Drummond.⁵⁷ Krishan and Selim studied the interactions of transverse waves,⁵⁸ and of transverse and longitudinal waves⁵⁹ in an unmagnetized cold plasma. Three- and four-wave interactions in a cold plasma with no magnetic field have been analyzed by Zakharov.⁶⁰

The advantages of the method are that, if the Hamiltonian is obtained in the quantized quantities, obtaining the transition probability is in principle straightforward. While the classical treatment is usually confronted with the conflicting choice of coordinate systems, i.e., Eulerian coordinates for fields and Lagrangian coordinates for particles, such complications from the coordinate systems do not occur in this method. The classical limiting procedure is sometimes extremely tedious, however.⁶¹ Furthermore, unnecessary quantum mechanical steps have to be followed even for problems which can easily be solved by classical methods. If there are only a few monochromatic waves in a plasma, then the use of the Fermi golden rule to calculate transition probabilities is no longer justified. Therefore, this method is primarily for the case where the random phase approximation has to be made.

1.3 Review of Lagrangian Method for Warm Plasmas

For nonlinear wave interactions in cold plasmas, the efficiency of the averaged Lagrangian method has been demonstrated by many authors. The algebraic manipulation in the iterative method is much more tedious for warm plasmas than for cold plasmas because the Vlasov equation has to be solved to second order in the perturbations. It follows that the

Lagrangian method should be even more advantageous for warm plasmas than for cold plasmas. No comprehensive development of the method has been presented so far for nonlinear wave interactions in warm plasmas. This is the principal aim of the present work.

1.3.1 The Microscopic Treatment

Although a microscopic Lagrangian function for warm collisionless plasmas was first developed by Low more than ten years ago,⁶² only a few authors have made use of it. There are certain obscurities in the original derivation. For example, Low showed that his Lagrangian gives the correct dynamical equations. In order to obtain small signal equations in correct form from the second order Lagrangian, however, he had to impose a consistency condition. Any solution derived from the expanded Lagrangian which would not satisfy this consistency condition was to be considered an incorrect solution. No discussion of the origin of the condition was given.

Low's Lagrangian has been used for several analyses of wave-wave interactions in warm plasmas: Suramlishvili obtained the coupling coefficients for three- and four-wave interactions involving Alfvén, helicon and ion acoustic waves.⁵² Boyd and Turner⁶³ considered the interaction of two ordinary waves and a cyclotron harmonic wave.⁶⁴ Dewar⁶⁵ used Low's Lagrangian to show the important role of nonlinear Landau damping in the modulational instability. The emphasis of these contributions is rather on solving specific cases than providing general theories. Furthermore, they have not discussed Low's consistency condition for the second order Lagrangian, nor for those for orders higher than second. One may therefore question whether any of their solutions are

incorrect. This point has been cleared up by Galloway and Kim,⁶⁶ who have shown that the consistency condition is not necessary.

1.3.2 The MHD Approximation

So far, most applications of the Lagrangian approach to plasmas described by the hydrodynamic equations utilize the MHD approximation, i.e., the plasmas are able to maintain a local isotropic velocity distribution by virtue of an assumed short mean free path for ion-neutral collisions. This model is used mainly to study low frequency phenomena that are usually associated with ion motions; the electrons serve primarily to maintain electrical neutrality. Higher frequency phenomena, such as plasma oscillations, are neglected. An immediate consequence of the restriction to low-frequency phenomena is the simplification that the displacement current term in Maxwell's equations can be dropped. The current due to free electrons is supposed to obey Ohm's law, because of the high electron-ion collision frequency.

The Lagrangian appropriate to the MHD approximation gives only the second moment equation (the momentum transfer equation); other moment equations, Ohm's law, and Maxwell's equations, are used as subsidiary conditions. This Lagrangian has been used by Eckart⁶⁷ who showed that the equations of motion for both incompressible and compressible fluids could be derived from variational principles. Katz⁶⁸ derived the equation for a one-fluid, inviscid, perfectly compressible plasma in an electromagnetic field. Newcomb,⁶⁹ on the other hand, treated a perfectly conducting fluid governed by the conventional hydrodynamic equations, for which the displacement and convection current are negligible under most conditions of interest. Using the minimum potential energy

principle, he studied the stability of steady flow for isotropic plasmas in toroidal and poloidal fields. Similar systems consisting of fluid, vacuum and perfectly conducting solid walls were considered by Lundgren.⁷⁰ Application of this Lagrangian has been extended recently to nonlinear problems by Dewar,⁴² who has studied the nonlinear interactions of hydromagnetic waves.

1.4 Aim of the Research

The foregoing review will serve to demonstrate the need to develop a unified and general theory of the Lagrangian method for nonlinear wave interactions in warm plasmas. This is the main purpose of our work, and theory will be presented for both the microscopic treatment and the hydrodynamic approximation to it.

1.4.1 The Microscopic Treatment

In what follows, we shall discuss first the microscopic treatment. This treatment is parallel to the averaged Lagrangian method developed in cold plasmas. A rigorous theory is presented to show how the method has to be modified in phase space. Results of some illustrative examples are compared to those previously obtained by other methods. In Chapter II, a Lagrangian for a warm plasma is formulated based on the work by Galloway and Kim.⁶⁶ The origin of Low's consistency condition is pointed out, and the lack of necessity for it is proved. This Lagrangian is used in Chapter III for the linear theory to obtain the equation of the particle trajectories and the dispersion relations. By proving the equivalence of the conservation of action and the conservation of energy, the equivalence of the procedures used by Dougherty⁴³ and Dysthe,³⁵ and by Galloway,³⁹ for obtaining the coupling coefficient

is shown. In Chapter IV, nonlinear wave-wave interaction is considered. The averaged Lagrangian method is applied to obtain the action transfer equation and the coupled mode equations. As some illustrative examples, the coupled mode equations are obtained in explicit forms for all possible interactions among waves propagating nearly parallel, and among those propagating exactly perpendicular, to the static magnetic field. These examples serve well to demonstrate the advantages of the method.

1.4.2 The Hydrodynamic Approximation

A plasma is well defined if the three velocity distribution functions for electrons, ions, and neutral molecules are known. However, they are often very difficult to determine. One must then be satisfied with certain mean values describing less perfectly the state of a plasma in the hydrodynamic approximation. This approximation is especially appropriate for a plasma where particle velocities are not comparable to the phase velocity of waves.

Chapter V is devoted to the Lagrangian appropriate to the hydrodynamic approximation. It is formulated by two different methods. The first is by integrating the exact microscopic Lagrangian in velocity space. For the linear and nonlinear theory for small perturbations, the Lagrangian is expanded. The expansion, however, involves some mathematical difficulties. The other method is to obtain the expanded hydrodynamic Lagrangian directly from the expanded microscopic Lagrangian by integrating in velocity space. In this way the difficulties involved in the expansion of the hydrodynamic Lagrangian can be avoided. As applications, interactions among two electron plasma waves and one ion acoustic wave, and among one electron plasma wave, one ordinary wave, and one ion acoustic

are considered by the first method, and that among two electron plasma waves and one Alfvén wave by the second method.

The Lagrangian to be described is more general than that of the MHD approximation mentioned in Section 1.3.2: first, it includes both the moment equation and Maxwell's equations; second, while only the ion motion is considered in the MHD approximation, our Lagrangian includes both electron and ion motions so as to account for both electron and ion waves; third, the MHD Lagrangian can be obtained from ours as the special case of spherical adiabatic compression.

Chapter II

A LAGRANGIAN FORMULATION FOR WARM PLASMAS

The complete set of equations for describing a warm collisionless plasma consists of the equation of motion for a charged particle, Maxwell's equations, and the Vlasov equation. The dispersion properties of plasma waves are determined by combining the expressions for current and charge densities obtained from the Vlasov equation with Maxwell's equations.

An alternative method for describing plasma waves is to use Hamilton's principle. This can be done by formulating the Lagrangian in phase space to include the equation of motion, and Maxwell's equations. In the course of carrying out the variational procedure, a conservation law in phase space (Liouville's theorem) must be imposed on the evolution of the electron distribution. For the collisionless plasma, this is the Vlasov equation.

In Section 2.1, the exact expression for the Lagrangian will be formulated rigorously. It will be proved that this is the desired form by showing that the Euler equations give the equation of motion and Maxwell's equations. In Section 2.2, we shall deal with the Lagrangian for a slightly perturbed system. The required approximate form is obtained by expanding the Lagrangian about the equilibrium state. In Section 2.1, Lagrangian and Eulerian coordinate systems are adopted simultaneously for the particle part and the field part of the Lagrangian, respectively. In order to overcome the difficulties arising from using mixed coordinate systems, a field-like Lagrangian function will be derived in Section 2.2, by making use of displacement vector³⁷ (sometimes called polarization vector⁷¹) coordinates. In Section 2.3,

it is shown that the first order Euler equations, obtained from the second order Lagrangian function, are the desired first order equation of motion and Maxwell's equations. Comments on Low's procedure are made in Section 2.4.

2.1 The Field-Particle Lagrangian

The Lagrangian for a system of electrons in an electromagnetic field consists of three contributions: that of the field, that of the electrons, and that of the interaction between the electrons and the field. In obtaining the last two, a Lagrangian coordinate system has to be adopted, while an Eulerian coordinate system is used for the first. We will first consider the last two parts separately. The total Lagrangian will then be obtained by adding the field contribution to the electron contribution.

The contribution of a single electron at $(\underline{r}_\sigma, \underline{v}_\sigma)$, at time t , including its interaction effects with the field, will be the classical expression for the charged-particle Lagrangian,³⁶

$$\mathcal{L}_\sigma^P = \frac{1}{2} m v_\sigma^2(t) + q[\varphi(\underline{r}_\sigma, t) - \underline{v}_\sigma(t) \cdot \underline{A}(\underline{r}_\sigma, t)] , \quad (2.1)$$

where q is the charge of an electron; φ and \underline{A} are the scalar and vector potentials, and the subscript σ indicates a specific electron. Equation (2.1) holds everywhere along its trajectory.

Now consider an infinitesimal box, cell σ , in phase space. It is considered to contain many electrons in the vicinity of electron σ , and defined to move in time such that no electrons flow in or out, as shown in Figure 2.1. This conservation law implies that the phase volume and the electron density of cell σ remain constant along its

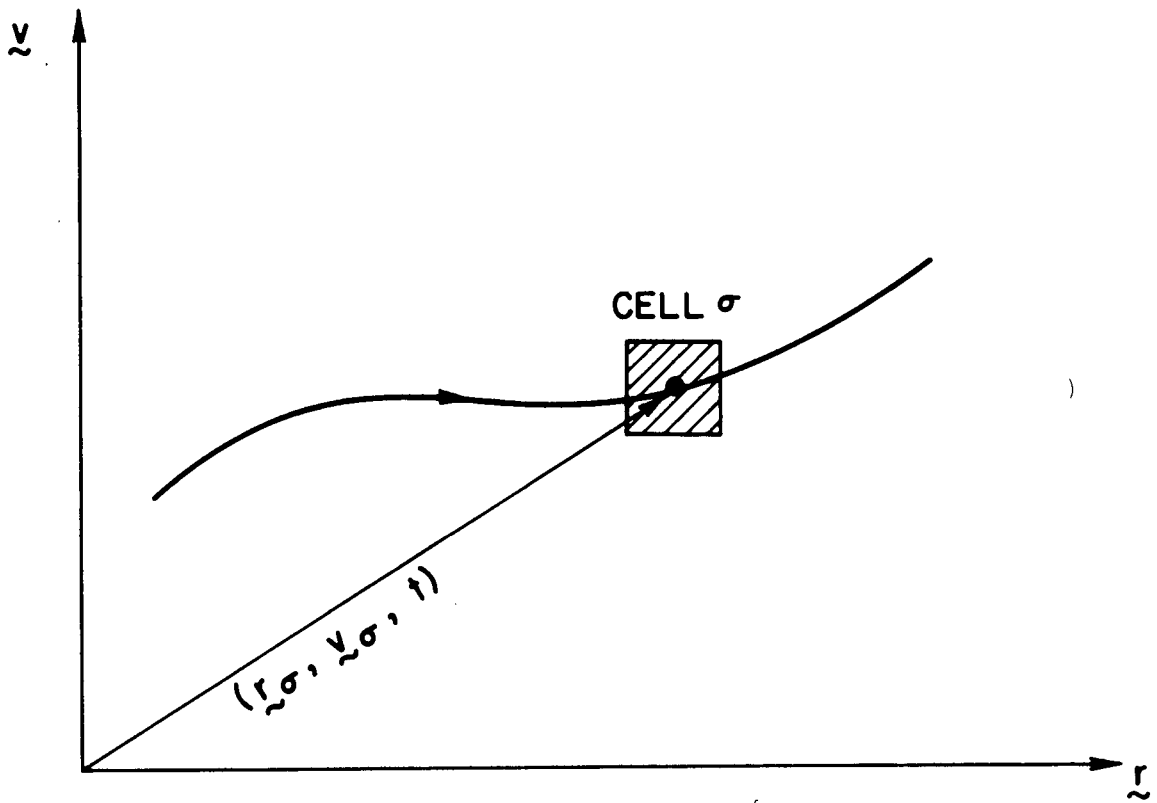


Figure 2.1 Conservation in phase space [the number of electrons, n_σ , inside cell σ is constant along its trajectory].

trajectory. Since the electromagnetic fields can be considered uniform over the cell, the contribution of cell σ to the Lagrangian function may be written as $n_{\sigma} \mathcal{L}_{\sigma}^P$, where n_{σ} represents the number of electrons in the cell, normalized to the total number of electrons in the system. The total contribution of the electrons in the system can be found by summing \mathcal{L}_{σ}^P over all σ . The discrete summation of cells can be transformed to integration by introducing continuous variables for σ , so that $\sum_{\sigma} n_{\sigma} \mathcal{L}_{\sigma}^P \rightarrow \int d^3r' \int d^3v' F(\underline{r}', \underline{v}') \mathcal{L}^P$, where F is a continuous function. The continuous set of variables $(\underline{r}', \underline{v}')$ are known as Lagrangian coordinates, and are usually chosen to indicate the initial position in phase space of cell σ .⁷²

Adding the contribution of the field, \mathcal{L}^F , to that of the electrons, \mathcal{L}^P , we obtain the total Lagrangian,

$$L = \int d^3r' \int d^3v' F(\underline{r}', \underline{v}') \mathcal{L}^P + \int d^3r \mathcal{L}^F, \quad (2.2)$$

where \mathcal{L}^P and \mathcal{L}^L are defined as

$$\mathcal{L}^P = \frac{1}{2} m v^2(\underline{r}', \underline{v}', t) + q[\varphi - \underline{v}(\underline{r}', \underline{v}', t) \cdot \underline{A}], \quad (2.3)$$

$$\mathcal{L}^F = \frac{1}{2} \left[\epsilon_0 \left(\nabla \varphi + \frac{\partial \underline{A}}{\partial t} \right)^2 - \frac{1}{\mu_0} (\nabla \times \underline{A})^2 \right].$$

Here, all quantities for which the independent variables are not explicitly shown in \mathcal{L}^P are to be considered as functions of $\underline{r}', \underline{v}'$ and t , through their dependence on the Eulerian variables \underline{r} , e.g., $\varphi = \varphi[\underline{r}(\underline{r}', \underline{v}', t), t]$.

The quantities to be independently varied are the position of an electron, \underline{r} , and the fields, \underline{A} and φ , all of which satisfy the

usual restrictions on their variation,

$$\delta\psi(t_1) = \delta\psi(t_2) = 0 , \quad (2.4)$$

where t_1 and t_2 are the initial and final times. Before taking variations, it should be noted that the conservation law in phase space requires

$$\frac{dF}{dt} = 0 . \quad (2.5)$$

Some authors^{67,68} have modified the problem by introducing a Lagrange multiplier for this subsidiary condition, but this is not essential.

By letting $F(\underline{r}', \underline{v}') = f(\underline{r}, \underline{v}, t)$, and changing from the variable set $(\underline{r}', \underline{v}')$ to $(\underline{r}, \underline{v})$, we see that Equation (2.5) can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{F}{m} \cdot \nabla_{\underline{v}} f = 0 , \quad (2.6)$$

which is the Vlasov equation.

With the help of Equations (2.4) and (2.6), the Euler-Lagrange equations from \underline{r} , φ , and \underline{A} variations of Equation (22), respectively, yield⁷³

$$m \frac{d\underline{v}}{dt} = -q(\underline{E} + \underline{v} \times \underline{B}) , \quad (2.7)$$

$$\epsilon_0 \nabla \cdot \underline{E} = -q \int d^3v f(\underline{r}, \underline{v}, t) , \quad (2.8)$$

$$\frac{1}{\mu_0} \nabla \times \underline{B} = -q \int d^3v \underline{v} f(\underline{r}, \underline{v}, t) + \epsilon_0 \frac{\partial \underline{E}}{\partial t} , \quad (2.9)$$

where

$$\underline{E} = -\nabla\varphi - \frac{\partial \underline{A}}{\partial t} , \quad \underline{B} = \nabla \times \underline{A} . \quad (2.10)$$

Equation (2.7) describes the motion of an electron in the electromagnetic field. Equations (2.8) - (2.10) constitute the Maxwell equations.

2.2 Approximation to the Lagrangian

In setting up a variational principle for the trajectories of an assembly of electrons, it is quite natural to use Lagrangian coordinates. As one may appreciate from Section 2.1, however, this would involve some difficulties in integrating by parts, since the differential operators are in Eulerian coordinates. Furthermore, the Eulerian coordinate system is overwhelmingly popular in describing plasmas. It is therefore convenient to choose Eulerian coordinates for the independent variables. Fortunately, the conflicting requirements can be satisfied by the introduction of a 'displacement vector'.³⁷

Figure 2.2 shows a cell in phase space at $(\underline{r}, \underline{v})$, at an instant of time t , following a trajectory in a perturbed field. It would be at $(\underline{r}_0, \underline{v}_0)$ at the same instant of time if the perturbation were absent. Let $(\underline{\xi}, \dot{\underline{\xi}})$ be the displacement of the cell due to the perturbation, then

$$\underline{r} = \underline{r}_0 + \underline{\xi}, \quad \underline{v} = \underline{v}_0 + \dot{\underline{\xi}}. \quad (2.11)$$

Let $(\underline{r}_0, \underline{v}_0)$ be fixed for all time. The electrons occupying the cell at $(\underline{r}_0, \underline{v}_0)$ in equilibrium at t are flowing out of the cell as time passes, and another group of electrons occupies the cell at a later time. The cell at $(\underline{r}, \underline{v})$ in the perturbed state, which corresponds to the equilibrium cell at $(\underline{r}_0, \underline{v}_0)$, is then located at a different place from the one at t . However, the new location $(\underline{r}, \underline{v})$ still satisfies Equation (2.11).

From the definition of two corresponding cells, the number of electrons in the equilibrium cell is always the same as that of the perturbed cell. Furthermore, the number of electrons remains constant for all time. It therefore follows that

$$f(\underline{r}, \underline{v}, t) d^3r d^3v = f_0(\underline{r}_0, \underline{v}_0) d^3r_0 d^3v_0, \quad (2.12)$$

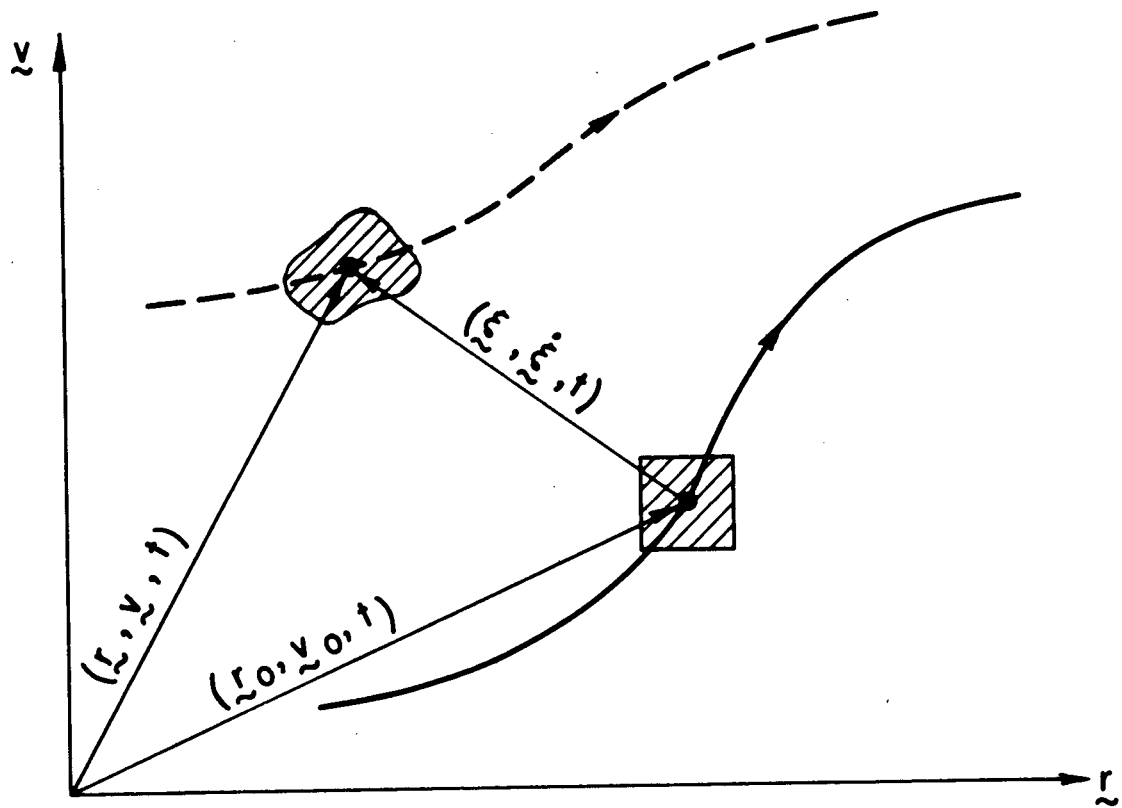


Figure 2.2 Perturbed trajectory in phase space [without perturbation, the cell would follow the solid curve. Its perturbed trajectory is shown dashed].

where f_0 is the electron distribution function in the equilibrium state. But the conservation of density along the perturbed trajectory requires

$$F(\underline{r}', \underline{v}') = f(\underline{r}, \underline{v}, t) , \quad d^3r' d^3v' = d^3r d^3v . \quad (2.13)$$

Separating the fields into equilibrium and time varying quantities as

$$\varphi = \varphi_0 + \varphi_1 , \quad \underline{A} = \underline{A}_0 + \underline{a} , \quad (2.14)$$

and substituting Equations (2.11) - (2.14) into Equation (2.2) gives the Lagrangian

$$L = \int d^3r_0 \int d^3v_0 f_0(\underline{r}_0, \underline{v}_0) \mathcal{L}^P + \int d^3r \mathcal{L}^F , \quad (2.15)$$

where we have

$$\mathcal{L}^P = \frac{1}{2} m(\underline{v}_0 + \dot{\underline{\xi}})^2 + q[\varphi_0 + \varphi_1 - (\underline{v}_0 + \dot{\underline{\xi}}) \cdot (\underline{A}_0 + \underline{a})] , \quad (2.16)$$

$$\mathcal{L}^F = \frac{1}{2} \epsilon_0 \left(\nabla_0 \varphi_0 + \frac{\partial \underline{A}_0}{\partial t} + \nabla_0 \varphi_1 + \frac{\partial \underline{a}}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla_0 \times \underline{A}_0 + \nabla_0 \times \underline{a})^2 . \quad (2.17)$$

The field quantities in Equation (2.16) should be evaluated at $(\underline{r}_0 + \underline{\xi}, \underline{v}_0 + \dot{\underline{\xi}})$, and the subscript 0 on the differential operators represents the $(\underline{r}_0, \underline{v}_0)$ - coordinate system. In Appendix A it is shown that

$$\dot{\underline{\xi}} = D \underline{\xi} , \quad (2.18)$$

where the operator D is defined by

$$D \equiv \frac{\partial}{\partial t} + \underline{v}_0 \cdot \nabla_0 + \underline{a}_0 \cdot \nabla_{v_0} . \quad (2.19)$$

The Lagrangian in Equation (2.15) is now a 'field-like' one, and the independent variables in Equation (2.15) can be considered as Eulerian coordinates.

If the perturbation is small, then the expansion of the Lagrangian in Equation (2.15) generates a series in which L_n , the Lagrangian of n^{th} order in the perturbation, is given by

$$L_n = \int d^3r \int d^3v f_0(\underline{r}, \underline{v}) \mathcal{L}_n^P + \int d^3r \mathcal{L}_n^F. \quad (2.20)$$

Here the subscript 0 on \underline{r} and \underline{v} has been dropped, and, unless specified, the subscripts 0 and 1 will be dropped from variables hereafter up to Chapter V. The first few terms of \mathcal{L}_n^P and \mathcal{L}_n^F are as follows:

Zero Order: For the unperturbed terms we have

$$\begin{aligned} \mathcal{L}_0^P &= \frac{1}{2} m v^2 + q(\Phi - \underline{v} \cdot \underline{A}), \\ \mathcal{L}_0^F &= \frac{\epsilon_0}{2} (\nabla \Phi)^2 - \frac{1}{2\mu_0} (\nabla \times \underline{A}_0)^2. \end{aligned} \quad (2.21)$$

First Order: We have the separate expressions,

$$\begin{aligned} \mathcal{L}_1^P &= m \underline{v} \cdot D \underline{\xi} + q[\varphi + (\underline{\xi} \cdot \nabla) \Phi - \underline{v} \cdot (\underline{\xi} \cdot \nabla) \underline{A} - D \underline{\xi} \cdot \underline{A} - \underline{v} \cdot \underline{a}], \\ \mathcal{L}_1^F &= \epsilon_0 \nabla \Phi \cdot \left(\nabla \varphi + \frac{\partial \underline{a}}{\partial t} \right) - \frac{1}{\mu_0} (\nabla \times \underline{A}) \cdot (\nabla \times \underline{a}). \end{aligned} \quad (2.22)$$

It is obvious that

$$L_1 = 0, \quad (2.23)$$

since $\int L_0 dt$ has been made stationary by choice of the zeroth order solution.

Second Order: In this case, the separate expressions are

$$\begin{aligned} \mathcal{L}_2^P &= \frac{1}{2} m (D \underline{\xi})^2 + q[(\underline{\xi} \cdot \nabla) \varphi + \frac{1}{2} (\underline{\xi} \underline{\xi} : \nabla \nabla) \Phi] - q \underline{v} \cdot [(\underline{\xi} \cdot \nabla) \underline{a} + \frac{1}{2} (\underline{\xi} \underline{\xi} : \nabla \nabla)^2 \underline{A}] \\ &\quad - q D \underline{\xi} \cdot [\underline{a} + (\underline{\xi} \cdot \nabla) \underline{A}], \quad \mathcal{L}_2^F = \frac{\epsilon_0}{2} \left(\nabla \varphi + \frac{\partial \underline{a}}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla \times \underline{a})^2, \end{aligned} \quad (2.24)$$

where the double dot product is such that $\underline{\underline{ab:cd}} = \underline{a} \cdot (\underline{b} \cdot \underline{c})\underline{d}$. Substituting them in Equation (2.20), and making use of the relations,

$$\underline{e} = -\nabla\varphi - \frac{\partial \underline{a}}{\partial t}, \quad \underline{b} = \nabla \times \underline{a}, \quad (2.25)$$

gives, after some manipulation,

$$\begin{aligned} L_2 = \int d^3r \int d^3v f_0 \left\{ \frac{1}{2} m (D\underline{\xi})^2 - q \underline{\xi} \cdot (\underline{e} + \underline{v} \times \underline{b}) + \frac{1}{2} q D\underline{\xi} \cdot \underline{\xi} \times \underline{B} \right. \\ \left. - \frac{1}{2} q \underline{\xi} \cdot (\underline{\xi} \cdot \nabla) (\underline{E} + \underline{v} \times \underline{B}) \right\} + \frac{1}{2} \int d^3r (\epsilon_0 e^2 - \frac{1}{\mu_0} b^2). \end{aligned} \quad (2.26)$$

Third Order: The field contribution vanishes at this order, leaving

$$\begin{aligned} \mathcal{L}_3^P = \frac{1}{2} q [(\underline{\xi\xi\xi}:\nabla\nabla)\varphi + \frac{1}{3} (\underline{\xi\xi\xi\xi}:\nabla\nabla\nabla)\Phi] - \frac{1}{2} q \underline{v} \cdot [(\underline{\xi\xi\xi}:\nabla\nabla)\underline{a} + \frac{1}{3} (\underline{\xi\xi\xi\xi}:\nabla\nabla\nabla)\underline{A}] \\ - q D\underline{\xi} \cdot [(\underline{\xi} \cdot \nabla)\underline{a} + \frac{1}{2} (\underline{\xi\xi\xi}:\nabla\nabla)\underline{A}], \quad \mathcal{L}_3^F = 0, \end{aligned} \quad (2.27)$$

where the triple dot product is defined such that $\underline{\underline{abc:def}} = \underline{a} \cdot [\underline{b} \cdot (\underline{c} \cdot \underline{d})\underline{e}]\underline{f}$. Substitution in Equation (2.20) gives

$$\begin{aligned} L_3 = q \int d^3r \int d^3v f_0 \left\{ \frac{1}{2} (\underline{\xi\xi\xi}:\nabla\nabla)\varphi + \frac{1}{6} (\underline{\xi\xi\xi\xi}:\nabla\nabla\nabla)\Phi - \underline{v} \cdot \left[\frac{1}{2} (\underline{\xi\xi\xi}:\nabla\nabla)\underline{a} + \frac{1}{6} (\underline{\xi\xi\xi\xi}:\nabla\nabla\nabla)\underline{A} \right] \right. \\ \left. - D\underline{\xi} \cdot \left[(\underline{\xi} \cdot \nabla)\underline{a} + \frac{1}{2} (\underline{\xi\xi\xi}:\nabla\nabla)\underline{A} \right] \right\}. \end{aligned} \quad (2.28)$$

2.3 First Order Equations

Before proceeding further, we must demonstrate that variation of our second order Lagrangian gives the small signal equations correctly.

Variation of $\underline{\xi}$: Taking the variation of L_2 [Equation (2.26)] with respect to $\underline{\xi}$ yields

$$m D^2 \underline{\xi} = \underline{f} + (\underline{\xi} \cdot \nabla) \underline{F} - q D\underline{\xi} \times \underline{B}, \quad (2.29)$$

where \underline{F} and \underline{f} are given by

$$\underline{\mathbf{F}} = -q(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) , \quad \underline{\mathbf{f}} = -q(\underline{\mathbf{e}} + \underline{\mathbf{v}} \times \underline{\mathbf{b}}) . \quad (2.30)$$

Now, expansion of Equation (2.7) about $(\underline{\mathbf{r}}_0, \underline{\mathbf{v}}_0)$ in equilibrium yields

$$m \frac{d}{dt} (\underline{\mathbf{v}}_0 + \underline{\xi}) = -q[\underline{\mathbf{E}} + (\underline{\xi} \cdot \nabla)\underline{\mathbf{E}} + \underline{\mathbf{e}} + (\underline{\mathbf{v}}_0 + \underline{\xi}) \times \{\underline{\mathbf{B}} + (\underline{\xi} \cdot \nabla)\underline{\mathbf{B}} + \underline{\mathbf{b}}\}] . \quad (2.31)$$

Extracting the first order quantities from Equation (2.31), and dropping the subscript 0 from $\underline{\mathbf{v}}_0$, gives Equation (2.29), and thus confirms that it is the correct small signal equation of motion.

Variation of φ : Carrying out this variation leads to the expression

$$\epsilon_0 \nabla \cdot \underline{\mathbf{e}} = q \int d^3v \nabla \cdot (f_0 \underline{\xi}) , \quad (2.32)$$

which we must show to be the first order Poisson equation. To do this, it is sufficient to show that the first order charge density is

$$\rho = q \int d^3v \nabla \cdot (f_0 \underline{\xi}) . \quad (2.33)$$

From the definition of a cell, described in Section 2.2, it follows that the number of electrons in the cell at $(\underline{\mathbf{r}}, \underline{\mathbf{v}})$ in the perturbed field is the same as that in the cell at $(\underline{\mathbf{r}}_0, \underline{\mathbf{v}}_0)$ in the equilibrium state

$$f(\underline{\mathbf{r}}, \underline{\mathbf{v}}, t) d^3v d^3r = f_0(\underline{\mathbf{r}}_0, \underline{\mathbf{v}}_0) d^3v_0 d^3r_0 , \quad (2.34)$$

where $d^3v d^3r$, the volume of the cell at $(\underline{\mathbf{r}}, \underline{\mathbf{v}})$ (see Figure 2.3), is related to $d^3v_0 d^3r_0$, the volume of the cell at $(\underline{\mathbf{r}}_0, \underline{\mathbf{v}}_0)$, by

$$d^3v d^3r = d^3v_0 d^3r_0 , \quad \mathcal{J} = \frac{\partial(\underline{\mathbf{r}}, \underline{\mathbf{v}})}{\partial(\underline{\mathbf{r}}_0, \underline{\mathbf{v}}_0)} , \quad (2.35)$$

where \mathcal{J} is the Jacobian. Substitution of Equation (2.11) for the Jacobian reduces Equation (2.35) to the first order equation

$$d^3v d^3r = (1 + \nabla_0 \cdot \underline{\xi})(1 + \nabla_{v_0} \cdot \underline{\xi}) d^3v_0 d^3r_0 . \quad (2.36)$$

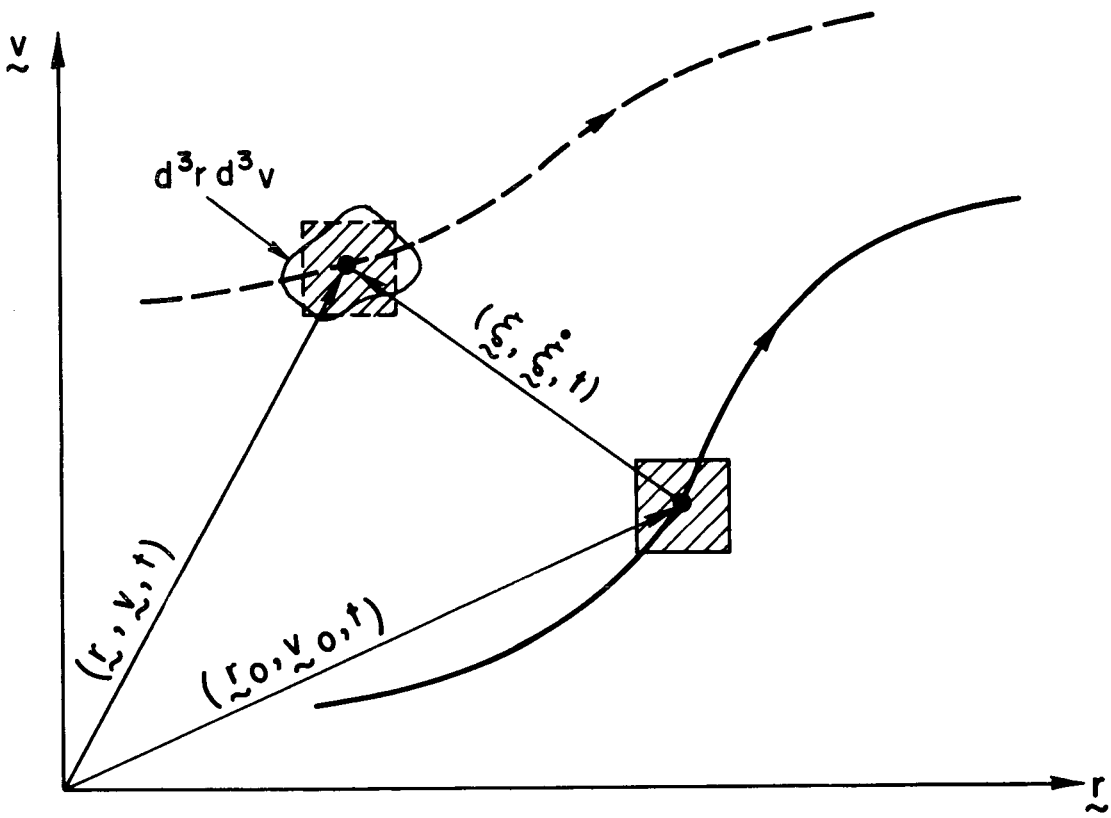


Figure 2.3 Phase space elements [the volumes of the shaded elements are both $d^3r_0 d^3v_0$. The volume of the cell in the perturbed system is $d^3r d^3v$].

From Equation (2.34), the number of electrons contained in the perturbed cell at $(\underline{r}, \underline{v})$, of volume $d^3r_0 d^3v_0$, may be written to first order as

$$f(\underline{r}, \underline{v}, t) d^3v_0 d^3r_0 = f_0(\underline{r}_0, \underline{v}_0) (1 - \nabla_0 \cdot \underline{\xi} - \nabla_{v_0} \cdot \dot{\underline{\xi}}) d^3v_0 d^3r_0 . \quad (2.37)$$

On the other hand, the number of particles in the cell at $(\underline{r}, \underline{v})$, of volume $d^3r_0 d^3v_0$ in equilibrium, is to first order

$$f_0(\underline{r}, \underline{v}) d^3v_0 d^3r_0 = [f_0(\underline{r}_0, \underline{v}_0) + (\underline{\xi} \cdot \nabla_0) f_0(\underline{r}_0, \underline{v}_0) + (\dot{\underline{\xi}} \cdot \nabla_{v_0}) f_0(\underline{r}_0, \underline{v}_0)] d^3v_0 d^3r_0 . \quad (2.38)$$

Subtracting Equation (2.38) from Equation (2.37) gives the first order distribution function

$$f_1(\underline{r}, \underline{v}) = -\nabla_0 \cdot (f_0 \underline{\xi}) - \nabla_{v_0} \cdot (f_0 \dot{\underline{\xi}}) . \quad (2.39)$$

Integrating f_1 over \underline{v}_0 yields

$$\rho = q \int d^3v_0 \nabla_0 \cdot (f_0 \underline{\xi}) . \quad (2.40)$$

Dropping the subscript 0 from $(\underline{r}_0, \underline{v}_0)$ in Equation (2.40) gives Equation (2.33). Equation (2.32) is, therefore, the correct first order Poisson equation.

Variation of \underline{a} : Finally, variation of Equation (2.26) with respect to \underline{a} , and use of the conservation law for the equilibrium density,

$$\underline{v} \cdot \nabla f_0 + \underline{a} \cdot \nabla_{\underline{v}} f_0 = 0 , \quad (2.41)$$

yields

$$\frac{1}{\mu_0} \nabla \times \underline{b} = \epsilon_0 \frac{\partial \underline{e}}{\partial t} - q \int d^3v [f_0 D \underline{\xi} - \underline{v} \nabla \cdot (f_0 \underline{\xi})] . \quad (2.42)$$

It only remains to show that the first order current density is

$$\underline{j} = -q \int d^3v_0 [f_0 \dot{\underline{\xi}} - \underline{v}_0 \nabla_0 \cdot (f_0 \underline{\xi})] , \quad (2.43)$$

to confirm that Equation (2.42) is one of Maxwell's equations.

By following an argument similar to that used above, we may write

$$\underline{v} f(\underline{r}, \underline{v}, t) d^3 v d^3 r = (\underline{v}_0 + \dot{\underline{\xi}}) f_0(\underline{r}_0, \underline{v}_0) d^3 r_0 d^3 v_0, \quad (2.44)$$

for the perturbed cell at $(\underline{r}, \underline{v})$. Substitution from Equation (2.36) gives

$$\begin{aligned} \underline{v} f(\underline{r}, \underline{v}, t) d^3 v_0 d^3 r_0 &= (\underline{v}_0 + \dot{\underline{\xi}}) f_0(\underline{r}_0, \underline{v}_0) d^3 v_0 d^3 r_0 \\ &\quad - \underline{v}_0 f_0(\underline{r}_0, \underline{v}_0) [\nabla_0 \cdot \underline{\xi} + \nabla_{v_0} \cdot \dot{\underline{\xi}}] d^3 v_0 d^3 r_0. \end{aligned} \quad (2.45)$$

The term similar to Equation (2.44) for the cell at $(\underline{r}, \underline{v})$ in the equilibrium state may be written as

$$\begin{aligned} \underline{v} f_0(\underline{r}, \underline{v}) d^3 v_0 d^3 r_0 &= (\underline{v}_0 + \dot{\underline{\xi}}) f_0 d^3 v_0 d^3 r_0 \\ &\quad + \underline{v}_0 (\underline{\xi} \cdot \nabla_0 + \dot{\underline{\xi}} \cdot \nabla_{v_0}) f_0(\underline{r}_0, \underline{v}_0) d^3 v_0 d^3 r_0. \end{aligned} \quad (2.46)$$

The first order current density is

$$\underline{j} = q \int d^3 v_0 \underline{v} [f(\underline{r}, \underline{v}, t) - f_0(\underline{r}, \underline{v})]. \quad (2.47)$$

Substitution of Equations (2.45) and (2.46) reduces Equation (2.47) to Equation (2.43), as required.

2.4 Comments on Low's Procedure

Low (1958) expanded the Lagrangian about the equilibrium state according to the scheme illustrated in Figure 2.4. Because of the initial conditions

$$\underline{\xi} = \dot{\underline{\xi}} = 0, \quad (2.48)$$

the initial electron density, $F(\underline{r}', \underline{v}')$, and volume, $d^3 r' d^3 v'$, in the perturbed system can be considered as those at $t = 0$ in equilibrium, as shown in Figure 2.4. The conservation law along the unperturbed

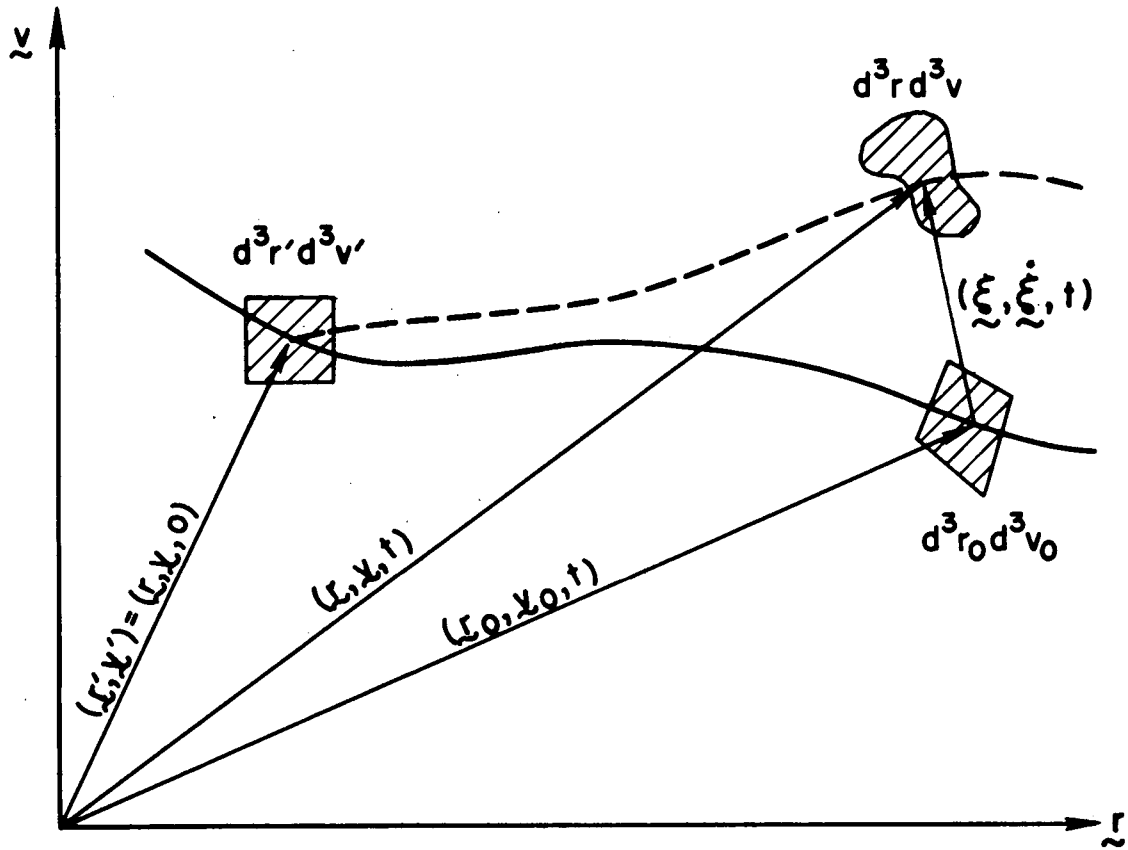


Figure 2.4 Low's model for the expansion of the Lagrangian.

trajectory then gives

$$F(\underline{r}', \underline{v}') = f(\underline{r}, \underline{v}, t) = f_0(\underline{r}_0, \underline{v}_0) , \quad d^3r d^3v = d^3r' d^3v' = d^3r_0 d^3v_0 , \quad (2.49)$$

and it is this equation which Low used for the expansion of the Lagrangian. Equation (2.49) requires unit Jacobian, which use of Equation (2.11) shows is equivalent to

$$\nabla \cdot \underline{\xi} + \nabla_v \cdot \dot{\underline{\xi}} = 0 . \quad (2.50)$$

This is the so-called "consistency condition", used by Low in proving the first order Euler-Lagrange equations [Equations (2.32) and (2.42)] to be the first order Maxwell equations. Low argued that any solution from his Lagrangian which does not satisfy Equation (2.50) must be thrown out as an improper one.

If the consistency condition is accepted, the misleading impression is gained that the Lagrangian would be invalid in a system with different initial conditions, or in a steady state which is not affected by any initial conditions. Furthermore, a higher order "consistency condition" would presumably be required in order to check the validity of solutions to nonlinear problems, i.e. corresponding to the inclusion of \mathcal{L}_3 and higher order terms in the series approximation to \mathcal{L} .

On the other hand, our transformation of coordinates via Equations (2.12) and (2.13) does not require any specification of initial conditions. Neither do we need any restriction in proving Equations (2.32) and (2.42) to be Maxwell's equations. Low's Lagrangian [Equation (2.2)] and the supplementary conservation laws [Equation (2.5)] are in fact the complete set of equations to describe a warm, collisionless plasma with arbitrary initial conditions. All solutions from his Lagrangian are therefore proper without invoking his consistency condition.

Chapter III

MICROSCOPIC TREATMENT

I. LINEAR THEORY

To describe the small-signal behavior of warm plasmas, we may use the first order Euler-Lagrange equations [Equations (2.29), (2.32) and (2.42)] obtained from the Lagrangian second order in perturbation in Chapter II. Fourier transformation of those equations should give the dispersion relation. An alternative procedure is to average the second order component of the Lagrangian density with the perturbing variables expanded in a Fourier series. Variations of the averaged Lagrangian with respect to the Fourier amplitudes of the variables give the Fourier transformed form of the first order equations. The advantage of the averaged Lagrangian method is that it gives an additional equation which cannot be derived in Chapter II. It is the equation for conservation of action obtained from the variation with respect to phase.

The above procedure is presented in Section 3.1; small signal equations and the equation for conservation of action are obtained for an infinite, collisionless, warm plasma in a uniform static magnetic field. It is also shown that the Euler-Lagrange equations obtained by variation with respect to the amplitudes are the first order equations obtained in Chapter II. In Section 3.2, the equation for the conservation of energy is derived from the averaged Hamiltonian, and is shown to be equivalent to conservation of action. The particle trajectory equation is solved, and wave dispersion relations are obtained in Section 3.3 for all possible magnetoplasma waves propagating at an oblique angle to the static magnetic field. Results of Section 3.3 are given for the limiting cases of waves propagating nearly parallel and exactly perpendicular to the magnetic field in Sections 3.4 and 3.5.

3.1 Euler-Lagrange Equations from the Averaged Lagrangian

3.1.1 Procedure

The perturbing dynamical variables can always be expressed as a sum of propagating waves

$$\begin{aligned} \xi(\underline{r}, \underline{v}, t) &= \sum'_k \{ \xi_k(\underline{v}) \exp[i\theta_k(\underline{r}, t)] + \xi_{-k}(\underline{v}) \exp[i\theta_{-k}(\underline{r}, t)] \} , \\ a(\underline{r}, t) &= \sum'_k \{ a_k \exp[i\theta_k(\underline{r}, t)] + a_{-k} \exp[i\theta_{-k}(\underline{r}, t)] \} , \end{aligned} \quad (3.1)$$

where a prime indicates that the summation extends over only half of \underline{k} -space, and the θ_k 's are integer multiples of a phase angle θ .

Then variables ξ_k and a_k satisfy

$$\begin{aligned} \xi_k(\underline{v}) &= \frac{1}{2\pi} \int_0^{2\pi} \xi(\underline{r}, \underline{v}, t) \exp(-ik\theta) d\theta , \\ a_k &= \frac{1}{2\pi} \int_0^{2\pi} a(\underline{r}, t) \exp(-ik\theta) d\theta . \end{aligned} \quad (3.2)$$

An equation similar to the second one in Equation (3.1) can be written for the scalar potential, φ . It is apparent then from the Fourier analysis, that

$$\xi_{-k} = \xi_k^* , \quad a_{-k} = a_k^* , \quad \theta_{-k} = -\theta_k . \quad (3.3)$$

The angular-frequency, ω_k , and the wave vector, \underline{k} , are given by

$$\omega_k = \frac{\partial \theta_k}{\partial t} , \quad \underline{k} = - \frac{\partial \theta_k}{\partial \underline{r}} . \quad (3.4)$$

The θ_k, ω_k , and \underline{k} are complex in general. However, their imaginary parts can be assumed to be very small compared with their real parts for stable or nearly stable waves, for which the linear theory is valid. The imaginary parts may therefore be neglected.

With the definition of Equation (3.1), the single particle Lagrangian second order in perturbation may be written in the form

$$\overline{\mathcal{L}}_2^P = \sum_k' \mathcal{L}_k^{P(2)}(\omega_k, \underline{k}_i, \xi_k^i, \xi_k^{j*}, \nabla_v \xi_k^i, \nabla_v \xi_k^{j*}), \quad (3.5)$$

where the bar over \mathcal{L}_2^P indicates that averages are taken in space and time. The Euler-Lagrange equation obtained from the ξ_k^{j*} variation of Equation (3.5) yields

$$\frac{\partial \mathcal{L}_k^{P(2)}}{\partial \xi_k^{j*}} - \sum_i \frac{\partial}{\partial v_i} \left[\frac{\partial \mathcal{L}_k^{P(2)}}{\partial (\partial \xi_k^{j*} / \partial v_i)} \right] = 0, \quad (3.6)$$

Solving Equation (3.6) gives the spatial displacement vector, ξ_k , in terms of field quantities.

The averaged Lagrangian density component, $\overline{\mathcal{L}}_2$, is also a function of ξ_k . However, since ξ_k can be eliminated with the help of Equation (3.6), it can be written as

$$\overline{\mathcal{L}}_2 = \sum_k' \mathcal{L}_k^{(2)}, \quad \mathcal{L}_k^{(2)} = \sum_{i,j} A_k^{ij}(\omega_k, \underline{k}) a_k^i a_k^{j*}. \quad (3.7)$$

The Euler-Lagrange equations obtained from the a_k^{j*} and θ_k variations of Equation (3.7) yield, respectively,

$$\sum_i A_k^{ij}(\omega_k, \underline{k}) a_k^i = 0, \quad (3.8)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}_k^{(2)}}{\partial \omega_k} \right) + \frac{\partial}{\partial \underline{r}} \cdot \left(- \frac{\partial \mathcal{L}_k^{(2)}}{\partial \underline{k}} \right) = 0. \quad (3.9)$$

The derivative $\partial/\partial t$ (or $\partial/\partial \underline{r}$) denotes that the differentiation is taken with \underline{r} (or t) fixed. It is, however, considered as 'total' in the sense that the dependence of $\mathcal{L}_k^{(2)}$ on t (or \underline{r}) through ω_k or \underline{k} is allowed for. The nontrivial solution of Equation (3.8) gives the eigenvalues determined by

$$\det A_k^{ij} = 0, \quad (3.10)$$

Its eigenvectors reduce Equation (3.7) to

$$\underline{f}_k^{(2)} = 0 . \quad (3.11)$$

Equation (3.9) may be written as

$$\frac{\partial}{\partial t} (N_k) + \frac{\partial}{\partial \underline{r}} \cdot (\underline{v}_k N_k) = 0 , \quad (3.12)$$

where

$$N_k \equiv \frac{\partial \mathcal{L}_k^{(2)}}{\partial \omega_k} , \quad \underline{v}_k \equiv - \frac{\partial \mathcal{L}_k^{(2)} / \partial \underline{k}}{\partial \mathcal{L}_k^{(2)} / \partial \omega_k} . \quad (3.13)$$

Equations (3.6) and (3.8) are the electron equation of motion and Maxwell's equations, respectively. Equation (3.10) is the small signal dispersion relation. Additional information is provided by Equation (3.12), which could not be obtained in Chapter II. It can be shown that N_k is the action variable for a system with periodic motion.⁴⁸ Equation (3.12) is therefore to be interpreted as the conservation of action.

3.1.2 Explicit Expressions for the Euler-Lagrange Equations

Explicit expressions for the single particle equation of motion [Equation (3.6)], Maxwell's equations [Equation (3.8)], and the equation for the conservation of action [Equation (3.12)] are required. However, that for the action conservation equation is not given below. It will be shown instead in Section 3.2 that the conservation of action is equivalent to the conservation of energy and momentum. Since the energy equation is more familiar to plasma physicists, the explicit expression for it is derived in Section 3.2. The explicit expressions for Equations (3.6) and (3.8) can be found by making use of the second order component of the Lagrangian function obtained in Chapter II.

Let a plasma in a uniform static magnetic field oriented in the z-direction be subject to a small perturbation. The zero order quantities may then be assumed to be homogeneous. With the gauge chosen such that $\bar{\phi} = \varphi = 0$, averaging the Lagrangian functions in Equations (2.24) and (2.26) gives the single particle Lagrangian and the Lagrangian density, respectively, as

$$\mathcal{L}_k^{P(2)} = \frac{1}{2} m |D_{k,0} \xi_k|^2 + \frac{1}{2} q (D_{k,0} \xi_k) \cdot \xi_k^* \times \underline{B} + q \xi_k^* \cdot (\underline{e}_k + \underline{v} \times \underline{b}_k) + \text{c.c.}, \quad (3.14)$$

$$\mathcal{L}_k^{(2)} = -\frac{1}{2} q \int d^3v f_0 \xi_k^* \cdot (\underline{e}_k + \underline{v} \times \underline{b}_k) + \frac{1}{2} \epsilon_0 |\underline{e}_k|^2 - \frac{1}{2\mu_0} |\underline{b}_k|^2 + \text{c.c.}, \quad (3.15)$$

where c.c. indicates complex conjugate and

$$\underline{e}_k = -i\omega_k \underline{a}_k, \quad \underline{b}_k = -i\mathbf{k} \times \underline{a}_k, \quad (3.16)$$

$$D_{k,\mu} \equiv \omega_c \frac{\partial}{\partial \phi} + i [\omega_{k,\mu} - k_{\perp} v_{\perp} \cos(\phi - \phi_k)], \quad \omega_{k,\mu} \equiv \omega_k - k_z v_z + \mu \omega_c, \quad (\mu = 0, \pm 1, \pm 2, \dots). \quad (3.17)$$

In Equation (3.17), ω_c is the electron cyclotron frequency; ϕ and ϕ_k are the azimuthal angles of \underline{v} and \underline{k} , respectively, and the subscript \perp indicates a component perpendicular to the static magnetic field (see Figure 3.1). It may be seen immediately that $D_{k,0}$ is the Fourier transform of the RHS of Equation (2.19). We note that we may set $\phi_k = 0$ without loss of generality. However, in nonlinear problems involving interactions among many waves, all of the waves do not necessarily propagate collinearly, i.e., the ϕ_k 's of all waves are not zero. In preparation for the following chapter, we shall assume $\phi_k \neq 0$ here. After substitution of

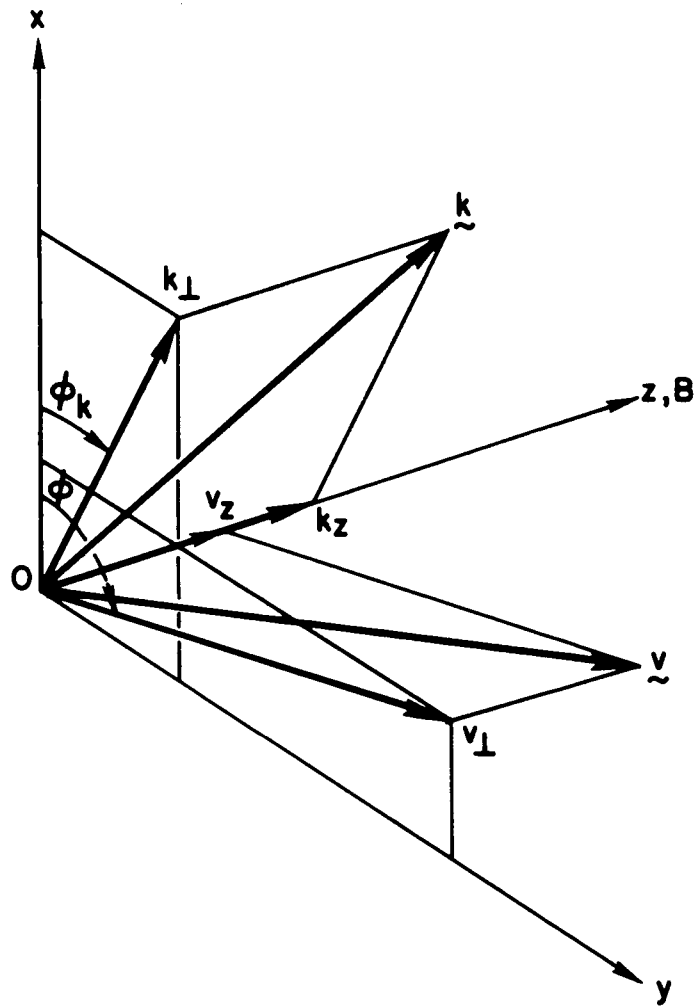


Figure 3.1 Coordinate system for particle velocity and wave vector.

of Equations (3.14) and (3.15), Equations (3.6) and (3.8) become

$$D_{k,0}^2 \underline{\xi}_k + D_{k,0} \underline{\xi}_k \times \underline{\omega}_c = -\frac{q}{m} \left[\left(1 - \frac{\underline{k} \cdot \underline{v}}{\omega_k} \right) \underline{e}_k + \frac{\underline{k}}{\omega_k} (\underline{v} \cdot \underline{e}_k) \right], \quad (3.18)$$

$$\underline{n} \times (\underline{n} \times \underline{e}_k) + \underline{\kappa}_k \cdot \underline{e}_k = 0, \quad (3.19)$$

where \underline{n} ($\equiv \underline{k}c/\omega_k$) is the refractive index vector and $\underline{\omega}_c$ is a vector with its magnitude ω_c directed along the z-axis. The equivalent plasma permittivity tensor, $\underline{\kappa}_k$, and the current density, \underline{j}_k , are given by

$$\underline{\kappa}_k \equiv 1 + \frac{\underline{j}_k}{i\epsilon_0 \omega_k \underline{e}_k}, \quad \underline{j}_k = -iq \int d^3v f_0 [\omega_k \underline{\xi}_k + \underline{k} \times (\underline{v} \times \underline{\xi}_k)]. \quad (3.20)$$

Equations (3.18) and (3.19) are the equation of motion of an electron and the wave equation, respectively. It should be noted that these equations are Fourier transforms of Equations (2.29) and (2.42).

3.2 Energy and Momentum Conservation

To each dynamical variable, Ψ_μ , corresponds a canonical momentum, π_μ , defined by

$$\pi_\mu \equiv \frac{\partial \mathcal{L}_2}{\partial (\partial \Psi_\mu / \partial t)}. \quad (3.21)$$

However, no canonical momentum exists for the Fourier amplitude of the dynamical variables a_k^i or a_k^{j*} , because $\mathcal{L}_k^{(2)}$ in Equation (3.7) does not involve \dot{a}_k^i or \dot{a}_k^{j*} . Only the phase, θ_k , has a canonical momentum. This is the action variable, N_k [defined in Equation (3.13)].

According to Hamilton's theory, and with the use of Equation (3.11), the generalized energy and momentum densities averaged in time and space are obtained as

$$\overline{\mathcal{H}}_2 = \sum_k' \mathcal{H}_k^{(2)}, \quad \mathcal{H}_k^{(2)} = \omega_k N_k, \quad (3.22)$$

$$\overline{G_k} = \sum_k' G_k^{(2)}, \quad G_k^{(2)} = \underline{v}_k N_k. \quad (3.23)$$

If we differentiate $f_k^{(2)}$ and substitute Equation (3.13), we obtain

$$\frac{\partial f_k^{(2)}}{\partial t} = N_k \left(\frac{\partial \omega_k}{\partial t} - \underline{v}_k \cdot \frac{\partial \underline{k}}{\partial t} \right), \quad \frac{\partial f_k^{(2)}}{\partial r_i} = N_k \left(\frac{\partial \omega_k}{\partial r_i} - \underline{v}_k \cdot \frac{\partial \underline{k}}{\partial r_i} \right). \quad (3.24)$$

By combining this with Equation (3.12), and the ray equation obtained from Equation (3.4)

$$\frac{\partial \underline{k}}{\partial t} = - \frac{\partial \omega_k}{\partial \underline{r}}, \quad (3.25)$$

we may express the total differentiation of $H_k^{(2)}$ and $G_k^{(2)}$ with respect to time as

$$\frac{\partial}{\partial t} (\omega_k N_k) + \frac{\partial}{\partial \underline{r}} \cdot (\underline{v}_k \omega_k N_k) = 0, \quad (3.26)$$

$$\frac{\partial}{\partial t} (k_i N_k) + \frac{\partial}{\partial \underline{r}} \cdot (\underline{v}_k k_i N_k) = 0. \quad (3.27)$$

As in Equation (3.9), the partial derivatives in Equations (3.24) - (3.27) merely indicate that the differential is taken with other coordinates fixed. They are, however, considered as 'total' in Hamilton's theory. Equations (3.26) and (3.27) are the equations for conservation of energy and momentum, respectively. It is important to note that the use of Equations (3.11), (3.24) and (3.25) reduces the conservation of energy [Equation (3.26)], or momentum [Equation (3.27)], to the conservation of action [Equation (3.12)].

From Equations (3.13), (3.15) and (3.19), explicit expressions may be obtained for the energy and power flow:

$$\omega_k N_k = \frac{1}{2\epsilon_0} |\underline{b}_k|^2 + \frac{1}{2} \underline{e}_k^* \cdot \epsilon_0 \frac{\partial}{\partial \omega_k} (\omega_k \underline{\kappa}_k) \cdot \underline{e}_k + \text{c.c.}, \quad (3.28)$$

$$\vec{v}_k \omega_k N_k = \frac{1}{\mu_0} \vec{e}_k \times \vec{b}_k^* - \frac{1}{2} \frac{\partial}{\partial k} (\vec{e}_k^* \cdot \epsilon_0 \omega_k \vec{K}_k \cdot \vec{e}_k) + \text{c.c.} \quad (3.29)$$

The first term on the RHS of Equation (3.28) is the magnetic energy in free space. The second contains the electric energy in free space, the particle energy, and the particle-field interaction energy. The first term on the RHS of Equation (3.29) is the Poynting vector, and the second is the power flow due to particles flowing out of a volume coherently with the wave.⁷³

3.3 Solutions of the Euler-Lagrange Equations

3.3.1 Perturbed Electron Trajectory

A difficulty in solving Equation (3.18) is that the x and y components of the LHS are coupled. It may easily be seen, however, that it can be diagonalized by a unitary matrix

$$\vec{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \quad (3.30)$$

New characteristic vectors for $\vec{\xi}_k$ are then $(\xi_k^R, \xi_k^L, \xi_k^Z)$, where

$$\xi_k^I = \sum_j U_{Ij} \xi_k^j, \quad (I = R, L, Z, \quad j = x, y, z). \quad (3.31)$$

Since the y component is equal in magnitude to the x component, and lags it by 90° , ξ_k^R is a vector rotating in a right-handed sense about the z -axis. Similarly ξ_k^L is rotating in a left-handed sense. The unitary transformation of Equation (3.18) gives

$$\vec{D}_k \cdot \vec{\xi}_k = \vec{M}_k \cdot \vec{e}_k, \quad (3.32)$$

where we have

$$\vec{D}_k = \begin{bmatrix} D_{k,-1} & D_{k,0} & 0 & 0 \\ 0 & & D_{k,1} & D_{k,0} & 0 \\ 0 & & 0 & & D_{k,0}^2 \end{bmatrix}, \quad (3.33)$$

$$\vec{M}_k = \begin{bmatrix} M_k^{RR} & M_k^{RL} & M_k^{RZ} \\ M_k^{LR} & M_k^{LL} & M_k^{LZ} \\ M_k^{ZR} & M_k^{ZL} & M_k^{ZZ} \end{bmatrix}. \quad (3.34)$$

The elements of \vec{M}_k are given by

$$\begin{aligned} M_k^{RR} &= M_k^{LL*} = -\frac{q}{m} \left\{ \frac{\omega_{k,0}}{\omega_k} - \frac{k_{\perp} v_{\perp}}{2\omega_k} \exp [i(\phi - \phi_k)] \right\}, \\ M_k^{ZZ} &= -\frac{q}{m} \left[1 - \frac{k_{\perp} v_{\perp}}{\omega_k} \cos (\phi - \phi_k) \right], \\ M_k^{RL} &= M_k^{LR*} = -\frac{q}{m} \frac{k_{\perp} v_{\perp}}{2\omega_k} \exp [i(\phi + \phi_k)], \\ M_k^{ZR} &= M_k^{ZL*} = -\frac{q}{m} \frac{k_z v_{\perp}}{\sqrt{2} \omega_k} \exp(-i\phi), \\ M_k^{RZ} &= M_k^{LZ*} = -\frac{q}{m} \frac{k_{\perp} v_z}{\sqrt{2} \omega_k} \exp(i\phi). \end{aligned} \quad (3.35)$$

After some manipulation, the solution to Equation (3.32) can be written explicitly as

$$\vec{\xi}_k = \vec{G}_k \cdot \vec{e}_k, \quad (3.36)$$

where the elements of the matrix \vec{G}_k are

$$\begin{aligned}
G_k^{RR} &= G_{-k}^{LL*} = \frac{q}{m} \left\{ \frac{\omega_{k,0}}{\omega_k} P_{0,-1}^n - \frac{k_{\perp} v_{\perp}}{2\omega_k} P_{0,1}^n \exp [i(\phi - \phi_k)] \right\} , \\
G_k^{ZZ} &= \frac{q}{m} \left\{ P_{0,0}^n - \frac{k_{\perp} v_{\perp}}{2\omega_k} \left[P_{-1,-1}^n \exp [i(\phi - \phi_k)] + P_{1,1}^n \exp [i(\phi - \phi_k)] \right] \right\} , \\
G_k^{RL} &= G_{-k}^{LR*} = \frac{q}{m} \frac{k_{\perp} v_{\perp}}{2\omega_k} P_{0,1}^n \exp [i(\phi + \phi_k)] , \\
G_k^{ZR} &= G_{-k}^{ZL*} = \frac{q}{m} \frac{k_z v_{\perp}}{\sqrt{2} \omega_k} P_{-1,-1}^n \exp (-i\phi) , \\
G_k^{RZ} &= G_{-k}^{LZ*} = \frac{q}{m} \frac{k_{\perp} v_z}{\sqrt{2} \omega_k} P_{0,-1}^n \exp (i\phi_k) , \tag{3.37}
\end{aligned}$$

and $P_{\mu,\nu}^n$ is defined by

$$P_{\mu,\nu}^n \equiv \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\omega_{k,n+\mu} \omega_{k,n+\nu}} J_n(\lambda_k) \exp \left\{ i[n(\phi - \phi_k) + \lambda_k \sin(\phi - \phi_k)] \right\} . \tag{3.38}$$

Here, λ_k is written for $k_{\perp} v_{\perp} / \omega_c$, and the following Bessel function identity has been introduced:

$$\exp (i\lambda_k \sin \phi) \equiv \sum_{n=-\infty}^{\infty} J_n(\lambda_k) \exp (in\phi) . \tag{3.39}$$

The subscripts on the G's in Equation (3.37) indicate the \underline{k} or $-\underline{k}$ wave, as shown in Figure 3.2. A change in sign of the subscript k makes sign changes in ω_k , k_{\perp} and k_z , but the sign of the azimuth, ϕ_k , does not change.

Equation (3.36) describes the perturbed electron trajectory. It may be seen that, if the rf electric field has a transverse variation

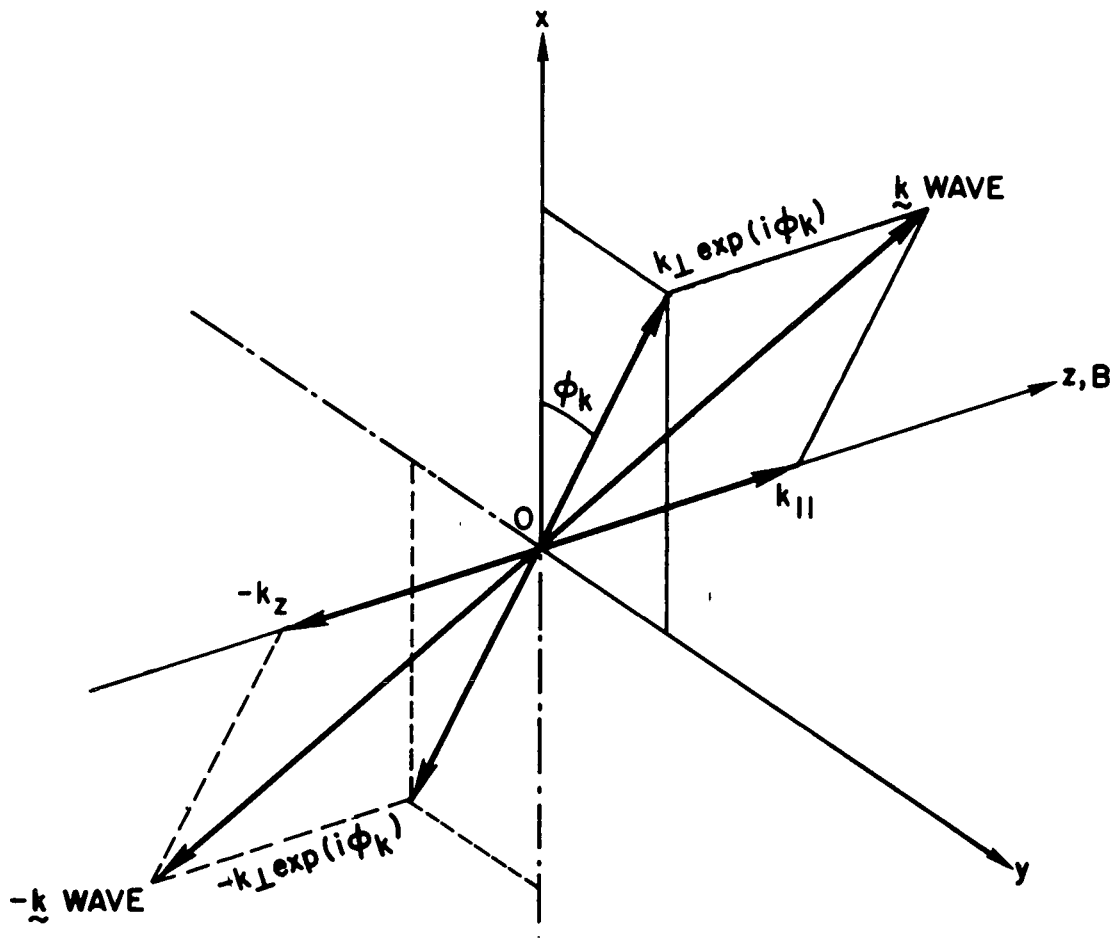


Figure 3.2 Wave vectors of \vec{k} and $-\vec{k}$ waves.

($k_{\perp} \neq 0$), strong interactions occur between the rf field and the electron when the Doppler-shifted frequency approaches a cyclotron harmonic.

3.3.2 Dispersion Relations

The unitary transformation of $\underline{e}_{\mathbf{k}} = (e_{\mathbf{k}}^x, e_{\mathbf{k}}^y, e_{\mathbf{k}}^z)$ to $\underline{e}_{\mathbf{k}} = (e_{\mathbf{k}}^R, e_{\mathbf{k}}^L, e_{\mathbf{k}}^Z)$ reduces Equation (3.19) to

$$\underline{\tilde{A}} \cdot \underline{e}_{\mathbf{k}} = 0, \quad (3.40)$$

where

$$\underline{\tilde{A}} = \begin{bmatrix} -\eta_z^2 - \eta_{\perp}^2/2 + \kappa_{\mathbf{k}}^{RR} & -(\eta_{\perp}^2/2) \exp(2i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{RL} & (\eta_z \eta_{\perp}/\sqrt{2}) \exp(i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{RZ} \\ -(\eta_{\perp}^2/2) \exp(-2i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{LR} & -\eta_z^2 - \eta_{\perp}^2/2 + \kappa_{\mathbf{k}}^{LL} & (\eta_z \eta_{\perp}/\sqrt{2}) \exp(-i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{LZ} \\ (\eta_z \eta_{\perp}/\sqrt{2}) \exp(-i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{ZR} & (\eta_z \eta_{\perp}/\sqrt{2}) \exp(i\phi_{\mathbf{k}}) + \kappa_{\mathbf{k}}^{ZL} & -\eta_{\perp}^2 + \kappa_{\mathbf{k}}^{ZZ} \end{bmatrix}. \quad (3.41)$$

Here, η_z and η_{\perp} are the components of the refractive index vector parallel and perpendicular to the z-axis. Using Equations (3.20) and (3.36), and the relations

$$J_n'(\lambda_{\mathbf{k}}) = J_{n-1}(\lambda_{\mathbf{k}}) - \frac{n}{\lambda_{\mathbf{k}}} J_n(\lambda_{\mathbf{k}}) = \frac{n}{\lambda_{\mathbf{k}}} J_n(\lambda_{\mathbf{k}}) - J_{n+1}(\lambda_{\mathbf{k}}), \quad (3.42)$$

$$\int_0^{2\pi} \exp[i(m\phi + \lambda_{\mathbf{k}} \sin\phi)] d\phi = 2\pi J_{-m}(\lambda_{\mathbf{k}}), \quad (3.43)$$

the equivalent permittivity matrix can be obtained explicitly as

$$\kappa_{\mathbf{k}}^{\text{RR}} = \kappa_{-\mathbf{k}}^{\text{LL}} = 1 - \left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}}\right)^2 \sum_{\mathbf{n}} \left\langle \frac{1}{\omega_{\mathbf{k},\mathbf{n}-1}} \left[\frac{\omega_{\mathbf{k},0}^2}{\omega_{\mathbf{k},\mathbf{n}}} + \frac{(k_{\perp} v_{\perp})^2}{2\omega_{\mathbf{k},\mathbf{n}-1}} + \frac{(k_{\perp} v_{\perp})^2}{2\omega_{\mathbf{k},\mathbf{n}+1}} \right] J_{\mathbf{n}}^2 + \frac{\omega_{\mathbf{k},0} k_{\perp} v_{\perp}}{\omega_{\mathbf{k},\mathbf{n}} \omega_{\mathbf{k},\mathbf{n}+1}} J_{\mathbf{n}} J_{\mathbf{n}+1} \right\rangle,$$

$$\kappa_{\mathbf{k}}^{\text{ZZ}} = 1 - \left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}}\right)^2 \sum_{\mathbf{n}} \left\langle \left[\left(\frac{\omega_{\mathbf{k}+\mathbf{n}\mathbf{c}}}{\omega_{\mathbf{k},\mathbf{n}}} \right)^2 + \frac{(k_{\perp} v_{\perp})^2}{\omega_{\mathbf{k},\mathbf{n}+1} \omega_{\mathbf{k},\mathbf{n}-1}} \right] J_{\mathbf{n}}^2 \right\rangle,$$

$$\begin{aligned} \kappa_{\mathbf{k}}^{\text{RL}} = \kappa_{\mathbf{k}}^{\text{LR}*} &= -\left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}}\right)^2 \exp(2i\phi_{\mathbf{k}}) \sum_{\mathbf{n}} \left\langle \frac{1}{2} \left(\frac{k_{\perp} v_{\perp}}{\omega_{\mathbf{k},\mathbf{n}}} \right)^2 J_{\mathbf{n}+1} J_{\mathbf{n}-1} - \frac{(k_{\perp} v_{\perp})^2}{2\omega_{\mathbf{k},\mathbf{n}+1} \omega_{\mathbf{k},\mathbf{n}-1}} J_{\mathbf{n}}^2 \right. \\ &\quad \left. - \frac{\omega_{\mathbf{k},0} k_{\perp} v_{\perp}}{\omega_{\mathbf{k},\mathbf{n}} \omega_{\mathbf{k},\mathbf{n}+1}} J_{\mathbf{n}} J_{\mathbf{n}+1} \right\rangle, \end{aligned}$$

$$\begin{aligned} \kappa_{\mathbf{k}}^{\text{RZ}} = \kappa_{\mathbf{k}}^{\text{ZR}*} &= \kappa_{-\mathbf{k}}^{\text{ZL}} = \kappa_{-\mathbf{k}}^{\text{LZ}*} \\ &= -\left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}}\right)^2 \exp(i\phi_{\mathbf{k}}) \sum_{\mathbf{n}} \left\langle \frac{\omega_{\mathbf{k},0} k_{\perp} v_{\perp}}{\sqrt{2} \omega_{\mathbf{k},\mathbf{n}} \omega_{\mathbf{k},\mathbf{n}-1}} J_{\mathbf{n}}^2 - \frac{(\omega_{\mathbf{k}+\mathbf{n}\mathbf{c}}) k_{\perp} v_{\perp}}{\sqrt{2} \omega_{\mathbf{k},\mathbf{n}}^2} J_{\mathbf{n}} J_{\mathbf{n}+1} \right\rangle, \end{aligned} \quad (3.44)$$

where ω_{N} is the electron plasma frequency; $\langle \rangle$ denotes the mean value over velocity space, and the argument of the Bessel functions is $\lambda_{\mathbf{k}}$. Unless otherwise indicated, the summations over \mathbf{n} contain all negative and positive integers. The small-signal dispersion relation is expressed by $\det \vec{\mathbf{A}} = 0$ [Equation (3.10)].

The solution of the perturbed trajectory of an electron [Equation (3.36)], the dispersion relation [Equation (3.40)], and the equations for the energy conservation [Equations (3.26), (3.28) and (3.29)] are very complicated. Simpler forms can be obtained for the special cases of waves propagating nearly parallel or exactly perpendicular to the magnetic field. These two limits will be considered in the following sections.

3.4 Quasiparallel Propagation ($k_{\perp} \ll k_z$)

3.4.1 Perturbed Electron Trajectory

To first order in k_{\perp} , Equation (3.37) reduces to

$$\begin{aligned}
 G_{\mathbf{k}}^{RR} &= G_{-\mathbf{k}}^{LL*} = \frac{q}{m\omega_{\mathbf{k}}\omega_{\mathbf{k},-1}} \left\{ 1 + \frac{k_{\perp} v_{\perp}}{2\omega_{\mathbf{k},0}} \exp[i(\phi - \phi_{\mathbf{k}})] + \frac{k_{\perp} v_{\perp}}{\omega_{\mathbf{k},-2}} \exp[-i(\phi - \phi_{\mathbf{k}})] \right\}, \\
 G_{\mathbf{k}}^{ZZ} &= \frac{q}{m\omega_{\mathbf{k},0}^2} \left\{ 1 + \frac{k_{\perp} v_{\perp}}{2\omega_{\mathbf{k}}} \left[\frac{\omega_{\mathbf{k}}(\omega_{\mathbf{k}} - \omega_c) - (k_z v_z)^2}{\omega_{\mathbf{k},-1}^2} \exp[-i(\phi - \phi_{\mathbf{k}})] \right. \right. \\
 &\quad \left. \left. + \frac{\omega_{\mathbf{k}}(\omega_{\mathbf{k}} + \omega_c) - (k_z v_z)^2}{\omega_{\mathbf{k},1}^2} \exp[i(\phi - \phi_{\mathbf{k}})] \right] \right\}, \\
 G_{\mathbf{k}}^{RL} &= G_{-\mathbf{k}}^{LR*} = \frac{q}{2m} \frac{k_{\perp} v_{\perp}}{\omega_{\mathbf{k}}\omega_{\mathbf{k},0}\omega_{\mathbf{k},1}} \exp[i(\phi + \phi_{\mathbf{k}})], \\
 G_{\mathbf{k}}^{ZR} &= G_{-\mathbf{k}}^{ZL*} = \frac{q}{\sqrt{2} m} \frac{k_z v_{\perp}}{\omega_{\mathbf{k}}\omega_{\mathbf{k},-1}} \exp(-i\phi) \left\{ 1 + \frac{k_{\perp} v_{\perp} (\omega_{\mathbf{k},-1} + \omega_{\mathbf{k},0})}{2\omega_{\mathbf{k},0}^2} \exp[i(\phi - \phi_{\mathbf{k}})] \right. \\
 &\quad \left. + \frac{k_{\perp} v_{\perp} (\omega_{\mathbf{k},-1} + \omega_{\mathbf{k},-2})}{2\omega_{\mathbf{k},-2}^2} \exp[-i(\phi - \phi_{\mathbf{k}})] \right\}, \\
 G_{\mathbf{k}}^{RZ} &= G_{-\mathbf{k}}^{LZ*} = \frac{q}{\sqrt{2} m} \frac{k_{\perp} v_z}{\omega_{\mathbf{k}}\omega_{\mathbf{k},0}\omega_{\mathbf{k},-1}} \exp(i\phi_{\mathbf{k}}). \tag{3.45}
 \end{aligned}$$

We shall make use of these expressions in Section 4.3.

In the case of all waves propagating precisely parallel to the magnetic field ($k_{\perp} = 0$), Equation (3.36) reduces to

$$\begin{aligned}
 \xi_{\mathbf{k}}^R &= \frac{q}{m\omega_{\mathbf{k}}\omega_{\mathbf{k},-1}} e_{\mathbf{k}}^R, & \xi_{\mathbf{k}}^L &= \frac{q}{m\omega_{\mathbf{k}}\omega_{\mathbf{k},1}} e_{\mathbf{k}}^L, \\
 \xi_{\mathbf{k}}^Z &= -i \frac{q}{\sqrt{2} m} \frac{b_{\mathbf{k}v_{\perp}}^R}{\omega_{\mathbf{k},-1}} \exp(-i\phi) + i \frac{q}{\sqrt{2} m} \frac{b_{\mathbf{k}v_{\perp}}^L}{\omega_{\mathbf{k},1}} \exp(i\phi) + \frac{q}{m\omega_{\mathbf{k},0}} e_{\mathbf{k}}^Z. \tag{3.46}
 \end{aligned}$$

The displacement of an electron in the xy-plane coincides in direction with the electric field of the right-hand or left-hand polarized wave, which interacts strongly with electrons traveling such that $\omega_{\mathbf{k}} - \mathbf{k}_{\perp} \cdot \mathbf{v}_{\perp} \approx \pm \omega_c$. The first two terms of the displacement in the z-direction are due to the Lorentz force of the transverse magnetic field ($\propto \mathbf{v}_{\perp} \times \mathbf{b}_{\mathbf{k}}$). However, averaging over velocity space causes these terms to vanish. The average displacement in the z-direction is therefore only due to the longitudinal plasma (Langmuir) wave, which interacts strongly with electrons traveling with velocities close to the z-component of the phase velocity, i.e., $\omega_{\mathbf{k}} - \mathbf{k}_{\parallel} v_{\parallel} \approx 0$.

3.4.2 Dispersion Relations

To first order in k_{\perp} , the elements of matrix $\vec{\tilde{A}}$ in Equation (3.41) become

$$A_{\mathbf{k}}^{\text{RR}} = A_{-\mathbf{k}}^{\text{LL}} = 1 - \eta_z^2 - \left(\frac{\omega_N}{\omega_{\mathbf{k}}} \right)^2 \left\langle \frac{\omega_{\mathbf{k},0}}{\omega_{\mathbf{k},-1}} + \frac{1}{2} \left(\frac{k_z v_{\perp}}{\omega_{\mathbf{k},-1}} \right)^2 \right\rangle ,$$

$$A_{\mathbf{k}}^{\text{ZZ}} = 1 - \left\langle \left(\frac{\omega_N}{\omega_{\mathbf{k},0}} \right)^2 \right\rangle , \quad A_{\mathbf{k}}^{\text{RL}} = A_{\mathbf{k}}^{\text{LR}*} = 0 ,$$

$$A_{\mathbf{k}}^{\text{RZ}} = A_{\mathbf{k}}^{\text{ZR}*} = A_{-\mathbf{k}}^{\text{ZL}} = A_{-\mathbf{k}}^{\text{LZ}*}$$

$$= \eta_z \frac{\eta_{\perp}}{\sqrt{2}} \exp(i\phi_{\mathbf{k}}) - \left(\frac{\omega_N}{\omega_{\mathbf{k}}} \right)^2 \frac{k_{\perp}}{\sqrt{2}} \exp(i\phi_{\mathbf{k}}) \left\langle \frac{v_z}{\omega_{\mathbf{k},-1}} - \frac{k_z v_{\perp}^2}{2} \left(\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k},0}} - \frac{\omega_{\mathbf{k}} - \omega_c}{\omega_{\mathbf{k},-1}} \right) \right\rangle . \quad (3.47)$$

In the limit of purely parallel propagation ($k_{\perp} = 0$), the off-diagonal terms of $\vec{\tilde{A}}$ vanish. The nonzero diagonal terms are then the dispersion relations of the right-hand and left-hand polarized transverse waves, and the longitudinal Langmuir wave.⁷⁴

3.5 Perpendicular Propagation ($k_z = 0$)

3.5.1 Perturbed Electron Trajectory

For this case, a rectangular coordinate system is more convenient than the rotating coordinate system used in Sections 3.3 and 3.4. Putting $k_z = 0$, and transforming the electric field (e_k^R, e_k^L, e_k^Z) to ($e_k^{\parallel}, e_k^{\perp}, e_k^Z$), as indicated in Figure 3.3, reduces Equation (3.36) to

$$\begin{bmatrix} \mathfrak{E}_k^{\parallel} \\ \mathfrak{E}_k^{\perp} \\ \mathfrak{E}_k^Z \end{bmatrix} = \begin{bmatrix} G_k^{\parallel\parallel} & G_k^{\parallel\perp} & G_k^{\parallel Z} \\ G_k^{\perp\parallel} & G_k^{\perp\perp} & G_k^{\perp Z} \\ G_k^{Z\parallel} & G_k^{Z\perp} & G_k^{ZZ} \end{bmatrix} \begin{bmatrix} e_k^{\parallel} \\ e_k^{\perp} \\ e_k^Z \end{bmatrix}, \quad (3.48)$$

where

$$\begin{aligned} G_k^{\parallel\parallel} &= \frac{q}{m} P_{-1,1}^n, & G_k^{ZZ} &= \frac{q}{m} P_{0,-n}^n, \\ G_k^{\perp\perp} &= G_k^{\parallel\parallel} - \frac{q}{m} \frac{k_{\perp} v_{\perp}}{2\omega_k} \left\{ P_{0,-1}^n \exp[-i(\phi - \phi_k)] + P_{0,1}^n \exp[i(\phi - \phi_k)] \right\}, \\ G_k^{\perp\parallel} &= -i \frac{q}{m} P_{0,1,-1}^n, & G_k^{Z\parallel} &= G_k^{Z\perp} = 0, \\ G_k^{\parallel\perp} &= -G_k^{\perp\parallel} + i \frac{q}{m} \frac{k_{\perp} v_{\perp}}{2\omega_k} \left\{ P_{0,-1}^n \exp[-i(\phi - \phi_k)] - P_{0,1}^n \exp[i(\phi - \phi_k)] \right\}, \\ G_k^{\parallel Z} &= \frac{q}{m} \frac{k_{\perp} v_z}{\omega_k} P_{1,-1}^n, & G_k^{\perp Z} &= -i \frac{q}{m} \frac{k_{\perp} v_z}{\omega_k} P_{0,1,-1}^n, \end{aligned} \quad (3.49)$$

and, similar to Equation (3.38), $P_{0,\mu,\nu}^n$ is defined as

$$P_{0,\mu,\nu}^n \equiv \sum_{n=-\infty}^{\infty} (-1)^n \frac{\omega_c}{\omega_{k,n} \omega_{k,n+\mu} \omega_{k,n+\nu}} J_n(\lambda_k) \exp \left\{ i[n(\phi - \phi_k) + \lambda_k \sin(\phi - \phi_k)] \right\}. \quad (3.50)$$

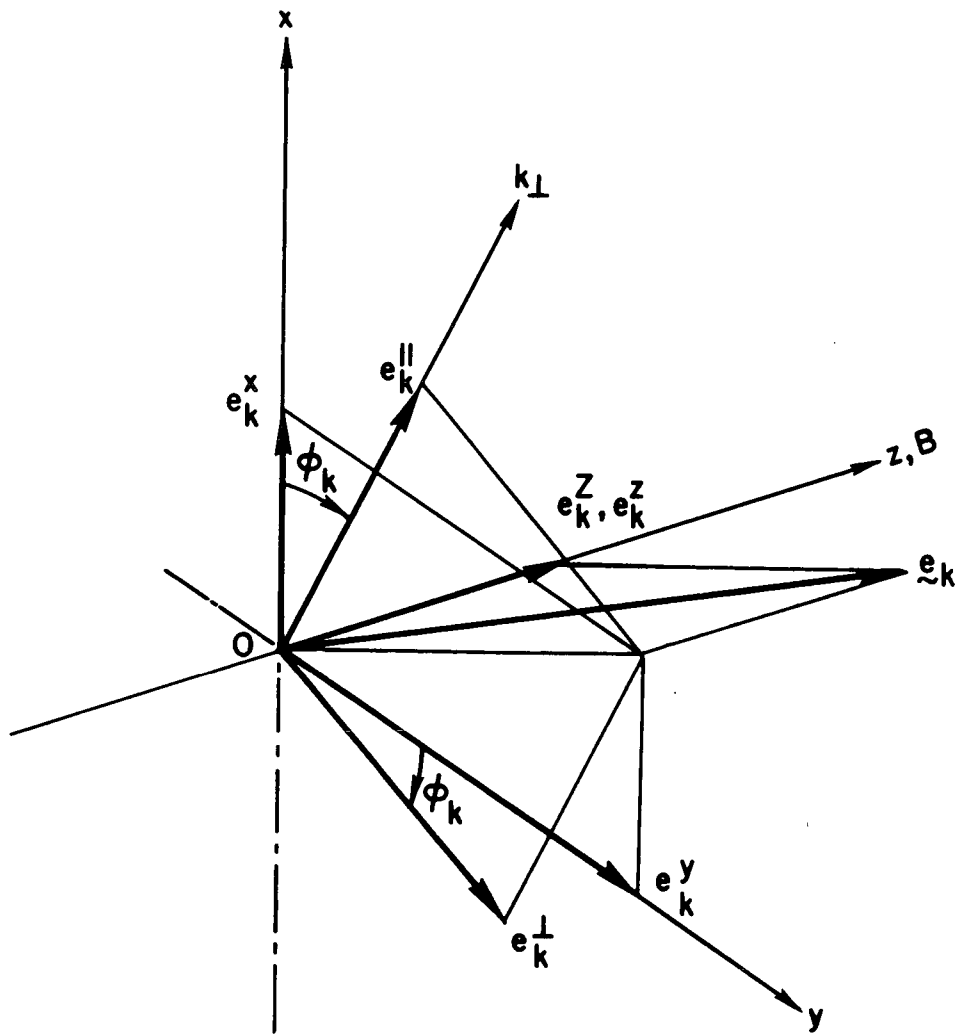


Figure 3.3 Electric field described by (e_k^x, e_k^y, e_k^z) and $(e_k^{\parallel}, e_k^{\perp}, e_k^z)$.

It should be noted that superscripts \perp and \parallel indicate the components perpendicular and parallel to the wave vector, respectively. On the other hand, the subscripts \perp and \parallel represent the components perpendicular and parallel to the static magnetic field.

To see how each component of the rf field contributes to ξ in Equation (3.48), let us first consider the effect of the longitudinal component (e_k^\parallel). The displacement of ξ_k^\parallel due to e_k^\parallel is related by $\xi_k^\parallel = G_k^{\parallel\parallel} e_k^\parallel$, the expression for which corresponds to that previously obtained by Crawford.⁷⁵ This shows that a perturbation due to e_k^\parallel becomes very large when ω_k approaches $(n \pm 1) \omega_c$, i.e., when the wave provides space harmonics of the electric field over the direction of the electron cyclotron motion. It may be seen that the contribution of e_k^\parallel to ξ_k^\perp is due to an effect similar to the electron drift in a static magnetic field ($\tilde{v}_D \sim \tilde{e}_k^\parallel \times B/B^2$).⁷⁶ The electric field component e_k^\parallel makes no contribution to ξ_k^Z . The perpendicular components (e_k^\perp and e_k^Z) exert effects on ξ_k similar to those of e_k^\parallel . However, they have perpendicular rf magnetic fields. Therefore, additional contributions are made by the Lorentz force ($\propto \tilde{v} \times \tilde{b}_k$). This can be seen by noting that the terms in Equation (3.49) giving these additional contributions all involve $k_\perp v_\perp / \omega_k$ or $k_\perp v_z / \omega_k$. With the help of Equation (3.16), $k_\perp v_\perp e_k^\perp / \omega_k$, $k_\perp v_\perp e_k^Z / \omega_k$ and $k_\perp v_z e_k^Z / \omega_k$ can be rewritten as $v_\perp b_{\perp k}^Z$, $-v_\perp b_{\perp k}^\perp$ and $-v_z b_{z k}^\perp$, respectively.

3.5.2 Dispersion Relations

For this case, the elements of matrix \vec{A} are obtained from Equation (3.41), with the base vector ($e_k^\parallel, e_k^\perp, e_k^Z$), as

$$\begin{aligned}
A_{\mathbf{k}}^{\parallel\parallel} &= 1 - \omega_N^2 \sum_n \frac{\langle J_n^2 \rangle}{\omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} , \\
A_{\mathbf{k}}^{\perp\perp} &= 1 - \eta_{\perp}^2 - \left(\frac{\omega_N}{\omega_{\mathbf{k}}} \right)^2 \sum_n \left\langle \frac{\omega_{\mathbf{k}}^2 + (k_{\perp} v_{\perp})^2}{\omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} J_n^2 + \frac{2\omega_{\mathbf{k}} k_{\perp} v_{\perp}}{\omega_{\mathbf{k},n} \omega_{\mathbf{k},n-1}} J_n J_{n-1} \right\rangle , \\
A_{\mathbf{k}}^{\text{ZZ}} &= 1 - \eta_{\perp}^2 - \left(\frac{\omega_N}{\omega_{\mathbf{k}}} \right)^2 \left[1 + k_{\perp}^2 \sum_n \frac{\langle (v_z J_n)^2 \rangle}{\omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} \right] , \\
A_{\mathbf{k}}^{\parallel\perp} &= A_{\mathbf{k}}^{\perp\parallel*} = -i \omega_N^2 \omega_c \sum_n \frac{\langle J_n^2 \rangle}{\omega_{\mathbf{k},n} \omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} , \\
A_{\mathbf{k}}^{\perp\text{Z}} &= A_{\mathbf{k}}^{\text{Z}\perp*} = i \frac{\omega_N^2 \omega_c k_{\perp}}{\omega_{\mathbf{k}}} \sum_n \frac{\langle v_z J_n^2 \rangle}{\omega_{\mathbf{k},n} \omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} , \\
A_{\mathbf{k}}^{\parallel\text{Z}} &= A_{\mathbf{k}}^{\text{Z}\parallel} = - \frac{\omega_N^2 k_{\perp}}{\omega_{\mathbf{k}}} \sum_n \frac{\langle v_z J_n^2 \rangle}{\omega_{\mathbf{k},n-1} \omega_{\mathbf{k},n+1}} . \tag{3.51}
\end{aligned}$$

The elements of the matrix $\vec{\tilde{A}}$ for perpendicular propagation have been obtained previously by Baldwin et al.⁷⁷ Their results can easily be reduced to Equation (3.51) by making use of suitable Bessel function identities [Equation (3.42)].

For an electron velocity distribution function even in v_z , the elements $A_{\mathbf{k}}^{\perp\text{Z}}$, $A_{\mathbf{k}}^{\text{Z}\perp}$, $A_{\mathbf{k}}^{\parallel\text{Z}}$ and $A_{\mathbf{k}}^{\text{Z}\parallel}$ all vanish, and $e_{\mathbf{k}}^{\text{Z}}$ is decoupled from $e_{\mathbf{k}}^{\parallel}$ and $e_{\mathbf{k}}^{\perp}$. The electric field $e_{\mathbf{k}}^{\text{Z}}$ is therefore a normal mode whose dispersion relation is determined by $A_{\mathbf{k}}^{\text{ZZ}} = 0$. In the cold plasma limit this reduces to the well known dispersion relation of the ordinary mode. For a nonzero temperature plasma, however, $A_{\mathbf{k}}^{\text{ZZ}}$ has poles at $\omega^2 = n^2 \omega_c^2$; $n = 1, 2, 3, \dots$, which lead to propagation in the neighborhoods of the cyclotron harmonics. For this reason, we shall refer to $e_{\mathbf{k}}^{\text{Z}}$ as the ordinary cyclotron harmonic wave. The case where longitudinal waves ($e_{\mathbf{k}}^{\parallel}$) are almost normal modes has been considered in a great detail by Baldwin et al.⁷⁷ By this we mean that there exist solutions of the

dispersion relation other than $e_{\mathbf{k}}^Z$; namely $A_{\mathbf{k}}^{\parallel\parallel} A_{\mathbf{k}}^{\perp\perp} - A_{\mathbf{k}}^{\parallel\perp} A_{\mathbf{k}}^{\perp\parallel} = 0$. To a good approximation these are the solutions of $A_{\mathbf{k}}^{\parallel\parallel} = 0$ and are known as longitudinal cyclotron harmonic waves (or Bernstein modes).

For the special case of an isotropic Maxwellian electron velocity distribution where

$$f_0 = n_0 \left(\frac{m}{2\pi\kappa T} \right)^{3/2} \exp \left[- \frac{m(v_{\perp}^2 + v_z^2)}{2\kappa T} \right], \quad (3.52)$$

one can readily obtain, upon substituting the integration identity⁷⁸

$$\int_0^{\infty} \lambda J_n^2(\lambda) \exp \left(- \frac{\lambda^2}{2a} \right) d\lambda = a \exp(-a) I_n(a), \quad (3.53)$$

into Equation (3.51), the dispersion relations for the longitudinal and ordinary cyclotron harmonic waves, respectively in the forms

$$1 = 2\omega_N^2 \frac{\exp(-k_{\perp}^2 \kappa T / m\omega_c^2)}{k_{\perp}^2 \kappa T / m\omega_c^2} \sum_{n=1}^{\infty} \frac{n^2 I_n(k_{\perp}^2 \kappa T / m\omega_c^2)}{\omega_{\mathbf{k},n} \omega_{\mathbf{k},-n}},$$

$$1 - \mathbf{N}_{\perp}^2 = \left(\frac{\omega_N}{\omega_c} \right)^2 \left[1 + 2\omega_c^2 \exp(-k_{\perp}^2 \kappa T / m\omega_c^2) \sum_{n=1}^{\infty} \frac{n^2 I_n(k_{\perp}^2 \kappa T / m\omega_c^2)}{\omega_{\mathbf{k},n} \omega_{\mathbf{k},-n}} \right]. \quad (3.54)$$

Dispersion curves corresponding to the first and second expressions in Equation (3.54) have been computed by Crawford⁷⁵ and Lee⁷⁹, respectively, for a range of plasma parameters. The second term on the RHS of the dispersion relation for the ordinary cyclotron harmonic wave is negligible except very near to cyclotron harmonics, i.e., at nearly all frequencies the dispersion relation is essentially that of the ordinary wave in a cold plasma.

Chapter IV

MICROSCOPIC TREATMENT

II. NONLINEAR WAVE-WAVE INTERACTIONS

The linear theory of Chapter III assumes that an arbitrary perturbation can be expressed as a superposition of non-interacting eigenmodes. A characteristic result of nonlinearity is the occurrence of coupling among these modes. If the coupling is weak, then arbitrary perturbations may still be expressed approximately as a superposition of eigenmodes, but whose amplitudes are now assumed to vary slowly in time and space. By 'slowly' we mean that the scale of the variation due to the wave-wave interaction is much longer in time and space than the period and wavelength of the disturbance. We shall assume that the ratio of this scale is comparable to the ratio of the amplitude of a perturbation to that of the zero order quantity, and denote them by a small parameter ϵ . We shall also assume that interaction occurs coherently, i.e., we shall not treat interaction in the random phase approximation described in Section 1.2. We shall consider all possible types of three-wave interactions in a warm, collisionless magnetoplasma, with an immobile ion background.

The procedure to be followed is similar to that used by Dysthe in the analysis of nonlinear wave interaction in a cold plasma.³⁵ The averaged Lagrangian is first obtained in Section 4.1 for waves propagating at an oblique angle to the static magnetic field. It was demonstrated in Chapter III that action is conserved in the linear theory. When nonlinear interaction occurs, action is exchanged among waves due to the wave coupling. The action transfer equations and the coupled mode equations are derived in Section 4.2. In Section 4.3, these equations are specialized to the case of waves propagating nearly parallel to the

magnetic field. In Section 4.4, the interaction among three cyclotron harmonic waves is considered. Coupled mode equations are presented in explicit forms suitable for computation.

4.1 Averaged Lagrangian Density

Taking the slow variation of wave amplitudes due to the mode coupling into account, we may modify Equation (3.1) to the form

$$\begin{aligned}\xi(\underline{r}, \underline{v}, t) &= \sum_{\underline{k}} \Xi_{\underline{k}}(\underline{e}\underline{r}, \underline{v}, \epsilon t) \exp[i\theta_{\underline{k}}(\underline{r}, t)] , \\ a(\underline{r}, t) &= \sum_{\underline{k}} A_{\underline{k}}(\underline{e}\underline{r}, \epsilon t) \exp[i\theta_{\underline{k}}(\underline{r}, t)] ,\end{aligned}\quad (4.1)$$

where the slowly varying amplitudes, $\Xi_{\underline{k}}(\underline{e}\underline{r}, \underline{v}, \epsilon t)$ and $A_{\underline{k}}(\underline{e}\underline{r}, \epsilon t)$, may be written in terms of the Fourier amplitudes of linear theory, $\xi_{\underline{k}}(\underline{v})$ and $a_{\underline{k}}$, as

$$\begin{aligned}\Xi_{\underline{k}}(\underline{e}\underline{r}, \underline{v}, \epsilon t) &= \alpha_{\underline{k}}(\underline{e}\underline{r}, \epsilon t) [\xi_{\underline{k}}(\underline{v}) + \epsilon \beta_{\underline{k}}(\underline{v})] , \\ A_{\underline{k}}(\underline{e}\underline{r}, \epsilon t) &= \alpha_{\underline{k}}(\underline{e}\underline{r}, \epsilon t) a_{\underline{k}} .\end{aligned}\quad (4.2)$$

Here $\alpha_{\underline{k}}$ and $\beta_{\underline{k}}$ can be regarded as correction factors accounting for the nonlinear effects, and

$$\alpha_{-\underline{k}} = \alpha_{\underline{k}}^* , \quad \beta_{-\underline{k}} = \beta_{\underline{k}}^* .\quad (4.3)$$

Consideration of the Lagrangian density expanded to second order in the perturbations sufficed for the linear theory of Chapter III. Information on nonlinear plasma phenomena is contained in the terms higher than second order. For sufficiently weak plasma nonlinearity, the third order term in the Lagrangian is much larger than those of higher orders, so only the second and third order terms need be considered here.

The second and the third order terms in the averaged Lagrangian

density are obtained by substituting Equation (4.1) into Equations (2.26) and (2.28), and taking the average over a scale much longer in time and space than the period of the fast varying quantities, but much smaller than the scale of the slow amplitude variation due to nonlinear interaction. All terms in $\bar{\mathcal{L}}_3^P$ vanish, except those satisfying the resonance condition

$$\theta_k + \theta_{k'} + \theta_{k''} = 0, \quad (4.4)$$

which implies the synchronism conditions

$$k + k' + k'' = 0, \quad \omega_k + \omega_{k'} + \omega_{k''} = 0. \quad (4.5)$$

For notational convenience, we will define $\Lambda_k^{(2)}$ such that

$$\bar{\mathcal{L}}_k^{(2)} = \Lambda_k^{(2)} + \Lambda_{-k}^{(2)}, \quad \Lambda_{-k}^{(2)} = \Lambda_k^{(2)*}. \quad (4.6)$$

Although $\bar{\mathcal{L}}_k$ is always a real quantity, Λ_k is not necessarily real. To first order in ϵ , the part of the averaged Lagrangian density governing a weakly nonlinear plasma, i.e., the sum of the second and third order terms, is given by

$$\begin{aligned} \bar{\mathcal{L}} = & \sum_k |\alpha_k|^2 \Lambda_k^{(2)} + \epsilon \sum_k \alpha_{-k} \left[\frac{\partial \alpha_k}{\partial t} \frac{\partial}{\partial(i\omega_k)} + \frac{\partial \alpha_k}{\partial r} \cdot \frac{\partial}{\partial(-ik)} \right] \Lambda_k^{(2)} \\ & + \epsilon \sum_{k+k'+k''=0} \alpha_k \alpha_{k'} \alpha_{k''} \Lambda_{kk'k''}^{(3)} \exp[i(\theta_k + \theta_{k'} + \theta_{k''})], \end{aligned} \quad (4.7)$$

where the Λ 's are given by

$$\Lambda_{\mathbf{k}}^{(2)} = \int d^3v f_0 \Lambda_{\mathbf{k}}^{P(2)} + \Lambda_{\mathbf{k}}^{F(2)}, \quad \Lambda_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{(3)} = \int d^3v f_0 \Lambda_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{P(3)},$$

$$\Lambda_{\mathbf{k}}^{P(2)} = \frac{m}{2} |D_{\mathbf{k},0} \tilde{\xi}_{\mathbf{k}}|^2 + \frac{q}{2} (D_{\mathbf{k},0} \tilde{\xi}_{\mathbf{k}}) \cdot \tilde{\xi}_{\mathbf{k}}^* \times \tilde{\mathbf{B}} - q \tilde{\xi}_{\mathbf{k}}^* \cdot (\tilde{\mathbf{e}}_{\mathbf{k}} + \tilde{\mathbf{v}} \times \tilde{\mathbf{b}}_{\mathbf{k}}),$$

$$\Lambda_{\mathbf{k}}^{F(2)} = \frac{1}{2} \epsilon_0 |\mathbf{e}_{\mathbf{k}}|^2 - \frac{1}{2\mu_0} |\mathbf{b}_{\mathbf{k}}|^2,$$

$$\Lambda_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{P(3)} = \frac{q}{2} (D_{\mathbf{k},0} \tilde{\xi}_{\mathbf{k}}) \cdot \tilde{\xi}_{\mathbf{k}'} \times \tilde{\mathbf{b}}_{\mathbf{k}''} + i \frac{q}{2} \tilde{\xi}_{\mathbf{k}} \tilde{\xi}_{\mathbf{k}'} : \tilde{\mathbf{k}}'' (\tilde{\mathbf{e}}_{\mathbf{k}''} + \tilde{\mathbf{v}} \times \tilde{\mathbf{b}}_{\mathbf{k}''}). \quad (4.8)$$

In Equation (4.7), the term involving $\beta_{\mathbf{k}}$ has disappeared due to the first order equation of motion.

4.2 Action Transfer and Coupled Mode Equations

Variations of $\bar{\mathcal{L}}$ in Equation (4.7) with respect to $\theta_{\mathbf{k}}, \theta_{\mathbf{k}'},$ and $\theta_{\mathbf{k}''}$ give

$$\begin{aligned} \frac{\partial}{\partial t} (|\alpha_{\mathbf{k}}|^2 N_{\mathbf{k}}) + \frac{\partial}{\partial \tilde{\mathbf{r}}} \cdot (\tilde{\mathbf{v}}_{\mathbf{k}} |\alpha_{\mathbf{k}}|^2 N_{\mathbf{k}}) &= \frac{\partial}{\partial t} (|\alpha_{\mathbf{k}'}|^2 N_{\mathbf{k}'}) + \frac{\partial}{\partial \tilde{\mathbf{r}}} \cdot (\tilde{\mathbf{v}}_{\mathbf{k}'} |\alpha_{\mathbf{k}'}|^2 N_{\mathbf{k}'}) \\ &= \frac{\partial}{\partial t} (|\alpha_{\mathbf{k}''}|^2 N_{\mathbf{k}''}) + \frac{\partial}{\partial \tilde{\mathbf{r}}} \cdot (\tilde{\mathbf{v}}_{\mathbf{k}''} |\alpha_{\mathbf{k}''}|^2 N_{\mathbf{k}''}) = i \left(\alpha_{\mathbf{k}} \alpha_{\mathbf{k}'} \alpha_{\mathbf{k}''} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{k}''} \Lambda_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{(3)} + \text{c.c.} \right). \end{aligned} \quad (4.9)$$

where the summation is understood to contain all permutations of $\mathbf{k}, \mathbf{k}', \mathbf{k}''$.

The first two equations are the so-called Manley-Rowe relations,⁸

giving information about the relative rates of transfer of action density $(|\alpha_{\mathbf{k}}|^2 N_{\mathbf{k}})$ between the waves. The third equation gives the direction and the absolute value of the rate of transfer.

Variation of $\bar{\mathcal{L}}$ with respect to $\alpha_{-\mathbf{k}}$ gives

$$iN_k \left(\frac{\partial}{\partial t} + \underline{v}_k \cdot \frac{\partial}{\partial \underline{r}} \right) \alpha_k = \alpha_{-k'} \alpha_{-k''} \sum_{k, k', k''} \Lambda_{-k-k'-k''}^{(3)} \quad (4.10)$$

Similar equations for the k' and k'' -waves are derived from the $\alpha_{-k'}$ and $\alpha_{-k''}$ variations of $\bar{\mathcal{L}}$, respectively. Equation (4.10) is the coupled mode equation. It should be noted that the action transfer equation [Equation (4.9)] can be derived from the coupled mode equation by using Equation (3.12).

Since N_k , \underline{v}_k and $\Lambda_{-k-k'-k''}^{(3)}$ are known for any wave, Equation (4.10) can be solved in principle. The expressions for N_k , \underline{v}_k and $\Lambda_{-k,-k',-k''}^{(3)}$ in terms of field quantities are extremely difficult to evaluate in general, as indicated in Chapter III. A simpler form of Equation (4.10) is obtained in the limiting cases of waves propagating either nearly parallel or exactly perpendicular to the static magnetic field. These will be considered in the following sections.

4.3 Quasiparallel Propagation ($k_{\perp} \ll k_z$)

Substitution of Equations (3.36), (3.45) and (4.8), and some tedious but straightforward algebra, reduces Equation (4.10) to the following form first order in k_{\perp} ,

$$\begin{aligned} & iN_k \left(\frac{\partial}{\partial t} + \underline{v}_k \cdot \frac{\partial}{\partial \underline{r}} \right) \alpha_k \\ &= \alpha_{-k'} \alpha_{-k''} \sum_{k, k', k''} \left(Q_{kk'k''}^Z e_k^{Z*} e_{k'}^{Z*} e_{k''}^{Z*} + Q_{kk'k''}^R e_k^{L*} e_{k'}^{R*} e_{k''}^{Z*} + Q_{kk'k''}^L e_k^{R*} e_{k'}^{L*} e_{k''}^{Z*} \right. \\ & \left. + Q_{kk'k''}^{RR} e_k^{L*} e_{k'}^{R*} e_{k''}^{R*} + Q_{kk'k''}^{LL} e_k^{R*} e_{k'}^{L*} e_{k''}^{L*} + Q_{kk'k''}^{RZ} e_k^{L*} e_{k'}^{Z*} e_{k''}^{Z*} + Q_{kk'k''}^{LZ} e_k^{R*} e_{k'}^{Z*} e_{k''}^{Z*} \right) \end{aligned} \quad (4.11)$$

where the Q's are given by

$$Q_{kk'k''}^R = Q_{-k-k'-k''}^{L*} = i\epsilon_0 \frac{q}{m} k_z \frac{\omega_N^2}{\omega_k \omega_{k'}} \left\langle \frac{\omega_{k',0}}{\omega_{k',0}^2 \omega_{k',-1}} - \frac{k_z' v_{\perp}^2}{2\omega_{k',-1}^2} \left(\frac{k_z}{\omega_{k',0}^2} + \frac{k_z''}{2\omega_{k,1}^2} \right) \right\rangle,$$

$$Q_{kk'k''}^Z = -i \frac{\epsilon_0 q}{2m} k_z'' \omega_N^2 \left\langle \frac{1}{\omega_{k,0} \omega_{k',0}} \right\rangle,$$

$$\begin{aligned} Q_{kk'k''}^{RR} = Q_{-k-k'-k''}^{LL*} = & -\frac{i\epsilon_0 q}{\sqrt{2m}} \frac{k_z'' \omega_N^2 \exp(i\phi_{k''})}{\omega_k \omega_{k'} \omega_{k''}} \left\langle \frac{1}{\omega_{k,1} \omega_{k',-1} \omega_{k'',-1}} \left\{ \omega_{k,0} \omega_{k',0} \right. \right. \\ & + \frac{1}{4} k_z k_z' k_z'' v_{\perp}^4 \left[k_z'' \left(\frac{1}{\omega_{k,2}^2} + \frac{1}{\omega_k \omega_{k,2}} - \frac{1}{\omega_{k,1} \omega_{k',0}} - \frac{1}{\omega_{k,2} \omega_{k'',-1}} + \frac{1}{\omega_{k'',0}^2} + \frac{1}{\omega_{k'',0} \omega_{k'',-1}} \right) \right. \\ & + \left. k_z' \left(\frac{1}{\omega_{k,2}^2} + \frac{1}{\omega_{k,1} \omega_{k,2}} - \frac{1}{\omega_{k,1} \omega_{k'',0}} - \frac{1}{\omega_{k,2} \omega_{k',-1}} + \frac{1}{\omega_{k',-1} \omega_{k'',0}} \right) \right] \\ & + \frac{1}{2} (k_z'' v_{\perp})^2 \left(\frac{\omega_{k',0}}{\omega_{k,1}} - \frac{\omega_{k',0}}{\omega_{k'',-1}} + \frac{\omega_{k',0} \omega_{k,1}}{\omega_{k'',0} \omega_{k'',-1}} + \frac{\omega_{k',0} \omega_{k,1}}{\omega_{k'',0}^2} \right) + \frac{1}{2} (k_z' v_{\perp})^2 \left(\frac{\omega_{k',0}}{\omega_{k,1}} + \frac{\omega_{k,0}}{\omega_{k',-1}} \right) \\ & \left. \left. - \frac{1}{2} k_z' k_z'' v_{\perp}^2 \left(\frac{2\omega_{k'',-1}}{\omega_{k,1}} + \frac{2\omega_{k,0}}{\omega_{k,2}} + \frac{\omega_{k',1}}{\omega_{k'',0}} \right) \right\} \right\rangle, \end{aligned}$$

$$\begin{aligned} Q_{kk'k''}^{RZ} = Q_{-k-k'-k''}^{LZ*} = & \frac{i\epsilon_0 q}{\sqrt{2m}} \frac{k_z'' \omega_N^2 \exp(i\phi_{k''})}{\omega_k} \left\langle \frac{1}{\omega_{k''} \omega_{k,1} \omega_{k',0}} \left(1 + \frac{k_z' v_{\perp}}{\omega_{k',0}} - \frac{k_z v_{\perp}}{\omega_{k'',-1}} \right) \right. \\ & + \frac{1}{2} k_z k_z'' v_{\perp}^2 \left[\frac{1}{\omega_{k',0}^2 \omega_{k'',0} \omega_{k'',-1}} \left(\frac{1}{\omega_{k'',-1}} + \frac{1}{\omega_{k'',0}} \right) + \frac{1}{\omega_{k,0} \omega_{k,1} \omega_{k',0}^2} \left(\frac{1}{\omega_{k,1}} + \frac{1}{\omega_{k,0}} \right) \right. \\ & \left. + \frac{1}{\omega_{k''} \omega_{k,1} \omega_{k',0} \omega_{k'',-1}} \left(\frac{1}{\omega_{k'',-1}} - \frac{1}{\omega_{k,1}} \right) \right] \\ & + \frac{1}{2} k_z k_z' v_{\perp}^2 \left[\frac{1}{\omega_{k',0}^2 \omega_{k'',0} \omega_{k'',-1}} \left(\frac{1}{\omega_{k'',-1}} + \frac{1}{\omega_{k'',0}} \right) + \frac{1}{\omega_{k,0} \omega_{k,1} \omega_{k',0}^2} \left(\frac{1}{\omega_{k,1}} + \frac{1}{\omega_{k,0}} \right) \right. \\ & \left. \left. + \frac{1}{\omega_{k''} \omega_{k,1} \omega_{k',0} \omega_{k'',-1}} \left(\frac{1}{\omega_{k',0}} - \frac{1}{\omega_{k,1}} \right) - \frac{1}{\omega_{k,1}^2 \omega_{k'',0} \omega_{k'',-1}} \left(\frac{1}{\omega_{k'',-1}} + \frac{1}{\omega_{k'',0}} \right) \right] \right\rangle. \end{aligned}$$

(4.12)

For illustrative purposes, the synchronism relations [Equation (4.5)] may be written as

$$\tilde{k}_p = \tilde{k}_s + \tilde{k}_i, \quad \omega_p = \omega_s + \omega_i. \quad (4.13)$$

In conventional parametric amplification terminology, the highest frequency component is called the 'pump' wave, and the lower frequencies the 'signal' and 'idler' waves.¹⁰ With the use of Equation (4.13), the coupled mode equations for all possible cases can then be written in the form

$$\begin{aligned} I_p^U \left(\frac{d}{dt} + \tilde{v}_p^U \cdot \frac{d}{dr} \right) e_p^U &= c_{UVW}^* e_s^V e_i^W, \\ I_s^V \left(\frac{d}{dt} + \tilde{v}_s^V \cdot \frac{d}{dr} \right) e_s^V &= c_{UVW} e_i^{W*} e_p^U, \\ I_i^W \left(\frac{d}{dt} + \tilde{v}_i^W \cdot \frac{d}{dr} \right) e_i^W &= c_{UVW} e_s^{V*} e_p^U, \end{aligned} \quad (4.14)$$

from Equation (4.11), where U, V and W can be either R, L or P; the coupling coefficient c_{UVW} is the proper linear combination of Q's in the RHS of Equation (4.11); \tilde{v}_k^U (U = R, L, P) is the group velocity of the U-wave, and I_k^U and the slowly varying electric field, e_k^U , are defined by

$$I_k^U \equiv i \frac{N_k^U}{|e_k^U|^2}, \quad e_k^U(\epsilon r, \epsilon t) = \alpha_k(\epsilon r, \epsilon t) e_k^U, \quad \begin{aligned} &(U = R, L \text{ or } P; \\ &k = s, i \text{ or } p). \end{aligned} \quad (4.15)$$

Here e_k^Z must be replaced by e_k^P when it represents the electron plasma wave. The explicit expression for I_k^U for right-hand polarized, left-hand polarized, and plasma waves can be obtained from Equations (3.28) and (3.44) to yield

$$\begin{aligned}
I_k^R = I_{-k}^{L*} &= 2i\epsilon_0 \frac{1}{\omega_k} \left\{ 1 + \frac{\omega_N^2}{2\omega_k} \left\langle \frac{1}{\omega_{k,-1}} \left[\omega_c + \frac{(k_z v_\perp)^2}{\omega_{k,-1}} \right] \right\rangle \right\}, \\
I_k^P &= 2i\epsilon_0 \omega_N^2 \left\langle \frac{1}{\omega_{k,0}} \right\rangle. \tag{4.16}
\end{aligned}$$

Equation (4.11) indicates that nonlinear interaction is possible (i) among three plasma waves, (ii) among one plasma wave and two circularly-polarized waves, (iii) among three circularly-polarized waves, and (iv) among two plasma waves and one circularly-polarized wave. The problem now remaining is to find the coupling coefficient in the coupled mode equation [Equation (4.14)] for each of these cases:

Case (i) (C_{PPP}): To first order in k_\perp , all terms but the first one on the RHS of Equation (4.11) can be dropped in this case. The coupling coefficient then becomes

$$C_{PPP} = \sum_{-p,s,i} Q_{-psi}^Z, \tag{4.17}$$

where Q_{-psi}^Z is $Q_{kk'k''}^Z$ in Equation (4.12), with $k = -p$, $k' = s$, and $k'' = i$. Since $Q_{kk'k''}^Z$ does not involve k_\perp , the coupling coefficient remains unchanged for waves propagating exactly parallel to the magnetic field.

Case (ii) ($C_{RRP}, C_{RPR}, C_{LLP}, C_{LPL}, C_{PRL}, C_{PLR}$): In this case, only the second and third terms on the RHS of Equation (4.11) contribute to the coupling coefficient, which becomes

$$C_{RRP}(p,s,i) = C_{LLP}^*(-p,-s,-i) = (Q_{-psi}^R - Q_{-sp-i}^R), \tag{4.18}$$

$$C_{PRL} = (Q_{is-p}^R + Q_{si-p}^L). \tag{4.19}$$

C_{RPR} , C_{LPL} and C_{PLR} are obtained from C_{RRP} , C_{LLP} and C_{PRL} , respectively, by interchanging s and i . As in Case (i), the coupling coefficients are the same for $k_{\perp} \approx 0$ and $k_{\perp} = 0$.

Case (iii) (C_{RRR} , C_{LLL} , C_{RRL} , C_{RLR} , C_{LLR} , C_{LRL}): C_{LLL} is not of interest because the topology of the dispersion characteristics does not allow the synchronism conditions of Equation (4.5) to be satisfied.²⁷ To obtain the coupling coefficients, the e^Z 's have to be eliminated from the RHS of Equation (4.11) by use of the small signal wave equation [Equations (3.40) and (3.47)],

$$e_k^Z = -\frac{1}{A_k^{ZZ}} (A_k^{ZR} e_k^R + A_k^{ZL} e_k^L). \quad (4.20)$$

When this is done, the coupling coefficients C_{RRR} , C_{RRL} and C_{LLR} are found to be

$$C_{RRR} = \sum_{s,i} \left[Q_{-psi}^{RR} - \frac{A_i^{RZ}}{A_i^{ZZ}} (Q_{-psi}^R + Q_{s-pi}^L) \right], \quad (4.21)$$

$$C_{RRL}(p,s,i) = C_{LLR}^*(-p,-s,-i) = \frac{1}{2} \sum_{-p,i} \left[Q_{si-p}^{LL} - \frac{A_p^{ZR}}{A_p^{ZZ}} (Q_{si-p}^L + Q_{is-p}^R) \right], \quad (4.22)$$

where A_k^{ZZ} , A_k^{RZ} and A_k^{ZR} are given by Equation (3.47). The coupling coefficients C_{RLR} and C_{LRL} are obtained from C_{RRL} and C_{LLR} , respectively, by interchanging s and i . The coupling vanishes as $k_{\perp} \rightarrow 0$ in this case.

Case (iv) (C_{RPP} , C_{LPP} , C_{PRP} , C_{PPR} , C_{PLP} , C_{PPL}): All terms on the RHS of Equation (4.11) must be changed into the form involving two e^Z 's with the use of the wave equation [Equation (3.40)]. This yields

$$C_{RPP}(p, s, i) = C_{LPP}^*(-p, -s, -i)$$

$$= \frac{1}{2} \sum_{s, i} \left[Q_{-psi}^{RZ} - \frac{A_s^{ZR}}{A_s^{RR}} (Q_{-psi}^R + Q_{s-pi}^L) - \frac{A_p^{ZR}}{A_p^{ZZ}} Q_{si-p}^Z \right], \quad (4.23)$$

$$C_{PRP}(p, s, i) = C_{PLP}^*(-p, -s, -i)$$

$$= \frac{1}{2} \sum_{-p, i} \left[Q_{si-p}^{LZ} - \frac{A_i^{ZL}}{A_i^{LL}} (Q_{si-p}^L + Q_{is-p}^R) - \frac{A_s^{RZ}}{A_s^{ZZ}} Q_{i-ps}^Z \right], \quad (4.24)$$

where A_k^{RR} , A_k^{LL} , A_k^{RZ} , A_k^{ZR} and A_k^{ZL} are given by Equation (3.47). We may obtain C_{PPR} and C_{PPL} from C_{PRP} and C_{PLP} , respectively, by interchanging s and i . The coupling vanishes as $k_{\perp} \rightarrow 0$.

4.4 Perpendicular Propagation ($k_z = 0$)

As illustrated in Chapter III, for propagation oblique to the static magnetic field the polarizations of normal modes are extremely complicated. We shall study the relatively simple case of an electron velocity distribution even in v_z . In this situation, e_k^{\parallel} and e_k^Z can be considered as normal modes. These were introduced and termed the longitudinal and ordinary cyclotron harmonic waves, respectively, in Section 3.5.2. As remarked at the end of that subsection, thermal effects on the propagation of the ordinary cyclotron harmonic wave are generally negligible. Therefore, for the coupling coefficients for the interaction involving the ordinary cyclotron harmonic wave, we shall approximate the wave by the ordinary mode for a cold plasma (see end of Section 4.5).

Since the phase velocity of the ordinary wave is much greater than the particle velocity, Equation (3.48) becomes

$$\xi_{\mathbf{k}}^{\parallel} = G_{\mathbf{k}}^{\parallel\parallel} e_{\mathbf{k}}^{\parallel} , \quad \xi_{\mathbf{k}}^{\perp} = G_{\mathbf{k}}^{\perp\parallel} e_{\mathbf{k}}^{\parallel} , \quad \xi_{\mathbf{k}}^{\mathbf{Z}} = G_{\mathbf{k}}^{\mathbf{ZZ}} e_{\mathbf{k}}^{\mathbf{Z}} , \quad (4.25)$$

where the G's are given by

$$G_{\mathbf{k}}^{\parallel\parallel} = \frac{q}{m} P_{-1,1}^n , \quad G_{\mathbf{k}}^{\perp\parallel} = i \frac{q}{m} P_{0,1,-1}^n , \quad G_{\mathbf{k}}^{\mathbf{ZZ}} = \frac{q}{m\omega_{\mathbf{k}}^2} . \quad (4.26)$$

Equation (4.10) is then written as

$$iN_{\mathbf{k}} \left(\frac{\partial}{\partial t} + \mathbf{v}_{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{r}} \right) \alpha_{\mathbf{k}} = \alpha_{-\mathbf{k}} \alpha_{-\mathbf{k}''} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{k}''} \left(Q_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\parallel} e_{\mathbf{k}}^{\parallel*} e_{\mathbf{k}'}^{\parallel*} e_{\mathbf{k}''}^{\parallel*} + Q_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\mathbf{O}} e_{\mathbf{k}}^{\parallel*} e_{\mathbf{k}'}^{\mathbf{O}*} e_{\mathbf{k}''}^{\mathbf{O}*} \right) , \quad (4.27)$$

where the superscript Z has changed to O to represent the ordinary cyclotron harmonic wave, and the Q's are given by

$$\begin{aligned} Q_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\parallel} &= -\frac{i}{2} n_0 q k_{\perp} \left\langle \left[G_{-\mathbf{k}}^{\parallel\parallel} \cos(\phi_{\mathbf{k}} - \phi_{\mathbf{k}''}) - G_{-\mathbf{k}}^{\perp\parallel} \sin(\phi_{\mathbf{k}} - \phi_{\mathbf{k}''}) \right] \right. \\ &\quad \times \left. \left[G_{-\mathbf{k}'}^{\parallel\parallel} \cos(\phi_{\mathbf{k}'} - \phi_{\mathbf{k}''}) - G_{-\mathbf{k}'}^{\perp\parallel} \sin(\phi_{\mathbf{k}'} - \phi_{\mathbf{k}''}) \right] \right\rangle , \\ Q_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\mathbf{O}} &= \frac{\epsilon_0}{2} k_{\perp} \frac{\omega_N^2}{\omega_{\mathbf{k}}^2 \omega_{\mathbf{k}''}^2} \left\langle \left[D_{-\mathbf{k},0} + i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}''}) \right] \left[G_{-\mathbf{k}}^{\parallel\parallel} \cos(\phi_{\mathbf{k}} - \phi_{\mathbf{k}''}) - G_{-\mathbf{k}}^{\perp\parallel} \sin(\phi_{\mathbf{k}} - \phi_{\mathbf{k}''}) \right] \right\rangle . \end{aligned} \quad (4.28)$$

Substituting Equations (4.5) and (4.26), using the Bessel function identities of Equation (3.42), and noting that

$$(D_{\mathbf{k},0} - i\omega_{\mathbf{k},n}) \exp\{i[n(\phi - \phi_{\mathbf{k}}) + \lambda_{\mathbf{k}} \sin(\phi - \phi_{\mathbf{k}})]\} = 0 ,$$

$$\int_0^{2\pi} \exp\{i[n\phi + \lambda_{\mathbf{k}} \sin(\phi - \phi_{\mathbf{k}}) + \lambda_{\mathbf{k}'} \sin(\phi - \phi_{\mathbf{k}'})]\} d\phi = 2\pi J_n(\lambda_{\mathbf{k}'}) \exp(in\phi_{\mathbf{k}''}) , \quad (4.29)$$

we can reduce Equation (4.28) to the form

$$\begin{aligned}
Q_{kk'k''}^{\parallel} &= i \frac{\epsilon_0 q}{8m} k''_{\perp} \omega_N^2 \sum_{m,n} (-1)^{m+n} \frac{\langle J_m(\lambda_k) J_n(\lambda_{k'}) J_{m+n}(\lambda_{k''}) \rangle}{\omega_{k,m} \omega_{k',n}} \\
&\times \left(\frac{\exp[i(\phi_k - \phi_{k''})]}{\omega_{k,m+1}} + \frac{\exp[-i(\phi_k - \phi_{k''})]}{\omega_{k,m-1}} \right) \\
&\times \left(\frac{\exp[-i(\phi_{k'} - \phi_{k''})]}{\omega_{k',n+1}} + \frac{\exp[-i(\phi_{k'} - \phi_{k''})]}{\omega_{k',n-1}} \right) \exp\{i[m(\phi_k - \phi_{k''}) + n(\phi_{k'} - \phi_{k''})]\}, \\
Q_{kk'k''}^{\circ} &= i \frac{\epsilon_0 q}{4m} k''_{\perp} \frac{\omega_N^2}{\omega_k \omega_{k''}} \sum_n \left(2-n \frac{\omega_c}{\omega_{k'}} \right) \frac{\langle J_n^2(\lambda_k) \rangle}{\omega_{k,n}} \\
&\times \left(\frac{\exp[i(\phi_k - \phi_{k''})]}{\omega_{k,n+1}} + \frac{\exp[-i(\phi_k - \phi_{k''})]}{\omega_{k,n-1}} \right). \tag{4.30}
\end{aligned}$$

As in Equation (4.10), the summation in Equation (4.27) contains all permutations of k, k', k'' , while those in Equation (4.30) are double summations of m and n over all positive and negative integers.

Equation (4.27) can be written in the form of Equation (4.14) with

$$\begin{aligned}
I_k^{\parallel} &= i \epsilon_0 \omega_N^2 \sum_n \left(\frac{1}{\omega_{k,n-1}} + \frac{1}{\omega_{k,n+1}} \right) \frac{\langle J_n^2(\lambda_k) \rangle}{\omega_{k,n-1} \omega_{k,n+1}}, \\
I_k^{\circ} &= i \frac{\epsilon_0}{\omega_k} \left[2 - \left(\frac{k_{\perp} c}{\omega_k} \right)^2 \right]. \tag{4.31}
\end{aligned}$$

These result form substitution of Equations (3.28) and (3.44) into Equation (4.15). Possible interactions are, from Equation (4.27), (i) among three longitudinal cyclotron harmonic waves, (ii) among one longitudinal and two ordinary cyclotron harmonic waves. The coupling

coefficient for each of these cases can readily be obtained from Equation (4.27):

Case (i) ($C_{\parallel\parallel\parallel}$): In this case, only the first term on the RHS of Equation (4.27) contributes to the coupling coefficient, which becomes

$$C_{\parallel\parallel\parallel} = \sum_{-p,s,i} Q_{-pis}^{\parallel} , \quad (4.32)$$

where Q_{-pis}^{\parallel} is $Q_{kk'k''}^{\parallel}$ with $k = -p$, $k' = s$, $k'' = i$.

Case (ii) ($C_{\parallel\parallel 00}$, $C_{O\parallel 0}$, $C_{00\parallel}$): In this case, only the second term on the RHS of Equation (4.27) contributes to the coupling coefficient, which becomes

$$C_{\parallel\parallel 00} = \sum_{s,i} Q_{-psi}^0 , \quad (4.33)$$

$$C_{O\parallel 0} = \sum_{-p,i} Q_{si-p}^0 . \quad (4.34)$$

We can obtain $C_{00\parallel}$ from $C_{O\parallel 0}$ by interchanging s and i .

The infinite series forms of the Q 's in Equation (4.28) are impractical for computation. Alternative expressions for them are obtained in Appendix B.

4.5 Discussion

We have been following essentially the averaged Lagrangian method developed in cold plasma wave theory. Since all variables are defined in phase space, the method is modified for warm plasma by adding the correction factor β_k , which accounts for nonlinear

effects on velocity. However, it has not produced any additional Euler-Lagrange equations because the term involving β_k has disappeared from the averaged Lagrangian in Equation (4.7).

General solutions of the coupled mode equations [Equation (4.14)] are complicated, but can be obtained in terms of elliptic integrals.⁴ For the special case where one wave is considerably stronger than the other two waves, however, solutions can be obtained in a simple form. If the pump wave is very much stronger than the signal and idler, then the rate of change of \mathcal{E}_p^U is of order $\mathcal{E}_s^V \mathcal{E}_i^W$, and \mathcal{E}_p^U may be assumed to be constant. The signal and idler waves are determined by the equation

$$\left[\left(\frac{\partial}{\partial t} + \tilde{v}_s^V \cdot \frac{\partial}{\partial \tilde{r}} \right) \left(\frac{\partial}{\partial t} + \tilde{v}_i^W \cdot \frac{\partial}{\partial \tilde{r}} \right) - \hat{\gamma}^2 \right] \mathcal{E}_{s,i}^{V,W} = 0, \quad (4.35)$$

where the growth (or decay) rate, $\hat{\gamma}$, is given by $\{ |C_{UVW} \mathcal{E}_p^U|^2 / I_s^V I_i^{W*} \}^{1/2}$. The electric fields \mathcal{E}_s^V and \mathcal{E}_i^W , are either growing or decaying, depending on the sign of $I_s^V I_i^{W*}$. From Equations (3.22) and (4.15), this sign is the same as that of $[\omega_s \omega_i K_s^V K_i^W]$ where K_s^V and K_i^W are the energies of the signal and idler waves, i.e., \mathcal{E}_s^V and \mathcal{E}_i^W are exponentially growing (decaying) when $[\omega_s \omega_i K_s^V K_i^W] > 0 (< 0)$. The growth (or decay) rate can be calculated from Equation (4.35) for a specific unperturbed electron velocity distribution, and wave numbers and frequencies satisfying the synchronism conditions.

Table 4.1 shows all possible nonlinear interactions among normal modes propagating nearly parallel or precisely perpendicular to the static magnetic field. The third column of Table (a) indicates the

Table 4.1.

All Possible Nonlinear Interactions Among Normal Modes*

(a) Quasiparallel Propagation

Involving Waves	Pump-Signal and Idler	Coupling for $k_{\perp}=0$	Previous Investigation	
			microscopic treatment	cold plasma or hydrodynamic approximation
(i) Three plasma waves	P P,P	yes	Refs.28,33,34	Refs. 17,20
(ii) One plasma wave and two circularly- polarized waves	R R,P L L,P P R,L	yes	Refs. 33,34	Ref. 26
(iii) Three circularly- polarized waves	R R,R	no	Ref. 34	Ref. 27
	R R,L			
	L R,L			
(iv) Two plasma waves and one circularly- polarized wave	R P,P L P,P P R,P P L,P	no	None	None

(b) Perpendicular Propagation

Involving Waves	Pump-Signal and Idler†	Previous Investigation	
		microscopic treatment	cold plasma or hydrodynamic approximation
(i) Three longitudinal cyclotron harmonic waves	,	Refs. 32,80	Ref. 25
(ii) One longitudinal and two ordinary cyclotron harmonic waves	O ,O	Refs. 30,65	Ref. 22
	O,O	None	Ref. 22

* (For the electron velocity distribution function even in v_z . Interactions between parallel propagating waves and perpendicularly propagating waves are not included.)

† (||: Longitudinal cyclotron harmonic wave - extraordinary wave in cold plasmas,
O: Ordinary cyclotron harmonic wave - ordinary wave in cold plasmas.)

interaction in the limit of $k_{\perp} = 0$. As shown in Section 4.3, when three plasma waves or one plasma wave and two circularly-polarized waves are involved, interactions still occur for all waves propagating exactly parallel to the static magnetic field, and their coupling coefficients are the same as those for quasiparallel propagation. The remainder of the interactions in the table vanish for exact parallel propagation because their coupling coefficients are first order in k_{\perp} .

The last two columns indicate the interactions previously investigated by other methods for cold or warm plasmas. The coupling coefficients obtained in Section 4.3 for the first five situations in Table (a) are identical to those derived by Kim, Harker and Crawford.³⁴ The first case in Table (b) involving three cyclotron harmonic waves was first investigated by Harker and Crawford³², and later by Wagers.⁸⁰ The former considered collinearly propagating waves ($\phi_k = \phi_{k'} = \phi_{k''}$). The latter generalized the analysis to allow for waves propagating at arbitrary angles in the plane normal to the magnetic field ($\phi_k \neq \phi_{k'} \neq \phi_{k''}$). The coupling coefficient $C_{\parallel\parallel\parallel}$ in Equation (4.32) is in a form different from those obtained by these authors. While they could not derive the Manley-Rowe relations analytically from the coupled mode equations, the present coefficient is written in a form symmetric in $-p, s, i$ and automatically satisfies the relationship; a useful indication of the power of the Lagrangian method. The coupling coefficient $C_{O\parallel O}$ has been derived by Boyd and Turner⁶³ for a Maxwellian velocity distribution. Their result follows readily from Equation (4.34).

Table 4.1 also shows the new interactions obtained in this chapter. Those involving three circularly-polarized waves are the combinations of R-R, L and L-R, L. Typical synchronism conditions for these interactions are shown for a cold plasma in Figure 4.1. For a warm plasma,

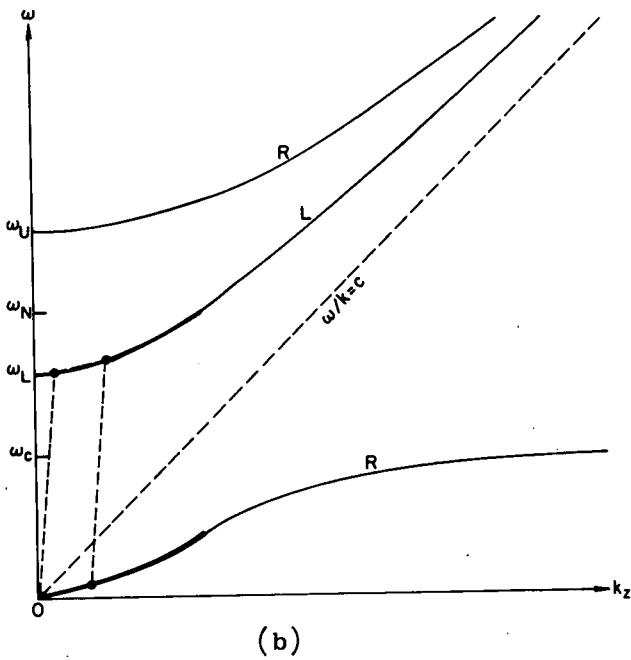
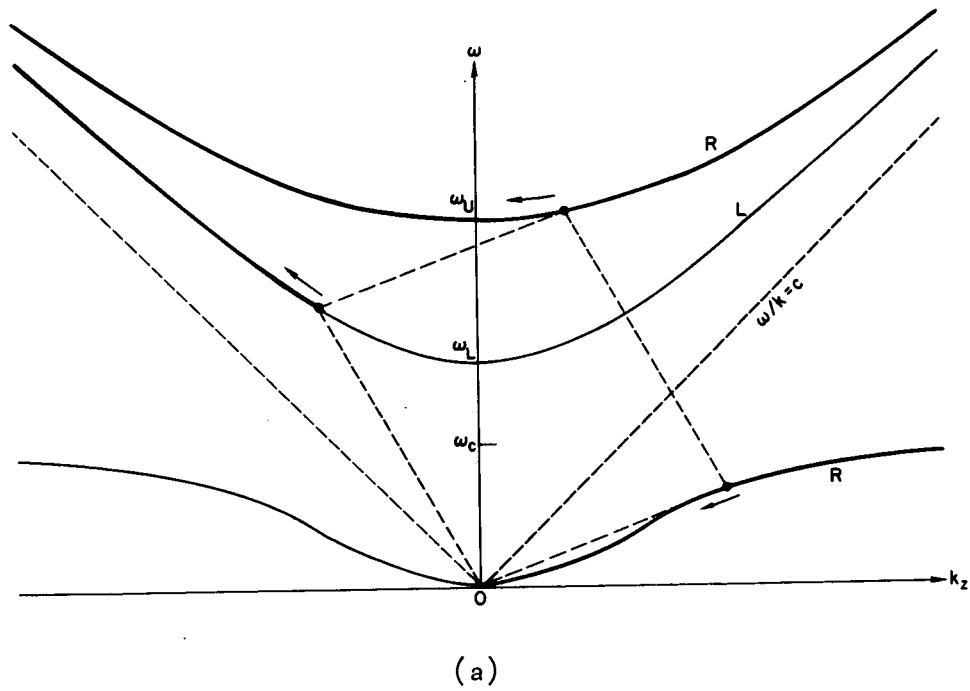
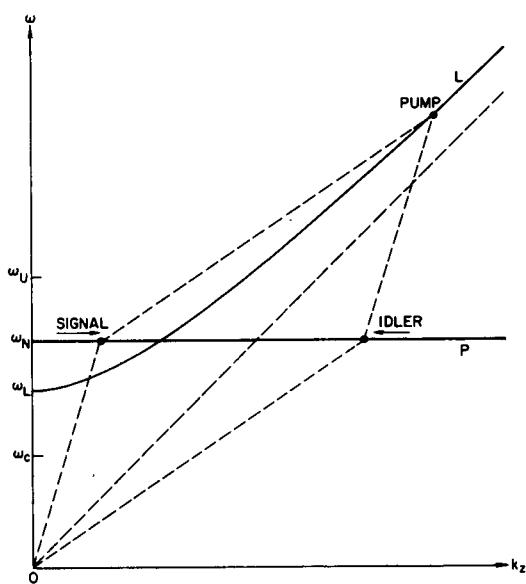


Figure 4.1 Synchronism conditions for nonlinear interaction among three circularly-polarized waves in a cold plasma. [Cut-off frequencies $\omega_{U,L} \equiv (\omega_N^2 + \omega_c^2/4)^{1/2} \pm \omega_c/2$]. (a) Pump R-signal and idler R,L; (b) Pump L-signal and idler R,L.

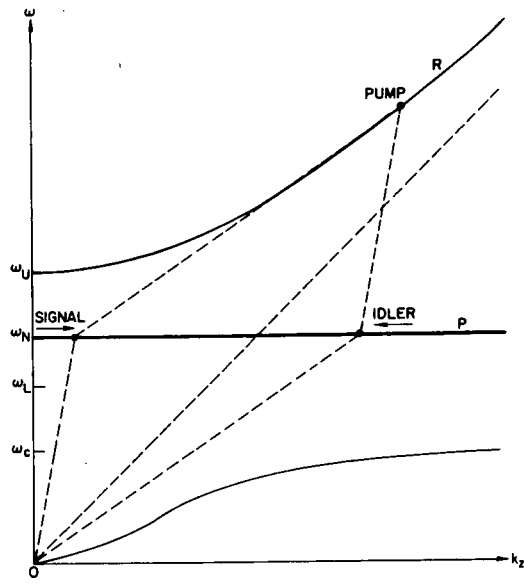
thermal effects make only a very slight difference to the shape of the upper branch of the R-wave and the L-wave branch. The lower branch of the R-wave can be strongly modified, and the three interacting modes in Figure 4.1(a) can all be in the first quadrant of the ω -k plane, as shown in Figure 4.1(b)

As shown in Table 4.1, the interaction involving two plasma waves and one circularly-polarized wave has four cases: R-P,P; L-P,P; P-R,P; and P-L,P. The topology of synchronism conditions of the L-P,P case is illustrated for a cold plasma in Figure 4.2(a). Figures 4.2(b) and (c) shows synchronism conditions of the R-P,P case for a cold plasma. When $\omega_U < 2\omega_N$ (or $\omega_c < 3\omega_N/2$), the right-hand polarized wave is on the upper branch while it is on the lower branch when $\omega_c > 2\omega_N$. As mentioned above, the lower branch of the R-wave can be strongly modified by thermal effects and the interacting R-wave can always be on the lower branch. The last two cases (P-R,P and P-L,P) have no solution in cold plasma theory since the synchronism conditions obviously cannot be satisfied. For a warm plasma, the whistler and the plasma waves can be unstable for a nonMaxwellian-velocity distribution. The possibility also exists of these waves propagating with either positive or negative small-signal energy. It would therefore be of great interest to make further detailed studies of the R-P,P and P-R,P cases to determine the relative growth rates of the small-signal and nonlinear instabilities, and to see which should dominate in experimentally reliable situations.

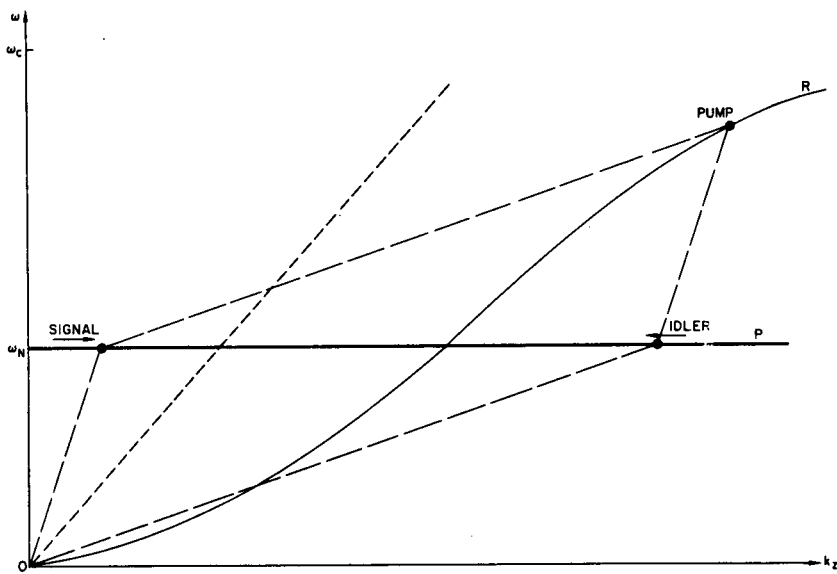
A new interaction of perpendicularly propagating waves treated here is the case \parallel -O,O. Figure 4.3(a) shows its synchronism conditions for the O-wave representing the ordinary wave in a cold plasma.



(a)



(b)



(c)

Figure 4.2 Synchronism conditions for interaction among one circularly-polarized wave and two plasma waves. (a) Pump L-signal and idler P,P; (b) Pump R-signal and idler P,P for $\omega_c < 3\omega_N/2$; (c) Pump R-signal and idler P,P for $\omega_c > 2\omega_N$.

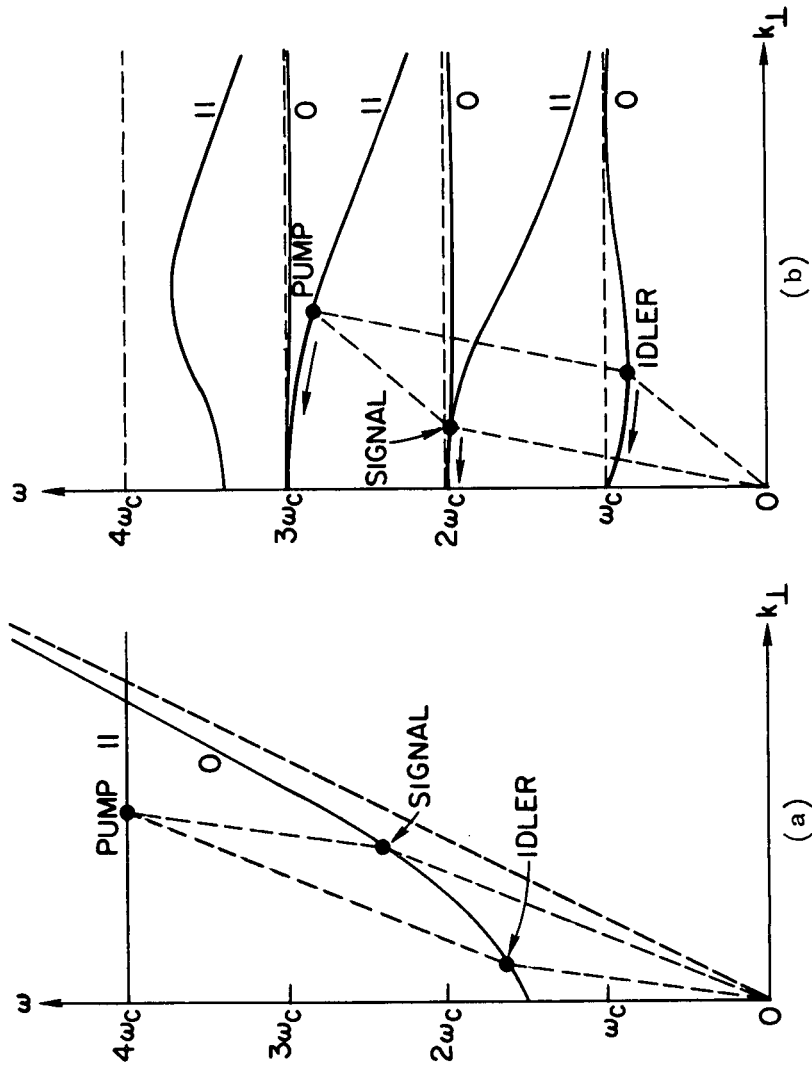


Figure 4.3 Synchronism conditions for nonlinear interaction of pump \parallel -signal and idler O, O . (a) O is the ordinary wave in a cold plasma; (b) O is the ordinary cyclotron harmonic wave.

Since the dispersion curve of the ordinary wave is inside the light cone, so is the pump wave as shown in the figure, where the \parallel -wave is non-dispersive. Unless the magnetic field and plasma were homogeneous, synchronism would not be maintained. Furthermore, because of the zero group velocity, collisional effects exert a strong influence and may invalidate the theory. Therefore, this region of the \parallel -wave is not of great interest. Figure 4.3(b) shows the synchronism conditions for the O-wave representing the ordinary cyclotron harmonic wave. In the region satisfying the synchronism conditions, the waves are dispersive. How good our approximation is for $C_{\parallel 00}$ [Equation (4.33)] open to doubt since we assumed cold plasma when simplifying it. The whole question is probably somewhat hypothetical in practice; the interaction would be extremely difficult to detect experimentally because the propagation bands of the O-wave are very narrow.⁷⁹

Chapter V

HYDRODYNAMIC APPROXIMATION

The hydrodynamic approximation for plasma wave propagation is appropriate when the thermal velocities of the charged particles are not comparable to the wave phase velocity. The plasma can then be described in terms of macroscopic quantities such as its density, drift velocity, pressure, etc. These quantities are related to each other by moment equations obtained by carrying out the velocity integration of the Vlasov equation. If the adiabatic assumption is made, the infinite set of moment equations so obtained can be truncated at the third. The first, second and third moment equations, known as the continuity, momentum and state equations, respectively, together with Maxwell's equations, then provide a closed set of equations.

It should be possible to describe the macroscopic properties of plasmas by Hamilton's principle by formulating a suitable hydrodynamic Lagrangian. In this chapter, such a Lagrangian will be derived by two different methods. In the first, the hydrodynamic Lagrangian density is obtained for the momentum equation and Maxwell's equations by integrating the exact microscopic Lagrangian of Chapter II in velocity space. In applying Hamilton's principle, the continuity and energy equations are then used as subsidiary conditions. To describe linear propagation and wave-wave interactions, this Lagrangian is expanded about the equilibrium state. Because of some mathematical difficulties, the expansion will be carried out here for only three illustrative cases: one-dimensional compression parallel to the static magnetic field, two-dimensional compression perpendicular to it, and three-dimensional compression. In the second method, the expanded hydrodynamic Lagrangian

is obtained directly from the expanded microscopic Lagrangian of Section 2.2. In contrast to the first method, this does not require any subsidiary condition, and avoids the mathematical difficulties involved in the expansion of the Lagrangian.

In Sections 5.1 and 5.2, the Lagrangian is formulated by the first method. The second and third order Lagrangians are then obtained for the three illustrative cases mentioned above. The second method is presented in Section 5.3. Section 5.4 describes how the ion contribution modifies the first order Euler-Lagrange equations and the coupled mode equations. In Section 5.6, the nonlinear interaction of two electron plasma waves and an ion acoustic wave, and of an electron plasma wave, an ordinary wave and an ion acoustic wave, are studied as an application of the Lagrangian by the first method. The application of the second method is carried out in Section 5.6, where dispersion relations are obtained from the second order Lagrangian of the second method and are compared to those in Chapter III. As an example of nonlinear wave interaction, the interaction of an Alfvén wave with two electron plasma waves is examined.

5.1 Formulation of the Lagrangian Density - Method I

The hydrodynamic equations obtained by integrating the Vlasov equation [Equation (2.6)] in velocity space with the adiabatic approximation are⁸¹

$$\frac{dn}{dt} + n \frac{\partial}{\partial \tilde{\mathbf{r}}} \cdot \tilde{\mathbf{u}} = 0 , \quad (5.1)$$

$$\frac{d\tilde{\mathbf{u}}}{dt} + \frac{\partial}{\partial \tilde{\mathbf{r}}} \cdot \tilde{\mathbf{s}} = - \frac{q}{m} (\tilde{\mathbf{E}} + \tilde{\mathbf{u}} \times \tilde{\mathbf{B}}) , \quad (5.2)$$

$$\frac{d\vec{s}}{dt} = - \left\{ \frac{q}{m} \vec{s} \times \vec{B} + \vec{s} \cdot \frac{\partial}{\partial \vec{r}} \vec{u} \right\} - \text{transpose} , \quad (5.3)$$

where the local plasma density, n , the pressure tensor, \vec{P} , and the total time derivatives, d/dt , are respectively defined by

$$n = \int f(\vec{r}, \vec{v}, t) d^3v , \quad \vec{P} \equiv m \int \vec{w} \vec{w} f(\vec{r}, \vec{v}, t) d^3v , \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \frac{\partial}{\partial \vec{r}} , \quad (5.4)$$

and the tensor \vec{s} , and the random velocity, \vec{w} , are related to the pressure tensor, \vec{P} , and the drift velocity, \vec{u} , respectively by

$$\vec{s} \equiv \frac{\vec{P}}{mn} , \quad \vec{w} \equiv \vec{v} - \vec{u} . \quad (5.5)$$

The diagonal elements of \vec{s} are the squares of the thermal velocities in each coordinate direction.⁸² Equations (5.1) - (5.3) are the equations of continuity, momentum, and state, respectively. Together with Maxwell's equations, they constitute a complete set of equations describing adiabatic processes in plasmas in terms of macroscopic quantities. A Lagrangian for the hydrodynamic approximation should include these equations.

Analogously to integrating the Vlasov equation in velocity space in order to obtain the moment equations, we may attempt to integrate the microscopic Lagrangian in velocity space to obtain the Lagrangian for the hydrodynamic approximation. Transforming the Lagrangian coordinates (\vec{r}', \vec{v}') to Eulerian coordinates (\vec{r}, \vec{v}) , and integrating Equation (2.2) over the velocity, yields

$$\mathcal{L} = \frac{nm}{2} u^2 + \frac{nm}{2} \text{Tr} \vec{s} + nq(\varphi - \vec{u} \cdot \vec{A}) + \frac{\epsilon_0}{2} \left(\nabla \varphi + \frac{\partial \vec{A}}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla \times \vec{A})^2 . \quad (5.6)$$

However, it should be remembered that the conservation law in phase

space was used as a subsidiary condition in connection with Equation (2.2). The information contained in the Vlasov equation has therefore been lost from this integrated Lagrangian. It has been shown in other Stanford work by Peng⁸³ that changing the sign of the term $\text{Tr } \vec{\underline{s}}$ is necessary to obtain the required Lagrangian,

$$\mathcal{L} = \frac{nm}{2} u^2 - \frac{nm}{2} \text{Tr } \vec{\underline{s}} + nq(\varphi - \underline{u} \cdot \underline{A}) + \frac{\epsilon_0}{2} \left(\nabla\varphi + \frac{\partial \underline{A}}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla \times \underline{A})^2. \quad (5.7)$$

It is easily seen through the use of Equations (5.1) and (5.3) as subsidiary conditions, that the variation of this form of \mathcal{L} with respect to \underline{r} gives Equation (5.2). As in Chapter II, the variations of φ and \underline{A} give the Maxwell equations.

5.2 Expansion of the Lagrangian Density

To follow the procedure of Chapters III and IV for small-signal wave propagation and wave-wave interactions, it is necessary to expand the Lagrangian. As in Chapter II, two corresponding cells satisfy

$$\underline{r} = \underline{r}_0 + \underline{\xi}, \quad n(\underline{r}, t) d^3r = n_0(\underline{r}_0) d^3r_0, \quad (5.8)$$

where $n_0(\underline{r}_0)$ is the equilibrium density. For a small perturbation, the local drift velocity and the tensor $\vec{\underline{s}}$ of the perturbed cell may be expanded about the cell in equilibrium (see Section 2.2) to give

$$\underline{u}(\underline{r}, t) = \underline{U}(\underline{r}_0) + \dot{\underline{\xi}}(\underline{r}_0, t), \quad \dot{\underline{\xi}} = \frac{\partial}{\partial t} + (\underline{U} \cdot \nabla_0) \underline{\xi}, \quad (5.9)$$

$$\vec{\underline{s}}(\underline{r}, t) = \vec{\underline{s}}_0(\underline{r}_0) + \vec{\underline{s}}_1(\underline{r}_0, t) + \vec{\underline{s}}_2(\underline{r}_0, t) + \vec{\underline{s}}_3(\underline{r}_0, t) + \dots \quad (5.10)$$

where \underline{U} is the drift velocity of the cell at \underline{r}_0 , and the subscripts on $\vec{\underline{s}}$ indicate the order in $\underline{\xi}$. The second relation in Equation (5.9) has been obtained by an argument similar to that given in Appendix A.

The expanded Lagrangian can be obtained from Equation (5.7) by expanding Equation (2.14) about \underline{r}_0 in a Taylor series, and substituting it together with Equations (5.8) - (5.10). However, it is still required that \vec{s}_n be expressed in terms of ξ , φ_1 , and \underline{a} . In principle, this could be done by expanding the ten simultaneous differential equations expressed by Equations (5.1) - (5.3) (one for n , three for \underline{u} , and six for \vec{s}) to the desired order in perturbation. In practice, obtaining the solution for \vec{s}_n is extremely complicated. For \vec{s}_3 , for example, the number of differential equations to be solved simultaneously after expansion is 26 (2 for n to the second order, 6 for \underline{u} to the second order, and 18 for \vec{s} to the third order). The solutions for \vec{s}_n are very much simplified for the special cases considered by Delcroix,⁸⁴ i.e., linear adiabatic compression parallel to the magnetic field, cylindrical adiabatic compression perpendicular to the magnetic field, and spherical adiabatic compression. In what follows in this section, we shall restrict the discussion to these three illustrative cases.

5.2.1 Linear Adiabatic Compression Parallel to the Magnetic Field

Let the static magnetic field be uniform and directed along the z -axis; the velocity distribution at every point have rotational symmetry about the z -axis, and all perturbations be independent of x and y . Thus,

$$\begin{aligned}
 B_z = B_0, \quad B_x = B_y = 0, \quad \xi_x = \xi_y = \dot{\xi}_x = \dot{\xi}_y = 0, \\
 s_{xx} = s_{yy} \equiv s_{\perp}, \quad s_{zz} \equiv s_{\parallel}, \quad s_{xy} = s_{yz} = s_{zx} = 0
 \end{aligned}
 \tag{5.11}$$

where s_{\perp} and s_{\parallel} can be related to the electron thermal velocity, \underline{v}_t , and temperature, T , by⁸²

2

$$s_{\perp} = \frac{v_{t\perp}^2}{2} = \frac{\kappa T_{\perp}}{2m}, \quad s_{\parallel} = \frac{v_{t\parallel}^2}{2} = \frac{\kappa T_{\parallel}}{m}. \quad (5.12)$$

This case is the same as that of electrostatic waves in a uniform plasma propagating parallel to the magnetic field. The six equations for \vec{s} in Equation (5.3) reduce to two,

$$\frac{ds_{\perp}}{dt} = 0, \quad \frac{d}{dt} \left(\frac{s_{\parallel}}{n} \right) = 0. \quad (5.13)$$

Expansion of Equation (5.13) is carried out in Appendix C.

Substitution of Equations (5.8) - (5.10), and (C.8), into Equation (5.7) gives the second and third order Lagrangians as

$$\begin{aligned} \mathcal{L}_2 &= \frac{n_0 m}{2} \xi_z^2 - \frac{3}{2} n_0 m s_{\parallel} \left(\frac{\partial \xi_z}{\partial z} \right)^2 - n_0 q \xi_z e_z + \frac{\epsilon_0}{2} e_z^2, \\ \mathcal{L}_3 &= 2n_0 m s_{\parallel} \left(\frac{\partial \xi_z}{\partial z} \right)^3 - \frac{n_0 q}{2} \xi_z^2 \frac{\partial e_z}{\partial z}, \end{aligned} \quad (5.14)$$

where e_z is given by Equation (3.16), and the subscript 0 has been dropped from s_{\parallel} . By summing \mathcal{L}_2 and \mathcal{L}_3 , the averaged Lagrangian can be written as Equation (4.7) with $\Lambda_k^{(2)}$ and $\Lambda_{kk'k''}^{(3)}$ given by

$$\begin{aligned} \Lambda_k^{(2)} &= \frac{n_0 m}{2} \omega_{k,0}^2 |\xi_k^z|^2 - \frac{3}{2} n_0 m s_{\parallel} |k_z \xi_k^z|^2 - n_0 q \xi_k^z e_k^{z*} + \frac{\epsilon_0}{2} |e_k^z|^2, \\ \Lambda_{kk'k''}^{(3)} &= i2n_0 m s_{\parallel} k_z k_z' k_z'' \xi_k^z \xi_{k'}^z \xi_{k''}^z + \frac{i}{2} n_0 q k_z e_k^z \xi_{k'}^z \xi_{k''}^z, \end{aligned} \quad (5.15)$$

where $\omega_{k,\mu}$ has been written for $\omega_k - k_z U_z + \mu \omega_c$. Here it should be noted that $\vec{\Phi} = 0$ implies $\vec{U}_{\perp} = 0$, from the zeroth order of Equation (5.2) for a uniform plasma in equilibrium.

5.2.2 Cylindrical Adiabatic Compression Perpendicular to the Magnetic Field

For compression purely perpendicular to the magnetic field, we may put

$$\xi_z = \dot{\xi}_z = 0, \quad \frac{\partial}{\partial z} = 0, \quad s_{xz} = s_{yz} = 0,$$

$$B_z = B_0 + b_z, \quad B_x = B_y = 0. \quad (5.16)$$

If we further assume a symmetric compression about the z-axis, we may also put

$$s_{xx} = s_{yy} \equiv s_{\perp}, \quad s_{zz} \equiv s_{\parallel}, \quad s_{xy} = 0. \quad (5.17)$$

The six equations for \vec{s} in Equation (5.3) then reduce to

$$\frac{d}{dt} \left(\frac{s_{\perp}}{n} \right) = 0, \quad \frac{ds_{\parallel}}{dt} = 0, \quad (5.18)$$

$$\frac{\partial \dot{\xi}_x}{\partial x} = \frac{\partial \dot{\xi}_y}{\partial y}, \quad \frac{\partial \dot{\xi}_x}{\partial y} = - \frac{\partial \dot{\xi}_y}{\partial x}. \quad (5.19)$$

Expansion of Equation (5.18) is carried out in Appendix C.

Of Equations (5.16) - (5.19), we only need Equation (5.18) for the expansion of the Lagrangian in Equation (5.7). Let us consider the implications of Equations (5.16) and (5.19). In Equation (5.16), the second relation implies that the waves under consideration are propagating perpendicular to the magnetic field. Furthermore, since the time-varying magnetic field of the wave is in the z-direction, it can be seen from Equation (3.16) that \vec{e}_k is polarized in the xy-plane. It follows that there is no time varying displacement in the z-direction. Equation (5.19) expresses the rate of strain relations,⁸⁵ which become

$\partial \xi_x / \partial x = \partial \xi_y / \partial y$ and $\partial \xi_x / \partial y = -\partial \xi_y / \partial x$ for the steady state. They imply that the motion due to a wave is of uniform dilatation in the xy-plane with rotation about the z-axis.⁸⁶ This motion is only possible for a wave whose frequency is well below the cyclotron frequency, so that the displacement of a fluid element due to the wave is sufficiently randomized by the gyration for symmetrical compression to be approached. A wave satisfying both Equations (5.16) and (5.19) is the compressional Alfvén wave with a frequency well below the ion cyclotron frequency, and propagating perpendicular to the static magnetic field.

Although compressional Alfvén waves propagating perpendicular to the magnetic field can satisfy the synchronism conditions for wave-wave interaction among themselves, we shall only consider the second order Lagrangian here. Substituting Equations (5.8) - (5.10), and (C.10), into Equation (5.7) gives the second order Lagrangian as

$$\begin{aligned}
 \mathcal{L}_2 = & \frac{n_0^m}{2} \xi_{\perp}^2 - n_0 m s_{\perp} \left[(\nabla_{\perp} \cdot \xi_{\perp})^2 - \frac{\partial \xi_x}{\partial x} \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_x}{\partial y} \frac{\partial \xi_y}{\partial x} \right] \\
 & - n_0 q \xi_{\perp} \cdot e_{\perp} + \frac{n_0 q}{2} \xi_{\perp} \cdot \xi_{\perp} \times \tilde{B} + \frac{\epsilon_0}{2} e^2 - \frac{1}{2\mu_0} b_z^2, \quad (5.20)
 \end{aligned}$$

where the subscript 0 has been dropped from \tilde{B} and s_{\perp} . The averaged Lagrangian then becomes

$$\begin{aligned}
 \mathcal{L}_k^{(2)} = & \sum_k \left\{ \frac{n_0^m}{2} \omega_{k,0}^2 |\xi_{k1}|^2 - n_0 m s_{\perp} |\tilde{k}_{\perp} \cdot \xi_{k1}|^2 \right. \\
 & \left. - n_0 q \xi_{k1}^* \cdot e_{k1} + i \frac{n_0 q}{2} \omega_{k,0} \xi_{k1} \cdot \xi_{k1}^* \times \tilde{B} + \frac{\epsilon_0}{2} |e_k|^2 - \frac{1}{2\mu_0} |b_k^z|^2 \right\}. \quad (5.21)
 \end{aligned}$$

5.2.3 Spherical Adiabatic Compression

In this case, even though perturbations occur, the plasma preserves isotropic properties. The elements of \vec{s} can therefore be written as

$$s_{xx} = s_{yy} = s_{zz} \equiv s, \quad s_{xy} = s_{yz} = s_{zx} = 0. \quad (5.22)$$

Equation (5.3) then yields

$$\frac{d}{dt} \left(\frac{s}{n^{2/3}} \right) = 0, \quad (5.23)$$

$$\frac{\partial \dot{\xi}_x}{\partial x} = \frac{\partial \dot{\xi}_y}{\partial y} = \frac{\partial \dot{\xi}_z}{\partial z}, \quad \frac{\partial \dot{\xi}_x}{\partial y} + \frac{\partial \dot{\xi}_y}{\partial x} = \frac{\partial \dot{\xi}_y}{\partial z} + \frac{\partial \dot{\xi}_z}{\partial y} = \frac{\partial \dot{\xi}_z}{\partial x} + \frac{\partial \dot{\xi}_x}{\partial z} = 0. \quad (5.24)$$

If the first relation of Equation (5.24) is identically zero, then Equation (5.24) describes a combined translation and rotation as in the case of rigid bodies.⁸⁷ The first relation is not zero in general. This implies the addition of a uniform extension (or contraction)⁸⁶ to the translational and rotational motion.⁸⁷ Fluid motion of a uniform extension (or contraction) is possible in a plasma in which elastic collisions conserve energy and momentum, and in which the frequency of the wave is much lower than that of collisions. The motion of the fluid elements due to the wave is then randomized by collisions during a period of the wave. The translational and rotational motion is due to the static electric and magnetic fields.

Substitution of Equations (5.8) - (5.10), and (C.11), into Equation (5.7) gives the second and third order Lagrangians as

$$\begin{aligned}
\mathcal{L}_2 &= \frac{n_0^m}{2} \dot{\xi}^2 - \frac{5}{6} n_0 m s (\nabla \cdot \xi)^2 + \frac{n_0^q}{2} \dot{\xi} \cdot \xi \times B \\
&\quad - n_0^q \xi \cdot (\underline{e} + \underline{U} \times \underline{b}) + \frac{\epsilon_0}{2} e^2 - \frac{1}{2\mu_0} b^2, \\
\mathcal{L}_3 &= -\frac{5}{54} n_0 m s (\nabla \cdot \xi)^3 + \frac{5}{6} n_0 m s (\xi \xi : \nabla \nabla) (\nabla \cdot \xi) \\
&\quad - \frac{n_0^q}{2} \xi \xi : \nabla (\underline{e} + \underline{U} \times \underline{b}) + \frac{n_0^q}{2} \dot{\xi} \times (\xi \times \underline{b}), \tag{5.25}
\end{aligned}$$

where the subscript 0 has again been dropped from s , and in obtaining the first two terms in \mathcal{L}_3 from Equation (C.11), the total divergence terms are neglected. The averaged Lagrangians are then written as Equation (4.7) with

$$\begin{aligned}
\Lambda_k^{(2)} &= \frac{n_0^m}{2} \omega_{k,0}^2 |\xi_k|^2 - \frac{5}{6} n_0 m s |k \cdot \xi|^2 + i \frac{n_0^q}{2} \omega_{k,0} \xi_k \cdot \xi_k^* \times B \\
&\quad - n_0^q \xi_k^* \cdot (\underline{e}_k + \underline{U} \times \underline{b}_k) + \frac{\epsilon_0}{2} |\underline{e}_k|^2 - \frac{1}{2\mu_0} |\underline{b}_k|^2, \\
\Lambda_{kk',k''}^{(3)} &= -i \frac{5}{54} n_0 m s \underline{kk}'k'' : \xi_k \xi_{k'} \xi_{k''} + i \frac{5}{6} n_0 m s \underline{kkk} : \xi_k \xi_{k'} \xi_{k''} \\
&\quad + i \frac{n_0^q}{2} \xi_k \xi_{k''} : k (\underline{e}_k + \underline{U} \times \underline{b}_k) + i \frac{n_0^q}{2} \omega_{k,0} \xi_k \times (\xi_{k'} \times \underline{b}_{k''}). \tag{5.26}
\end{aligned}$$

5.3 Formulation of the Lagrangian Density - Method II

The hydrodynamic Lagrangian in Section 5.1 is formulated from the exact microscopic Lagrangian expressed by Equation (2.2). Because of the mathematical difficulties involved in the expansion, however, its practical use is restricted to very special cases. In order to avoid these difficulties, we will obtain the expanded Lagrangians for the hydrodynamic approximation directly from the expanded microscopic

Lagrangian of Equations (2.26) and (2.28).

Let the random velocity at equilibrium be \underline{W} . The velocity \underline{v} -coordinate system in Equations (2.26) and (2.28) may be changed to the random velocity \underline{W} -coordinate system via

$$\underline{v} = \underline{U} + \underline{W}, \quad (5.27)$$

where the subscript 0 is dropped from the equilibrium velocity, \underline{v}_0 .

This change of coordinate system results in the following transformation:

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \underline{r}} \rightarrow \frac{\partial}{\partial \underline{r}} - \left(\frac{\partial}{\partial \underline{r}} \underline{U} \right) \cdot \frac{\partial}{\partial \underline{W}}, \quad \frac{\partial}{\partial \underline{v}} \rightarrow \frac{\partial}{\partial \underline{W}}. \quad (5.28)$$

For a plasma uniform at equilibrium, the second term on the RHS of the second transformation in Equation (5.28) can be dropped. In the absence of a static electric field, the time differential operator, D , defined in Equation (2.19), may be written as

$$D = \frac{d}{dt} - (\underline{W} \times \underline{\omega}_c) \cdot \nabla_{\underline{W}} + (\underline{W} \cdot \nabla), \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + U_z \frac{\partial}{\partial z}. \quad (5.29)$$

Here, the subscript 0 is dropped again from t , \underline{r} and ∇ . Let the plasma temperature be very low, i.e., the mean value of the Larmor radius, and the mean thermal velocity of particles are very small compared to the wavelength and phase velocity, respectively. Since the spread of the velocity distribution function is small for this situation, the displacement $\underline{\xi}$ may be expanded about $\underline{v} = \underline{U}$ as

$$\underline{\xi}(\underline{U} + \underline{W}) = \underline{\xi}(\underline{U}) + (\underline{W} \cdot \nabla_{\underline{W}}) \underline{\xi}(\underline{U}) + \frac{1}{2} (\underline{W} \underline{W} : \nabla_{\underline{W}} \nabla_{\underline{W}}) \underline{\xi}(\underline{U}) + \dots \quad (5.30)$$

Substituting Equations (5.27) - (5.30), dropping the terms of $\nabla_{\underline{W}} \nabla_{\underline{W}} \underline{\xi}$ and those of higher order than second in \underline{W} , and using $\langle \underline{W} \rangle = 0$,

reduces Equations (2.26) and (2.28), to

$$\begin{aligned}
\mathcal{L}_2 = n_0 & \left\langle \frac{m}{2} \left| \frac{d\tilde{\xi}}{dt} \right|^2 - q\tilde{\xi} \cdot (\tilde{e} + \tilde{U} \times \tilde{b}) + \frac{q}{2} \frac{d\tilde{\xi}}{dt} \cdot \tilde{\xi} \times \tilde{B} \right. \\
& + \frac{m}{2} \left[\tilde{W} \cdot \nabla \tilde{\xi} + \tilde{W} \cdot \frac{d}{dt} (\nabla_{\tilde{W}} \tilde{\xi}) - \frac{q}{m} \tilde{W} \times \tilde{B} \cdot \nabla_{\tilde{W}} \tilde{\xi} \right]^2 \\
& + m \frac{d\tilde{\xi}}{dt} \cdot (\overline{WW} : \nabla_{\tilde{W}}) \tilde{\xi} - q(\tilde{W} \cdot \nabla_{\tilde{W}}) \tilde{\xi} \cdot \tilde{W} \times \tilde{b} + \frac{q}{2} (\overline{WW} : \nabla_{\tilde{W}}) \tilde{\xi} \cdot \tilde{\xi} \times \tilde{B} \\
& \left. + \frac{q}{2} [\tilde{W} \cdot \nabla \tilde{\xi} + \tilde{W} \cdot \frac{d}{dt} (\nabla_{\tilde{W}} \tilde{\xi}) - \frac{q}{m} \tilde{W} \times \tilde{B} \cdot \nabla_{\tilde{W}} \tilde{\xi}] \cdot \tilde{\xi} \times \tilde{B} \right\rangle + \frac{1}{2} \epsilon_0 e^2 - \frac{1}{2\mu_0} b^2, \quad (5.31)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_3 = \frac{n_0 q}{2} & \left\langle \frac{d\tilde{\xi}}{dt} \cdot \tilde{\xi} \times \tilde{b} - \overline{\tilde{\xi}\tilde{\xi}} : \nabla(\tilde{e} + \tilde{U} \times \tilde{b}) \right. \\
& + [\tilde{W} \cdot \nabla \tilde{\xi} + \tilde{W} \cdot \frac{d}{dt} (\nabla_{\tilde{W}} \tilde{\xi}) - \frac{q}{m} \tilde{W} \times \tilde{B} \cdot \nabla_{\tilde{W}} \tilde{\xi}] \cdot (\tilde{W} \cdot \nabla_{\tilde{W}}) \tilde{\xi} \times \tilde{b} + (\overline{WW} : \nabla_{\tilde{W}}) \tilde{\xi} \cdot \tilde{\xi} \times \tilde{b} \\
& \left. - (\tilde{W} \cdot \nabla_{\tilde{W}}) \tilde{\xi} (\tilde{W} \cdot \nabla_{\tilde{W}}) \tilde{\xi} : \nabla(\tilde{e} + \tilde{U} \times \tilde{b}) - (\tilde{W} \cdot \nabla_{\tilde{W}}) (\overline{\tilde{\xi}\tilde{\xi}}) : \nabla(\tilde{W} \times \tilde{b}) \right\rangle. \quad (5.32)
\end{aligned}$$

Though it is not a necessary restriction, the velocity distribution is now assumed symmetric for the sake of simplicity, so that the off-diagonal elements of $\vec{s}_0 \equiv \langle \overline{WW} \rangle$ vanish. The averaged Lagrangian then yields Equation (4.7) with

$$\Lambda_{\mathbf{k}}^{(2)} = \lambda_0^{(2)} + \lambda_{00}^{(2)} + \lambda_{10}^{(2)} + \lambda_{11}^{(2)} + \text{c.c.}, \quad (5.33)$$

$$\Lambda_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{(3)} = \lambda_0^{(3)} + \lambda_{00}^{(3)} + \lambda_{10}^{(3)} + \lambda_{11}^{(3)} + \text{c.c.} \quad (5.34)$$

The $\lambda^{(2)}$'s in Equations (5.33) are given by

$$\begin{aligned}
\lambda_0^{(2)} &= \frac{1}{2} n_0 m \omega_{k,0}^2 |\underline{\xi}_k|^2 - n_0 q \underline{\xi}_k \cdot (\underline{e}_k^* + \underline{u} \times \underline{b}_k^*) + \frac{1}{2} n_0 q \omega_{k,0} \underline{\xi}_k \cdot (\underline{\xi}_k^* \times \underline{B}) \\
&\quad + \frac{1}{2} \epsilon_0 |\underline{e}_k|^2 - \frac{1}{2\mu_0} |\underline{b}_k|^2, \\
\lambda_{00}^{(2)} &= \frac{1}{2} n_0 m k_{\alpha}^2 s_{\alpha\alpha} |\underline{\xi}_k|^2, \\
\lambda_{10}^{(2)} &= -2n_0 m \omega_{k,0} k_{\alpha} s_{\alpha\alpha} \underline{\xi}_k^* \cdot \frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} - i n_0 q \underline{\xi}_k^* \cdot \epsilon_{\alpha\beta\gamma} k_{\alpha} s_{\alpha\alpha} B_{\beta} \frac{\partial \underline{\xi}_k}{\partial W_{\gamma}} + i n_0 q k_{\alpha} \epsilon_{\alpha\alpha\gamma} \underline{\xi}_k^* \cdot \frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} \times \underline{B}, \\
\lambda_{11}^{(2)} &= \frac{1}{2} n_0 m \omega_{k,0}^2 s_{\alpha\alpha} \frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} \cdot \left(\frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} \right)^* + \frac{n_0 q^2}{2m} \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} B_{\beta}^2 \left| \frac{\partial \underline{\xi}_k}{\partial W_{\gamma}} \right|^2 \\
&\quad - i q n_0 \omega_{k,0} \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} B_{\beta} \frac{\partial \underline{\xi}_k}{\partial W_{\gamma}} \cdot \left(\frac{\partial \underline{\xi}_k}{\partial W_{\gamma}} \right)^* + \frac{i}{2} n_0 q \omega_{k,0} s_{\alpha\alpha} \frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} \cdot \frac{\partial \underline{\xi}_k^*}{\partial W_{\alpha}} \times \underline{B} \\
&\quad - \frac{n_0 q^2}{2m} \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} B_{\beta} \frac{\partial \underline{\xi}_k}{\partial W_{\gamma}} \cdot \left(\frac{\partial \underline{\xi}_k}{\partial W_{\alpha}} \right)^* \times \underline{B} - n_0 q \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} b_{\beta}^k \left(\frac{\partial \underline{\xi}_k^{\gamma}}{\partial W_{\alpha}} \right)^*, \quad (5.35)
\end{aligned}$$

where α, β , and γ represent in any order the variables x, y , and z , the subscript 0 has been dropped from $s_{\alpha\alpha}$, and the property of the skew-symmetric tensor, $\epsilon_{\alpha\beta\gamma}$, that $\underline{a} \cdot \underline{b} \times \underline{c} = \epsilon_{\alpha\beta\gamma} a_{\alpha} b_{\beta} c_{\gamma}$, has been used. The rules for the subscripts and superscripts are as follows: the subscripts on \underline{k} and zero order variables indicate the components of the coordinates; the subscripts on ω and first order variables indicate the participating wave vectors. The coordinate components of these variables are indicated by superscripts. The only exceptions are \parallel and \perp , the use of which was explained in Section 3.5.1.

The λ 's in Equation (5.34) are given by

$$\lambda_0^{(3)} = \frac{i}{n} n_0 q \omega_{k,0} \xi_{\mathbf{k}} \cdot \xi_{\mathbf{k}'} \times \mathbf{b}_{\mathbf{k}''} + \frac{i}{2} n_0 q \mathbf{k} (\mathbf{e}_{\mathbf{k}} + \mathbf{U} \times \mathbf{b}_{\mathbf{k}}) : \xi_{\mathbf{k}} \xi_{\mathbf{k}''} ,$$

$$\lambda_{00}^{(3)} = 0 ,$$

$$\begin{aligned} \lambda_{10}^{(3)} = & -\frac{i}{2} n_0 q s_{\alpha\alpha} (\mathbf{k}_{\alpha} - \mathbf{k}'_{\alpha}) \xi_{\mathbf{k}} \cdot \frac{\partial \xi_{\mathbf{k}'}}{\partial W_{\alpha}} \times \mathbf{b}_{\mathbf{k}''} + \frac{i}{2} n_0 q \mathbf{k} \cdot \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} b_{\mathbf{k}}^{\beta} \xi_{\mathbf{k}}', \frac{\partial \xi_{\mathbf{k}''}}{\partial W_{\alpha}} \\ & + \frac{i}{2} n_0 q \mathbf{k} \cdot \xi_{\mathbf{k}}', \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha} b_{\mathbf{k}}^{\beta} \frac{\partial \xi_{\mathbf{k}''}}{\partial W_{\alpha}} , \end{aligned}$$

$$\begin{aligned} \lambda_{11}^{(3)} = & \frac{i}{2} n_0 q \omega_{k,0} s_{\alpha\alpha} \frac{\partial \xi_{\mathbf{k}}}{\partial W_{\alpha}} \cdot \frac{\partial \xi_{\mathbf{k}'}}{\partial W_{\alpha}} \times \mathbf{b}_{\mathbf{k}''} - \frac{n_0 q^2}{2m} \epsilon_{\alpha\beta\gamma} s_{\alpha\alpha}^B \frac{\partial \xi_{\mathbf{k}}}{\partial W_{\gamma}} \cdot \frac{\partial \xi_{\mathbf{k}'}}{\partial W_{\alpha}} \times \mathbf{b}_{\mathbf{k}''} \\ & + \frac{i}{2} n_0 q s_{\alpha\alpha} \mathbf{k} (\mathbf{e}_{\mathbf{k}} + \mathbf{U} \times \mathbf{b}_{\mathbf{k}}) : \frac{\partial \xi_{\mathbf{k}'}}{\partial W_{\alpha}} \frac{\partial \xi_{\mathbf{k}''}}{\partial W_{\alpha}} . \end{aligned} \quad (5.36)$$

5.4 Inclusion of Ion Motion

Since the second and third order Lagrangians have now been obtained in Sections 5.2 and 5.3, we are in a position to follow the procedure described in Chapters III and IV to obtain linear dispersion relations and the coupled mode equations. Before doing this, however, since ions play an important role for the hydromagnetic waves, we must consider how the inclusion of ion motion modifies the Lagrangians and the Euler-Lagrange equations.

Including the ion motion, the Lagrangian may be written as

$$\mathcal{L} = \mathcal{L}^{\text{Pe}} + \mathcal{L}^{\text{Pi}} + \mathcal{L}^{\text{F}} , \quad (5.37)$$

where the electron contribution, \mathcal{L}^{Pe} , is written for the terms in the Lagrangians in the previous sections which contain the displacement vector ξ . In order to denote the electron contribution, superscripts

(e) must be appended to m , \underline{u} , \underline{U} , $\omega_{k,\mu}$, ω_c , \vec{s} , $\underline{\xi}$ and $\nabla_{\underline{w}}\underline{\xi}$ of Sections 5.2 and 5.3. The ion contribution, \mathcal{L}^{Pi} , is obtained from \mathcal{L}^P by changing the superscripts (e) on these variables to (i) with the following changes in sign:

$$q \rightarrow -q, \quad \omega_c^{(e)} \rightarrow -\omega_c^{(i)}, \quad \omega_{k,\mu}^{(e)} \rightarrow \omega_{k,-\mu}^{(i)}. \quad (5.38)$$

In the linear theory, the equations for the ion motion to be derived from the variations in $\underline{\xi}^{(i)}$ and $\nabla_{\underline{w}}\underline{\xi}^{(i)}$ are therefore the same as those for the electron motion, except for the superscripts and sign changes just noted. In Maxwell's equations, which are obtained by variation of the fields, the current and charge densities consist of the contributions of both electrons and ions, i.e.,

$$\underline{j} = \underline{j}^{(e)} + \underline{j}^{(i)}, \quad \rho = \rho^{(e)} + \rho^{(i)}, \quad (5.39)$$

where $\underline{j}^{(i)}$ and $\rho^{(i)}$ are respectively obtained from $\underline{j}^{(e)}$ and $\rho^{(e)}$ with the same changes as noted for \mathcal{L}^{Pi} . The action, N_k , and Hamiltonian, \mathcal{H}_k , should also be written in analogous form to Equation (5.37).

In the nonlinear theory, the correction factor α_k , defined in Chapter IV, is the same for both electron and ion motions. The other nonlinear correction factor, β_k , does not appear in the hydrodynamic approximation because $\underline{\xi}$ and $\nabla_{\underline{w}}\underline{\xi}$ are no longer functions of \underline{y} . The Lagrangian, the action transfer equation, and the coupled mode equation can then be written as Equations (4.7), (4.9) and (4.10) with N_k , $\Lambda_k^{(2)}$ and $\Lambda_{kk',k''}^{(3)}$ including the effects of ion motion.

5.5 Application of the Lagrangian by Method I

As an application of the Lagrangian formulated by the first method, we shall consider the interaction of waves which belong to the parallel

compression case treated in Section 5.2.1. Cylindrical compression need not be studied since the waves do not satisfy the synchronism conditions. The case of spherical compression is for collision-dominated plasmas, and is outside the scope of our work. It is under extensive investigation at Stanford by Peng.⁸³

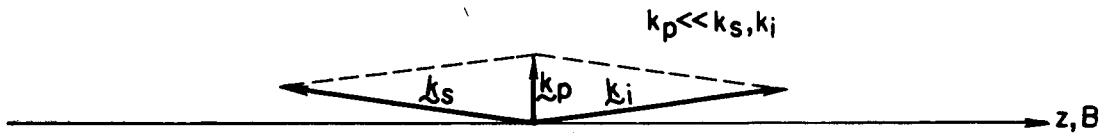
The waves in the parallel compression category are electrostatic waves: electron plasma and ion acoustic waves. In this subsection, we shall obtain the linear dispersion relations and the coupled mode equations for these waves. The ordinary mode does not belong to this category since it has variations in the xy-plane ($k_{\perp} \neq 0$), and has a time varying magnetic field. It thus violates the assumptions of Equation (5.11). However, since the wavelength of the ordinary wave is usually much longer than those of electron plasma and ion acoustic waves, as shown in Figure 5.1, the dipole approximation for this wave may be satisfactory when it interacts with waves propagating axially. Consequently, we shall also consider interaction involving the ordinary wave.

5.5.1 Linear Theory

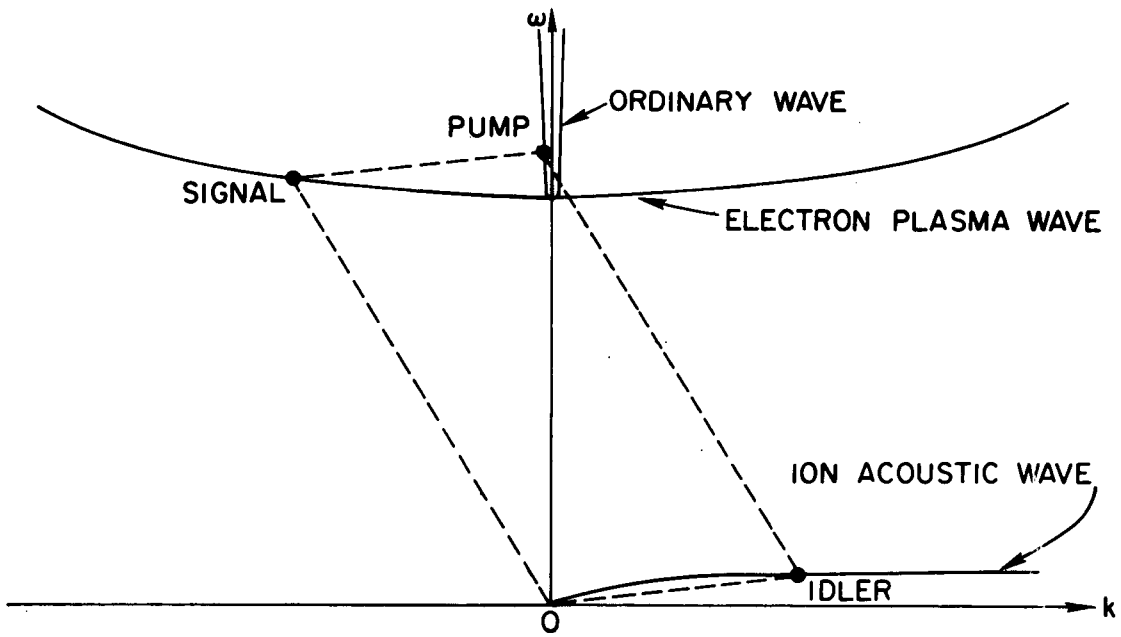
For small-signal propagation, the terms of order $O(\epsilon)$ in Equation (4.7) can be neglected. Since only a single wave number, \underline{k} , is necessary, the Lagrangian may be written as

$$\mathcal{L}_{\underline{k}}^{(2)} = \Lambda_{\underline{k}}^{(2)} + \Lambda_{\underline{k}}^{(2)*}, \quad (5.40)$$

where $\Lambda_{\underline{k}}^{(2)}$ is given by Equation (5.15). Variations of $\mathcal{L}_{\underline{k}}^{(2)}$ with respect to $\xi_{-\underline{k}}^{z(e)}$ and $a_{-\underline{k}}^{z(e)}$ give, respectively,



(a)



(b)

Figure 5.1 Synchronism parallelograms for interaction of one ordinary wave (pump), one electron plasma wave (signal), and one ion acoustic wave (idler).

$$\xi_{\mathbf{k}}^{z(e)} = \frac{q}{m^{(e)}} \left(\frac{e_{\mathbf{k}}^z}{\omega_{\mathbf{k},0}^{(e)2} - 3k_{z\parallel}^2 s_{\parallel}^2(e)} \right), \quad \omega_{\mathbf{k},0}^{(e)} \equiv \omega_{\mathbf{k}} - k_z U_z^{(e)},$$

$$-ik_z \epsilon_0 e_{\mathbf{k}}^z = \rho^{(e)} + \rho^{(i)}, \quad \rho^{(e)} = -iqn_0 k_z \xi_{\mathbf{k}}^{z(e)}. \quad (5.41)$$

The first expression in Equation (5.41) is the equation of electron fluid motion. A similar relation for ions is obtained by the proper changes in Equation (5.38). The third expression is Poisson's equation. Combining the relations in Equation (5.41) gives the dispersion relation as

$$\kappa_{\mathbf{k}} = 1 - \sum_{\alpha} \frac{\omega_N^{(\alpha)2}}{\omega_{\mathbf{k},0}^{(\alpha)2} - 3k_{z\parallel}^2 s_{\parallel}^2(\alpha)} = 0, \quad (\alpha = e, i), \quad (5.42)$$

where $\kappa_{\mathbf{k}}$ is the equivalent plasma permittivity. This is the familiar dispersion relation for an electrostatic wave. A simpler form of dispersion relation for an electron plasma wave can be obtained by assuming the frequency and the phase velocity of the wave to be very high compared to the ion plasma frequency and the average ion thermal velocity, respectively, i.e.,

$$\omega_{\mathbf{k},0}^{(i)} \gg \omega_N^{(i)} \quad \text{and} \quad k_{z\parallel}^2 s_{\parallel}^2(i). \quad (5.43)$$

Equation (5.42) then reduces to

$$\omega_{\mathbf{k},0}^{(e)2} = \omega_N^{(e)2} + 3k_{z\parallel}^2 s_{\parallel}^2(e), \quad s_{\parallel}^2(e) = \frac{\kappa T_{\parallel}^{(e)}}{m^{(e)}}. \quad (5.44)$$

On the other hand, the assumption that the wavelength of the ion acoustic wave and the electron thermal velocity are much greater than the Debye length and the phase velocity of the wave, respectively, i.e.,

$$\omega_N^{(e)2} \gg k_z^2 s_{\parallel}^{(e)} \gg \omega_{k,0}^{(e)2}, \quad (5.45)$$

gives the dispersion relation for an ion acoustic wave as

$$\omega_{k,0}^{(i)2} = \frac{3k_z^2}{m^{(i)}} \sum_{\alpha} m^{(\alpha)} s_{\parallel}^{(\alpha)}, \quad \sum_{\alpha} m^{(\alpha)} s_{\parallel}^{(\alpha)} = \kappa \left(T_{\parallel}^{(e)} + T_{\parallel}^{(i)} \right). \quad (5.46)$$

The variation of $\Gamma_k^{(2)}$ with respect to θ_k gives the equation for the conservation of action,

$$\frac{\partial N_k}{\partial t} + \frac{\partial (v_k^z N_k)}{\partial z} = 0,$$

$$N_k = \frac{1}{2} \frac{\partial}{\partial \omega_k} \left(e_k^{z*} \epsilon_0 \kappa_k e_k^z \right) + \text{c.c.}, \quad v_k^z N_k = -\frac{1}{2} \frac{\partial}{\partial k_z} \left(e_k^{z*} \epsilon_0 \kappa_k e_k^z \right) + \text{c.c.}, \quad (5.47)$$

where the action, N_k , and the group velocity, v_k^z , are defined in Equation (3.13). Explicit expressions for N_k and v_k^z can be obtained by substituting Equation (5.42) into (5.47).

5.5.2 The Coupled Mode Equations

By using $\Lambda_{kk'k''}^{(3)}$ in Equation (5.15) and the first order equation of motion [the first expression in Equation (5.41)], we can write Equation (4.10) as

$$iN_k \left(\frac{\partial}{\partial t} + v_k^z \frac{\partial}{\partial z} \right) \alpha_k = \alpha_{-k'} \alpha_{-k''} \sum_{k, k', k''} \left(\sum_{\alpha} Q_{kk'k''}^{(\alpha)} \right) e_k^{z*} e_k^{z*} e_{k''}^{z*}, \quad (5.48)$$

where the first summation is over all permutations of k, k', k'' ; the second is over the electron and ion species, and $Q_{kk'k''}^{(\alpha)}$ is given by

$$Q_{kk'k''}^{(\alpha)} = (-)^{\alpha} \epsilon_0 \frac{q\omega_N^{(\alpha)2}}{i2m^{(\alpha)}} \frac{k_z}{\left(\omega_{k,0}^{(\alpha)2} - 3k_z^2 s_{\parallel}^{(\alpha)} \right) \left(\omega_{k',0}^{(\alpha)2} - 3k_z'^2 s_{\parallel}^{(\alpha)} \right) \left(\omega_{k'',0}^{(\alpha)2} - 3k_z''^2 s_{\parallel}^{(\alpha)} \right)} \left(1 + \frac{4s_{\parallel}^{(\alpha)} k_z' k_z''}{\omega_{k,0}^{(\alpha)2} - 3k_z^2 s_{\parallel}^{(\alpha)}} \right). \quad (5.49)$$

where $(-)^{\alpha}$ is 1 for electrons and -1 for ions. Equation (5.48) can be written in the form of Equation (4.14) with

$$I_{\mathbf{k}}^z \equiv i \frac{N_{\mathbf{k}}^z}{|e_{\mathbf{k}}^z|^2} = 2i\epsilon_0 \sum_{\alpha} \frac{\omega_{\mathbf{k},0}^{(\alpha)2} \omega_{\mathbf{k},0}^{(\alpha)}}{\left(\omega_{\mathbf{k},0}^{(\alpha)2} - \beta_{\mathbf{k},0}^2 \omega_{\mathbf{k},0}^{(\alpha)2}\right)^2}, \quad (5.50)$$

$$C_{zzz} = \sum_{-p,s,i} \left(\frac{\sum_{\alpha} Q_{-psi}^{(\alpha)}}{\alpha} \right). \quad (5.51)$$

Equation (5.50) has been obtained from Equations (5.42) and (5.47).

The role of the ordinary mode can also be considered in the non-linear interactions, as pointed out at the beginning of this section. Equation (4.14) is therefore the coupled mode equation for any combination of the electron plasma wave (P), the ion acoustic wave (I), and the ordinary wave (O). The combinations which satisfy the synchronism conditions are (i) pump P-signal and idler P, I (see Figure 5.2), and (ii) pump O-signal and idler P, I (see Figure 5.1).

Case (i) (C_{PPI}): Using Equation (5.43) for the electron plasma wave and Equation (5.45) for the ion acoustic wave together with their linear dispersion relations given by Equations (5.44) and (5.46), the quantities $I_{\mathbf{k}}$ and C in Equations (5.50) and (5.51) can be written as

$$I_{\mathbf{k}}^P = 2i\epsilon_0 \frac{\omega_{\mathbf{k},0}^{(e)}}{\omega_N^{(e)2}}, \quad I_{\mathbf{k}}^I = \frac{2i}{9} \epsilon_0 \left(\frac{k_D}{k_z} \right)^4 \sum_{\alpha} \frac{\omega_{\mathbf{k},0}^{(\alpha)}}{\omega_N^{(\alpha)2}}, \quad (5.52)$$

$$C_{PPI} = -i \frac{\epsilon_0^q}{\beta_m^{(e)}} \frac{1}{k_i \omega_N^{(e)2}} \left(k_D^2 + 12 k_p^z k_s^z \right), \quad (5.53)$$

where k_D is defined by

$$k_D = (\text{Debye length})^{-1}. \quad (5.54)$$

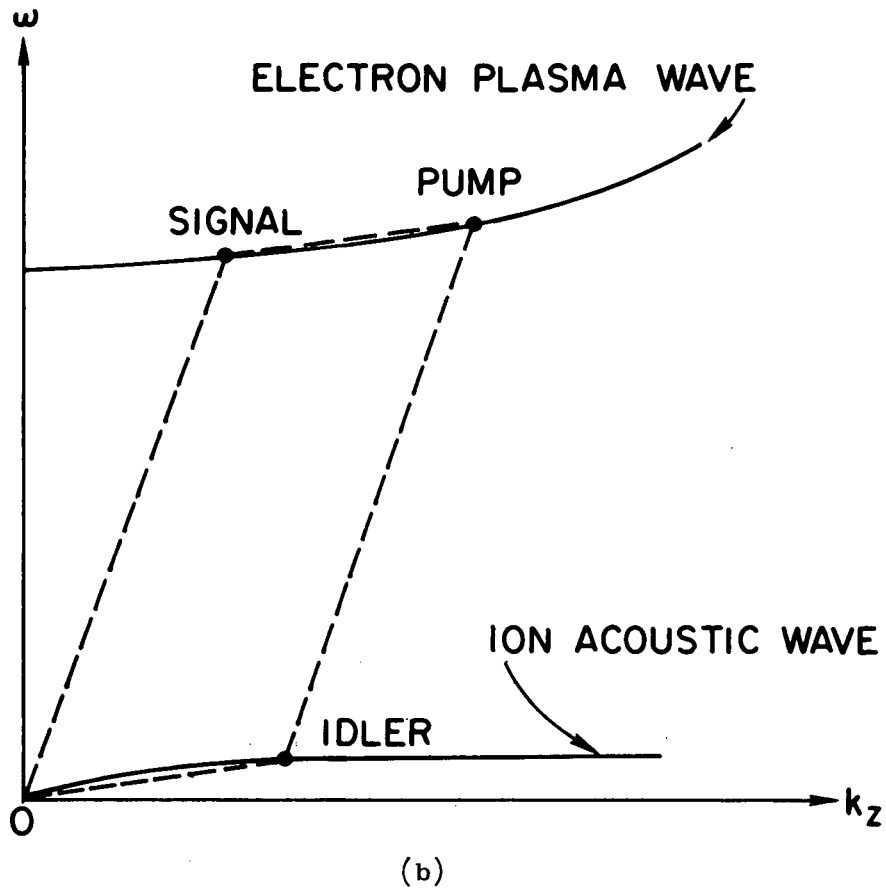
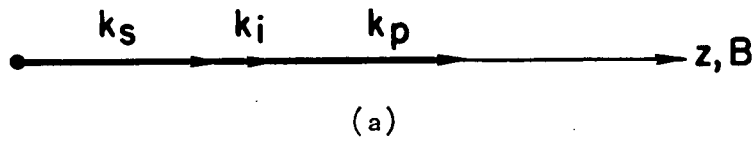


Figure 5.2 Synchronism parallelogram for interaction of two plasma waves (pump and signal) and one ion acoustic wave (idler).

In deriving Equation (5.53), the synchronism condition of Equation (4.13) has been used. For the special case where the Debye length is much shorter than the wavelengths of the ion acoustic waves (pump and signal), the coupling coefficient in Equation (5.53) becomes that obtained earlier by Tsytovich.⁸⁸

Case (ii) (C_{OPI}): In the limit of $k_p^z \rightarrow 0$ for the ordinary wave, the wave vectors of the signal and idler waves are equal and opposite [see Figure 5.1(a)], i.e.,

$$k \equiv k_s^z = k_i^z . \quad (5.55)$$

By using the approximations and dispersion relations expressed by Equations (5.43) - (5.46) for the electron plasma and ion acoustic waves, the quantities I_k and C in Equations (5.50) and (5.51) can be reduced to

$$I_p^0 = -2i\epsilon_0 \sum_{\alpha} \frac{\omega^{(\alpha)2} N}{\omega^{(\alpha)3} k_{\alpha,0}} , \quad (5.56)$$

$$C_{\text{OPI}} = i \frac{\epsilon_0 q}{\beta} \frac{k_D^2}{k} \sum_{\alpha} \frac{1}{m^{(\alpha)} \omega_{p,0}^{(\alpha)2}} . \quad (5.57)$$

The coupling coefficient in Equation (5.57) has been obtained by other methods by Goldman⁸⁹ and Harker.⁹⁰

5.6 Application of the Lagrangian by Method II

The purpose of the following linear theory is to demonstrate that the second order Lagrangian obtained by the second method [Equations (5.33) and (5.34)] is indeed the correct one. To do so, we shall prove that the linear dispersion relations obtained from the hydrodynamic

Lagrangian are the same, within the low temperature approximation, as those obtained by the microscopic treatment of Chapter III. For simplicity, only the electron contribution will be considered in the linear calculation. A similar demonstration can be made for the non-linear theory. Since the procedure is the same as for linear theory, the demonstration for an electron plasma is omitted, and we shall consider instead the interaction of two electron plasma waves propagating nearly parallel to the static magnetic field with an Alfvén wave propagating precisely parallel. For this interaction the ion motions will be included, since they play the dominant role in determining Alfvén wave properties.

5.6.1 Linear Theory

As in Section 5.5.1, the Lagrangian can be written as Equation (5.40) for the linear theory, with $\Lambda_k^{(2)}$ as in Equation (5.33). The first order variables to be independently varied are ξ_k , $\nabla_{\mathbf{W}} \xi_k$ and \tilde{a}_k , with the gauge $\varphi = 0$. After some algebraic manipulation, the Euler-Lagrange equation from the variation in $\nabla_{\mathbf{W}} \xi_{-k}$ yields

$$\begin{aligned} \frac{\partial \xi_k^R}{\partial W_L} &= \left(\frac{\partial \xi_{-k}^L}{\partial W_R} \right)^* = \frac{2k_R}{\omega_{k,-2}} \xi_k^R, & \frac{\partial \xi_k^Z}{\partial W_Z} &= \frac{2k_z}{\omega_{k,0}} \xi_k^Z, \\ \frac{\partial \xi_k^R}{\partial W_R} &= \left(\frac{\partial \xi_{-k}^L}{\partial W_L} \right)^* = \frac{2k_L}{\omega_{k,1}} \xi_k^R - i \frac{q}{m} \frac{b_k^Z}{\omega_{k,0} \omega_{k,1}}, \\ \frac{\partial \xi_k^R}{\partial W_Z} &= \left(\frac{\partial \xi_k^L}{\partial W_Z} \right)^* = \left(\frac{1}{\omega_{k,0}} + \frac{1}{\omega_{k,-1}} \right) k_z \xi_k^R, \\ \frac{\partial \xi_k^Z}{\partial W_L} &= \left(\frac{\partial \xi_k^Z}{\partial W_R} \right)^* = \left(1 + \frac{\omega_{k,0}}{\omega_{k,-1}} \right) \frac{k_R}{\omega_{k,-1}} \xi_k^Z - i \frac{q}{m} \frac{b_k^R}{\omega_{k,-1}^2}. \end{aligned} \quad (5.58)$$

Cartesian coordinates (x,y,z) were used in Equation (5.35). For simplicity, the quantities \underline{k} , $\underline{\xi}_k$, \underline{a}_k (hence \underline{e}_k and \underline{b}_k), and \underline{W} in Equation (5.58) are expressed in the rotating coordinates (R, L, Z) , via the transformation of Equation (3.31). It should be noted that the transformation for $\partial/\partial \underline{W}$ yields

$$\partial/\partial W_I = \sum_j U_{Ij}^{-1} \partial/\partial W_j, \quad (I = R,L,Z, \quad j = x,y,z), \quad (5.59)$$

where U_{Ij}^{-1} is the inverse of U_{Ij} .

The variation of $\underline{f}_k^{(2)}$ in Equation (5.40) with respect to $\underline{\xi}_k$ gives

$$\begin{aligned} (\omega_{k,0}^2 + \omega_{k,\mp 1}^2 + k_{\perp}^2 + k_z^2) \underline{\xi}_k^{R,L} &= 2\omega_{k,0} k_{R,L} s_{\perp} \frac{\partial \underline{\xi}_k^{R,L}}{\partial W_{R,L}} + 2\omega_{k,\mp 1} k_{L,R} s_{\perp} \frac{\partial \underline{\xi}_k^{R,L}}{\partial W_{L,R}} \\ &+ (\omega_{k,0} + \omega_{k,\mp 1}) k_z s_{\parallel} \frac{\partial \underline{\xi}_k^{R,L}}{\partial W_Z} + \frac{q}{m} \left(\underline{e}_k^{R,L} \pm i U_z \underline{b}_k^{R,L} \right), \\ (\omega_{k,0}^2 + k_{\perp}^2 + k_z^2) \underline{\xi}_k^Z &= (\omega_{k,0} + \omega_{k,1}) k_{R,\perp} s_{\perp} \frac{\partial \underline{\xi}_k^Z}{\partial W_R} + (\omega_{k,0} + \omega_{k,-1}) k_{L,\perp} s_{\perp} \frac{\partial \underline{\xi}_k^Z}{\partial W_L} \\ &+ 2\omega_{k,0} k_z s_{\parallel} \frac{\partial \underline{\xi}_k^Z}{\partial W_Z} + \frac{q}{m} \underline{e}_k^Z. \end{aligned} \quad (5.60)$$

The Euler-Lagrange equation resulting from the \underline{a}_k -variation yields the wave equation, as in Equation (3.19),

$$\underline{\eta} \times (\underline{\eta} \times \underline{e}_k) + \underline{\kappa}_k \cdot \underline{e}_k = 0, \quad \kappa_k^{ij} = 1 + \frac{j_k^i}{i \epsilon_0 \omega_k e_k^j}, \quad (5.61)$$

where the current density is given by

$$\underline{j}_{\mathbf{k}} = -in_0q (\omega_{\mathbf{k}} - \vec{\Omega}_{\mathbf{k}}) \underline{\xi}_{\mathbf{k}} ,$$

$$\begin{aligned} \Omega_{\mathbf{k}}^{\text{RR}} &= k_{\text{R}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{R}}} + k_{\text{Z}} s_{\parallel} \frac{\partial}{\partial W_{\text{Z}}} , & \Omega_{\mathbf{k}}^{\text{LL}} &= k_{\text{L}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{L}}} + k_{\text{Z}} s_{\parallel} \frac{\partial}{\partial W_{\text{Z}}} , \\ \Omega_{\mathbf{k}}^{\text{ZZ}} &= -k_{\text{Z}} U_{\text{Z}} + k_{\text{R}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{R}}} + k_{\text{L}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{L}}} , & \Omega_{\mathbf{k}}^{\text{RL}} &= -k_{\text{R}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{L}}} , \\ \Omega_{\mathbf{k}}^{\text{LR}} &= -k_{\text{L}\perp} s_{\perp} \frac{\partial}{\partial W_{\text{R}}} , & \Omega_{\mathbf{k}}^{\text{RZ}} &= -k_{\text{Z}} s_{\perp} \frac{\partial}{\partial W_{\text{L}}} , & \Omega_{\mathbf{k}}^{\text{LZ}} &= -k_{\text{Z}} s_{\perp} \frac{\partial}{\partial W_{\text{R}}} , \\ \Omega_{\mathbf{k}}^{\text{ZR}} &= -k_{\text{L}} \left(U_{\text{Z}} + s_{\parallel} \frac{\partial}{\partial W_{\text{Z}}} \right) , & \Omega_{\mathbf{k}}^{\text{ZL}} &= -k_{\text{R}} \left(U_{\text{Z}} + s_{\parallel} \frac{\partial}{\partial W_{\text{Z}}} \right) . \end{aligned} \quad (5.62)$$

Equations (5.58), (5.60) and (5.61) are the required first order hydrodynamic equations.

If the mean value of the Larmor radius and the thermal velocity of the electrons are very small compared to the wavelength and the phase velocity of the waves, i.e.,

$$k_{\perp}^2 s_{\perp}^2 \ll \omega_c^2 , \quad k_{\text{Z}}^2 s_{\parallel}^2 \ll \omega_{\mathbf{k},\mu} \omega_{\mathbf{k},\nu} \quad (\mu, \nu = 0, \pm 1, \text{ or } \pm 2), \quad (5.63)$$

for a wave whose frequency is far from the cyclotron frequency ($\omega_{\mathbf{k},0} \gg \omega_c$ or $\ll \omega_c$), the second relation in Equation (5.63) may be replaced by

$$k_{\text{Z}}^2 s_{\parallel}^2 \ll \omega_{\mathbf{k},0}^2 . \quad (5.64)$$

Elimination of $\nabla_{\mathbf{W}} \underline{\xi}$ from Equations (5.58) and (5.60) gives the equation of motion to first order in k_{\perp}^2/ω_c^2 and $k_{\text{Z}}^2 s_{\parallel}^2/\omega_{\mathbf{k},\mu} \omega_{\mathbf{k},\nu}$, as

$$\underline{\xi}_{\mathbf{k}} = \underline{\underline{G}}_{\mathbf{k}} \cdot \underline{e}_{\mathbf{k}} , \quad (5.65)$$

where the elements of the matrix \vec{G}_k are given by

$$\begin{aligned}
G_k^{RR} &= G_{-k}^{LL*} = \frac{q}{m} \frac{1}{\omega_k \omega_{k,-1}} \left(1 + 2 \frac{k_{\perp}^2 s_{\perp}}{\omega_{k,0} \omega_{k,-2}} + \frac{k_z^2 s_{\parallel}}{\omega_{k,-1}^2} \right) \\
G_k^{ZZ} &= \frac{q}{m} \frac{1}{\omega_{k,0}^2} \left\{ 1 + \frac{k_{\perp}^2 s_{\perp}}{\omega_{k,1} \omega_{k,-1}} \left[1 + \frac{2\omega_{k,0}^2}{\omega_{k,1} \omega_{k,-1}} \left(1 - \frac{\omega_{k,0}}{\omega} \right) \right] + 3 \frac{k_z^2 s_{\parallel}}{\omega_{k,0}^2} \right\}, \\
G_k^{RL} &= G_{-k}^{LR*} = 2 \frac{q}{m} \frac{k_R^2 s_{\perp}}{\omega_k \omega_{k,0} \omega_{k,1} \omega_{k,-1}}, \\
G_k^{ZR} &= G_{-k}^{ZL*} = \frac{q}{m} \frac{s_{\perp} k_{\perp} k_z}{\omega_k \omega_{k,0} \omega_{k,-1}} \left(\frac{1}{\omega_{k,0}} + \frac{1}{\omega_{k,-1}} \right), \\
G_k^{RZ} &= G_{-k}^{LZ*} = \frac{q}{m} \frac{1}{\omega_k \omega_{k,0} \omega_{k,-1}} \left\{ k_R U_Z + 3 \frac{k_{\perp}^2 k_R s_{\perp} U_z}{\omega_{k,1} \omega_{k,-2}} \right. \\
&\quad \left. + k_z k_R s_{\parallel} \left[\frac{\omega_k}{\omega_{k,0}} \left(\frac{1}{\omega_{k,0}} + \frac{1}{\omega_{k,-1}} \right) + \frac{k_z U_z}{\omega_{k,-1}^2} \right] \right\}. \tag{5.66}
\end{aligned}$$

With the use of Equations (5.58) and (5.64), the equivalent plasma permittivity matrix can be expressed as

$$\begin{aligned}
\kappa_{\mathbf{k}}^{\text{RR}} = \kappa_{-\mathbf{k}}^{\text{LL}} &= 1 - \frac{\omega_{\text{N}}^2}{\omega_{\mathbf{k}}^2 \omega_{\mathbf{k},-1}} \left[\omega_{\mathbf{k},0} + s_{\perp} \left(\frac{2k_{\perp}^2}{\omega_{\mathbf{k},-2}} + \frac{k_z^2}{\omega_{\mathbf{k},-1}} \right) + \frac{k_z^2 s_{\parallel}}{\omega_{\mathbf{k},-1}} \left(\frac{\omega_{\mathbf{k},0}}{\omega_{\mathbf{k},-1}} - 1 \right) \right], \\
\kappa_{\mathbf{k}}^{\text{ZZ}} &= 1 - \left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}} \right)^2 \left\{ \left(\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k},0}} \right)^2 \left(1 + \frac{k_z^2 s_{\parallel}^2}{\omega_{\mathbf{k},0}^2} \right) + \frac{k_{\perp}^2 U_z^2}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}} \right. \\
&\quad \left. + \frac{k_{\perp}^2 s_{\perp}}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}} \left[3 \frac{k_{\perp}^2 U_z^2}{\omega_{\mathbf{k},2} \omega_{\mathbf{k},-2}} + \frac{\omega_{\mathbf{k}}^2 (3 - \omega_c^2 / \omega_{\mathbf{k},0}^2) - 4 \omega_{\mathbf{k}} \omega_{\mathbf{k},0} + \omega_{\mathbf{k},0}^2 + \omega_c^2}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}} \right] \right\} \\
&\quad \left. + \frac{k_{\perp}^2 s_{\parallel}}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}} \left[1 + 4 \frac{k_z U_z \omega_{\mathbf{k},0}}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}} + \frac{k_z^2 U_z^2 (3 \omega_{\mathbf{k},0}^2 + \omega_c^2)}{\omega_{\mathbf{k},1}^2 \omega_{\mathbf{k},-1}^2} \right] \right\}, \\
\kappa_{\mathbf{k}}^{\text{RL}} = \kappa_{\mathbf{k}}^{\text{LR}*} &= -2 \left(\frac{\omega_{\text{N}}}{\omega_{\mathbf{k}}} \right)^2 \frac{s_{\perp} k_{\perp}^2}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-1}}, \\
\kappa_{\mathbf{k}}^{\text{RZ}} = \kappa_{\mathbf{k}}^{\text{ZR}*} = \kappa_{-\mathbf{k}}^{\text{ZL}} = \kappa_{-\mathbf{k}}^{\text{LZ}*} &= - \frac{\omega_{\mathbf{k}}^2}{\omega_{\mathbf{k}}^2 \omega_{\mathbf{k},-1}} \left[U_z + s_{\perp} \left(\frac{k_z \omega_{\mathbf{k}}}{\omega_{\mathbf{k},0}^2} + 3 \frac{k_{\perp}^2 U_z}{\omega_{\mathbf{k},1} \omega_{\mathbf{k},-2}} + \frac{k_z^2 U_z}{\omega_{\mathbf{k},0} \omega_{\mathbf{k},-1}} \right) \right. \\
&\quad \left. + \frac{k_z s_{\parallel}}{\omega_{\mathbf{k},-1}} \left(1 + \frac{k_z U_z}{\omega_{\mathbf{k},-1}} \right) \right]. \tag{5.67}
\end{aligned}$$

The wave dispersion relations are then obtained by substitution of Equation (5.67) into Equation (5.61).

The equivalent permittivity in Equation (5.67) can also be obtained from Equation (3.44), with the low temperature approximations of Equation (5.63), by using the following series representation of the J_n Bessel function:

$$J_n(\lambda) = \left(\frac{\lambda}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-\lambda^2/4)^m}{m! \Gamma(m+n+1)} . \quad (5.68)$$

Thus, when the low temperature approximation is made, the dispersion relations from the microscopic treatment reduce to those obtained in this section.

5.6.2 Interaction of Electron Plasma and Alfvén Waves

The interaction mechanism of two plasma waves and an Alfvén wave is as follows. In the magnetosonic régime, the Alfvén wave propagating parallel to the static magnetic field is a linearly polarized transverse wave.⁹¹ If its electric field is polarized in the direction of the x-axis, this electric field can interact with the x-component of the electric field of an electron plasma wave propagating nearly parallel to the magnetic field to produce rf space charge variations in the x-direction. This rf space charge may in turn excite another electron plasma wave also propagating nearly parallel to the magnetic field. These electron plasma waves have the same azimuthal angle which may, of course, be chosen arbitrarily. Since the strongest interaction obviously occurs when the electric field components of these electron plasma waves normal to the magnetic field are aligned along the x-axis, we shall assume that the electron plasma waves propagate in the xz-plane as shown in Figure 5.3. Therefore, all vectors are in either x, y or z-directions. For algebraic simplicity, drift velocities are ignored in the following.

Linear Equations: First, for the Alfvén wave, we have

$$\omega_k \ll \omega_c^{(i)}, \quad k_x = k_y = 0, \quad e_k^y = e_k^z = b_k^x = b_k^z = 0, \quad b_y = \frac{k_z}{\omega_k} e_x . \quad (5.69)$$

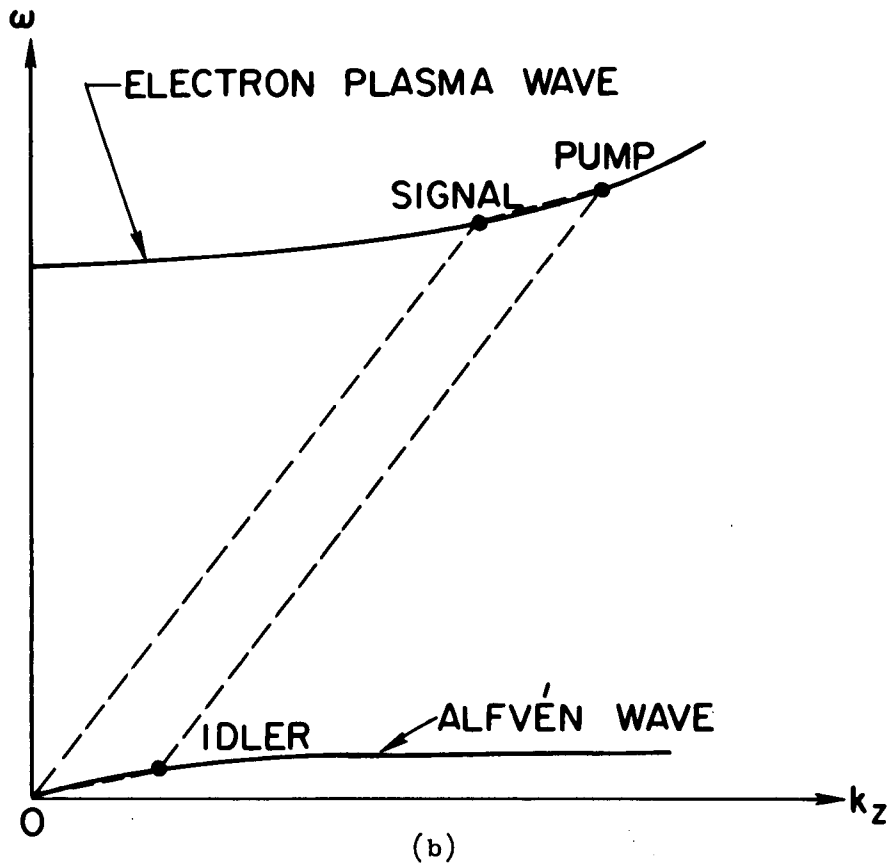
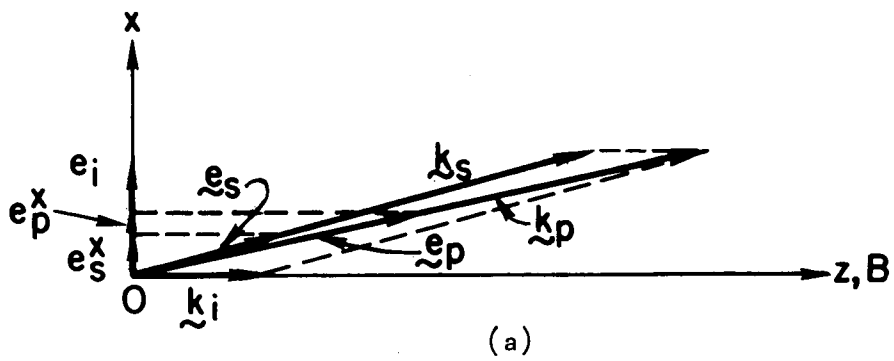


Figure 5.3 Synchronism parallelograms for interaction of two plasma waves (pump and signal) and one Alfvén wave (idler).

To lowest order in $\omega_k/\omega_c^{(i)}$, transformation from (R,L,Z) to (x,y,z) coordinates reduces Equations (5.58) and (5.65), respectively, to

$$\begin{aligned} \frac{\partial \xi_k^x(e)}{\partial W_x(e)} &= \frac{\partial \xi_k^x(e)}{\partial W_y(e)} = \frac{\partial \xi_k^y(e)}{\partial W_x(e)} = \frac{\partial \xi_k^y(e)}{\partial W_y(e)} = \frac{\partial \xi_k^z(e)}{\partial W_z(e)} = 0, \\ \frac{\partial \xi_k^x(e)}{\partial W_z(e)} &= \frac{k_z}{\omega_k} \xi_k^x(e) - i \frac{k_z}{\omega_c} \xi_k^x(e) + \frac{q}{m(e)} \frac{b_k^y}{\omega_c^2}, \\ \frac{\partial \xi_k^y(e)}{\partial W_z(e)} &= i \frac{k_z}{\omega_c} \xi_k^x(e) + \frac{k_z}{\omega_k} \xi_k^y(e) - i \frac{q}{m(e)} \frac{b_k^y}{\omega_c \omega_k}, \\ \frac{\partial \xi_k^z(e)}{\partial W_x(e)} &= \frac{q}{m(e)} \frac{b_k^y}{\omega_c^2}, \quad \frac{\partial \xi_k^z(e)}{\partial W_y(e)} = -2i \frac{q}{m(e)} \frac{\omega_k b_k^y}{\omega_c^3}, \end{aligned} \quad (5.70)$$

$$\begin{aligned} \xi_k^x(e) &= -\frac{q}{m(e)} \frac{1}{\omega_c^2} \left(1 + \gamma \frac{k_z^2(e)}{\omega_c^2} \right) e_k^x, \quad \xi_k^z(e) = 0, \\ \xi_k^y(e) &= i \frac{q}{m(e)} \frac{1}{\omega_c \omega_k} \left[1 + \left(\frac{\omega_k}{\omega_c} \right)^2 + \frac{k_z^2(e)}{\omega_c^2} \right] e_k^x. \end{aligned} \quad (5.71)$$

Similar equations for the ions are obtained from Equations (5.70) and (5.71) by the procedure described in Section 5.4. Including the ion motion in Equations (5.61) and (5.67) gives the dispersion relation as

$$n_z^2 - \kappa_k = 0, \quad \kappa_k = 1 + \sum_{\alpha} \omega_N^{(\alpha)2} \left[\frac{1}{\omega_c^{(e)} \omega_c^{(i)}} + \frac{k_z^2 (s_{\parallel}^{(\alpha)} - s_{\perp}^{(\alpha)})}{\omega_c^{(\alpha)2} \omega_k^2} \right]. \quad (5.72)$$

Next, for the electron plasma wave, we have the relations

$$k_x \ll k_z, \quad k_y = 0, \quad e_k^x \ll e_k^z, \quad e_k^y = 0, \\ b_k^x = b_k^z = 0, \quad b_k^y = \frac{1}{\omega_k} (k_z e_k^x - k_x e_k^z). \quad (5.73)$$

To first order in k_x/k_z , Equation (5.58) becomes

$$\frac{\partial \xi_k^x(e)}{\partial W_x(e)} = \frac{\partial \xi_k^x(e)}{\partial W_y(e)} = \frac{\partial \xi_k^y(e)}{\partial W_x(e)} = \frac{\partial \xi_k^y(e)}{\partial W_y(e)} = 0, \quad \frac{\partial \xi_k^z(e)}{\partial W_z(e)} = 2 \frac{k_z}{\omega_k} \xi_k^z(e), \\ \frac{\partial \xi_k^x(e)}{\partial W_z(e)} = \frac{k_z}{\omega_k} \left(1 + \frac{\omega_k^2}{\omega_{k,1}(e)\omega_{k,-1}(e)} \right) \xi_k^x(e) + i \frac{k_z \omega(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} \xi_k^y(e) - \frac{q}{m(e)} \frac{1}{\omega_{k,1}(e)\omega_{k,-1}(e)} b_k^y, \\ \frac{\partial \xi_k^y(e)}{\partial W_z(e)} = -i \frac{k_z \omega(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} \xi_k^x(e) + \frac{k_z}{\omega_k} \left(1 + \frac{\omega_k^2}{\omega_{k,1}(e)\omega_{k,-1}(e)} \right) \xi_k^y(e) + i \frac{q}{m(e)} \frac{\omega(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} b_k^y, \\ \frac{\partial \xi_k^z(e)}{\partial W_x(e)} = \frac{k_x \omega_k}{\omega_{k,1}(e)\omega_{k,-1}(e)} \left(1 + \frac{\omega_k^2 + \omega(e)^2}{\omega_{k,1}(e)\omega_{k,-1}(e)} \right) \xi_k^z(e) + \frac{q}{m(e)} \frac{\omega_k^2 + \omega(e)^2}{\omega_{k,1}(e)^2 \omega_{k,-1}(e)^2} b_k^y, \\ \frac{\partial \xi_k^z(e)}{\partial W_y(e)} = -i \frac{k_z \omega(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} \left(1 + \frac{2\omega_k^2}{\omega_{k,1}(e)\omega_{k,-1}(e)} \right) \xi_k^z(e) - 2i \frac{q}{m(e)} \frac{\omega_k \omega(e)}{\omega_{k,1}(e)^2 \omega_{k,-1}(e)^2} b_k^y, \quad (5.74)$$

and Equation (5.65) becomes

$$\begin{aligned}
\mathbb{E}_k^x(e) &= \frac{q}{m(e)} \left[\frac{1}{\omega_{k,1}(e)\omega_{k,-1}(e)} + \frac{k_{z\parallel}^2 s(e)}{2\omega_k} \left(\frac{1}{\omega_{k,1}(e)^3} + \frac{1}{\omega_{k,-1}(e)^3} \right) \right] e_k^x + \frac{q}{m(e)} \frac{k_x k_z s(e)}{\omega_{k,1}(e)^2 \omega_{k,-1}(e)^2} e_k^z, \\
\mathbb{E}_k^y(e) &= -i \frac{q}{m(e)} \frac{1}{\omega_k} \left[\frac{\omega_c(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} + \frac{k_{z\parallel}^2 s(e)}{2} \left(\frac{1}{\omega_{k,-1}(e)^3} - \frac{1}{\omega_{k,1}(e)^3} \right) \right] e_k^x \\
&\quad - i \frac{q}{m(e)} \frac{k_x k_z s(e) \omega_c(e) (3\omega_k^2 - \omega_c^2)}{\omega_{k,1}(e)^3 \omega_{k,-1}(e)^2 \omega_c^2}, \\
\mathbb{E}_k^z(e) &= \frac{q}{m(e)} \frac{1}{\omega_k^2} \left(1 + 3 \frac{k_{z\parallel}^2 s(e)}{\omega_k^2} \right) e_k^z. \tag{5.75}
\end{aligned}$$

Substitution of Equations (5.67) and (5.73) reduces the wave equation [Equation (5.61)] to

$$A_k^{xx} e_k^x + A_k^{xz} e_k^z = 0, \quad \kappa_k = 0, \tag{5.76}$$

where A_k^{xx} , A_k^{xz} , and κ_k are given by

$$\begin{aligned}
A_k^{xx} &= 1 - \eta_z^2 - \left(\frac{\omega(e)}{\omega_k} \right)^2 \left[\frac{\omega_k^2}{\omega_{k,1}(e)\omega_{k,-1}(e)} + \frac{(\omega_k^2 + \omega_c^2) k_{z\parallel}^2 s(e)}{\omega_{k,1}(e)^2 \omega_{k,-1}(e)^2} \right. \\
&\quad \left. + \frac{\omega_c^2 k_{z\parallel}^2 s(e)}{2} \left(\frac{1}{\omega_{k,1}(e)^3} + \frac{1}{\omega_{k,-1}(e)^3} \right) \right], \\
A_k^{xz} &= \eta_x \eta_z - \left(\frac{\omega(e)}{\omega_k} \right)^2 \frac{k_x k_z}{\omega_{k,1}(e)\omega_{k,-1}(e)} \left[s_{\perp}(e) + \frac{(\omega_k^2 + \omega_c^2) s_{\parallel}(e)}{\omega_{k,1}(e)\omega_{k,-1}(e)} \right], \\
\kappa_k &= 1 - \left(\frac{\omega(e)}{\omega_k} \right)^2 \left(1 + \frac{3k_{z\parallel}^2 s(e)}{\omega_k^2} \right). \tag{5.77}
\end{aligned}$$

In Equation (5.77), the ion contributions have disappeared because

$$\omega_N^{(i)} \ll \omega_k \quad [\text{Equation (5.43)}].$$

Coupled Mode Equations: In order to obtain the coupled mode equations, the general expression for $\Lambda_{kk',k''}^{(3)}$ in Equation (5.34) has to be reduced according to the conditions of the present case. Let the Alfvén wave be the idler. Equation (5.69) for the Alfvén wave, and Equation (5.73) for the electron plasma wave then reduce $\lambda_0^{(3)}$, $\lambda_{10}^{(3)}$ and $\lambda_{11}^{(3)}$ in Equation (5.35) to the forms

$$\begin{aligned} \sum_{kk',k''} \lambda_0^{(3)}(e) &= i \frac{n_0 q}{2} \sum_{k,k'} \left\{ k_x e_{k'}^z \xi_i^x(e) + \sum_{k,i} \left[k_z e_k^x + (\omega_k' - \omega_i) b_k^y \right] \xi_i^x(e) \right\} \xi_k^z(e), \\ \sum_{kk',k''} \lambda_{10}^{(3)}(e) &= in_0 q \sum_{k,k'} \sum_{k,i} \left[k_z s_i^z(e) b_k^y \xi_k^y(e) \frac{\partial \xi_i^z(e)}{\partial \omega_x(e)} \right. \\ &\quad \left. + k_z s_{\parallel}^z(e) \left(\xi_k^x(e) \frac{\partial \xi_k^x(e)}{\partial \omega_z(e)} + \frac{\partial \xi_k^x(e)}{\partial \omega_z(e)} \xi_k^z(e) \right) b_i^y \right], \\ \sum_{kk',k''} \lambda_{11}^{(3)}(e) &= -in_0 q s_{\parallel}^z(e) \sum_{k,k'} \sum_{k,i} \omega_k \frac{\partial \xi_k^x(e)}{\partial \omega_z(e)} \frac{\partial \xi_k^z(e)}{\partial \omega_z(e)} b_i^y. \end{aligned} \quad (5.78)$$

Since substitution of Equations (5.70), (5.71), (5.74) and (5.75) into Equation (5.78) gives a rather complicated expression, an additional simplifying assumption will be made that

$$\omega_k, \omega_{k'} \gg \omega_c^{(i)}, \quad (k, k' = \text{electron plasma waves}). \quad (5.79)$$

After eliminating e_k^x by use of Equation (5.76), the coupled mode equation can be written in the form of Equation (4.14) with

$$I_k^A = 2i\epsilon_0 \frac{1}{\omega_k} \left(1 + \frac{\sum \omega(\alpha)^2}{\omega_c^{(e)} \omega_c^{(i)}} \right), \quad I_k^P = 2i\epsilon_0 \frac{1}{\omega_k} \left(2 - \frac{\omega^{(e)2}}{\omega_k^2} \right),$$

$$C_{PPA} = \sum_{-p, s} \left(\sum_{\alpha} Q_{-ps}^z(\alpha) - \frac{A_p^{xz}}{A_p^{zz}} \sum_{\alpha} Q_{-ps}^x(\alpha) \right), \quad (5.80)$$

where A_p^{xz} and A_p^{zz} are given in Equation (5.77) with $k = p$, and $Q_{kk'}^z(\alpha)$ and $Q_{kk'}^x(\alpha)$ are defined by

$$\begin{aligned} Q_{kk'}^z(\alpha) = & (-)^{\alpha} \frac{\epsilon_0^q}{im(\alpha)} k_x \omega_N^{(\alpha)2} \left\{ \frac{1}{\omega_c^{(\alpha)2} \omega_k \omega_{k'}} - \frac{k_{is}^z(\alpha)}{\omega_k \omega_i} \left(\frac{k_i^z}{\omega_{k,1}(\alpha) \omega_{k,-1}(\alpha) \omega^2} - \frac{k_z}{\omega_c^{(\alpha)2} \omega_k^2} \right. \right. \\ & + \left. \frac{k'_z}{\omega_c^{(\alpha)2} \omega_{k,1}(\alpha) \omega_{k,-1}(\alpha)} \right) + \frac{k_{is}^z(\alpha)}{\omega_{k,1}(\alpha) \omega_{k,-1}(\alpha) \omega_k^2 \omega_i} \left(\frac{k_z}{\omega_k} - \frac{2k'_z}{\omega_{k'}} - \frac{k_z \omega}{\omega_{k,1}(\alpha) \omega_{k,-1}(\alpha)} \right) \\ & \left. + \frac{s(\alpha)}{\omega_c^{(\alpha)2} \omega_k \omega_{k'}} \left(\frac{3k_z'^2}{\omega_k^2} - \frac{k_z^2}{\omega_{k'} \omega_i} - \frac{2k'_z k_i}{\omega_{k'} \omega_i} + \frac{3k_i^z}{\omega_c^{(\alpha)2}} \right) \right\}, \\ Q_{kk'}^x(\alpha) = & (-)^{\alpha} \frac{\epsilon_0^q}{m(\alpha)} \omega_N^{(\alpha)2} \left\{ \frac{1}{\omega_{k'}^2} \left(\frac{k_i^z \omega}{\omega_{k,1}(\alpha) \omega_{k,-1}(\alpha) \omega_i} - \frac{k_z \omega_i}{\omega_c^{(\alpha)2} \omega_k} \right) \right. \\ & - \left. \frac{k_{is}^z(\alpha)}{\omega_k \omega_i} \left[\frac{k_i}{2\omega_{k'}^2} \left(\frac{1}{\omega_{k,1}(\alpha)2} + \frac{1}{\omega_{k,-1}(\alpha)2} \right) + \frac{1}{2\omega_c^{(\alpha)2}} \left(\frac{2k_z}{\omega_{k'}^2} + \frac{k'_z}{\omega_{k,1}(\alpha)2} + \frac{k'_z}{\omega_{k,-1}(\alpha)2} \right) \right] \right. \\ & + \frac{k_{is}^z(\alpha)}{\omega_{k'}^2 \omega_i} \left[\frac{3k_z' \omega}{\omega_{k,1}(\alpha) \omega_{k,-1}(\alpha) \omega_k^2} - \frac{k_z^2}{2\omega_k} \left(\frac{1}{\omega_{k,1}(\alpha)2} + \frac{1}{\omega_{k,-1}(\alpha)2} \right) + \frac{k_z k'_z}{\omega_k} \left(\frac{1}{\omega_{k,1}(\alpha)} - \frac{1}{\omega_{k,-1}(\alpha)} \right)^2 \right. \\ & \left. \left. + \frac{k_z^2}{2} \left(\frac{1}{\omega_{k,1}(\alpha)3} + \frac{1}{\omega_{k,-1}(\alpha)3} \right) \right] + \frac{k_{is}^z(\alpha)}{\omega_c^{(\alpha)2} \omega_k \omega_{k'}} \left(\frac{4k'_z k_i^z}{\omega_{k'}} - \frac{3k_z'^2 \omega_i}{\omega_{k'}^2} - \frac{k_i^z}{\omega_i} \right) \right\}. \quad (5.81) \end{aligned}$$

Here $(-)^{\alpha}$ is 1 for electrons and -1 for ions. It should be noted that Equation (5.81) has two kinds of terms, one of order $O(\omega_c^0)$ and the other of order $O(\omega_c^{-2})$. Since $\omega_N^{(e)} \gg \omega_N^{(i)}$, the former terms for the ion equation can be neglected compared to those for the electron equation. Therefore, for $\sum_{\alpha} Q^{(\alpha)}$, the summation need be taken only for the terms of order $O(\omega_c^{-2})$. It should also be noted that the x-components of the electric fields of the electron plasma waves, and hence the coupling coefficient, C_{PPA} , vanish when $k_x \rightarrow 0$.

5.7 Discussion

In this chapter, two methods of formulation of the hydrodynamic Lagrangian have been presented and applied to several specific cases of wave-wave interactions. Lagrangians previously used by others for the MHD approximation include only the second moment equation.^{42,69,70} Since our hydrodynamic Lagrangians include Maxwell's equations and the second moment equation (first method), or three moment equations (second method), they are more general. Fundamental equations for the MHD approximation can be obtained from our Lagrangian in Section 5.2.3 for spherical compression by introducing Ohm's Law.⁸³

The second method in this chapter has very important advantages over the first. In particular, as demonstrated in Section 5.6, it does not involve the mathematical difficulties associated with the expansion of the Lagrangian in the first method, which effectively restrict us to the three special cases treated in Section 5.2. Another significant advantage is that in the case of nonlinear interaction it can be applied when all interacting waves do not have the

same value of γ . This situation cannot be handled by the first Lagrangian method. For example, although the small-signal properties of electron plasma waves and compressional Alfvén waves in an isotropic plasma can be studied by the first method using two different second order Lagrangians [Equations (5.15) and (5.21)], for interactions between them we must use only one Lagrangian; since these two kinds of waves have different Lagrangians for third order and above, a satisfactory higher order Lagrangian cannot be chosen. In the second Lagrangian method, however, this difficulty does not occur; the interaction of any combination of waves can be treated provided that the basic assumption that the wave phase velocities are greater than the electron thermal velocity is valid.

Finally, we note that the second Lagrangian method includes all necessary moment equations, whereas the first contains only the second moment equation (the momentum equation), and must employ the other moment equations as subsidiary conditions.

Chapter VI

CONCLUSIONS

6.1 Review of the Research

The main purpose of our work has been to develop a general theory of the averaged Lagrangian method to describe warm, collisionless plasmas, and to demonstrate the effectiveness of the method in analyzing wave-wave interactions. This has involved formulation of appropriate Lagrangians, and extension of previous work on the averaged Lagrangian method for cold plasmas. Treatment of specific cases has demonstrated the power of the method in analyzing wave-wave interactions by confirming some results obtained previously by other methods, and providing coupling coefficients for a number of new interactions. Since our developments cover both the microscopic treatment and the hydrodynamic approximation to it, the averaged Lagrangian method is now available for use in all of the usual plasma descriptions: microscopic, hydrodynamic, and cold plasma.

Among the advantages of the averaged Lagrangian method, as compared with the iterative method, are first, as demonstrated in Section 4.2, that the coupled mode equations can be obtained in canonical form from α_k -variation of the Lagrangian without specifying the detailed form of the Lagrangian for a particular system. In the iterative method, on the other hand, the coupled mode equations can be obtained only after detailed and lengthy algebra specific to the case under study. In the Lagrangian method, tedious algebra is required only in obtaining the coupling coefficients. Second, the Manley-Rowe relations can be obtained by variation with respect to phase, and are automatically consistent with the coupled mode equations.

The second formulation of the hydrodynamic Lagrangian proposed in Section 5.3 has significant advantages over the iterative method (here we mean direct use of the moment equations and Maxwell's equations). As described at the beginning of Section 5.2, expansion of the moment equations involves mathematical difficulties. Consequently, even for small-signal propagation, most of the work by direct use of the moment equations and Maxwell's equations has been limited to either the three cases in Section 5.2, or the special case of a plasma in a strong static magnetic field,⁹² and the expansion has been carried out using an adiabatic equation of state of the form of Equation (C.1). For cases more general than those just mentioned, the validity of the adiabatic equation of state is in doubt. For example, an electrostatic wave has been shown [Section 5.2.1] to satisfy Equation (C.1) with $\gamma = 3$. As soon as its propagation direction departs from the z-direction, the assumptions in Equation (5.11) are violated, and Equation (5.12) is no longer justified. As demonstrated in Sections 5.3 and 5.6, and discussed in Section 5.7, such difficulties are avoided in the second formulation of the hydrodynamic Lagrangian.

The averaged method has been used to obtain the coupled mode equations for all possible interactions of waves propagating nearly parallel, or precisely perpendicular to the static magnetic field, using the microscopic equations. Several interactions have also been treated in the hydrodynamic approximation. The new interactions that we have studied by the microscopic treatment are those among three circularly-polarized waves (except that among three right-hand circularly polarized waves); among two plasma waves and one circularly-polarized wave, and among one longitudinal cyclotron harmonic wave

and two ordinary cyclotron harmonic waves (pump \parallel - signal and idler 0,0). In the hydrodynamic approximation, the only new interaction is that among two electron plasma waves and one Alfvén wave. Since one of the principal aims of our applications of the method has been to compare the results, and the ease of obtaining them, with previous analyses by other methods, we have not carried out numerical calculations. We will simply speculate here on some future directions that appear to be worth following.

It is evident that rf space charge plays an important role in mode coupling. In order for nonlinear interaction to occur, at least one of the participating waves must have nonzero rf space charge. This being so, we may divide all interactions into four cases: those among three longitudinal waves; among two longitudinal and one transverse wave; among one longitudinal and two transverse waves, and among three quasiperpendicular waves which have nonzero axial components. Previous analyses have shown that relatively strong interactions occur among three longitudinal waves (e.g., plasma waves or longitudinal cyclotron harmonic waves).^{28,32} For interaction of three cyclotron harmonic waves, for instance, Harker and Crawford calculated the nonlinear growth rate due to a pump wave of 0.4 volt to be order of 1 - 10 db/cm over the range of cyclotron frequencies for a typical laboratory plasma of $n_0 = 1.25 \times 10^{10} \text{ cm}^{-3}$, when the electron temperature is 4 volts.³² The interactions of this category studied in our work are the P-P,P and $\parallel\text{-}\parallel\text{-}\parallel$ in the microscopic treatment, and the P-P,I for the hydrodynamic approximation. In the P-P,I interaction we note that the range of frequency for synchronism is narrow, and the electron plasma waves and the ion acoustic wave have very low group velocity, so that plasma

inhomogeneity or collisional effects may invalidate our results.

The next important interactions should be those involving two longitudinal waves and one transverse wave. Examples treated here are the R-P,P; L-P,P; P-R,P; and P-L,P for the microscopic description, and the O-P,P and P-P,A in the hydrodynamic approximation. The mechanism of interaction by the microscopic treatment may be considered to be similar to that of interaction among three longitudinal waves, since longitudinal components of circularly-polarized waves can produce rf space charge in the direction of propagation when they propagate slightly oblique to the static magnetic field. In the case of precisely parallel propagation the longitudinal components vanish, and so does the interaction, as shown in Table 4.1. Because these longitudinal components are small, we may expect these interactions to be weaker in general than those involving three longitudinal waves. However, as mentioned in Section 4.5, the whistler and plasma waves can be unstable for a non-Maxwellian plasma, and can also carry positive or negative energy. It would consequently be of great interest to make further detailed analyses of the R-P,P and P-R,P cases when the R-wave represents a whistler. The character of the O-P,P interaction is essentially the same as that of the P-P,P interaction since, in the dipole approximation, the ordinary wave behaves like a plasma wave of infinite wavelength. The interaction may be expected to be correspondingly strong. In fact, this interaction is currently considered to be an important process in ionospheric modification experiments with high power sounders⁹³, and in producing plasmas by use of lasers.⁹⁴ As shown in Figure 5.3(a), the P-P,A interaction occurs due to coupling

of the components of electric field perpendicular to the static magnetic field. The interaction consequently vanishes for parallel propagation. Since the analysis in Section 5.6 was for the Alfvén wave within the magnetosonic regime ($\omega_k \ll \omega_c^{(i)}$), the range of frequencies of the electron plasma waves for synchronism is relatively narrow. In practice, slight inhomogeneity of the magnetic field and plasma would cause strong deviation from the theoretical predictions.

Interactions considered in our work involving one longitudinal wave and two transverse waves are the R-R,P; L-L,P and P-P,L, for parallel propagation, and the O- \parallel ,0 and \parallel -O,0 for perpendicular propagation. One of the features of these interactions is that the second order electron gyrations due to two transverse waves ($\propto \dot{\underline{E}}_k \times \underline{b}_k$) are coupled to the rf space charge of the longitudinal wave. Therefore, contrary to the case of interactions involving two longitudinal waves, interactions can occur among waves propagating precisely parallel to the magnetic field. Since only one longitudinal wave is involved, the interaction may be expected to be very weak.

The final case of interaction is among circularly-polarized waves propagating slightly oblique to the static magnetic field, whose electric field components in the direction of wave propagation produce small amounts of rf space charge. Examples are the R-R,R; R-R,L and L-R,L. The rf space charge is much weaker than for the first two, and hence no strong interactions can be expected. In the ionosphere, the growth rates due to the R-R,P; L-L,P or P-R,L are of order 10^{-3} - 10^{-6} db/m for a pump wave field strength of $e_p \sim 0.1$ m volt/m,⁹⁵ and that due to interaction among three whistlers is of order 10^{-6} db/m or less for a pump field of $e_p \sim 1$ m volt/m.²⁷ Previous calculations have, however, been performed assuming cold plasma, or by use of moment theory. It

should be remembered that electron plasma waves and whistlers can be unstable, and can also carry either positive or negative energy when the electron velocity distribution is nonMaxwellian. Further detailed studies are consequently necessary for interactions involving these waves under magnetospheric conditions.

6.2 Suggestions for Future Work

The next important step following our work is to obtain numerical solutions for the growth rates for comparison with observations made in the laboratory and the ionosphere. Most previous numerical solutions have been obtained for cold plasmas, or within the hydrodynamic approximation. Those obtained by the microscopic treatment for warm plasmas are potentially of greater interest because the microscopic description generally offers a broader range of possibilities for synchronism. Also, interactions involving longitudinal waves or whistlers, which would otherwise be weak, may be much more important in plasmas having nonMaxwellian particle velocity distributions.

A large number of interactions which can be treated in a straightforward manner by the averaged Lagrangian method still remain to be studied. In the microscopic treatment, for example, the interactions between waves propagating parallel to the static magnetic field, and those propagating perpendicular should be examined. In the hydrodynamic approximation, there are numerous combinations of electromagnetic waves and ion waves which are likely to couple strongly, and to be observable experimentally.

In our work, the averaged Lagrangian method has been developed for coherent three-wave interaction in an infinite, collisionless warm plasma in a uniform static magnetic field. Some extensions of the

present theory to less restricted conditions would be of interest:

1. Higher Order Wave-Wave Interactions: Waves which do not satisfy the synchronism conditions for three-wave interaction can still be coupled by higher order wave-wave interactions in which more than three waves interact. Even when three-wave interaction occurs among existing modes, such higher order wave-wave interactions may not be negligible, since they may have growth rates comparable to those of three-wave interaction. To describe n-wave interaction, the Lagrangian must be expanded up to n^{th} order in perturbation.

2. Background Reaction: If the reaction on the assumed homogeneous and time-invariant plasma is taken into account, the Euler-Lagrange equations for the perturbed state should be obtainable from variations of the expanded Lagrangian with respect to $\underline{r}(= \underline{r}_0 + \underline{\xi})$, $\varphi(= \Phi + \varphi_1)$ and $\underline{A}(= \underline{A}_0 + \underline{a})$, instead of $\underline{\xi}$, φ_1 and \underline{a} of Section 2.2.⁴³ The Euler-Lagrange equations from \mathcal{L}_2 should then yield equations equivalent to those of quasilinear theory.⁴ An alternative way to take account of the reaction on the background is to assume the equilibrium velocity distribution function, f_0 , to vary slowly in time and space, and to take variations in $\underline{\xi}$, φ_1 and \underline{a} . This method is being developed at Stanford by Galloway.⁹⁶

3. Nonlinear Wave-Particle Interaction: Next in order of complexity is nonlinear wave-particle interaction. This corresponds to the beating of two waves to form a sum or difference wave which then interacts with particles in resonance with it. As for wave-wave interaction, the Lagrangian must be expanded beyond second order in perturbation to take such effects into account. The main difference from wave-wave interaction is that the particle velocity distribution itself is modified by

the waves. Consequently, the technique developed to include the background reaction has to be extended.

4. Nonuniform Plasmas: The expanded Lagrangians in Section 2.2 are applicable to inhomogeneous plasmas. Homogeneity has not been assumed in formulating the hydrodynamic Lagrangian by the second method in Section 5.3. Therefore, microscopic or macroscopic properties of inhomogeneous plasmas can be described by using these Lagrangians. Among the subjects of interest is stability analysis. Suydam obtained a stability criterion for pinch in the MHD approximation.⁹⁷ Similar criteria for microscopic and macroscopic plasmas could be derived by obtaining the Hamiltonians from the Lagrangians, and by finding the conditions for minimizing the potential energies. It is noted, in passing, that, since the field quantities are functions of $\underline{r}_0 + \underline{\xi}$, the Lagrangian in Equation (2.15) is in the form of a correlation function.⁹⁸ Interesting properties of the Fourier transform of the correlation function would suggest the possibility of simpler analysis than the direct use of Maxwell's equations and the Vlasov equation for the microscopic treatment or the moment equations for the hydrodynamic approximation.

5. Bounded Plasmas: Amongst the complications not treated comprehensively is that of plasma boundaries. So far, Laval et al,⁹⁹ and Spithas and Manheimer¹⁰⁰, have considered the three-wave interaction of three coherent plasma waves in a waveguide with an infinite magnetic field, while Pérulli¹⁰¹ has made an extensive study of such processes from both a theoretical and experimental viewpoint. All of these studies have used the conventional iterative method. The averaged Lagrangian

method has been employed in recent Stanford work by Larsen¹⁰² for the interaction of three coherent plasma waves in an infinitely long cold plasma column with and without an infinite static magnetic field. The three-wave interaction work should be extended to the case of a plasma column of finite length in a finite magnetic field.

6. Random Phase Approximation: So far, we have discussed only interaction of coherent waves. As mentioned in Section 1.2, however, the random phase approximation must be used for turbulent plasmas. For randomly phased waves, only quantities averaged over a statistical ensemble are meaningful. Since simple averaging makes the LHS of Equation (4.10) vanish, the coupled mode equations cannot be used. Instead, the wave kinetic equation, which describes the rate of change of the plasmon distribution function or quantum density, is used.¹⁰³ The wave kinetic equation is the ensemble average of the action transfer equation [Equation (4.9)],¹⁰⁴ and the plasmon distribution function or quantum density is the action, N_k . Therefore, the coupling coefficients obtained in this work lead directly to those for turbulent plasmas.¹⁰⁵

Appendix A

DERIVATION OF EQUATION 2.18

Figure A.1 shows two pairs of corresponding cells defined in Section 2.2. One is at time t , and the other is the same pair after the infinitesimal time Δt . The equilibrium cell and nonequilibrium cell are located at $(\underline{r}_0, \underline{v}_0)$ and at $(\underline{r}, \underline{v})$, respectively, at time t , and at $(\underline{r}'_0, \underline{v}'_0)$ and $(\underline{r}', \underline{v}')$ at time $t' = t + \Delta t$. The displacement vectors are $(\underline{\xi}, \dot{\underline{\xi}})$ at t , and $(\underline{\xi}', \dot{\underline{\xi}}')$ at t' .

The displacement vectors at t and t' are given by

$$\underline{\xi} = \underline{r} - \underline{r}_0, \quad \underline{\xi}' = \underline{r}' - \underline{r}'_0. \quad (\text{A.1})$$

The position vectors \underline{r}' and \underline{r}'_0 at t' can be written as

$$\underline{r}' = \underline{r} + (\underline{v}_0 + \dot{\underline{\xi}})\Delta t, \quad \underline{r}'_0 = \underline{r}_0 + \underline{v}_0 \Delta t, \quad (\text{A.2})$$

where use has been made of $\underline{v} = \underline{v}_0 + \dot{\underline{\xi}}$. Substituting Equation (A.2) into Equation (A.1), and subtracting the first expression in Equation (A.1) from the second, gives

$$\Delta \underline{\xi} \equiv \underline{\xi}' - \underline{\xi} = \dot{\underline{\xi}} \Delta t. \quad (\text{A.3})$$

The velocity displacement vector, $\dot{\underline{\xi}}$, is therefore the time rate of change of the displacement vector $\underline{\xi}$. The time rate of change of $\underline{\xi}$, on the other hand, can be found by the Taylor expansion of $\underline{\xi}'$ about $\underline{\xi}$, and substitution of $\underline{\xi}$ from $\underline{\xi}'$. It yields

$$\Delta \underline{\xi} = \left[\frac{\partial \underline{\xi}}{\partial t} + (\underline{v}_0 \cdot \nabla) \underline{\xi} + \left(\frac{\underline{F}_0}{m} \cdot \nabla_v \right) \underline{\xi} \right] \Delta t. \quad (\text{A.4})$$

Equation (2.18) results from Equations (A.3) and (A.4).

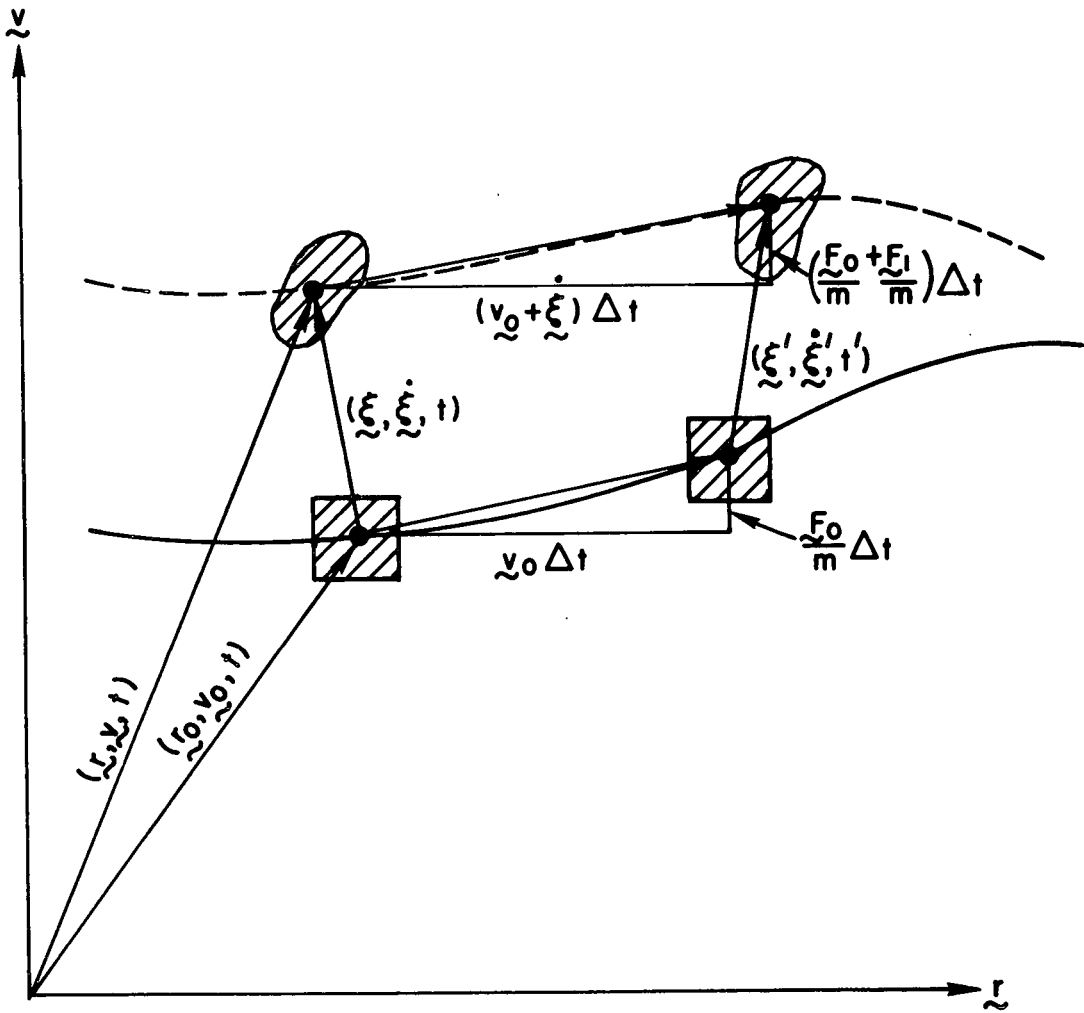


Figure A.1 Time evolution of a corresponding cell pair.

Appendix B

INTEGRAL REPRESENTATION OF THE COUPLING COEFFICIENTS
FOR PERPENDICULARLY PROPAGATING WAVES

In this appendix we shall derive expressions for the coupling coefficients in Equations (4.32)-(4.34) in convenient forms for computation. This will be done by expressing the Q's in Equation (4.28) in terms of integrals. To obtain integral expressions for the DG's and G's we must solve the equation of motion expressed by Equation (3.18) or (3.32).

In the absence of e_k^Z , solving Equation (3.32) for $D_{k,0} \xi_k$ gives

$$D_{k,0} \xi_k = D_{k,0} \vec{G}_k \cdot \vec{e}_k \quad (\text{B.1})$$

where

$$\begin{aligned} D_{k,0} G_k^{RR} &= D_{-k,0}^* G_{-k}^{LL*} = \frac{q}{2im} \frac{\exp\{-i[\omega_k - 1(\phi - \phi_k) - k_{\perp} v_{\perp} \sin(\phi - \phi_k)]/\omega_c\}}{\omega_c \sin(\pi\omega_k/\omega_c)} \\ &\times \int_{\phi - \phi_k - \pi}^{\phi - \phi_k + \pi} d\bar{\phi} \left[\exp(-i\bar{\phi}) + \frac{k_{\perp} v_{\perp}}{2\omega_k} \right] \exp\left[i(\omega_k \bar{\phi} + k_{\perp} v_{\perp} \sin\bar{\phi})/\omega_c \right], \\ D_{k,0} G_k^{RL} &= D_{-k,0}^* G_{-k}^{LR*} = -\frac{q}{4im} \frac{k_{\perp} v_{\perp} \exp\{i(\phi + \phi_k) - i[\omega_k(\phi - \phi_k) - k_{\perp} v_{\perp} \sin(\phi - \phi_k)]/\omega_c\}}{\omega_c \omega_k \sin(\pi\omega_k/\omega_c)} \\ &\times \int_{\phi - \phi_k - \pi}^{\phi - \phi_k + \pi} d\bar{\phi} \exp\left[i(\omega_k \bar{\phi} + k_{\perp} v_{\perp} \sin\bar{\phi})/\omega_c \right]. \end{aligned} \quad (\text{B.2})$$

Equation (B.2) in turn gives

$$\begin{aligned}
G_k^{RR} &= G_{-k}^{LL*} = -\frac{q}{4m} \frac{\exp\{-i[\omega_k(\phi-\phi_k)-k_{\perp}v_{\perp}\sin(\phi-\phi_k)]/\omega_c\}}{\omega_c^2 \sin^2(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \exp(i\bar{\phi}) \int_{\bar{\phi}-\pi}^{\bar{\phi}+\pi} d\bar{\bar{\phi}} \left[\exp(-i\bar{\bar{\phi}}) - \frac{k_{\perp}v_{\perp}}{2\omega_k} \right] \exp[i(\omega_k\bar{\bar{\phi}}-k_{\perp}v_{\perp}\sin\bar{\bar{\phi}})/\omega_c] , \\
G_k^{RL} &= G_{-k}^{LR*} = -\frac{q}{8m} \frac{k_{\perp}v_{\perp} \exp\{2i\phi_k - i[\omega(\phi-\phi_k)-k_{\perp}v_{\perp}\sin(\phi-\phi_k)]/\omega_c\}}{\omega_c^2 \omega_k \sin^2(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \exp(i\bar{\phi}) \int_{\bar{\phi}-\pi}^{\bar{\phi}+\pi} d\bar{\bar{\phi}} \exp[i(\omega_k\bar{\bar{\phi}}-k_{\perp}v_{\perp}\sin\bar{\bar{\phi}})/\omega_c] . \tag{B.3}
\end{aligned}$$

The unitary transformation from (e_k^R, e_k^L, e_k^Z) to $(e_k^{\parallel}, e_k^{\perp}, e_k^Z)$, described in Section 3.5.1, requires that

$$\begin{aligned}
G_k^{\parallel\parallel} &= \frac{1}{2} \left(G_k^{RR} + G_k^{LL} + G_k^{RL} \exp(-2i\phi_k) + G_k^{LR} \exp(2i\phi_k) \right) , \\
G_k^{\perp\parallel} &= -\frac{1}{2} \left(G_k^{RR} - G_k^{LL} + G_k^{RL} \exp(-2i\phi_k) - G_k^{LR} \exp(2i\phi_k) \right) . \tag{B.4}
\end{aligned}$$

Equation (B.4) also applies to $D_{k,0}G_k$. Substitution of Equations (B.2) and (B.3) into Equation (B.4) gives

$$\begin{aligned}
D_{k,0} G_k^{\parallel} &= \frac{q}{2im} \frac{\exp\{-i(\omega_k(\phi-\phi_k)-k_{\perp} v_{\perp} \sin(\phi-\phi_k))/\omega_c\}}{\omega_c \sin(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \cos(\phi-\bar{\phi}-\phi_k) \exp[i(\omega_k \bar{\phi} + k_{\perp} v_{\perp} \sin \bar{\phi})/\omega_c] , \\
D_{k,0} G_k^{\perp} &= \frac{q}{2im} \frac{\exp\{-i[\omega_k(\phi-\phi_k)-k_{\perp} v_{\perp} \sin(\phi-\phi_k)]/\omega_c\}}{\omega_c \sin(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \sin(\phi-\bar{\phi}-\phi_k) \exp[i(\omega_k \bar{\phi} + k_{\perp} v_{\perp} \sin \bar{\phi})/\omega_c] , \\
G_k^{\parallel} &= -\frac{q}{4m} \frac{\exp\{-i[\omega_k(\phi-\phi_k)-k_{\perp} v_{\perp} \sin(\phi-\phi_k)]/\omega_c\}}{\omega_c^2 \sin^2(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \int_{\bar{\phi}-\pi}^{\bar{\phi}+\pi} d\bar{\phi} \cos(\bar{\phi}-\bar{\phi}) \exp[i(\omega_k \bar{\phi} - k_{\perp} v_{\perp} \sin \bar{\phi})/\omega_c] , \\
G_k^{\perp} &= -\frac{q}{4m} \frac{\exp\{-i[\omega_k(\phi-\phi_k)-k_{\perp} v_{\perp} \sin(\phi-\phi_k)]/\omega_c\}}{\omega_c^2 \sin^2(\pi\omega_k/\omega_c)} \\
&\times \int_{\phi-\phi_k-\pi}^{\phi-\phi_k+\pi} d\bar{\phi} \int_{\bar{\phi}-\pi}^{\bar{\phi}+\pi} d\bar{\phi} \sin(\bar{\phi}-\bar{\phi}) \exp[i(\omega_k \bar{\phi} - k_{\perp} v_{\perp} \sin \bar{\phi})/\omega_c] . \quad (B.5)
\end{aligned}$$

Equation (4.28) is then written as

$$\begin{aligned}
Q_{kk'k''}^{\parallel} &= -\frac{i}{32} \epsilon_0 k'' \frac{\omega_N^2}{\omega_c} \left\langle T_k T_{k'} \exp\{-i[\omega_{k''}(\phi-k'' v_{\perp} \sin(\phi-\phi_{k''}))/\omega_c]\right\rangle , \\
Q_{kk'k''}^{\perp} &= \frac{i}{8} \epsilon_0 k'' \frac{\omega_N^2}{\omega_c^2} \left\langle [(2\omega_c/\omega_{k'}) S_k - (1-\omega_{k''}/\omega_{k'}) T_k] \right\rangle , \quad (B.6)
\end{aligned}$$

where S_k and T_k are defined by

$$S_k = \frac{\exp(-i\omega_k \phi_k / \omega_c)}{\sin(\pi\omega_k / \omega_c)} \int_{\phi - \phi_k - \pi}^{\phi - \phi_k + \pi} d\bar{\phi} \cos(\phi - \bar{\phi} - \phi_k) \exp\{-i[\omega_k \bar{\phi} + k_{\perp} v_{\perp} \sin \bar{\phi}] / \omega_c\},$$

$$T_k = \frac{\exp(-i\omega_k \phi_k / \omega_c)}{\sin^2(\pi\omega_k / \omega_c)} \int_{\phi - \phi_k - \pi}^{\phi - \phi_k + \pi} d\bar{\phi} \int_{\bar{\phi} - \pi}^{\bar{\phi} + \pi} d\bar{\phi}' \cos(\bar{\phi} - \bar{\phi}' + \phi_k - \phi_k) \exp\{-i[\omega_k \bar{\phi} - k_{\perp} v_{\perp} \sin \bar{\phi}] / \omega_c\}.$$

(B.7)

Substitution of Equations (B.6) and (B.7) into Equations (4.32)-(4.34) gives an integral representation for the coupling coefficients $C_{\parallel\parallel\parallel}$, $C_{\parallel 00}$, $C_{0\parallel 0}$, and $C_{00\parallel}$, as desired.

Appendix C

EXPANSION OF THE STATE EQUATIONS

The purpose of this appendix is to obtain the elements of the tensor \vec{s} second and third order in terms of the perturbation ξ , for each of the three illustrative cases in Section 5.2. For all three, the state equation [Equations (5.13), (5.18) and (5.23)] is of the form

$$\frac{d}{dt} \left(\frac{s}{n^\Gamma} \right) = 0, \quad \Gamma = \gamma - 1, \quad (\text{C.1})$$

where γ is the adiabatic compression constant. In order to expand Equation (C.1), we should first transform the total time derivative d/dt , in the (r, t) coordinates to that in the (r_0, t_0) coordinates. Equation (5.8) immediately gives

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \dot{\xi} \cdot \frac{\partial}{\partial r_0}, \quad \frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial r_0}. \quad (\text{C.2})$$

Therefore, it follows that

$$\frac{d}{dt} \rightarrow \frac{d}{dt_0}. \quad (\text{C.3})$$

Equation (C.3) states that the rate of change of a function along the perturbed trajectory is the same as that along the equilibrium trajectory.

From Equation (5.8) it follows that

$$n = \mathcal{J}^{-1} n_0, \quad \mathcal{J} = 1 + \nabla \cdot \xi + \frac{1}{2} [(\nabla \cdot \xi)^2 - \nabla \xi_i \nabla \xi_i] + \frac{\partial \xi}{\partial x} \cdot \left(\frac{\partial \xi}{\partial y} \times \frac{\partial \xi}{\partial z} \right), \quad (\text{C.4})$$

where the subscript 0 on x, y, z has been dropped. In order to expand Equation (C.1), let

$$s = s_0 + s_1 + s_2 + \dots, \quad J^\Gamma = 1 + \zeta_1 + \zeta_2 + \dots \quad (\text{C.5})$$

Here ζ_n can be expressed in terms of ξ from Equation (C.4). Substituting Equations (C.4) and (C.5) into Equation (C.1), and expanding it, gives the s_n as

$$s_1 = -s_0 \zeta_1, \quad s_2 = -s_0 (\zeta_2 - \zeta_1^2), \quad s_3 = -s_0 [\zeta_3 - \zeta_1 (2\zeta_2 - \zeta_1^2)], \dots \quad (\text{C.6})$$

(i) Parallel Compression: From Equation (5.13), $\Gamma = 0$ and 2 for s_\perp and s_\parallel , respectively. By Equation (5.11), the Jacobian in Equation (C.4) reduces to

$$J = 1 + \frac{\partial \xi_z}{\partial z} \quad (\text{C.7})$$

Equation (C.6) then becomes

$$s_{\perp 1} = s_{\perp 2} = s_{\perp 3} = \dots = 0, \\ s_{\parallel 1} = -2s_{\parallel 0} \frac{\partial \xi_z}{\partial z}, \quad s_{\parallel 2} = 3s_{\parallel 0} \left(\frac{\partial \xi_z}{\partial z} \right)^2, \quad s_{\parallel 3} = -4s_{\parallel 0} \left(\frac{\partial \xi_z}{\partial z} \right)^3, \quad (\text{C.8})$$

(ii) Cylindrical Compression: From Equation (5.18), $\Gamma = 1$ and 0 for s_\perp and s_\parallel , respectively. By the assumptions in Equation (5.16), the Jacobian for this case becomes

$$J = 1 + \nabla_\perp \cdot \xi + \frac{\partial \xi_x}{\partial x} \frac{\partial \xi_y}{\partial y} - \frac{\partial \xi_x}{\partial y} \frac{\partial \xi_y}{\partial x} \quad (\text{C.9})$$

Equation (C.6) then yields

$$s_{11} = -s_{10}(\nabla_1 \cdot \xi) , \quad s_{12} = s_{10} \left[(\nabla_1 \cdot \xi)^2 - \frac{\partial \xi_x}{\partial x} \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_x}{\partial y} \frac{\partial \xi_y}{\partial x} \right] .$$

$$s_{13} = -s_{10}(\nabla_1 \cdot \xi) \left[(\nabla_1 \cdot \xi)^2 - 2 \frac{\partial \xi_x}{\partial x} \frac{\partial \xi_y}{\partial y} + 2 \frac{\partial \xi_x}{\partial y} \frac{\partial \xi_y}{\partial x} \right] ,$$

$$s_{\parallel 1} = s_{\parallel 2} = s_{\parallel 3} = \dots = 0 . \quad (\text{C.10})$$

(iii) Spherical Compression: From Equation (5.23), $\Gamma = 2/3$. From Equations (C.4) - (C.6), the s_n 's are obtained as

$$s_1 = -\frac{2}{3} s_0 \nabla \cdot \xi , \quad s_2 = \frac{1}{3} s_0 \left[\frac{2}{3} (\nabla \cdot \xi)^2 + \nabla \xi : \nabla \xi \right] ,$$

$$s_3 = \frac{1}{3} s_0 \left[\frac{5}{27} (\nabla \cdot \xi)^3 - \frac{5}{3} (\nabla \cdot \xi) (\nabla \xi : \nabla \xi) - 2 \frac{\partial \xi}{\partial x} \cdot \left(\frac{\partial \xi}{\partial y} \times \frac{\partial \xi}{\partial z} \right) \right] . \quad (\text{C.11})$$

Appendix D

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Nonlinear Interaction Between Circularly Polarized Waves and Electron Plasma Waves*

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This paper presents a general analysis of nonlinear three-wave interaction in a hot magnetoplasma between circularly polarized waves propagating at small angles to the static magnetic field, and electron plasma (Langmuir) waves propagating parallel. First, the coupled mode equations are derived by iterative solution of the Vlasov equation. Simplified expressions for the coupling coefficients are then obtained for all of the wave combinations for which the frequency and wave number synchronism conditions can be satisfied, and the wave coupling coefficients are nonzero. These comprise interactions among three right-hand polarized waves, two circularly polarized waves and one Langmuir wave, and three Langmuir waves. The paper concludes with a brief discussion of the specific cases most likely to be worth subjecting to detailed numerical analysis.

I. INTRODUCTION

The nonlinear interaction of three or more waves in a magnetoplasma has been a subject of extensive investigations in recent years.¹ Initially, for simplicity, most of these investigations employed either cold plasma theory, or macroscopic theory based on moments of the Vlasov equation. More recently, interest has progressed to studies in which the Vlasov equation itself is used. The primary reason for this is the fact that even certain important small-signal phenomena, such as collisionless Landau and cyclotron damping, and the propagation of cyclotron harmonic waves, only emerge from a microscopic treatment. It is consequently essential to begin studies of nonlinear interactions involving these phenomena from the Vlasov equation. An additional reason is that certain new effects appear which have no counterpart in cold-plasma theory. In particular, for nonMaxwellian charged particle velocity distributions, the possibility of nonlinear interaction between waves carrying positive and negative small-signal energies exists. This can lead to explosive instability,^{2,3} in which all of the wave components grow at the expense of the charged particle energy.

Nonlinear plasma wave propagation is a notoriously complicated subject because of the wide variety of phenomena that may occur⁴: Individual waves may suffer nonlinear damping due to wave-particle interaction, and two or more waves may interact with the particles, or with each other in pure wave-wave interaction. In each case, for the interaction to be strong, certain synchronism conditions must be satisfied between such quantities as the wave frequencies, wave numbers, cyclotron frequency, and charged particle velocities. It seems likely that progress in such a complex situation will be made most easily by studying each effect separately in its region of greatest strength, and comparing the results for a given set of plasma parameters to determine which interaction effects should dominate. The final predictions, and if feasible the separate analyses leading to them, should then be subjected to experimental verification.

In this scheme, the authors have opted to concentrate on wave-wave interactions, in which the charged particle velocity distribution is assumed to remain steady in time. To further restrict the field, attention is limited to the high-frequency propagation branches obtained ignoring ion motions, and to propagation at

$\theta = \pi/2$ or $\theta \approx 0$, i.e., corresponding to interaction between principal modes propagating either perpendicular or parallel to the static magnetic field. So far, the case of perpendicular propagation has been examined in most detail by the authors and others. Theoretical studies have been published of three-wave interactions between cold plasma, transverse (extraordinary) waves,^{5,6} and between hot plasma, longitudinal (cyclotron harmonic) waves,^{7,8} and have received encouraging support from laboratory experiments.^{9,10} For approximately parallel propagation, a cold-plasma treatment has been given for the interaction of right-hand polarized transverse waves,¹¹ and the interaction between circularly polarized modes and a longitudinal electron plasma (Langmuir) wave has been studied, using moment theory for the longitudinal wave.^{12,13} The purpose of the present paper is to extend this work to include hot-plasma effects stemming from the Vlasov equation.

The general procedure for analyzing three-wave interactions adopted here is as follows. Maxwell's equations and the Vlasov equation are first expanded to first order. This yields equations for the field quantities and a small-signal dispersion relation. The expansion is then taken to second order, and all second-order terms are approximated by substituting first-order quantities in them. This can be done for three waves provided that the small-signal dispersion relations allow their frequencies and wave numbers to satisfy certain synchronism conditions. This procedure is carried

through in Sec. II for a general charged particle velocity distribution, and leads to a set of coupled mode equations.

In Sec. III, these coupled mode equations are specialized to three cases. The first is that of three right-hand polarized waves propagating almost parallel to the static magnetic field (there is no interaction in the limit of parallel propagation). Three left-hand polarized waves need not be considered since the topology of the dispersion characteristics does not allow the synchronism conditions to be satisfied. Mixtures of left- and right-hand polarization do not couple. The second is that of two circularly polarized waves interacting with a Langmuir wave, all three waves propagating strictly parallel to the magnetic field. The synchronism conditions can be satisfied for three different combinations of wave types, and these are treated individually. The case of one circularly polarized wave and two Langmuir waves propagating parallel is of no interest as coupling cannot occur. The third case considered is that of interaction among three Langmuir waves, and our treatment confirms the form of the relevant coupling coefficient given recently by Dysthe.¹⁴

Having obtained the general coupling coefficients and coupled mode equations, specific problems of mixing and parametric amplification may be attacked. Although no detailed numerical solutions are given in this paper, some comments on the procedure to obtain them are given in Sec. III.D, and the likely results are discussed in Sec. IV.

II. THEORY

A. Basic Equations

Our system is described by Maxwell's equations

$$\nabla \times \boldsymbol{\varepsilon} = -\partial \mathbf{H} / \partial t, \quad \nabla \times \mathbf{H} = \mathbf{g} + (\partial \boldsymbol{\varepsilon} / \partial T), \quad (1)$$

supplemented by an expression for the convection current density

$$\mathbf{g} = -\int F \mathbf{V} dV. \quad (2)$$

The evolution of the charged particle velocity distribution function F is described by the Vlasov equation

$$(\partial F / \partial t) + \mathbf{V} \cdot \nabla F - (\boldsymbol{\varepsilon} + \mathbf{V} \times \mathbf{H}) \cdot (\partial F / \partial \mathbf{V}) - \Omega_c (\partial F / \partial \phi) = 0, \quad (3)$$

where we have used cylindrical coordinates in velocity space defined such that

$$V_x = V \cos \phi, \quad V_y = -V \sin \phi, \quad (4)$$

and the static magnetic field B_0 is oriented along the z axis.

Equations (1)-(4) have been written in terms of the normalized variables

$$\begin{aligned} \boldsymbol{\varepsilon} &= -e\mathbf{E}/m\omega_0, & \mathbf{H} &= -e\mathbf{B}/m\omega_0, & \mathbf{g} &= -\mathbf{J}/n_0ec, & \mathbf{V} &= \mathbf{v}/c, \\ X &= \omega_0 x/c, & T &= \omega_0 t, & \Omega &= \omega/\omega_0, & \Omega_c &= -eB_0/m\omega_0, & \mathbf{K} &= \mathbf{k}c/\omega_0, \end{aligned} \quad (5)$$

where ω_0 is the electron plasma frequency, and the remaining symbols have their usual meanings.

Taking the spatial-temporal Fourier transforms of Eqs. (1) and (3) yields

$$\begin{aligned} \mathbf{K}_\gamma \times \boldsymbol{\varepsilon}_\gamma &= \Omega_\gamma \mathbf{H}_\gamma, & \mathbf{K}_\gamma \times \mathbf{H}_\gamma &= -\mathbf{g}_\gamma - j\Omega_\gamma \boldsymbol{\varepsilon}_\gamma, \\ j\Omega_\gamma F_\gamma - j(\mathbf{V} \cdot \mathbf{K}_\gamma) F_\gamma + \Omega_c (\partial F_\gamma / \partial \phi) - (\boldsymbol{\varepsilon}_\gamma + \mathbf{V} \times \mathbf{H}_\gamma) \cdot (\partial F_0 / \partial \mathbf{V}) &= \sum_{\alpha, \beta} \boldsymbol{\varepsilon}_\alpha + \mathbf{V} \times \mathbf{H}_\alpha \cdot (\partial F_\beta / \partial \mathbf{V}), \end{aligned} \quad (6)$$

where it has been assumed that the synchronism conditions

$$\Omega_\gamma = \Omega_\alpha + \Omega_\beta, \quad \mathbf{K}_\gamma = \mathbf{K}_\alpha + \mathbf{K}_\beta, \quad (7)$$

for three-wave interaction are satisfied. The first two expressions of Eq. (6) are linear. The third has linear terms on the LHS, and nonlinear driving terms on the RHS.

In what follows, we shall restrict our attention to circularly polarized waves propagating along or at small angles to the static magnetic field, and electron plasma waves propagating along it. Under these conditions, we may write the wave number components as $(K_\gamma \theta_{x\gamma} \mathbf{i}_x, K_\gamma \theta_{y\gamma} \mathbf{i}_y, K_\gamma \mathbf{i}_z)$, where $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ are unit vectors. If we introduce the complex transverse vectors

$$\mathcal{E}_\gamma = \mathcal{E}_{x\gamma} + j\mathcal{E}_{y\gamma}, \quad \mathcal{H}_\gamma = \mathcal{H}_{x\gamma} + j\mathcal{H}_{y\gamma}, \quad \mathcal{J}_\gamma = \mathcal{J}_{x\gamma} + j\mathcal{J}_{y\gamma} = -\int F_\gamma V^2 dV dV_z \exp(-i\phi), \quad (8)$$

appropriate to propagation of right-hand polarized waves, then to lowest order in θ Eq. (6) yields the expressions

$$(K_\gamma^2/\Omega_\gamma^2)(\mathcal{E}_{z\gamma}\theta_\gamma - \mathcal{E}_\gamma) + \mathcal{E}_\gamma = (j/\Omega_\gamma)\mathcal{J}_\gamma, \quad (K_\gamma^2/2\Omega_\gamma^2)(\mathcal{E}_\gamma\theta_\gamma^* + \mathcal{E}_\gamma^*\theta_\gamma) + \mathcal{E}_{z\gamma} = j\mathcal{J}_{z\gamma}/\Omega_\gamma, \quad (9)$$

and

$$\begin{aligned} & (\Omega_\gamma - K_\gamma V_z)F_\gamma + j\Omega_c \frac{\partial F_\gamma}{\partial \phi} \\ & + j \left\{ \mathcal{E}_{z\gamma} \left[a + \frac{K_\gamma}{2\Omega_\gamma} [\theta_\gamma \exp j\phi + \theta_\gamma^* \exp(-j\phi)] \left(V_z \frac{\partial}{\partial V} - V \frac{\partial}{\partial V_z} \right) \right] + \frac{1}{2} b_\gamma [\mathcal{E}_{-\gamma}^* \exp(-j\phi) + \mathcal{E}_\gamma \exp j\phi] \right\} F_0 \\ & = -j \sum_{\alpha, \beta} \left\{ \mathcal{E}_{z\alpha} \left[a + \frac{K_\alpha}{2\Omega_\alpha} [\theta_\alpha \exp j\phi + \theta_\alpha^* \exp(-j\phi)] \left(V_z \frac{\partial}{\partial V} - V \frac{\partial}{\partial V_z} \right) \right] + \frac{jK_\alpha}{2\Omega_\alpha} (\theta_\alpha \mathcal{E}_{-\alpha}^* - \theta_\alpha^* \mathcal{E}_\alpha) \frac{\partial}{\partial \phi} \right. \\ & \left. + \frac{1}{2} (\exp j\phi) (\mathcal{E}_\alpha b_\alpha + j\mathcal{E}_\alpha d_\alpha \frac{\partial}{\partial \phi}) + \frac{1}{2} [\exp(-j\phi)] (\mathcal{E}_{-\alpha}^* b_\alpha - j\mathcal{E}_{-\alpha}^* d_\alpha \frac{\partial}{\partial \phi}) \right\} F_\beta + \frac{1}{2} (VK_\gamma) [\theta_\gamma \exp j\phi + \theta_\gamma^* \exp(-j\phi)] F_\gamma. \end{aligned} \quad (10)$$

Here, F_0 is the unperturbed electron velocity distribution function, and θ_γ has been written for $\theta_{x\gamma} + j\theta_{y\gamma}$. The operators a and b_α , and the velocity-dependent quantity d_α , are defined by

$$a = \partial/\partial V_z, \quad b_\alpha = (\partial/\partial V) + (K_\alpha/\Omega_\alpha)[V(\partial/\partial V_z) - V_z(\partial/\partial V)], \quad d_\alpha = V^{-1}[1 - (V_z K_\alpha/\Omega_\alpha)]. \quad (11)$$

B. Iterative Solution

Equation (10) may be solved iteratively to any desired order of approximation. The first iterate is obtained by setting F_β and F_γ to zero on the RHS and solving to obtain

$$F_\gamma = \left\{ -j\mathcal{E}_{z\gamma} \left[\frac{a}{T_{\gamma 0}} + \frac{K_\gamma}{2\Omega_\gamma} \left(\frac{\theta_\gamma \exp j\phi}{T_{\gamma 1}} + \frac{\theta_\gamma^* \exp(-j\phi)}{T_{\gamma-1}} \right) \left(V_z \frac{\partial}{\partial V} - V \frac{\partial}{\partial V_z} \right) \right] + \frac{1}{2} b_\gamma \left(\frac{\mathcal{E}_\gamma \exp j\phi}{T_{\gamma 1}} + \frac{\mathcal{E}_{-\gamma}^* \exp(-j\phi)}{T_{\gamma-1}} \right) \right\} F_0, \quad (12)$$

where $T_{\gamma n}$ is defined by

$$T_{\gamma n} = \Omega_\gamma - K_\gamma V_z - n\Omega_c. \quad (13)$$

This result can be used as an improved guess for F_β and F_γ on the RHS of Eq. (10). After performing a total of three iterations, we obtain all terms quadratic in \mathcal{E} and linear in θ . We may then write F_γ as a power series in \mathcal{E} as

$$F_\gamma = (G_{\gamma 11} + G_{\gamma 21}) \exp j\phi + (G_{\gamma 10} + G_{\gamma 20}) + (G_{\gamma 1-1} + G_{\gamma 2-1}) \exp(-j\phi) + (\text{other terms}), \quad (14)$$

where $G_{\gamma 11}, G_{\gamma 10}, G_{\gamma 21}$, and $G_{\gamma 20}$ are given by

$$\begin{aligned} G_{\gamma 11} &= -j \left\{ \frac{b_\gamma \mathcal{E}_\gamma}{2T_{\gamma 1}} + \frac{K_\gamma \theta_\gamma \mathcal{E}_{z\gamma}}{2\Omega_\gamma T_{\gamma 0} T_{\gamma 1}} \left[T_{\gamma 0} \left(V_z \frac{\partial}{\partial V} - V \frac{\partial}{\partial V_z} \right) + \Omega_\gamma V a \right] \right\} F_0, \\ G_{\gamma 10} &= -j [\mathcal{E}_{z\gamma} (a/T_{\gamma 0}) + \theta_\gamma^* \mathcal{E}_\gamma (K_\gamma V/4T_{\gamma 1} T_{\gamma 0}) b_\gamma] F_0, \\ G_{\gamma 21} &= -\frac{1}{8} (\mathcal{E}_\alpha \mathcal{E}_{-\beta}^*) \left(\frac{K_\gamma \theta_\gamma}{T_{\gamma 1} T_{\gamma 0}} (b_\beta + d_\beta) \frac{V}{T_{\alpha 1}} b_\alpha + \frac{K_\gamma \theta_\gamma V}{T_{\gamma 1} T_{\gamma 0}} (b_\alpha + d_\alpha) T_{\beta-1}^{-1} b_\beta + \frac{K_\alpha \theta_\alpha}{T_{\gamma 1}} (b_\beta + 2d_\beta) \frac{V}{T_{\alpha 1} T_{\alpha 2}} b_\alpha \right. \\ & \quad \left. + \frac{K_\beta \theta_\beta}{T_{\gamma 1}} b_\alpha \frac{V'}{T_{\beta-1} T_{\beta 0}} b_\beta - \frac{2K_\beta \theta_\beta}{\Omega_\beta T_{\gamma 1} T_{\alpha 1}} b_\alpha \right) F_0 - \frac{1}{2} (\mathcal{E}_\alpha \mathcal{E}_{z\beta}) (T_{\gamma 1}^{-1} b_\alpha T_{\beta 0}^{-1} a + T_{\gamma 1}^{-1} a T_{\alpha 1}^{-1} b_\alpha) F_0, \\ G_{\gamma 20} &= \frac{1}{4} (\mathcal{E}_{-\beta}^* \mathcal{E}_\alpha) [T_{\gamma 0}^{-1} (b_\beta + d_\beta) T_{\alpha 1}^{-1} b_\alpha + T_{\gamma 0}^{-1} (b_\alpha + d_\alpha) T_{\beta-1}^{-1} b_\beta] F_0 - \mathcal{E}_{z\alpha} \mathcal{E}_{z\beta} T_{\gamma 0}^{-1} a (T_{\beta 0}^{-1} + T_{\alpha 0}^{-1}) a F_0. \end{aligned} \quad (15)$$

Substituting Eq. (14) into Eqs. (8) and (9), we obtain

$$\begin{aligned} (K_\gamma^2/\Omega_\gamma^2)(\theta_\gamma \varepsilon_{z\gamma} - \varepsilon_\gamma) + \varepsilon_\gamma &= (-2\pi j/\Omega_\gamma) \iint V^2 dV dV_z (G_{\gamma 11} + G_{\gamma 21}), \\ (K_\gamma^2/2\Omega_\gamma^2)(\varepsilon_\gamma \theta_\gamma^* + \varepsilon_{-\gamma}^* \theta_\gamma) + \varepsilon_{z\gamma} &= (-2\pi j/\Omega_\gamma) \iint V dV V_z dV_z (G_{\gamma 10} + G_{\gamma 20}). \end{aligned} \quad (16)$$

Substitution of the expressions defined in Eq. (15) finally reduces these to

$$\begin{aligned} [1 - (K_\gamma^2/\Omega_\gamma^2) + (2\pi/\Omega_\gamma) \iint V^2 dV dV_z (bF_0/2T_{\gamma 1})] \varepsilon_\gamma + \Psi_\gamma \theta_\gamma \varepsilon_{z\gamma} &= (-2\pi j/\Omega_\gamma) \iint V^2 dV dV_z G_{\gamma 21}, \\ \frac{1}{2} \Psi_\gamma \theta_\gamma^* \varepsilon_\gamma + [1 - 2\pi \iint (F_0 V dV dV_z / T_{\gamma 0}^2)] \varepsilon_{z\gamma} &= (-2\pi j/\Omega_\gamma) \iint V dV V_z dV_z G_{\gamma 20}, \end{aligned} \quad (17)$$

where Ψ_γ is given by

$$\Psi_\gamma = (K_\gamma^2/\Omega_\gamma^2) - 2\pi \iint F_0 V dV dV_z \{ (K_\gamma V_z / T_{\gamma 1} \Omega_\gamma^2) + (K_\gamma^2 V^2 / 2\Omega_\gamma) [(T_{\gamma 0}^2 T_{\gamma 1})^{-1} + (K_\gamma V_z / \Omega_\gamma T_{\gamma 0} T_{\gamma 1}^2)] \}. \quad (18)$$

C. Velocity-Space Integration

Equation (18) is the desired general form of the coupled mode equations, describing how the electric field components ε_γ and $\varepsilon_{z\gamma}$ of the γ wave are influenced by those of the α and β components. To solve a specific problem, it is necessary to specify the unperturbed velocity distribution F_0 and to carry out the integrations. Considerable progress can be made if the not unduly restrictive condition is imposed that F_0 is of the separable form

$$F_0(V, V_z) = F_{0\perp}(V) F_{0z}(V_z), \quad 2\pi \int F_{0\perp}(V) V dV = 1. \quad (19)$$

This will be assumed in what follows.

It is also necessary to obtain expressions analogous to those of Eq. (17) for the α and β waves. The method of doing so is as follows. If we wish γ to represent one of the lower frequency waves, rather than the highest as in Eq. (7), it is only necessary to change the sign of β in the foregoing analysis. To demonstrate the procedure, and in anticipation of the requirements of Sec. III, we shall make this change. After carrying out appropriate velocity-space integrations, Eq. (17) then reduces to

$$D_\gamma \varepsilon_\gamma + \Psi_\gamma \theta_\gamma \varepsilon_{z\gamma} = P_\gamma \varepsilon_\alpha \varepsilon_\beta^* + Q_{\gamma\beta} \varepsilon_\alpha \varepsilon_\beta^*, \quad D_\gamma^P \varepsilon_{z\gamma} + \Psi_\gamma \theta_\gamma^* \varepsilon_\gamma = Q_{\beta\gamma} \varepsilon_\alpha \varepsilon_\beta^* + R_\gamma \varepsilon_\alpha \varepsilon_\beta^*. \quad (20)$$

The quantities D_γ and D_γ^P , and the coupling coefficients P_γ , $Q_{\gamma\beta}$, and R_γ , are given by

$$\begin{aligned} D_\gamma &= 1 - (K_\gamma^2/\Omega_\gamma^2) - (1/\Omega_\gamma^2) \int dV_z F_{0z} [(T_{\gamma 0}/T_{\gamma 1}) + K_\gamma^2 (\bar{V}^2/2T_{\gamma 1}^2)], \\ D_\gamma^P &= 2[1 - \int dV_z F_{0z} (1/T_{\gamma 0}^2)], \\ P_\gamma &= -\frac{j}{8\Omega_\alpha \Omega_\beta \Omega_\gamma} \sum_{\beta, \gamma} K_\beta \theta_\beta \int dV_z F_{0z} \left\{ \frac{4T_{\alpha 0} T_{\gamma 0}}{T_{\alpha 1} T_{\beta 1} T_{\gamma 1}} + K_\alpha K_\beta K_\gamma \bar{V}^4 \frac{\partial}{\partial V_z} [(T_{\alpha 1} T_{\beta 1} T_{\gamma 1})^{-1} (T_{\alpha 2}^{-1} + T_{\beta 0}^{-1})] \right. \\ &\quad \left. + \frac{2K_\gamma^2 \bar{V}^2}{T_{\alpha 1} T_{\beta 1} T_{\gamma 1}} \left(\frac{T_{\gamma 0}}{T_{\alpha 1}} + \frac{T_{\alpha 0}}{T_{\gamma 1}} \right) + 2K_\beta^2 \bar{V}^2 \left[\frac{T_{\gamma 0}}{T_{\alpha 1} T_{\beta 1} T_{\gamma 1}} (T_{\alpha 1}^{-1} + T_{\beta 1}^{-1}) + \frac{T_{\gamma 0}}{T_{\beta 0} T_{\beta 1} T_{\gamma 1}} (T_{\beta 0}^{-1} + T_{\beta 1}^{-1}) \right] \right. \\ &\quad \left. + \frac{2K_\beta K_\gamma \bar{V}^2}{T_{\alpha 1} T_{\beta 1} T_{\gamma 1}} \left[\frac{T_{\alpha 0}}{T_{\beta 0}} - \frac{T_{\gamma 1}}{T_{\beta 0}} + \frac{2T_{\alpha 0}}{T_{\alpha 2}} - \frac{2T_{\beta 1}}{T_{\alpha 1}} \right] \right\}, \\ Q_{\gamma\beta} &= \frac{j}{2\Omega_\alpha \Omega_\gamma} \int dV_z F_{0z} \left[\frac{2}{T_{\beta 0}^2} \left(\frac{K_\gamma T_{\alpha 0}}{T_{\alpha 1}} - \frac{K_\alpha T_{\gamma 0}}{T_{\gamma 1}} \right) - K_\alpha K_\gamma \bar{V}^2 \frac{\partial}{\partial V_z} (T_{\alpha 1} T_{\beta 0} T_{\gamma 1})^{-1} \right], \\ R_\gamma &= -2j \int dV_z F_{0z} (\partial/\partial V_z) (T_{\alpha 0} T_{\beta 0} T_{\gamma 0})^{-1}. \end{aligned} \quad (21)$$

The coupling coefficient $Q_{\beta\gamma}$ is obtained from $Q_{\gamma\beta}$ by interchanging β and γ . In the above expressions, a bar denotes an average value of the quantity concerned, i.e., of V^2 and V^4 . As may readily be guessed, Eqs. (20) and (21) result from some very lengthy manipulations. Attention is drawn to some of the more important points in the derivation in an Appendix to the paper.

The significance of D_γ and D_γ^P is worth noting. If we consider small-signal waves propagating parallel to the static magnetic field, the expressions in Eq. (20) reduce to $D_\gamma \varepsilon_\gamma \approx 0 \approx D_\gamma^P \varepsilon_{z\gamma}$, and we recognize the coefficients as the well-known hot-plasma dispersion relations for circularly polarized transverse waves, and longitudinal electron plasma (Langmuir) waves, respectively.

III. APPLICATIONS

We are now ready to apply the general results of Eq. (20) to specific cases. Three possibilities immediately come to mind: first, that all three waves might be circularly polarized transverse waves; second, that two circularly polarized waves might interact with a Langmuir wave; and third, that three Langmuir waves might interact with each other. We will indicate the appropriate coupling coefficients for these interactions separately, and then discuss how they may be used in the solution of mixing or parametric amplification problems in inhomogeneous plasmas. To emphasize the type of wave concerned, we will introduce the variables

$$\begin{aligned} \epsilon_{\alpha}^R &= \epsilon_{x\alpha} + j\epsilon_{y\alpha}, & \epsilon_{\alpha}^L &= \epsilon_{x\alpha} - j\epsilon_{y\alpha}, \\ \epsilon_{\alpha}^P &= \epsilon_{z\alpha}, & (\alpha > 0), \end{aligned} \quad (22)$$

referring to right- and left-hand circularly polarized waves and Langmuir waves, respectively.

A. Three Circularly Polarized Waves

Eliminating ϵ_z from Eq. (20), and neglecting terms higher than first order in θ and second order in ϵ yields

$$D_{\gamma}^R \epsilon_{\gamma}^R = C_{RR} \epsilon_{\alpha}^R \epsilon_{\beta}^{R*}, \quad (23)$$

where the coupling coefficient C_{RR} is given by

$$C_{RR} = P_{\gamma} - \sum_{\gamma, \beta} (\Psi_{\beta} \theta_{\beta} / D_{\beta}^P) Q_{\gamma, \beta},$$

$$\Psi_{\beta} = 1 - \Omega_{\beta}^{-1} \int F_{0z} dV_z$$

$$\times [T_{\beta 1}^{-1} + (K_{\beta}^2 \bar{V}^2 / 2T_{\beta 0} T_{\beta 1}) (T_{\beta 0}^{-1} + T_{\beta 1}^{-1})]. \quad (24)$$

It is interesting to note that $C_{RR} \rightarrow 0$ as $\theta_{\alpha}, \theta_{\beta}, \theta_{\gamma} \rightarrow 0$. Thus, there is no interaction for purely parallel propagation,

1. Pump R—Signal and Idler R, P (or P, R)

The required coupled mode equations are obtained from Eq. (25) as

$$D_p^R \epsilon_p^R = -C_{RP} \epsilon_s^R \epsilon_i^P, \quad D_s^R \epsilon_s^R = C_{RP} \epsilon_i^P \epsilon_p^R, \quad D_i^P \epsilon_i^P = C_{RP} \epsilon_s^R \epsilon_p^R, \quad (26)$$

where D_s^R , D_p^R , and D_i^P are given by D_{γ} in Eq. (21) after replacing γ by s , p , and i , respectively. The coupling coefficient C_{RP} is given by

$$C_{RP} = \frac{j}{2\Omega_p \Omega_s} \int dV_z F_{0z} \left[\frac{2}{T_{i0}^2} \left(\frac{K_s T_{p0}}{T_{p1}} - \frac{K_p T_{s0}}{T_{s1}} \right) - K_p K_s \bar{V}^2 \frac{\partial}{\partial V_z} (T_{p1} T_{i0} T_{s1})^{-1} \right]. \quad (27)$$

The first and second expressions in Eq. (26) have been obtained from the first expression of Eq. (25) by replacing (α, β, γ) with $(s, -i, p)$ and (p, i, s) , respectively. The last expression in Eq. (26) derives from the second of Eq. (25) by replacing (α, β, γ) with (p, s, i) .

2. Pump L—Signal and Idler L, P (or P, L)

The results for this case can be obtained simply by changing the sign of K_p , K_s , K_i , Ω_p , Ω_s , and Ω_i in Eq. (26). When this is done, the electric field vectors change from ϵ_p^R , ϵ_s^R , ϵ_i^P , to ϵ_p^{L*} , ϵ_s^{L*} , ϵ_i^{P*} , respectively. Upon making these substitutions, and then taking the complex conjugate, Eq. (26) becomes

$$D_p^L \epsilon_p^L = -C_{LP} \epsilon_s^L \epsilon_i^P, \quad D_s^L \epsilon_s^L = C_{LP} \epsilon_i^{P*} \epsilon_p^L, \quad D_i^P \epsilon_i^P = C_{LP} \epsilon_s^{L*} \epsilon_p^L, \quad (28)$$

 TABLE I. Acceptable combinations of waves for interaction ($\theta=0$).

Pump (p)	Signal and idler (s, i)
R	R, P
L	L, P
P	R, L

and this property is independent of the precise form of the velocity distribution.

In previous work,¹¹ the authors have considered interaction in a cold plasma. It is possible to retrieve the expression for C_{RR} derived there by evaluating Eq. (24) for the delta-function velocity distribution $F_0(V, V_z) = \delta(V) \delta(V_z) / 2\pi V$ appropriate to a cold plasma.

B. Two Circularly Polarized Waves and a Langmuir Wave

In this case, interaction is possible for $\theta=0$. Equation (20) then simplifies to

$$D_{\gamma} \epsilon_{\gamma} = Q_{\gamma\beta} \epsilon_{\alpha} \epsilon_{\beta}^{P*}, \quad D_{\gamma}^P \epsilon_{\gamma}^P = Q_{\beta\gamma} \epsilon_{\alpha} \epsilon_{\beta}^*. \quad (25)$$

Of the several possible combinations of left- and right-hand polarized waves (L, R) and Langmuir waves (P), there are only the three given in Table I for which the synchronism conditions can be satisfied, and the coupling coefficients are nonzero.¹³ For illustration purposes, we have called the highest frequency component the "pump" wave, and the lower frequencies the "signal" and "idler," analogous to conventional parametric amplification terminology.¹⁵ We will now consider the acceptable combinations separately.

where D_γ^L is defined as

$$D_\gamma^L = 1 - (K_\gamma^2/\Omega_\gamma^2) - (1/\Omega_\gamma^2) \int dV_z F_{0z} [(T_{\gamma 0}/T_{\gamma-1}) + K_\gamma^2 (\bar{V}^2/2T_{\gamma-1}^2)], \quad (\gamma = p, s), \quad (29)$$

and the coupling coefficient C_{LP} is given by

$$C_{LP} = (j/2\Omega_p\Omega_s) \int dV_z F_{0z} \{ (2/T_{i0}^2) [(K_s T_{p0}/T_{p-1}) - (K_p T_{s0}/T_{s-1})] - K_p K_s \bar{V}^2 (\partial/\partial V_z) (T_{p-1} T_{i0} T_{s-1})^{-1} \}. \quad (30)$$

3. Pump P —Signal and Idler R, L (or L, R)

The results for this case are obtained from Eq. (26) by changing the sign of Ω_s and K_s , and then making the interchange $(p, s, i) \rightarrow (s, i, p)$. When this is done, the field vectors change from $\mathcal{E}_s^R, \mathcal{E}_p^R, \mathcal{E}_i^P$, to $\mathcal{E}_i^{L*}, \mathcal{E}_s^R, \mathcal{E}_p^P$, respectively. Upon making these substitutions, we obtain from Eq. (26) the coupled mode equations

$$D_p^P \mathcal{E}_p^P = -C_{RL} \mathcal{E}_i^L \mathcal{E}_s^R, \quad D_s^R \mathcal{E}_s^R = C_{RL} \mathcal{E}_i^{L*} \mathcal{E}_p^P, \quad D_i^L \mathcal{E}_i^L = C_{RL} \mathcal{E}_s^R \mathcal{E}_p^P, \quad (31)$$

where the coupling coefficient C_{RL} is given by

$$C_{RL} = -\frac{j}{2\Omega_s\Omega_i} \int dV_z F_{0z} \left[\frac{2}{T_{p0}^2} \left(\frac{K_s T_{i0}}{T_{i-1}} + \frac{K_i T_{s0}}{T_{s-1}} \right) + K_s K_i \bar{V}^2 \frac{\partial}{\partial V_z} (T_{s1} T_{p0} T_{i-1})^{-1} \right]. \quad (32)$$

C. Three Langmuir Waves

For this case, we put $\theta_\beta = 0 = \theta_\gamma$ in Eq. (20) to obtain

$$D_\gamma^P \mathcal{E}_\gamma^P = R_\gamma \mathcal{E}_\alpha^P \mathcal{E}_\beta^{P*}. \quad (33)$$

The coupled mode equations take the form

$$D_p^P \mathcal{E}_p^P = -C_{PP} \mathcal{E}_s^P \mathcal{E}_i^P, \quad D_s^P \mathcal{E}_s^P = C_{PP} \mathcal{E}_p^P \mathcal{E}_i^{P*}, \quad (34)$$

$$D_i^P \mathcal{E}_i^P = C_{PP} \mathcal{E}_p^P \mathcal{E}_s^{P*},$$

where $D_p^P, D_s^P,$ and $D_i^P,$ are given by Eq. (21) with γ replaced by $p, s,$ and $i,$ respectively, and the coupling coefficient C_{PP} is given by

$$C_{PP} = -2j \int dV_z F_{0z} (\partial/\partial V_z) (T_{p0} T_{s0} T_{i0})^{-1}. \quad (35)$$

Here, the expressions in Eq. (34) have been obtained by replacing (α, β, γ) in Eq. (33) with $(s, -i, p), (p, i, s),$ and $(p, s, i),$ respectively.

D. Procedure for Solution of Problems

The coupling coefficients derived in Eqs. (24), (27), (30), (32), and (35) constitute the primary information needed for detailed numerical studies of the nonlinear interactions involved. The picture is only completed, however, when one inserts in them wave numbers and frequencies satisfying the synchronism conditions and the small-signal dispersion relations. These have been given elsewhere for the cold-plasma approximation to the results of Sec. III.A,¹¹ and the hydrodynamic approximation to those of Sec. III.B.¹³ They would have to be recalculated for any other velocity distribution chosen.

If we assume that the synchronism problem has been solved at some point in a weakly inhomogeneous plasma for which the dispersion relation D_γ is of the form

$$D_\gamma = D_\gamma[K(X), \Omega, X], \quad (36)$$

then the most general problem is to determine how \mathcal{E}_γ

varies. It will be noted that all of our wave equations are in the general form

$$D_\gamma \mathcal{E}_\gamma = C_\gamma \mathcal{E}_\alpha \mathcal{E}_\beta. \quad (37)$$

Using the technique of Berk and Book,¹⁶ we can expand these equations around the point of noninteraction to

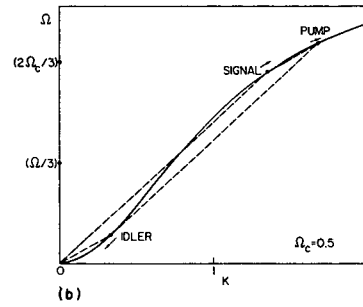
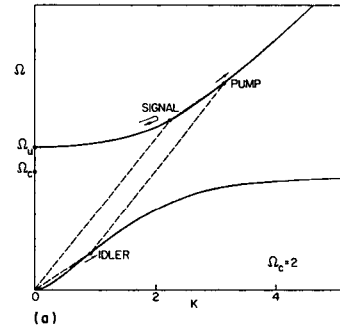


FIG. 1. Synchronism conditions for nonlinear interaction between three right-hand polarized waves in a cold plasma. (Heavy portions of curves indicate regions over which the pump and idler can vary. Arrows indicate signal and idler variation as pump increases.) (a) Pump in upper branch; (b) Pump in lower branch.

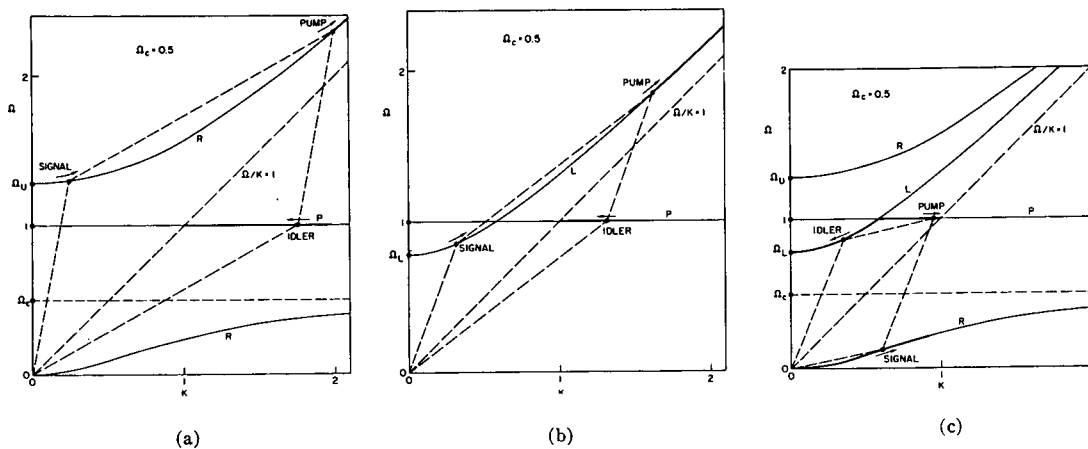


FIG. 2. Synchronism conditions for nonlinear interaction between two circularly polarized waves (*R* or *L*), and a plasma wave (*P*) in a cold plasma. [Cut-off frequencies $\Omega_{U,L} = (1 + \Omega_e^2/4)^{1/2} \pm \Omega_e/2$.] (a) Pump *R*—Signal and idler *R*, *P*; (b) Pump *L*—Signal and idler *L*, *P*; (c) Pump *P*—Signal and idler *R*, *L*.

obtain

$$\frac{\partial \epsilon_\gamma}{\partial X} \frac{\partial D_\gamma}{\partial K_\gamma} + \frac{1}{2} \epsilon_\gamma \left[\frac{\partial}{\partial X} \left(\frac{\partial D_\gamma}{\partial K_\gamma} \right) + \frac{\partial^2 D_\gamma}{\partial K_\gamma^2} \frac{\partial K_\gamma}{\partial X} \right] - \frac{\partial \epsilon_\gamma}{\partial T} \frac{\partial D_\gamma}{\partial \Omega_\gamma} = -j C_\gamma \epsilon_\alpha \epsilon_\beta. \quad (38)$$

Thus, once the dispersion relations and coupling coefficients are known, one may determine through Eq. (38) the variation of each wave amplitude in time and space, in homogeneous or weakly inhomogeneous plasmas, and for whatever type of problem is of interest, e.g., wave-wave mixing, or parametric amplification. Several examples of the procedure are given in Ref. 13.

IV. DISCUSSION

A very important point to note concerning the coupled mode equations [Eqs. (23), (26), (28), (31), and (34)] derived for the various cases discussed in Sec. III is that the coupling coefficients for the signal and idler equations are invariably equal in magnitude and sign, and equal in magnitude but opposite in sign to that for the pump equation. This is sufficient to guarantee that the Manley-Rowe relations are satisfied.¹⁵ The applicability of these relations to the problems studied here is not only of theoretical importance, but also constitutes a check on the accuracy of the extremely lengthy algebra and manipulations leading to the coupling coefficients.

Although these coupling coefficients [Eqs. (24), (27), (30), (32), and (35)] appear to be quite complicated, they are in fact straightforward to program for computer solution. Numerical results can then be obtained for wave growth due to mixing, or parametric amplification, as soon as the detailed form of the unperturbed charged particle velocity distribution F_0 is specified. We have not yet carried out such a program, and will

simply speculate here on some directions that appear to be worth following. We will do so by considering what additional features will appear when previous treatments are extended to include hot-plasma effects.

The first consideration in three-wave interaction problems is to determine what opportunities the small-signal dispersion relations offer for the frequency and wave number synchronism conditions to be satisfied. We shall examine first the case of three right-hand polarized waves propagating in a cold plasma. There are then two situations in which synchronism can be achieved.¹¹ These are shown in Fig. 1. In the first, the pump and signal lie on the upper branch, and the idler lies on the lower (whistler) branch. In the second, all three waves lie on the lower branch.

Suppose now that the plasma electron velocity distribution is Maxwellian. Thermal effects will make only a very slight difference to the shape of the upper branch, and will introduce no collisionless damping there, since the phase velocity exceeds the velocity of light. The lower branch will be susceptible to collisionless damping, however, and this will become particularly strong for $\Omega_e - \Omega \lesssim KV_e$. Here, V_e ($\ll 1$ for our non-relativistic theory to be valid) is the thermal velocity normalized to the velocity of light. We may reasonably conclude that as V_e increases from zero, the weak parametric growth predicted by the cold-plasma theory will be progressively quenched. The quenching effect will be greatest if any of the interaction frequencies approach Ω_e , so the case with all three waves on the lower branch will probably suffer most.

The situation becomes much more interesting if the plasma electron velocity distribution is non-Maxwellian. The upper branch will still be almost unchanged from its cold-plasma shape, but the lower branch can be

strongly modified and become unstable.¹⁷ The possibility exists of both positive and negative energy modes occurring, and these might lead to explosive instability. It would be of great interest to determine the relative growth rates of the small-signal and nonlinear instabilities to see which would dominate in experimentally realizable situations.

So far, we have only discussed the interaction of three right-hand polarized waves. We now consider the three cases for which coupling coefficients were derived in Sec. III. B. Typical synchronism conditions for a cold plasma are shown in Fig. 2. As shown in Figs. 2(a) and (b), the combinations of R - R , P and L - L , P do not look very interesting for further study: even the Langmuir wave has a phase velocity of the order of the velocity of light, so the effects of nonrelativistic thermal motions will probably be small for all three wave components.

The R - R , P case is actually richer in synchronism possibilities than has been indicated in Fig. 2(a). For the plasma frequency lower than the cyclotron frequency, i.e., $\Omega_c > 1$, it is possible for the pump and signal to lie on the lower branch. For an isotropic Maxwellian velocity distribution, all three waves would then be subject to collisionless damping. For anisotropic distributions, a wide variety of interesting combinations involving damped and growing waves would be possible, and would require extensive investigation. We may also remark that, although for cold plasma it is not possible to achieve synchronism with the pump and signal in the

upper and lower branches, respectively, this condition could be achieved with suitable velocity distributions, and is open to investigation.

The remaining combination in Fig. 2 is that of P - R , L . We note that the plasma wave has a phase velocity of the order of the velocity of light, so that it will be very little affected by the inclusion of thermal velocities. The whistler branch may be damped and/or unstable, depending on the velocity distribution chosen, but it seems unlikely that significant nonlinear effects will occur. It should, incidentally always be remembered that in cases such as this one, where one of the waves (P) has an extremely low group velocity, plasma inhomogeneity will severely limit the distance over which the synchronism conditions required for interaction can be satisfied.

The final case treated in Sec. III, i.e., that of three Langmuir waves, has no solution in cold-plasma theory, since the synchronism conditions obviously cannot be satisfied in this limiting case of dispersionless plasma oscillations. For non-Maxwellian velocity distributions, a wide variety of damped and growing solutions can occur, however, and present a very important topic for further detailed nonlinear studies. To take a specific example, it would be valuable to examine the case of an approximately monoenergetic beam interacting with a Maxwellian plasma to determine to what extent the fastest-growing small-signal component is damped nonlinearly by wave-wave interaction, as opposed to wave-particle interaction.

APPENDIX: DERIVATION OF THE COUPLING COEFFICIENTS P_γ , $Q_{\gamma\beta}$, $Q_{\beta\gamma}$

After performing double integrations with respect to V and V_z , and changing the sign of β , Eq. (17) reduces to the forms expressed by Eq. (20), but with coupling coefficients defined by

$$\begin{aligned}
 P_\gamma = & \frac{\pi j}{4\Omega_\gamma} \iint dV dV_z F_0 \left(K_{\gamma\theta_\gamma} b_\alpha T_{\alpha 1}^{-1} (b_\beta - d_\beta) \frac{V^3}{T_{\gamma 1} T_{\gamma 0}} - K_{\gamma\theta_\gamma} b_\beta T_{\beta 1}^{-1} (b_\alpha - d_\alpha) \frac{V^3}{T_{\gamma 1} T_{\gamma 0}} \right. \\
 & \left. + K_{\alpha\theta_\alpha} b_\alpha \frac{V}{T_{\alpha 1} T_{\alpha 2}} (b_\beta - 2d_\beta) \frac{V^2}{T_{\gamma 1}} - K_{\beta\theta_\beta} b_\beta \frac{V}{T_{\beta 1} T_{\beta 0}} b_\alpha \frac{V^2}{T_{\gamma 1}} + \frac{2K_{\beta\theta_\beta}}{\Omega_\beta} b_\alpha \frac{V^2}{T_{\alpha 1} T_{\gamma 1}} \right), \\
 Q_{\gamma\beta} = & (\pi j / \Omega_\gamma) \iint dV dV_z F_0 [b_\alpha T_{\alpha 1}^{-1} a(V^2 / T_{\gamma 1}) - a T_{\beta 0}^{-1} b_\alpha (V^2 / T_{\gamma 1})], \\
 Q_{\beta\gamma} = & (\pi j / \Omega_\gamma) \iint dV dV_z F_0 [b_\alpha T_{\alpha 1}^{-1} (b_\beta - d_\beta) (V_z V / T_{\gamma 0}) - b_\beta T_{\beta 1}^{-1} (b_\alpha - d_\alpha) (V_z V / T_{\gamma 1})]. \quad (A1)
 \end{aligned}$$

By use of Eqs. (11) and (19), the expression for $Q_{\gamma\beta}$ may be reduced to that of Eq. (21). It may also be shown that $Q_{\beta\gamma}$ is related to $Q_{\gamma\beta}$ simply by interchange of β and γ . The main problem is that of reducing P_γ . To accomplish this, we note that the wave number synchronism condition implies that

$$K_{\alpha\theta_\alpha} = K_{\beta\theta_\beta} + K_{\gamma\theta_\gamma}. \quad (A2)$$

It follows that the equation for P_γ can be separated into two parts: one involving $K_{\beta\theta_\beta}$, and the other involving

$K_\gamma \theta_\gamma$. After differentiation of the integrands, we have

$$\begin{aligned}
 P_\gamma = & j \frac{K_\beta \theta_\beta}{8\Omega_\alpha \Omega_\beta \Omega_\gamma} \int dV_z F_{0z} \left\{ \frac{4T_{\alpha 0}}{T_{\gamma 1}} (T_{\alpha 1}^{-1} - T_{\beta 1}^{-1}) + K_\alpha K_\beta K_\gamma \bar{V}^4 \frac{\partial}{\partial V_z} \left(\frac{1}{T_{\gamma 1}^2} [(T_{\alpha 1} T_{\alpha 2})^{-1} - (T_{\beta 1} T_{\beta 0})^{-1}] \right) \right. \\
 & + 2K_\alpha^2 \bar{V}^2 (T_{\alpha 1}^2 T_{\gamma 1})^{-1} + 2K_\alpha K_\beta \bar{V}^2 (T_{\beta 1} T_{\beta 0} T_{\gamma 1})^{-1} + 2K_\alpha K_\gamma \bar{V}^2 \frac{1}{T_{\gamma 1}^2} \left(T_{\alpha 1}^{-1} - \frac{2}{T_{\beta 1}} \right) - 2K_\beta^2 \bar{V}^2 \frac{T_{\alpha 0}}{T_{\beta 1} T_{\beta 0} T_{\gamma 1}} (T_{\beta 1}^{-1} + T_{\beta 0}^{-1}) \\
 & + 2K_\beta K_\gamma \bar{V}^2 \frac{T_{\alpha 0}}{T_{\beta 1} T_{\beta 0} T_{\gamma 1}^2} + 4K_\beta K_\gamma \bar{V}^2 \frac{T_{\alpha 0}}{T_{\gamma 1}^2} [(T_{\alpha 1} T_{\alpha 2})^{-1} - (T_{\beta 1} T_{\beta 0})^{-1}] \left. \right\} + j \frac{K_\gamma \theta_\gamma}{8\Omega_\alpha \Omega_\beta \Omega_\gamma} \int dV_z F_{0z} \left\{ \frac{4T_{\alpha 0} T_{\beta 0}}{T_{\gamma 0} T_{\gamma 1}} (T_{\alpha 1}^{-1} - T_{\beta 1}^{-1}) \right. \\
 & + K_\alpha K_\beta K_\gamma \bar{V}^4 \frac{\partial}{\partial V_z} \{ (T_{\alpha 1} T_{\gamma 1})^{-1} [(T_{\alpha 2} T_{\gamma 1})^{-1} - (T_{\beta 1} T_{\gamma 1})^{-1} - (T_{\beta 1} T_{\gamma 0})^{-1}] \} + 2K_\alpha \bar{V}^2 \frac{T_{\beta 0}}{T_{\alpha 1}^2 T_{\gamma 1} T_{\gamma 0}} \\
 & + 2K_\alpha K_\gamma \bar{V}^2 \frac{T_{\beta 0}}{T_{\gamma 1} T_{\gamma 0}} \left(T_{\alpha 1}^{-1} - \frac{2}{T_{\beta 1}} \right) (T_{\gamma 1}^{-1} + T_{\gamma 0}^{-1}) + 2K_\alpha K_\beta \bar{V}^2 (T_{\gamma 1} T_{\gamma 0})^{-1} (T_{\beta 1}^{-1} - T_{\alpha 1}^{-1}) - 2K_\beta^2 \bar{V}^2 \frac{T_{\alpha 0}}{T_{\beta 1}^2 T_{\gamma 1} T_{\gamma 0}} \\
 & \left. + 2K_\beta K_\gamma \bar{V}^2 \frac{T_{\alpha 0}}{T_{\beta 1} T_{\gamma 1} T_{\gamma 0}} (T_{\gamma 1}^{-1} + T_{\gamma 0}^{-1}) + 4K_\beta K_\gamma \bar{V}^2 \frac{T_{\alpha 0}}{T_{\alpha 1} T_{\gamma 1}} [(T_{\alpha 2} T_{\gamma 1})^{-1} - (T_{\beta 1} T_{\gamma 1})^{-1} - (T_{\beta 1} T_{\gamma 0})^{-1}] \right\}. \quad (A3)
 \end{aligned}$$

P_γ is to be rearranged so that it shows symmetry in β and γ . Each of the two integrands in Eq. (A3) has terms of zero, second, and fourth order in V . The symmetry in γ and β of the zero and fourth-order terms is comparatively easy to show in the form of the first two terms in the large curly bracket of Eq. (21). The symmetric property of the second-order terms, i.e., those in \bar{V}^2 , is not an easy matter to demonstrate, however. To do so, we note first that symmetry of the last terms in the integrands of Eq. (A3) is evident from comparison with the \bar{V}^4 terms. By use of the synchronism conditions to eliminate K_α , the rest of the \bar{V}^2 terms may be divided into three groups, involving K_β^2 , K_γ^2 , and $K_\beta K_\gamma$. After some tedious algebra, all of the \bar{V}^2 terms then reduce to the symmetric form expressed in Eq. (21).

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¹ The basic theoretical approach is described in Ref. 4, and a list of relevant articles is given in Ref. 5.

² P. A. Sturrock in *Plasma Hydromagnetics*, edited by D. Bershader (Stanford U. P., Stanford, California, 1962), p. 55.

³ M. N. Rosenbluth, B. Coppi, and R. N. Sudan, *Proceedings of the Third International Conference on Plasma Physics and Controlled Nuclear Fusion, Novosibirsk, U.S.S.R., 1968* (IAEA, Vienna, 1969), p. 771.

⁴ R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).

⁵ K. J. Harker and F. W. Crawford, *J. Appl. Phys.* **40**, 3247 (1969).

⁶ C. Etievant, S. Ossakow, E. Ozizmir, C. H. Su, and I. Fidone, *Phys. Fluids* **11**, 1778 (1968).

⁷ K. J. Harker and F. W. Crawford, *J. Appl. Phys.* **39**, 5959 (1968).

⁸ M. Porkolab and R. P. H. Chang, Princeton University Rept. Matt. 738, 1970.

⁹ R. Cano, C. Etievant, I. Fidone, and B. Zanfagna, *Proceedings of the Ninth International Conference on Phenomena in Ionized Gases, Bucharest, Romania, 1969* (Acad. Soc. Rep. Romania, Bucharest, 1969), p. 656.

¹⁰ R. P. H. Chang and M. Porkolab, Princeton University Rept. Matt. 750 (1970).

¹¹ K. J. Harker and F. W. Crawford, *J. Geophys. Res.* **74**, 5029 (1969).

¹² A. Sjölund and L. Stenflo, *Zeit. Phys.* **204**, 211 (1967).

¹³ K. J. Harker and F. W. Crawford, *J. Geophys. Res.* **75**, S459 (1970).

¹⁴ K. B. Dysthe, *Int. J. Elect.* **29**, 401 (1970).

¹⁵ W. H. Louisell, *Coupled Mode and Parametric Electronics* (Wiley, New York, 1960).

¹⁶ H. L. Berk and D. L. Book, *Phys. Fluids* **12**, 649 (1969).

¹⁷ References to the relevant literature, and computed small-signal dispersion characteristics for a variety of electron velocity distributions, are given by: J. C. Lee and F. W. Crawford, *J. Geophys. Res.* **75**, 85 (1970).

REFERENCES

1. L. Tonks and I. Langmuir, *Phys. Rev.* 33, 195 and 990 (1929).
2. F.W. Crawford, *Plasma Waves in Space and in the Laboratory*, Eds. J.O. Thomas and B.J. Landmark (Edinburgh Univ. Press, Edinburgh, 1969), Vol. 1, p. 125.
3. A.A. Vedenov, *Theory of Turbulent Plasmas* (American Elsevier Publishing Co., New York, N.Y., 1968).
4. R.F. Sagdeev and A.A. Galeev, *Nonlinear Plasma Theory* (W.A. Benjamin, Inc., New York, N.Y., 1969).
5. V.N. Tsytovich, *Nonlinear Effects in Plasma* (Plenum Press, New York, N.Y., 1970).
6. R.W. Gould, *Phys. Letters* 25A, 559 (1967); R.W. Gould, T.M. O'Neil, and J.H. Malmberg, *Phys. Rev. Letters* 19, 219 (1967); J.H. Malmberg, C.B. Wharton, R.W. Gould and T.M. O'Neil, *Phys. Rev. Letters* 20, 95 (1968).
7. J.R. Pierce, *Traveling Wave Tubes* (D. Van Nostrand Company, Inc., New York, N.Y., 1950).
8. J.M. Manley and H.E. Rowe, *Proc. IRE* 44, 904 (1956).
9. P. Penfield, Jr., *Frequency-Power Formulas* (MIT Press, Cambridge, Mass., 1960).
10. W.H. Louisell, *Coupled Mode and Parametric Electronics* (John Wiley and Sons, New York, N.Y., 1960).
11. N. Bloembergen, *Nonlinear Optics* (W.A. Benjamin, Inc., New York, N.Y., 1965); S.E. Harris, *Proc. IEEE* 57, 2096 (1969).
12. R.B. Thompson and C.F. Quate, *J. Appl. Phys.* 42, 907 (1971); W.R. Shreve, G.S. Kino and M.V. Luukkala, *Elect. Letters* 7, 26 (1971).
13. P.A. Sturrock, *Proc. Roy. Soc.* A242, 277 (1957).
14. G.S. Kino, *J. Appl. Phys.* 31, 1449 (1960).
15. P.A. Sturrock, *Ann. Phys.* 9, 422 (1960).
16. H.E. Petschek, Nonlinear Plasma Phenomena presented at Meeting of the Div. of Plasma Phys. of the American Phys. Soc., Colorado Springs, November 1961.
17. V.N. Oraevskii and R.Z. Sagdeev, *Zh. Tekh. Fiz.* 32, 1291 (1962), [*Sov. Phys. Tech. Phys.* 7, 955 (1963)].
18. A.A. Galeev and V.I. Karpman, *Zh. Eksp. Teor. Fiz.* 43, 2243 (1962), [*Sov. Phys. JETP* 17, 403 (1963)].

19. B.B. Kadomtsev, Plasma Turbulence (Academic Press, New York, N.Y., 1965).
20. A. Sjölund and L. Stenflo, *J. Appl. Phys.* 38, 2676 (1967);
C. Etievant, I. Fidone and M. Perulli, *Plasma Phys.* 9, 151 (1967);
R.E. Aamodt and M.O. Sloan, *Phys. Fluids* 11, 2218 (1968).
21. L.S. Danilkin, *Zh. Tekh. Fiz.* 35, 667 (1965), [*Sov. Phys. Tech. Phys.* 10, 524 (1965)].
22. C. Etievant, S. Ossakow, E. Ozizmir, C.H. Su and I. Fidone, *Phys. Fluids* 11, 1778 (1968).
23. K.F. Lee, *Phys. Rev.* 181, 447 (1969).
24. L. Stenflo, *J. Plasma Phys.* 5, 413 (1971).
25. K.J. Harker and F.W. Crawford, *J. Appl. Phys.* 40, 3247 (1969).
26. A. Sjölund and L. Stenflo, *Z. Phys.* 204, 211 (1967); H. Wilhelmsson, *J. Plasma Phys.* 3, 215 (1969); K.J. Harker and F.W. Crawford, *J. Geophys. Res.* 75, 5459 (1970).
27. K.J. Harker and F.W. Crawford, *J. Geophys. Res.* 74, 5029 (1969).
28. V.M. Dikasov, L.I. Rudakov and D.D. Ryutov, *Zh. Eksp. Teor. Fiz.* 48, 913 (1965), [*Sov. Phys. JETP* 21, 608 (1965)]; B. Coppi, M.N. Rosenbluth and R.N. Sudan, *Ann. Phys.* 55, 207 and 248 (1969); Proc. Third Conference on Plasma Phys. and Controlled Nuclear Fusion Research, Novosibirsk, Vienna (IAEA, Vienna, 1969), Vol. 1, p. 771; K.B. Dysthe and R.N. Franklin, *Plasma Phys.* 12, 705 (1970); K.B. Dysthe, *Int. J. Elect.* 29, 401 (1970).
29. D.F. DuBois and M.V. Goldman, *Phys. Rev.* 164, 207 (1967).
30. R. Cano, C. Etievant, I. Fidone and G. Granata, *Nuclear Fusion* 9, 223 (1969).
31. R.E. Aamodt, *Phys. Rev.* 138, A45 (1965); R.A. Ellis, Jr. and M. Porkolab, *Phys. Rev. Letters* 21, 529 (1968); M. Porkolab and R.P.H. Chang, *Phys. Fluids* 13, 2054 (1970).
32. K.J. Harker and F.W. Crawford, *J. Appl. Phys.* 39, 5959 (1968).
33. L. Stenflo, *J. Plasma Phys.* 4, 585 (1970).
34. H. Kim, K.J. Harker and F.W. Crawford, *J. Appl. Phys.* 42, 1997 (1971) [a copy of this paper attached to the thesis as Appendix D].
35. K.B. Dysthe, Stanford University Institute for Plasma Research Report No. 396 (June 1971).
36. L. Landau and E. Lifschitz, The Classical Theory of Fields (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965), Chaps. 3 and 4.

37. P.A. Sturrock, *Ann. Phys.* 4, 306 (1958).
38. P.A. Sturrock, *Ann. Phys.* 15, 250 (1961).
39. J.J. Galloway, Stanford University Institute for Plasma Research Rept. No. 362 (November 1970).
40. G.B. Whitham, *J. Fluid Mech.* 22, 273 (1965).
41. F.P. Bretherton and C.J.R. Garrett, *Proc. Roy. Soc.* A302, 529 (1968).
42. R.L. Dewar, *Phys. Fluids* 13, 2710 (1970).
43. J.P. Dougherty, *J. Plasma Phys.* 4, 761 (1970).
44. J.J. Galloway and F.W. Crawford, Proc. Fourth European Conference on Controlled Fusion and Plasma Physics, Rome, Italy, September 1970. (CNEN, Rome, 1970) p. 161.
45. A.A. Vedenov and L.I. Rudakov, *Dokl. Akad. Nauk SSSR* 159, 767 (1965), [*Sov. Phys. Dokl.* 9, 1073 (1965)].
46. J.P. Dougherty, Cornell Univ. Center for Radiophys. and Space Res. Rept. 398 (August 1970)
47. P.A. Sturrock, *Plasma Phys.* 2, 158 (1961).
48. H. Goldstein, Classical Mechanics (Addison-Wesley Publishing Co., Reading, Mass., 1959), p. 288ff.
49. K.J. Harker, Stanford University Institute for Plasma Research Rept. No. 395 (November 1970).
50. Ref. 3, Chap. 7; Ref. 4, p. 12; Ref. 5, p. 54.
51. V.N. Tsytovich, *Sov. Phys. Uspek.* 9, 805 (1967)
52. G.I. Suramlshvili, *Zh. Eksp. Teor. Fiz.* 52, 255 (1967), [*Sov. Phys. JETP* 25, 165 (1967)]; *Zh. Tekh. Fiz.* 39, 1774 (1969), [*Sov. Phys. Tech. Phys.* 14, 1334 (1970)]; *Zh. Tekh. Fiz.* 40, 1802 (1970), [*Sov. Phys. Tech. Phys.* 15, 1407 (1971)].
53. E.G. Harris, Advances in Plasma Physics, Eds. A. Simon and W.B. Thompson (Interscience Publishers, New York, N.Y., 1969), Vol. 3, p. 157.
54. L.I. Schiff, Quantum Mechanics, Third Ed. (McGraw-Hill Book Co., New York, N.Y., 1968), Chap. 14.
55. D. Pines and J.R. Schrieffer, *Phys. Rev.* 125, 804 (1962).
56. G.M. Walters and E.G. Harris, *Phys. Fluids* 11, 112 (1968).
57. R.E. Aamodt and W E. Drummond, *Phys. Fluids* 7, 1816 (1964).

58. S. Krishan, *Plasma Phys.* 10, 201 (1968).
59. S. Krishan and A.A. Selim, *Plasma Phys.* 10, 931 (1968).
60. V.E. Zakharov, *Zh. Eksp. Teor. Fiz.* 51, 688 (1966), [*Sov. Phys. JETP* 24, 455 (1967)].
61. D.W. Ross, *Phys. Fluids* 12, 613 (1969).
62. F.E. Low, *Proc. Roy. Soc.* A248, 282 (1958).
63. T.J.M. Boyd and J.G. Turner, Proc. Tenth International Conference on Phenomena in Ionized Gases, Oxford, England, September 1971 (Donald Parsons and Co., Oxford, England, 1971), p. 322.
64. I.B. Bernstein, *Phys. Rev.* 109, 10 (1958).
65. R.L. Dewar, *Univ. of Maryland Center for Theoretical Phys. Rept.* No. 71-080 (January 1971).
66. J.J. Galloway and H. Kim, *J. Plasma Phys.* 6, 53 (1971).
67. C. Eckart, *Phys. Fluids* 3, 421 (1969).
68. S. Katz, *Phys. Fluids* 4, 345 (1961).
69. W.A. Newcomb, *Nucl. Fusion (Supplement) Part 2*, 451 (1962).
70. T.S. Lundgren, *Phys. Fluids* 6, 898 (1963).
71. D.L. Bobroff, *IRE Trans. Elec. Devices* ED-6, 68 (1959).
72. H. Lamb, Hydrodynamics (Dover Publications, New York, N.Y., 1945), p. 12.
73. J.F. Denisse and J.L. Delcroix, Plasma Waves (Interscience Publishers, New York, N.Y., 1963), Chap. 5; W.P. Allis, S.J. Buchsbaum and A. Bers, Waves in Anisotropic Plasmas (MIT Press, Cambridge, Mass., 1963), Chap. 8.
74. T.H. Stix, The Theory of Plasma Waves (McGraw-Hill Book Co., New York, N.Y., 1962), p. 194.
75. F.W. Crawford, *J. Res. NBS* 69D, 789 (1965).
76. L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, New York, N.Y., 1962), p. 3.
77. D.E. Baldwin, I.B. Bernstein and M.P.H. Weenink, Advances in Plasma Physics, Eds. A. Simon and W.B. Thompson (Interscience Publishers, New York, N.Y., 1969), Vol. 3, p. 1.
78. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products (Academic Press, New York, N.Y., 1965), p. 718.

79. K.F. Lee, *J. Appl. Phys.* 42, 5418 (1971).
80. R.S. Wagers, *Phys. Fluids* 14, 1586 (1971).
81. O. Buneman, *Phys. Fluids* 4, 669 (1961).
82. S.I. Braginskii, *Reviews of Plasma Physics*, Ed. M.A. Leontovich (Consultanta Bureau, New York, N.Y., 1965), Vol. I, p. 205.
83. Y.M. Peng (private communication).
84. J.L. Delcroix, *Plasma Physics* (John Wiley and Sons Ltd., New York, N.Y., 1965), p. 221.
85. S. Chapman and T.G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge Univ. Press, Cambridge, England, 1970), p. 19.
86. A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, N.Y., 1944), p. 44.
87. Ref. 85, p. 76; Ref. 86, p. 50.
88. Ref. 5, p. 316.
89. M.V. Goldman, *Ann. Phys.* 38, 95 (1966).
90. K.J. Harker, *Int. J. Elect.* 32, 297 (1972).
91. Ref. 74, p. 197.
92. G.F. Chen, M.L. Goldberger and F.E. Low, *Proc. Roy. Soc.* 236, 112 (1956).
93. W.F. Utlaut, *J. Geophys. Res.* 75, 6402 (1970); W.F. Utlaut and R. Cohen, *Science* 174, 245 (1971); V.V. Pustovalov and V.P. Silin, *Zh. Eksp. Teor. Fiz.* 59, 2215 (1970), [*Sov. Phys. JETP* 32, 1198 (1971)]; E. Valeo, C. Oberman and F.W. Perkins, *Phys. Rev. Letters* 28, 340 (1972).
94. P.K. Kaw and J.M. Dawson, *Phys. Fluids* 12, 2586 (1969).
95. K.J. Harker and F.W. Crawford in Ref. 26.
96. J.J. Galloway (private communication).
97. B.R. Suydam, *Proc. Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, Switzerland, September 1958* (Columbia Univ. Press, New York, N.Y., 1959), Vol. 31, p. 157.
98. R.N. Bracewell, *The Fourier Transform and Its Applications* (McGraw-Hill Book Co., New York, N.Y., 1965), Chap. 6.
99. G. Laval, R. Pellat and M. Pérulli, *Plasma Phys.* 11, 579 (1969).

100. E.N. Spithas and W.M. Manheimer, *Phys. Fluids* 13, 1110 (1970).
101. M. Pérulli, *Doctoral Thesis for University of Paris, Paris, France* (July 1968).
102. J.M. Larsen (private communication).
103. Ref. 3, p. 88.
104. Ref. 3, Appendix 2.
105. Ref. 4, p. 90.