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16. Abstract

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TWO BONDED HALF PLANES WITH A
CRACK GOING THROUGH THE INTERFACE[*]

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ABSTRACT

The plane problem of two bonded elastic half planes containing a finite crack perpendicular to and going through the interface is considered. The problem is formulated as a system of singular integral equations with generalized Cauchy kernels. Even though the system has three irregular points, it is shown that the unknown functions are algebraically related at the irregular point on the interface and the integral equations can be solved by a method developed previously. The system of integral equations is shown to yield the same characteristic equation as that for two bonded quarter planes in the general case of the through crack, and the characteristic equation for a crack tip terminating at the interface in the special case. The numerical results given in the paper include the stress intensity factors at the crack tips, the normal and shear components of the stress intensity factors at the singular point on the interface, and the crack surface displacements.

1. INTRODUCTION

In considering the fracture of composite materials, one
may approach the problem from two different points of view.
In the first approach the primary interest is in studying and
in estimating the "bulk strength" of the given structure under
a known system of external loads and environmental conditions.
In this type of studies it is usually assumed that the existing
imperfections such as voids, cracks, and inclusions are ran-
domly distributed throughout the composite medium and the mate-
rial is statistically homogeneous. Thus, the very nature of
the problem requires that some kind of a statistical strength
theory be used as a guide in the investigations.

In the second approach to studying the fracture of com-
posites, one is basically interested in the initiation of frac-
ture from the "localized" imperfections which are known (or
assumed) to exist in the material. In this type of studies it
is usually assumed that the composite medium consists of per-
fectly bonded elastic components and the localized imperfection
may be idealized as a plane crack or as a flat elastic inclu-
sion. The problem of a finite crack lying parallel to or at
a bimaterial interface in composites with various geometries
was discussed in [1-5]. The similar problem for a flat inclu-
sion was studied in [6]. An up-to-day review of the general
fracture problems in composite materials and a summary of some
of the known results may be found in [7]. The distinguishing
feature of the solutions given in [1-7] as well as the other

known solutions which appeared in literature within the past decade is that the strength of the stress singularity at the imperfection front is -1/2 and, in the case of a crack, the quasi-static stress state in the neighborhood of the crack front remains autonomous as the crack propagates. That is, aside from a slight change in a multiplicative constant known as the stress intensity factor, the asymptotic nature of the stress state in the vicinity of the crack front remains unchanged.

On the other hand, when the crack front terminates at a bi-material interface in the composite medium the strength of the stress singularity is no longer -1/2 and the angular distribution of stresses differs considerably from that of a crack tip imbedded in a homogeneous medium. After reaching the interface, further propagation of the crack may be in the form of (a) a cleavage crack into the second medium (Figure 1), (b) a debonding crack along the interface, or (c) a "reflected" crack back into the first medium. The basic problem of a finite crack terminating at a bi-material interface was discussed in some detail in a previous paper [8]. [8] also presents a tentative fracture criterion which may be used in predicting the mode of the fracture propagation (such as (a)-(c), mentioned above) as well as the level of the external loads initiating the fracture provided the fracture strengths of the two adjoining materials and of the interface are known.

In this paper we consider the problem of a crack going

through the interface into the second medium. From the analytical viewpoint the unusual feature of the problem is that it has three irregular points (two crack tips and the intersection of the crack with the interface) at which the stress state is generally singular. One of the objectives of the paper is the development of the necessary theoretical and numerical techniques for handling problems of this nature.

2. THE INTEGRAL EQUATIONS

Consider the plane elasticity problem shown in Figure 1. In this paper only the symmetric problem will be considered; that is, it is assumed that in the composite medium without the crack and subjected to the same external loads as the cracked medium, the planes $(r > 0, \theta = \pi)$ and $(r > 0, \theta = 0)$ are free from shear and are acted upon by normal stresses $p_1(r)$ and $p_2(r)$, respectively. Thus, through a proper superposition, the singular part of the problem may be reduced to that shown in Figure 1, where the self-equilibrating crack surface tractions $-p_1(r)$ and $-p_2(r)$ are the only external loads. In the special case of uniform loading p_1 and p_2 are constant and the continuity conditions require that

$$\frac{p_1}{E_1} = \frac{p_2}{E_2} \ , \quad \text{for plane stress},$$

$$\frac{1 - \nu_1^2}{E_1} \, p_1 = \frac{1 - \nu_2^2}{E_2} \, p_2 \ , \quad \text{for plane strain.} \qquad (1.a,b)$$

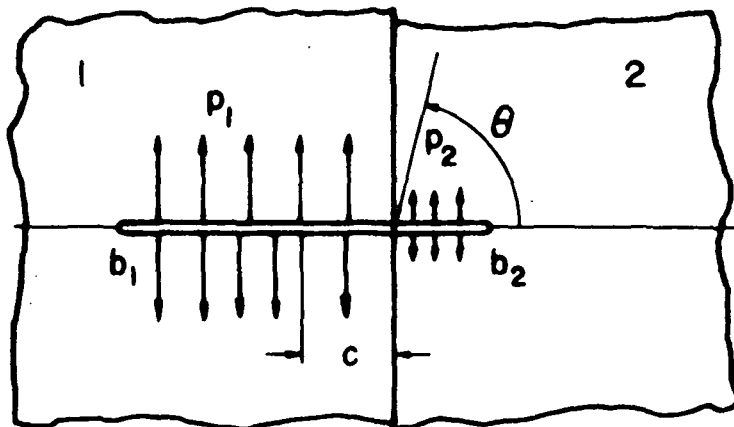To solve the problem shown in Figure 1 we will first

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Figure 1.   The geometry of the crack going through the
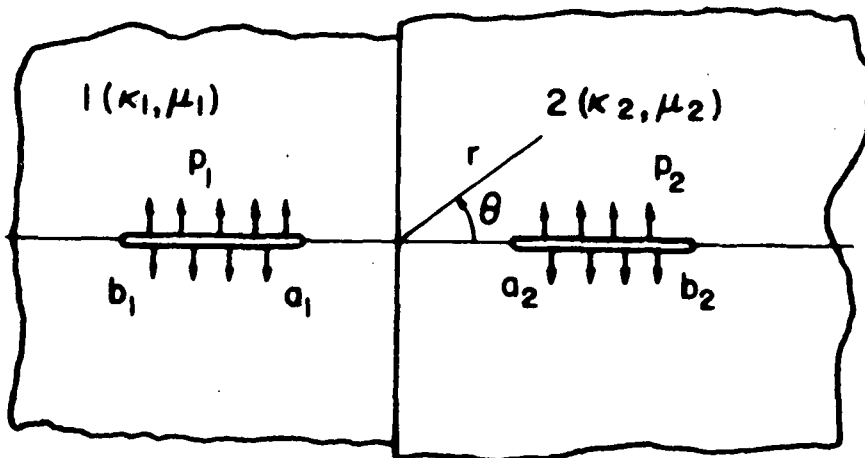interface of two bonded elastic half planes.

Figure 2.   Two cracks imbedded into adjacent half planes.

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consider the problem described in Figure 2 where each half plane contains a finite crack perpendicular to the interface, and then let the distances a_1 and a_2 go to zero.

Because of symmetry, it is sufficient to consider the problem in the half plane $0 \leq r < \infty$, $0 < \theta < \pi$ (Figure 2). The related elasticity problem has to be solved under the following boundary conditions:

$$\tau_{1r\theta}(r,\pi) = 0 , \qquad 0 \leq r < \infty,$$

$$\tau_{1\theta\theta}(r,\pi-0) = - p_1(r) , \qquad a_1 < r < b_1,$$

$$u_{1\theta}(r,\pi) = 0 , \qquad 0 \leq r < a_1, \quad b_1 < r < \infty; \qquad (2.a\text{-}c)$$

$$u_{1\theta} = u_{2\theta} , \quad u_{1r} = u_{2r} , \quad \tau_{1\theta\theta} = \tau_{2\theta\theta} ,$$

$$\tau_{1r\theta} = \tau_{2r\theta} , \qquad \theta = \pi/2, \quad 0 \leq r < \infty; \qquad (3.a\text{-}d)$$

$$\tau_{2r\theta}(r,0) = 0 , \qquad 0 \leq r < \infty,$$

$$\tau_{2\theta\theta}(r,+0) = - p_2(r) , \qquad a_2 < r < b_2,$$

$$u_{2\theta}(r,0) = 0 , \qquad 0 \leq r < a_2, \quad b_2 < r < \infty. \qquad (4.a\text{-}c)$$

Let the unknown functions f_1 and f_2 be defined as follows:

$$f_1(r) = - 2 \frac{\partial}{\partial r} u_{1\theta}(r,\pi-0) ,$$

$$f_2(r) = 2 \frac{\partial}{\partial r} u_{2\theta}(r,+0) . \qquad (5.a,b)$$

From (2.c), (4.c) and (5) it follows that

$$f_1(r) = 0 \, , \qquad 0 \le r < a_1, \quad b_1 < r < \infty,$$

$$f_2(r) = 0 \, , \qquad 0 \le r < a_2, \quad b_2 < r < \infty,$$

$$\int_{a_1}^{b_1} f_1(r)dr = 0 \, , \qquad \int_{a_2}^{b_2} f_2(r)dr = 0 \, . \qquad (6.a\text{-}d)$$

Using Mellin transforms and following the procedure out-lined in [8], after somewhat routine manipulations the problem can be reduced to the following system of integral equations for the functions f_1 and f_2:

$$\frac{1}{\pi} \int_{a_1}^{b_1} \frac{f_1(s)}{s-r} \, ds + \frac{1}{\pi} \int_{a_1}^{b_1} k_{11}(r,s)f_1(s)ds$$

$$+ \frac{1}{\pi} \int_{a_2}^{b_2} k_{12}(r,s)f_2(s)ds = -\frac{1+\kappa_1}{2\mu_1} \, p_1(r) \, , \quad a_1 < r < b_1,$$

$$\frac{1}{\pi} \int_{a_2}^{b_2} \frac{f_2(s)}{s-r} \, ds + \frac{1}{\pi} \int_{a_1}^{b_1} k_{21}(r,s)f_1(s)ds$$

$$+ \frac{1}{\pi} \int_{a_2}^{b_2} k_{22}(r,s)f_2(s)ds = -\frac{1+\kappa_2}{2\mu_2} \, p_2(r) \, , \quad a_2 < r < b_2,$$

$$(7.a,b)$$

where the kernels k_{ij} are given by

$$k_{11}(r,s) = \sum_{k=1}^{3} \frac{c_{1k}r^{k-1}}{(s+r)^k} \, , \qquad k_{22}(r,s) = \sum_{k=1}^{3} \frac{c_{2k}r^{k-1}}{(s+r)^k} \, ,$$

$$k_{12}(r,s) = \sum_{1}^{2} \frac{d_{1k}r^{k-1}}{(s+r)^k} \, , \qquad k_{21}(r,s) = \sum_{1}^{2} \frac{d_{2k}r^{k-1}}{(s+r)^k} \, ; \qquad (8.a\text{-}d)$$

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$$c_{11} = \frac{1}{2} - \frac{m_1(1+\kappa_1)}{2(m_1+\kappa_2)} - \frac{3(1-m_1)}{2(1+m_1\kappa_1)} ,$$

$$c_{12} = \frac{6(1-m_1)}{1+m_1\kappa_1} , \qquad c_{13} = -\frac{4(1-m_1)}{1+m_1\kappa_1} ;$$

$$c_{21} = \frac{1}{2} - \frac{m_2(1+\kappa_2)}{2(m_2+\kappa_1)} - \frac{3(1-m_2)}{2(1+m_2\kappa_2)} ,$$

$$c_{22} = \frac{6(1-m_2)}{1+m_2\kappa_2} , \qquad c_{23} = -\frac{4(1-m_2)}{1+m_2\kappa_2} ;$$

$$d_{11} = \frac{3(1+\kappa_1)}{2(m_2+\kappa_1)} - \frac{1+\kappa_1}{2(1+m_2\kappa_2)} , \qquad d_{12} = \frac{1+\kappa_1}{1+m_2\kappa_2} - \frac{1+\kappa_1}{m_2+\kappa_1} ;$$

$$d_{21} = \frac{3(1+\kappa_2)}{2(m_1+\kappa_2)} - \frac{1+\kappa_2}{2(1+m_1\kappa_1)} , \qquad d_{22} = \frac{1+\kappa_2}{1+m_1\kappa_1} - \frac{1+\kappa_2}{m_1+\kappa_2} ;$$

$$m_1 = \mu_2/\mu_1 , \qquad m_2 = \mu_1/\mu_2 . \qquad\qquad (9)$$

In (9) μ_1 and μ_2 are the shear moduli, $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for the generalized plane stress, $(i = 1,2)$, ν_i being the Poisson's ratio.

From (8) it is seen that for $a_1 > 0$, $a_2 > 0$ the kernels $k_{ij}(r,s)$, $(i,j = 1,2)$ are bounded and continuous. Hence (7) is a simple system of singular integral equations with an index $\kappa = 1$ and the additional conditions (6.c and d) which are required for a unique solution. These equations can be solved for the unknown functions f_1 and f_2 in a standard manner. A simple and highly effective numerical method has been described in [9]. After obtaining f_1 and f_2, all the desired field quantities may be expressed as and evaluated from the definite

integrals with kernels corresponding to the particular field quantity and the density functions f_i, (i = 1,2) (see, for example, [8] for the case of one crack). From the practical viewpoint of particular interest are the stress intensity factors defined by[*]

$$k(a_1) = \lim_{r \to a_1} \sqrt{2(a_1-r)} \; \tau_{1\theta\theta}(r,\pi) = \lim_{r \to a_1} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(r-a_1)} f_1(r),$$

$$k(b_1) = \lim_{r \to b_1} \sqrt{2(r-b_1)} \; \tau_{1\theta\theta}(r,\pi) = -\lim_{r \to b_1} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(b_1-r)} \; f_1(r),$$

$$k(a_2) = \lim_{r \to a_2} \sqrt{2(a_2-r)} \; \tau_{2\theta\theta}(r,0) = \lim_{r \to a_2} \frac{2\mu_2}{1+\kappa_2} \sqrt{2(r-a_2)} \; f_2(r),$$

$$k(b_2) = \lim_{r \to b_2} \sqrt{2(r-b_2)} \; \tau_{2\theta\theta}(r,0) = -\lim_{r \to b_2} \frac{2\mu_2}{1+\kappa_2} \sqrt{2(b_2-r)} \; f_2(r).$$

$$(10.a\text{-}d)$$

For the bonded planes Material 1 Aluminum ($E_1 = 10^7$ psi, $v_1 = 0.3$), and Material 2 Epoxy ($E_2 = 4.45 \times 10^5$ psi, $v_2 = 0.35$), Tables 1 and 2 show some of the calculated plane strain results. In these tables p_1 and p_2 refer to constant pressures applied to the crack surfaces and are related by (1). As a basis of comparison the special case of a homogeneous plane containing two collinear cracks may be mentioned. For $\mu_1 = \mu_2$, $v_1 = v_2$, $a_1 = a_2 = a$, $b_1 = b_2 = b$, the stress intensity factors are given by

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[*]See Section 4 below.

Table 1. Stress Intensity Factors for $a_1 > 0$, $a_2 > 0$.
Material 1: Aluminum, Material 2: Epoxy;
$a_2 = 1$, $\ell_2 = (b_2-a_2)/2 = 1/2$,
$c_1 = (b_1+a_1)/2 = 1.5$, $\ell_1 = (b_1-a_1)/2$ variable.

ℓ_1	$\dfrac{k(a_1)}{p_1\sqrt{\ell_1}}$	$\dfrac{k(b_1)}{p_1\sqrt{\ell_1}}$	$\dfrac{k(a_2)}{p_2\sqrt{\ell_2}}$	$\dfrac{k(b_2)}{p_2\sqrt{\ell_2}}$
0.05	1.0015	1.0014	0.97417	0.98165
0.10	1.0023	1.0021	0.97515	0.98241
0.15	1.0037	1.0033	0.97679	0.98369
0.25	1.0082	1.0069	0.98212	0.98775
0.50	1.0326	1.0231	1.0086	1.0079
0.75	1.0836	1.0492	1.0590	1.0454
1.00	1.1832	1.0869	1.1474	1.1089
1.25	1.4094	1.1437	1.3118	1.2210
1.40	1.8046	1.2012	1.5047	1.3451

Table 2. Stress Intensity Factors for $a_1 > 0$, $a_2 > 0$.
Material 1: Aluminum, Material 2: Epoxy;
$a_2 = 1$, $\ell_1 = \ell_2 = 0.5$, $c_1 = (b_1+a_1)/2$ variable.

c_1	$\dfrac{k(a_1)}{p_1\sqrt{\ell_1}}$	$\dfrac{k(b_1)}{p_1\sqrt{\ell_1}}$	$\dfrac{k(a_2)}{p_2\sqrt{\ell_2}}$	$\dfrac{k(b_2)}{p_2\sqrt{\ell_2}}$
1.50	1.0326	1.0231	1.0086	1.0079
1.25	1.0494	1.0326	1.0187	1.0143
1.00	1.0844	1.0499	1.0344	1.0236
0.75	1.1849	1.0881	1.0624	1.0393
0.60	1.4114	1.1454	1.0965	1.0576

$$\frac{k(a)}{p_0\sqrt{\ell}} = \frac{[b^2 E(m)/K(m) - a^2]\sqrt{2}}{(b-a)\sqrt{a(b+a)}} \, ,$$

$$\underline{} \quad \frac{k(b)}{p_0\sqrt{\ell}} = \frac{\sqrt{2}\, b^2[1 - E(m)/K(m)]}{(b-a)\sqrt{b(b+a)}} \, , \qquad\qquad (11.a,b)$$

$$(\ell = (b-a)/2, \quad m = 1 - a^2/b^2) \, ,$$

where p_0 is the constant pressure applied to the crack surfaces
and K and E are the complete elliptic integrals of first and
second kind, respectively. For example, for b = 2a which corre-
sponds to the row $\ell_1 = 0.5$ in Table 1 (or to $c_1 = 1.5$ in Table
2), we have k(a) = 1.01762 $p_0\sqrt{\ell}$, k(b) = 1.01249 $p_0\sqrt{\ell}$.

As $\ell_1 \to 1.5$ in Table 1 and $c_1 \to 0.5$ in Table 2, $k(a_1) \to \infty$
while the remaining three stress intensity factors approach
some finite values. The reason for this is that in this limit-
ing case $a_1 = 0$ and the singular behavior of the stresses around
the crack tip is of the form $R^{-\alpha}$ with $0.5 < \alpha < 1$ whereas the
definition of $k(a_1)$ is based on $R^{-0.5}$ type singularity (see (10))
where R is a small distance from the crack tip a_1 (see [8]).
The results for another special case are shown in Figure 3 where
$a_2 = b_2$. This is the problem considered in [8] where only one
of the adjoining materials contains a crack. Because of the
change in the power α of the stress singularity when the crack
terminates at the interface, again as $a \to 0$ $k(a) \to \infty$ for
$\mu_1 > \mu_2$ and $k(a) \to 0$ for $\mu_1 < \mu_2$, whereas k(b) in both cases
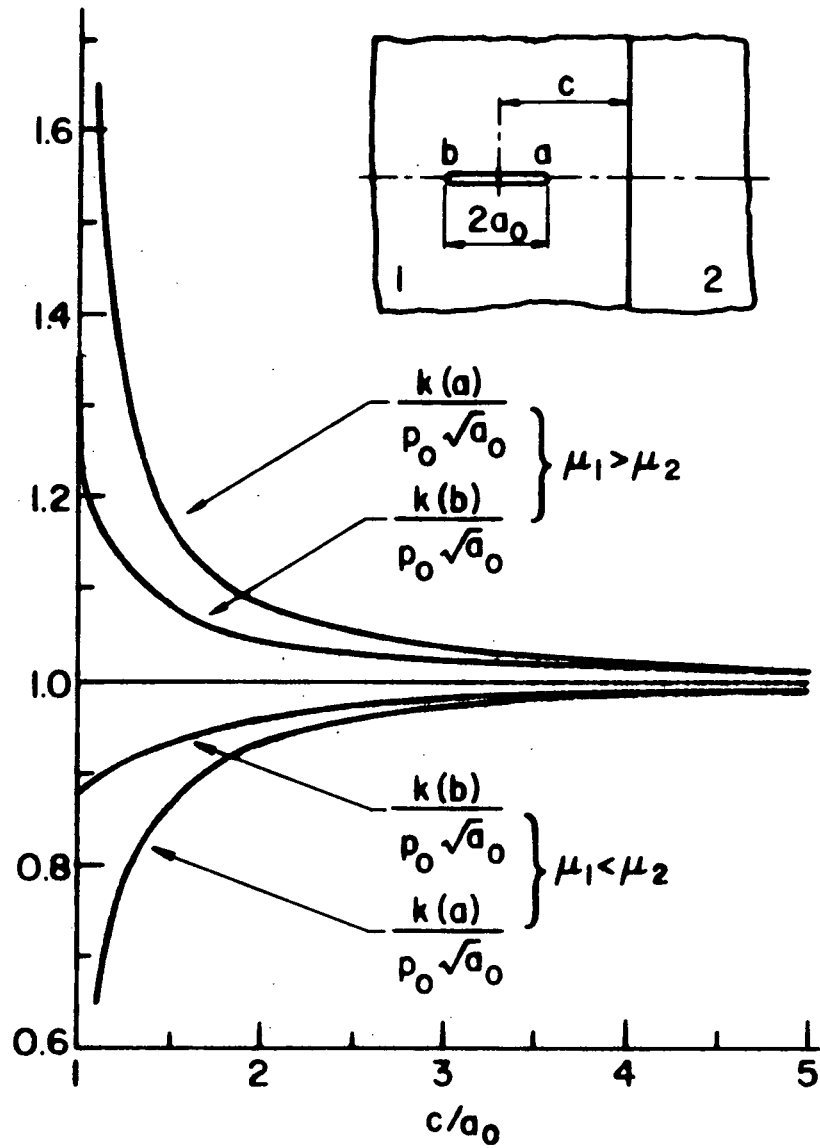remains bounded.

Figure 3.   Stress intensity factors for a crack perpendicular
to the interface (materials: aluminum and epoxy;
external load: crack surface pressure p_0).

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3. CRACK GOING THROUGH THE INTERFACE, $a_1 = 0 = a_2$

We now consider the main problem in which the crack goes through the interface, occupying both materials. In this case it is clear that the integral equations (7) are still valid, with $a_1 = 0 = a_2$. Thus, the equations have only three irregular points $(b_1, \pi; 0; b_2, 0)$, meaning that one has to be more careful in examining the singular behavior of the solution. Unlike the problem considered in the previous section for which $a_1 > 0$, $a_2 > 0$, as seen from (8), here the kernels k_{ij} are no longer bounded for all values of r and s in the closed intervals $[0, b_1]$ and $[0, b_2]$. At the irregular point 0 they become unbounded, indicating that at $r = 0$ the behavior of $f_i(r)$, $(i = 1, 2)$, will be affected by $k_{ij}(r, s)$.

To examine the behavior of f_1 and f_2 around the irregular points, following [10] we assume that the unknown functions may be expressed as

$$f_j(s) = \frac{g_j(s)}{s^\alpha (b_j - s)^{\beta_j}} = \frac{g_j(s) e^{\pi i \beta_j}}{s^\alpha (s - b_j)^{\beta_j}} , \qquad (12)$$

$$0 < Re(\alpha) < 1, \quad 0 < Re(\beta_j) < 1, \quad (j = 1, 2),$$

where α, β_1 and β_2 are the powers of the singularity at the three irregular points, the function g_j satisfies a Hölder condition in the closed interval $0 \leq s \leq b_j$, $(j = 1, 2)$, and the function $s^\alpha (s - b_j)^{\beta_j}$ is any definite branch which varies continuously on $0 < s < b_j$, $(j = 1, 2)$. Consider now the following sectionally holomorphic functions:

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$$G_j(z) = \frac{1}{\pi} \int_0^{b_j} \frac{f_j(s)}{s-z} \, ds = \frac{1}{\pi} \int_0^{b_j} \frac{g_j(s)e^{\pi i \beta_j} ds}{s^\alpha (s-b_j)^{\beta_j}(s-z)}, \quad (j=1,2),$$

$$(13)$$

where the complex variable $z = r + i\rho$ is defined in such a way that in each case the cut ($\rho = 0$, $0 < r < b_j$), ($j = 1,2$) lies along the positive real axis. Separating the leading terms at the end points of the cuts, (13) may be expressed as [10]

$$G_j(z) = \frac{g_j(0)e^{\pi i \alpha}}{b_j^{\beta_j} \sin\pi\alpha} \frac{1}{z^\alpha} - \frac{g_j(b_j)}{b_j^\alpha \sin\pi\beta_j} \frac{1}{(z-b_j)^{\beta_j}} + G_{0j}(z), \quad (14)$$

where G_{0j} is bounded everywhere except possibly at the ends, e_{jk}, near which

$$|G_{0j}(z)| < \frac{C_{jk}}{|z-e_{jk}|^{p_{jk}}}, \quad (j,k = 1,2) \tag{15}$$

$$e_{j1} = 0, \quad p_{j1} < \text{Re}(\alpha); \quad e_{j2} = b_j, \quad p_{j2} < \text{Re}(\beta_j),$$

C_{jk} and p_{jk} being real constants.

Using now the Plemelj formula [10]

$$\frac{1}{\pi} \int_0^{b_j} \frac{f_j(s)}{s-r} \, ds = \frac{1}{2}[G_j^+(r) + G_j^-(r)], \quad (0 < r < b_j) \tag{16}$$

the terms with the Cauchy singularity in (7) may be expressed as

$$\frac{1}{\pi} \int_0^{b_j} \frac{f_j(s)}{s-r} \, ds = \frac{g_j(0)}{b_j^{\beta_j}} \cot \pi \alpha \, \frac{1}{r^\alpha} - \frac{g_j(b_j)}{b_j^\alpha} \cot \pi \beta_j \, \frac{1}{(b_j-r)^{\beta_j}}$$

$$+ \, G_{1j}(r) \, , \qquad (0 < r < b_j, \quad j = 1,2), \qquad (17)$$

where the behavior of $G_{ij}(r)$ in the closed interval $[0,b_j]$ is similar to that of G_{0j} $(j = 1,2)$. The remaining terms in the integral equations (7) may be obtained directly from (14) through differentiation. From (7), (8) and (13) it is seen that in these terms $z = -r$ which is outside the cut $0 < r < b_j$, $j = 1,2$. As r varies in the interval $(0,b_j)$, z will vary in the interval $(-b_j,0)$ in which $G_j(z)$ is holomorphic, $(j = 1,2)$. Thus, from (13) and (14) we obtain

$$\frac{1}{\pi} \int_0^{b_j} \frac{f_j(s)ds}{s+r} = G_j(-r) = \frac{g_j(0)}{b_j^{\beta_j} \sin \pi \alpha} \frac{1}{r^\alpha} + G_{2j}(r) \, ,$$

$$\frac{1}{\pi} \int_0^{b_j} \frac{rf_j(s)ds}{(s+r)^2} = -r \frac{d}{dr} G_j(-r) = \frac{\alpha g_j(0)}{b_j^{\beta_j} \sin \pi \alpha} \frac{1}{r^\alpha} + G_{3j}(r) \, ,$$

$$\frac{1}{\pi} \int_0^{b_j} \frac{r^2 f_j(s)ds}{(s+r)^3} = \frac{r^2}{2} \frac{d^2}{dr^2} G_j(-r) = \frac{\alpha(\alpha+1)g_j(0)}{2b_j^{\beta_j} \sin \pi \alpha} \frac{1}{r^\alpha} + G_{4j}(r) \, ,$$

$$(0 < r < b_j, \quad j = 1,2), \qquad (18.a-c)$$

where it may easily be shown that the behavior of $G_{kj}(r)$ in the closed interval $[0,b_j]$, $(j = 1,2, \quad k = 2,3,4)$, will again be similar to that of G_{0j} given by (15).

If we now substitute from (17), (18) and (8) into (7),

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and multiply first both sides of each equation by r^α and let $r \rightarrow 0$, then (7.a) by $(b_1-r)^{\beta_1}$ and let $r \rightarrow b_1$, and (7.b) by $(b_2-r)^{\beta_2}$ and let $r \rightarrow b_2$, we obtain the following system of equations:

$$\left(\cos\pi\alpha + c_{11} + \alpha c_{12} + \frac{\alpha(\alpha+1)}{2} c_{13}\right) \frac{g_1(0)}{b_1^{\beta_1} \sin\pi\alpha}$$

$$+ (d_{11} + \alpha d_{12}) \frac{g_2(0)}{b_2^{\beta_2} \sin\pi\alpha} = 0 \ ,$$

$$(d_{21} + \alpha d_{22}) \frac{g_1(0)}{b_1^{\beta_1} \sin\pi\alpha}$$

$$+ \left(\cos\pi\alpha + c_{21} + \alpha c_{22} + \frac{\alpha(\alpha+1)}{2} c_{23}\right) \frac{g_2(0)}{b_2^{\beta_2} \sin\pi\alpha} = 0 \ ,$$

$$\frac{g_1(b_1)}{b_1^\alpha} \cot\pi\beta_1 = 0 \ ,$$

$$\frac{g_2(b_2)}{b_2^\alpha} \cot\pi\beta_2 = 0 \ . \tag{19.a-d}$$

$g_j(0)$ and $g_j(b_j)$, $(j = 1,2)$, are non-zero constants which will be shown to be related to the stress intensity factors. Thus, (19.c) and (19.d) give the following characteristic equations to determine β_1 and β_2:

$$\cot\pi\beta_1 = 0 \ , \qquad \cot\pi\beta_2 = 0 \ . \tag{20.a,b}$$

From (20) the values of β_1 and β_2 satisfying the condition $0 < Re(\beta_j) < 1$, $(j = 1,2)$, are found to be $\beta_1 = \beta_2 = 1/2$, which

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is the well-known result.

On the other hand, since $\alpha \neq 0$ and $\alpha \neq 1$, for (19.a) and (19.b) to give a non-zero solution for $g_j(0)$, the determinant of the coefficients must vanish, giving the characteristic equation to determine α as follows:

$$D(\alpha) \equiv (\cos\pi\alpha + c_{11} + \alpha c_{12} + \frac{\alpha(\alpha+1)}{2} c_{13})(\cos\pi\alpha + c_{21}$$

$$+ \alpha c_{22} + \frac{\alpha(\alpha+1)}{2} c_{23}) - (d_{11} + \alpha d_{12})(d_{21} + \alpha d_{22}) = 0 .$$

$$(21)$$

From (19) it is clear that the constants $g_1(0)$ and $g_2(0)$ are not independent and are related by

$$g_2(0) = - g_1(0) \sqrt{b_2/b_1} \; \frac{\cos\pi\alpha + c_{11} + \alpha c_{12} + \frac{\alpha(\alpha+1)}{2} c_{13}}{d_{11} + \alpha d_{12}} .$$

$$(22)$$

As will be pointed out later, the use of (22) will be necessary in order to obtain a unique solution for the integral equations (7).

In the special case of homogeneous medium, $m_j \to 1$, $\kappa_1 \to \kappa_2$, $c_{jk} \to 0$ ($j = 1,2; k = 1,2,3$), $d_{11} \to 1$, $d_{12} \to 0$, $d_{21} = 1$, $d_{22} \to 0$, the characteristic equation (20) becomes $\cos^2\pi\alpha - 1 = 0$, giving $\alpha = 0$ as the acceptable root and $g_2(0)/\sqrt{b_2} = - g_1(0)/\sqrt{b_1}$. Considering the definitions (5) and (12), this indicates that, as expected, at $r = 0$ the derivative of the crack surface displacement becomes finite and continuous.

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In the problem of two bonded half planes containing a through crack perpendicular to the interface (Figure 1), the singular behavior of the solution at and around $r = 0$ must be the same as that for two bonded quarter planes (see, for example, [11] and [12]). Defining the constants

$$K_1 = 2(m_1-1) , \qquad K_2 = m_1(1+\kappa_1) - (1+\kappa_2) ,$$

$$K_3 = m_1(1+\kappa_1) + (1+\kappa_2) , \qquad (23)$$

after some manipulations (21) may be reduced to

$$[(K_1-K_2) \cos^2 \frac{\pi\alpha}{2} - K_1(\alpha-1)^2]^2$$

$$+ K_3^2 \sin^2 \frac{\pi\alpha}{2} \cos^2 \frac{\pi\alpha}{2} - K_2^2 (\alpha-1)^2 = 0 . \qquad (24)$$

(24) is identical to the characteristic equation found in [11] and [12] which indicates that the two solutions will have the same singular behavior[*].

In the special case of only one of the cracks terminating at the interface, i.e., for $b_1 > a_1 = 0$, $b_2 > a_2 > 0$, the kernels k_{12}, k_{21} and k_{22} in (7) will be bounded and will have no contribution to the singularities. For this problem the solution is of the form

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[*]As shown in [11] the weak power singularity $0 < \text{Re}(\alpha) < 1$ as assumed in this paper is not the only possible singularity for two quarter planes. Theoretically it is possible to have a logarithmic singularity as well as bounded stresses. However, if one studies the conditions in detail, it is not difficult to show that for all practical material combinations, the power singularity is the only possibility.

$$f_1(s) = \frac{g_1(s)}{s^{\alpha_1}(b_1-s)^{\beta_1}} \; , \qquad f_2(s) = \frac{g_2(s)}{(s-a_2)^{\alpha_2}(b_2-s)^{\beta_2}} \; ,$$

$$(25.a,b)$$

and, following the procedure outlined above, the characteristic equations for α_i and β_i ($i = 1,2$) are found to be

$$\cos\pi\alpha_1 + c_{11} + \alpha c_{12} + \frac{\alpha(\alpha+1)}{2} c_{13} = 0 \; ,$$

$$\cot\pi\beta_1 = 0 \; , \qquad \cot\pi\alpha_2 = 0 \; , \qquad \cot\pi\beta_2 = 0 \; . \qquad (26.a\text{-}d)$$

(26.a) is the equation found in [8] for the single crack and (26.b-d) give $\beta_1 = \alpha_2 = \beta_2 = 1/2$.

4. STRESS INTENSITY FACTORS

Using the results of the previous section it can be shown that the strength of the stress singularity at the crack tips, conventionally known as the stress intensity factor, is related to and can be evaluated from the density functions or the displacement derivatives $f_j(r)$, ($j = 1,2$), defined by (5). First, consider the case of two cracks imbedded into the adjacent homogeneous materials (i.e., $b_j > a_j > 0$, $j = 1,2$) for which the functions f_j and the related holomorphic functions $G_j(z)$ may be expressed as

$$f_j(s) = \frac{g_j(s)}{(s-a_j)^{\alpha_j}(b_j-s)^{\beta_j}} \; , \qquad\qquad (27.a)$$

$$G_j(z) = \frac{1}{\pi} \int_{a_j}^{b_j} \frac{f_j(s)ds}{s-z} = \frac{g_j(a_j)e^{\pi i \alpha_j}}{(b_j-a_j)^{\beta_j} \sin\pi\alpha_j} \frac{1}{(z-a_j)^{\alpha_j}}$$

$$- \frac{g_j(b_j)}{(b_j-a_j)^{\alpha_j} \sin\pi\beta_j} \frac{1}{(z-b_j)^{\beta_j}} + G_{oj}(z) , \quad (j = 1,2)$$

(27.b)

where $\alpha_j = \beta_j = 1/2$. We now note that in equations (7) the left hand sides are the expressions for $[(1+\kappa_j)/2\mu_j]\tau_{j\theta\theta}$ for $(0 < r < a_j, \quad b_j < r < \infty)$ as well as for $(a_j < r < b_j)$ as indicated in (7). Thus, noting that the terms involving k_{ij} are bounded, outside the cuts $G_j(r)$ directly gives $\tau_{j\theta\theta}$ on the real axes, from which the stress intensity factors may be readily evaluated. For example, at the end $r = b_1$ from (27) we obtain

$$k(b_1) = \lim_{r \to b_1} \sqrt{2(r-b_1)} \tau_{1\theta\theta}(r,\pi) = \frac{2\mu_1}{1+\kappa_1} \lim_{r \to b_1} \sqrt{2(r-b_1)} G_1(r)$$

$$= - \frac{2\mu_1}{1+\kappa_1} \frac{\sqrt{2} g_1(b_1)}{\sqrt{b_1-a_1}} = - \frac{2\mu_1}{1+\kappa_1} \lim_{r \to b_1} \sqrt{2(b_1-r)} f_1(r) .$$

(28)

The other expressions given in (10) follow from (27) in a similar way which, in terms of g_j, become

$$k(b_2) = - \frac{2\mu_2}{1+\kappa_2} \frac{\sqrt{2} g_2(b_2)}{\sqrt{b_2-a_2}} ,$$

$$k(a_j) = \frac{2\mu_j}{1+\kappa_j} \frac{\sqrt{2} g_j(a_j)}{\sqrt{b_j-a_j}} , \quad (j = 1,2)$$

(29.a,b)

Consider now the second typical case, that is, a crack tip

terminating at the interface (e.g., $b_1 > a_1 = 0$, $b_2 > a_2 > 0$).
In this case the stress intensity factor will be defined as

$$k_1(0) = \lim_{r \to 0} \sqrt{2} \, r^{\alpha_1} \tau_{2\theta\theta}(r,0) \; , \tag{30}$$

where α_1 is obtained from (26.a) and $\tau_{2\theta\theta}(r,0)$ is given by
(7.b). Since $b_2 > a_2 > 0$, at $r = 0$ the first and third terms
on the left hand side of (7.b) will be bounded, and only the
second term will contribute to the stress singularity. In the
second term $a_1 = 0$, $0 < s < b_1$, $r > 0$, and $k_{21}(r,s)$ is given
by (8.d). Hence, the asymptotic value of the integral for small
r may be obtained from (8.d) and (18.a,b). Using the definition
(30) and ignoring the bounded terms, we then obtain

$$k_1(0) = \lim_{r \to 0} \sqrt{2} \, r^{\alpha_1} \frac{2\mu_2}{1+\kappa_2} \frac{1}{\pi} \int_0^{b_1} \sum_{k=1}^{2} \frac{d_{2k} r^{k-1}}{(s+r)^k} f_1(s) ds$$

$$= \sqrt{2} \, \mu_1^* \frac{g_1(0)}{\sqrt{b_1}} = \lim_{r \to 0} \sqrt{2} \, \mu_1^* r^{\alpha_1} f_1(r) \; ,$$

$$\mu_1^* = \frac{2\mu_1}{1+\kappa_1} \frac{d_{21} + \alpha d_{22}}{\sin \pi \alpha}$$

$$= m_1 \mu_1 \frac{(3-2\alpha_1)(1+m_1\kappa_1) - (1-2\alpha_1)(m_1+\kappa_2)}{(1+m_1\kappa_1)(m_1+\kappa_2)\sin \pi \alpha_1} \; . \tag{31.a,b}$$

Similarly for $b_1 > a_1 > 0$, $b_2 > a_2 = 0$, we find

$$k_2(0) = \lim_{r \to 0} \sqrt{2} \, r^{\alpha_2} \tau_{1\theta\theta}(r,\pi) = \sqrt{2} \, \mu_2^* \frac{g_2(0)}{\sqrt{b_2}} \tag{32}$$

where μ_2^* is obtained from (31.b) by interchanging the indices

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1 and 2 in μ_j, m_j, κ_j, ($j = 1,2$), and replacing α_1 by α_2 which, in turn, is obtained from (26.a) by substituting c_{2k} for c_{1k}, ($k = 1,2,3$). It should be noted that the bielastic constant μ_1^* is identical to μ^* given in [8] which was obtained by a different method. Needless to say, in this case for the remaining three end points the expressions given by (28) and (29) will remain valid.

For the third case in which the crack goes through the interface, i.e., for $a_1 = a_2 = 0$, $b_1 > 0$, $b_2 > 0$, we define the normal and the shear components of the stress intensity factor as follows*:

$$k_\theta = \lim_{r \to 0} r^\alpha \tau_{j\theta\theta}(r,\pi/2) , \qquad k_r = \lim_{r \to 0} r^\alpha \tau_{jr\theta}(r,\pi/2) ,$$

$$(j = 1 \text{ or } 2). \qquad (33)$$

Thus, to obtain k_θ and k_r the expressions for the interface stresses are needed. These expressions may be obtained as definite integrals with the density functions f_j, ($j = 1,2$),

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*In this case, the asymptotic expressions of the stresses for small values of r and for $0 < \theta < \pi$ are of the form

$$\tau_{ij}(r,\theta) = \frac{k(0)}{r^\alpha} h_{ij}(\theta) + O(r^{\alpha^1}), \quad (i,j = r,\theta; \text{Re}(\alpha) > \text{Re}(-\alpha^1)),$$

where $h_{\theta\theta}$ and $h_{r\theta}$ are bounded and continuous in $[0,\pi]$, including at $\theta = \pi/2$, and the bounded function h_{rr} has a discontinuity at $\theta = \pi/2$. Here, because of the physical importance of the interface and simplicity of the calculations, the stress intensity factors are defined in terms of the normal and shear stresses along the interface, i.e., $k_\theta = k(0)h_{\theta\theta}(\pi/2)$, $k_r = k(0)h_{r\theta}(\pi/2)$. As the results of this section show, within a multiplicative constant, $k(0)$ is nothing but the constant $g_1(0)$ (which is related to $g_2(0)$ through (22)).

and the appropriate kernels which are evaluated from the original Mellin Transform solution (see [8]). After some manipulations, the components of the stress vector at the interface are found to be

$$\tau_{1\theta\theta}(r,\tfrac{\pi}{2}) = \tau_{2\theta\theta}(r,\tfrac{\pi}{2}) = \frac{1}{\pi} \int_{0}^{b_1} [\; \frac{A_1 s}{s^2+r^2} + \frac{A_2 s(3r^2-s^2)}{(s^2+r^2)^2} \;]f_1(s)ds$$

$$+ \frac{1}{\pi} \int_{0}^{b_2} [\; \frac{A_3 s}{s^2+r^2} + \frac{A_4 s(3r^2-s^2)}{(s^2+r^2)^2} \;]f_2(s)ds \; ,$$

$$\tau_{1r\theta}(r,\tfrac{\pi}{2}) = \tau_{2r\theta}(r,\tfrac{\pi}{2}) = \frac{1}{\pi} \int_{0}^{b_1} [\; \frac{B_1 r}{s^2+r^2} + \frac{B_2 r(r^2-3s^2)}{(s^2+r^2)^2} \;]f_1(s)ds$$

$$+ \frac{1}{\pi} \int_{0}^{b_2} [\; \frac{B_3 r}{s^2+r^2} + \frac{B_4 r(r^2-3s^2)}{(s^2+r^2)^2} \;]f_2(s)ds \; , \qquad (34.a,b)$$

$$A_1 = \frac{\mu_2}{m_1+\kappa_2} \; , \quad A_2 = -\frac{\mu_2}{1+m_1\kappa_1} \; , \quad A_3 = -\frac{\mu_1}{m_2+\kappa_1} \; , \quad A_4 = \frac{\mu_1}{1+m_2\kappa_2} \; ,$$

$$B_1 = \frac{\mu_2(1+m_1\kappa_2)}{(1+m_1\kappa_1)(m_1+\kappa_2)} \; , \qquad B_2 = \frac{\mu_2(m_1+\kappa_2)}{(1+m_1\kappa_1)(m_1+\kappa_2)} \; ,$$

$$B_3 = \frac{\mu_1(1+m_2\kappa_1)}{(1+m_2\kappa_2)(m_2+\kappa_1)} \; , \qquad B_4 = \frac{\mu_1(m_2+\kappa_1)}{(1+m_2\kappa_2)(m_2+\kappa_1)} \; . \qquad (35)$$

The kernels in (34) can easily be expressed as the sum of terms containing

$$\frac{1}{s+ir} \; , \quad \frac{1}{s-ir} \; , \quad \frac{ir}{(s+ir)^2} \; , \quad \frac{ir}{(s-ir)^2} \; . \qquad (36)$$

It is clear that since $r > 0$, at $z = \pm ir$ the functions $G_j(z)$

defined by (13) will be holomorphic and, following the same procedure which led to (18) in the previous section, the integrals in (34) may be evaluated in terms of $G(z)$ and its derivative (at $z = \pm ir$). For example,

$$\frac{1}{\pi} \int_0^{b_1} \frac{s f_1(s) ds}{s^2 + r^2} = \frac{1}{2\pi} \int_0^{b_1} \left(\frac{1}{s+ir} + \frac{1}{s-ir} \right) f_1(s) ds$$

$$= \frac{1}{2} \left(e^{\pi i \alpha/2} + e^{-\pi i \alpha/2} \right) \frac{g_1(0)}{b_1^{\beta_1} \sin \pi \alpha} \frac{1}{r^\alpha} + F_1(r)$$

$$= \frac{g_1(0)}{2 b_1^{\beta_1} \sin \frac{\pi \alpha}{2}} \frac{1}{r^\alpha} + F_1(r) ,$$

$$\frac{1}{\pi} \int_0^{b_1} ir \left(\frac{1}{(s+ir)^2} - \frac{1}{(s-ir)^2} \right) f_1(s) ds$$

$$= \frac{\alpha g_1(0)}{b_1^{\beta_1} \sin \frac{\pi \alpha}{2}} \frac{1}{r^\alpha} + F_2(r) , \qquad (37.a,b)$$

where F_1 and F_2 are bounded functions. Carrying out the necessary manipulations, from (33-37) we obtain

$$k_\theta = \frac{1}{2 \sin \frac{\pi \alpha}{2}} \left\{ \frac{g_1(0)}{\sqrt{b_1}} [A_1 - (1-2\alpha) A_2] - \frac{g_2(0)}{\sqrt{b_2}} [A_3 - (1-2\alpha) A_4] \right\},$$

$$k_r = \frac{1}{2 \cos \frac{\pi \alpha}{2}} \left\{ \frac{g_1(0)}{\sqrt{b_1}} [B_1 - (1-2\alpha) B_2] - \frac{g_2(0)}{\sqrt{b_2}} [B_3 - (1-2\alpha) B_4] \right\},$$

$$(38.a,b)$$

where the constants $g_1(0)$ and $g_2(0)$ are linearly related through (22) and the constants A_i and B_i ($i = 1,..,4$) are given by (35).

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5. CRACK SURFACE DISPLACEMENTS

The crack surface displacements $u_{1\theta}$ and $u_{2\theta}$ are related to the density functions f_1 and f_2 through (5). Hence, noting that $u_{j\theta} = 0$ for $r > b_j$, $(j = 1,2)$, after evaluating f_1 and f_2 the displacements may be obtained from

$$u_{1\theta}(r,\pi-0) = \frac{1}{2} \int_r^{b_1} f_1(s)ds \ ,$$

$$u_{2\theta}(r,+0) = -\frac{1}{2} \int_r^{b_2} f_2(s)ds \ . \qquad (39.a,b)$$

6. SOLUTION OF THE INTEGRAL EQUATIONS AND NUMERICAL RESULTS

In deriving the integral equations (7), instead of the conditions (2.c) and (4.c) the conditions (6.a,b) are used; that is, the problem is formulated on the assumption that outside the cuts (a_j,b_j), the displacement derivatives f_j rather than the displacements $u_{j\theta}$, $(j = 1,2)$, are zero. Thus, in the case of nonintersecting cracks (i.e., for $a_1 > 0$, $a_2 > 0$, or $a_1 = 0$, $a_2 > 0$), the integral equations (7) must be solved under the single-valuedness conditions (6.c,d). Referring to [10], since the general solution of the system of singular integral equations [7] will contain two arbitrary constants, the two additional conditions (6.c,d) are needed for a unique solution. On the other hand for the intersecting cracks, that is, for $a_1 = 0 = a_2$, there is only one single-valuedness condition, namely, $-u_{1\theta}(0,\pi-0) = u_{2\theta}(0,+0)$, or

$$\int_0^{b_1} f_1(s)ds = \int_0^{b_2} f_2(s)ds \; . \tag{40}$$

The second condition which is necessary for a unique solution is provided by (22) relating $g_1(0)$ and $g_2(0)$, or the functions f_1 and f_2 at $r = 0$.

For $a_1 = a_2 = 0$, even though the special cases of the resulting system of singular integral equations with generalized Cauchy kernels (7) have been considered in [13] and [14], there is no known method which can be used to regularize the integral equations of this type. To solve these equations we first normalize the intervals (a_j, b_j), $(j = 1,2)$, through the following change in variables:

$$\xi_1 = \frac{2s - (b_1 + a_1)}{b_1 - a_1} \; , \qquad \xi_2 = \frac{2s - (b_2 + a_2)}{b_2 - a_2} \; ,$$

$$(a_j < s < b_j, \quad -1 < \xi_j < 1, \quad j = 1,2). \tag{41.a,b}$$

Next we let

$$f_j(s) = \frac{g_j(s)}{(s-a)^{\alpha_j}(b_j - s)^{\beta_j}} = \frac{\phi_j(\xi_j)}{(1+\xi_j)^{\alpha_j}(1-\xi_j)^{\beta_j}} \; , \tag{42}$$

giving

$$g_j(s) = \left(\frac{b_j - a_j}{2} \right)^{\alpha_j + \beta_j} \phi_j(\xi_j) \; . \tag{43}$$

Also defining

$$t_j = \frac{2r - (b_j + a_j)}{b_j - a_j} \ , \qquad P_j(t_j) = p_j(r) \ ,$$

$$(a_j < r < b_j, \quad -1 < t_j < 1, \quad j = 1,2), \tag{44}$$

and suppressing the subscripts in ξ_j and t_j, the system of integral equations (7) may be expressed as

$$\frac{1}{\pi} \int_{-1}^{1} \sum_{1}^{2} \left[\frac{\delta_{ij}}{\xi - t} + K_{ij}(t,\xi) \right] \phi_j(\xi) w_j(\xi) d\xi = - \frac{1 + \kappa_i}{2\mu_i} P_i(t) \ ,$$

$$(-1 < t < 1), \tag{45}$$

where

$$w_j(\xi) = (1+\xi)^{-\alpha_j}(1-\xi)^{-\beta_j} \ , \qquad (j = 1,2) \tag{46}$$

are the fundamental functions of the system, and the kernels K_{ij} may be obtained from k_{ij} by means of appropriate transformations. Now, observing that the fundamental functions $w_j(\xi)$ of the system of integral equations are weights of the Jacobi polynomials $P_n^{(-\beta_j, -\alpha_j)}(\xi)$, the equations may be solved numerically for the unknown functions ϕ_j by using the Gauss-Jacobi integration formulas described in [15]. After evaluating ϕ_j, the values of g_j at the end points, which are needed to calculate the stress intensity factors, may be obtained from (43) in terms of $\phi_j(\pm 1)$.

For the material pairs Aluminum-Epoxy some of the calculated results for the plane strain case are shown in Tables 3-5 and Figures 4-13. For $b_1 \to 0$ or $b_2 \to 0$ (i.e., for the case of a single crack terminating at the interface), the results

of [8] are obtained as the limiting values. In the numerical
examples given in this paper it is assumed that the composite
medium is loaded parallel to the interface and away from the
crack region in such a way that the stresses $p_1(r)$ and $p_2(r)$
are approximately constant. Thus, in the singular part of the
problem considered in this paper, the crack surface tractions
will be constant pressures $-p_1$ and $-p_2$ which are related
through (1).

In the example shown in Table 3 and Figures 4 and 5, the
material 1 is aluminum, material 2 is epoxy, one of the crack
tips b_1 is fixed at a unit distance from the interface, and
b_2 varies between 0 and $2b_1$. In Figure 4 as well as 6 and 8
the constants k_1 and k_2 refer to $k(b_1)$ and $k(b_2)$, respectively.

Table 3.   The Stress Intensity Factors for Through Crack,
$a_1 = a_2 = 0$. Material 1: Aluminum, Material 2:
Epoxy ($\nu_1 = 0.3$, $\nu_2 = 0.35$, $E_1/E_2 = 22.447$);
$b_1 = 1 =$ constant, $\ell = (b_1+b_2)/2$, b_2 variable,
$(\beta_1 = \beta_2 = 0.5$, $\alpha = 0.273692)$.

b_2/b_1	$k(b_1)/p_1\sqrt{\ell}$	$k(b_2)/p_2\sqrt{\ell}$	$k_\theta/p_1\ell^\alpha$	$k_r/p_1\ell^\alpha$
0.00	1.3552*	$\to \infty$	$\to -\infty$	$\to \infty$
0.05	1.40374	4.36065	-0.22477	0.08215
0.25	1.33240	2.13917	-0.08113	0.03184
0.50	1.23789	1.55615	-0.05906	0.02158
0.75	1.15887	1.31166	-0.04636	0.01694
1.00	1.09312	1.17874	-0.03839	0.01403
1.25	1.03767	1.09693	-0.03266	0.01194
1.50	0.99020	1.04261	-0.02822	0.01032
1.75	0.94902	1.00462	-0.02462	0.00900
2.00	0.91286	0.97701	-0.02160	0.00789

*Taken from Reference [8].

Table 4. The Stress Intensity Factors for Through Crack, $a_1 = a_2 = 0$. Material 1: Aluminum, Material 2: Epoxy; $2\ell = b_1+b_2 = 2 = $ constant, $c = (b_2-b_1)/2$ variable, ($\beta_1 = \beta_2 = 0.5$, $\alpha = 0.273692$).

c/ℓ	$k(b_1)/p_1\sqrt{\ell}$	$k(b_2)/p_2\sqrt{\ell}$	$k_\theta/p_1\ell^\alpha$	$k_r/p_1\ell^\alpha$
-1.00	1.3552*	$\to \infty$	$\to -\infty$	$\to \infty$
-0.95	1.40333	5.69903	-0.34124	0.12472
-0.75	1.37532	2.76781	-0.11960	0.04371
-0.50	1.29929	1.87137	-0.07423	0.02713
-0.25	1.20448	1.43765	-0.05310	0.01941
0.00	1.09312	1.17874	-0.03839	0.01403
0.25	0.96213	1.01590	-0.02575	0.00941
0.50	0.80281	0.91834	-0.01293	0.00473
0.75	0.59068	0.87340	0.00313	-0.00114
0.95	0.29854	0.87519	0.02903	-0.01061
1.00	$\to 0$	0.8827*	$\to \infty$	$\to -\infty$

*Taken from Reference [8].

Table 5. The Stress Intensity Factors for Through Crack, $a_1 = a_2 = 0$. Material 1: Epoxy, Material 2: Aluminum; $b_1 = 1 = $ constant, $\ell = (b_1+b_2)/2$, b_2 variable, ($\alpha = 0.273692$, $\beta_1 = \beta_2 = 0.5$).

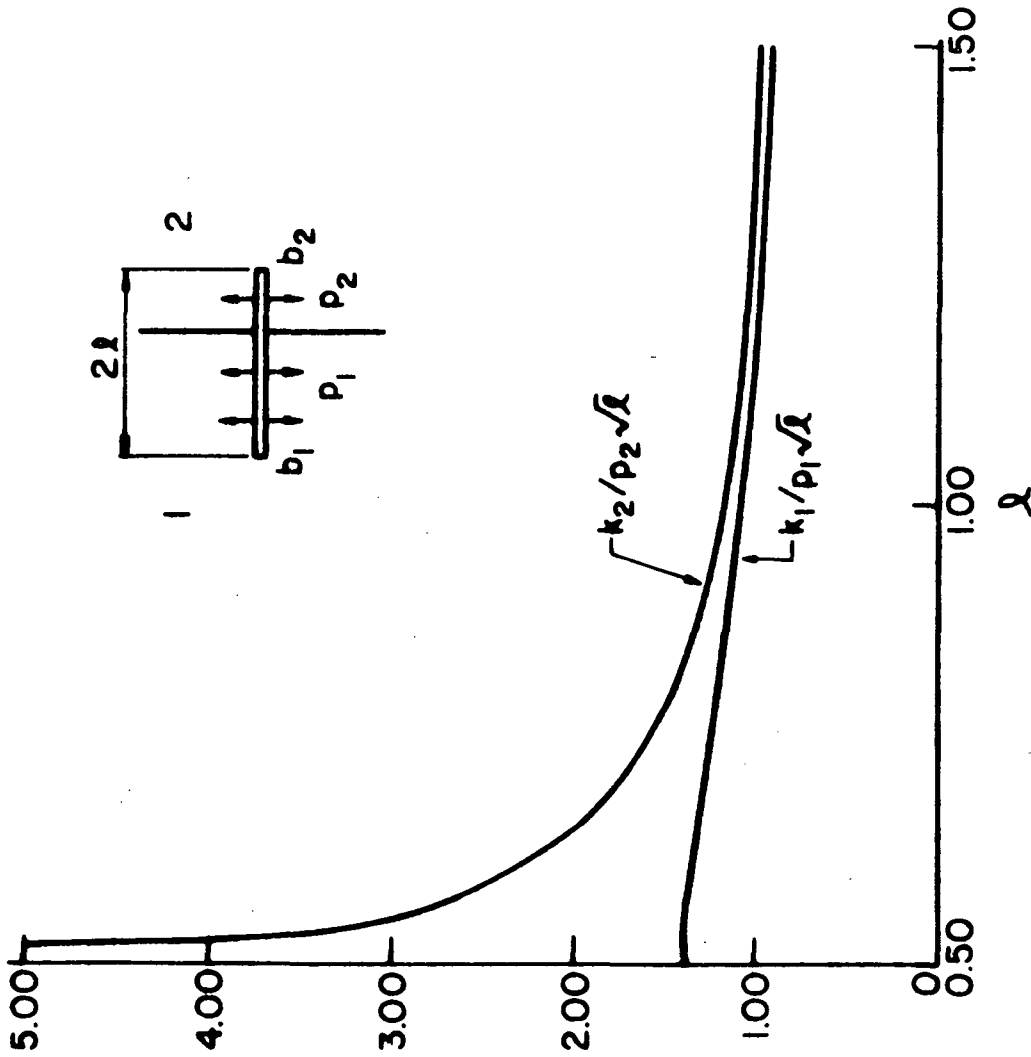| b_2/b_1	$k(b_1)/p_1\sqrt{\ell}$	$k(b_2)/p_2\sqrt{\ell}$	$k_\theta/p_1\ell^\alpha$	$k_r/p_1\ell^\alpha$
0.00	0.8827*	$\to 0$	$\to \infty$	$\to \infty$
0.05	0.87141	0.39001	0.44554	0.16283
0.25	0.89435	0.72689	-0.16129	-0.05895
0.50	0.97701	0.91286	-0.48485	-0.17720
0.75	1.07647	1.02106	-0.69736	-0.25487
1.00	1.17874	1.09312	-0.86170	-0.31493
1.25	1.27901	1.14476	-0.99895	-0.36509
1.50	1.37559	1.18357	-1.11885	-0.40891
1.75	1.46797	1.21377	-1.22669	-0.44832
2.00	1.55615	1.23789	-1.32567	-0.48449

*Taken from Reference [8].

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Figure 4.   Stress intensity factors for a crack going through the interface (materials: 1 aluminum, 2 epoxy; $b_1 = i = $ constant, $\ell = (b_1+b_2)/2$ variable).
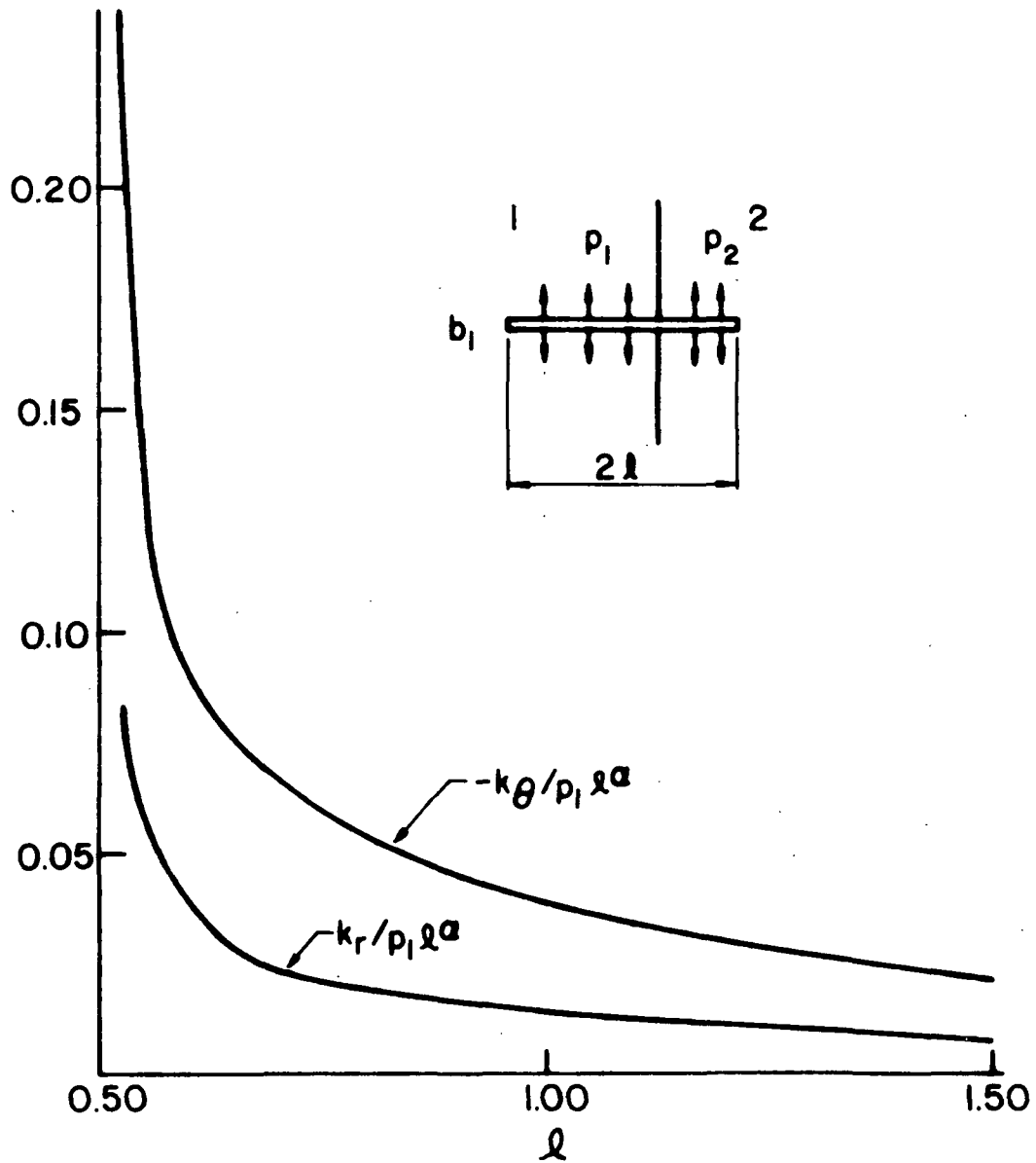
Figure 5. Stress intensity factors for normal and shear stresses on the interface for the through crack (materials: 1 aluminum, 2 epoxy; b_1 = constant).

Table 4 and Figures 6 and 7 show the results for the same material combination. In this second example it is assumed that the total length $2\ell = b_1 + b_2$ of the crack is constant, and the distance $c = (b_2 - b_1)/2$ from its center to the interface is variable. The stress intensity factors for the case of epoxy-aluminum are shown in Table 5 and Figures 8 and 9.

The tables also show the limiting values of the stress intensity factors for $b_2 \rightarrow 0$ and $c \rightarrow \pm 1$. These limiting values are obtained by observing that in the case of a through crack (i.e., for $b_j > a_j = 0$, ($j = 1,2$)), for the material pair aluminum-epoxy (21) gives the power of the stress singularity as $\alpha = 0.2737$. On the other hand, for the limiting case of a crack tip terminating at the interface, i.e., for $b_1 = 0$ or $b_2 = 0$, and for the material pair under consideration, from (26.a) the power α_1 of the stress singularity is calculated to be:

$$\alpha_1 = 0.8258 \ , \quad \text{crack in aluminum,}$$

$$\alpha_1 = 0.3381 \ , \quad \text{crack in epoxy.} \qquad (47.a,b)$$

Thus, for example, in the example shown in Table 3 and Figures 4 and 5, as $b_2 \rightarrow 0$ the powers of singularity become $\alpha \rightarrow \alpha_1$, $\beta_2 \rightarrow \alpha_1$. Therefore, for this limiting case, since $\alpha_1 = 0.8258 > \alpha = 0.2737$ and $\alpha_1 > \beta_2 = 0.5$, at $r = 0$ the stresses would have a stronger singularity and consequently the stress intensity factors k_θ and k_r, which are defined on the basis of $r^{-\alpha}$, and $k(b_2)$, which is defined on the basis of $r^{-0.5}$, would all go to infinity. Similar arguments apply to the limiting values
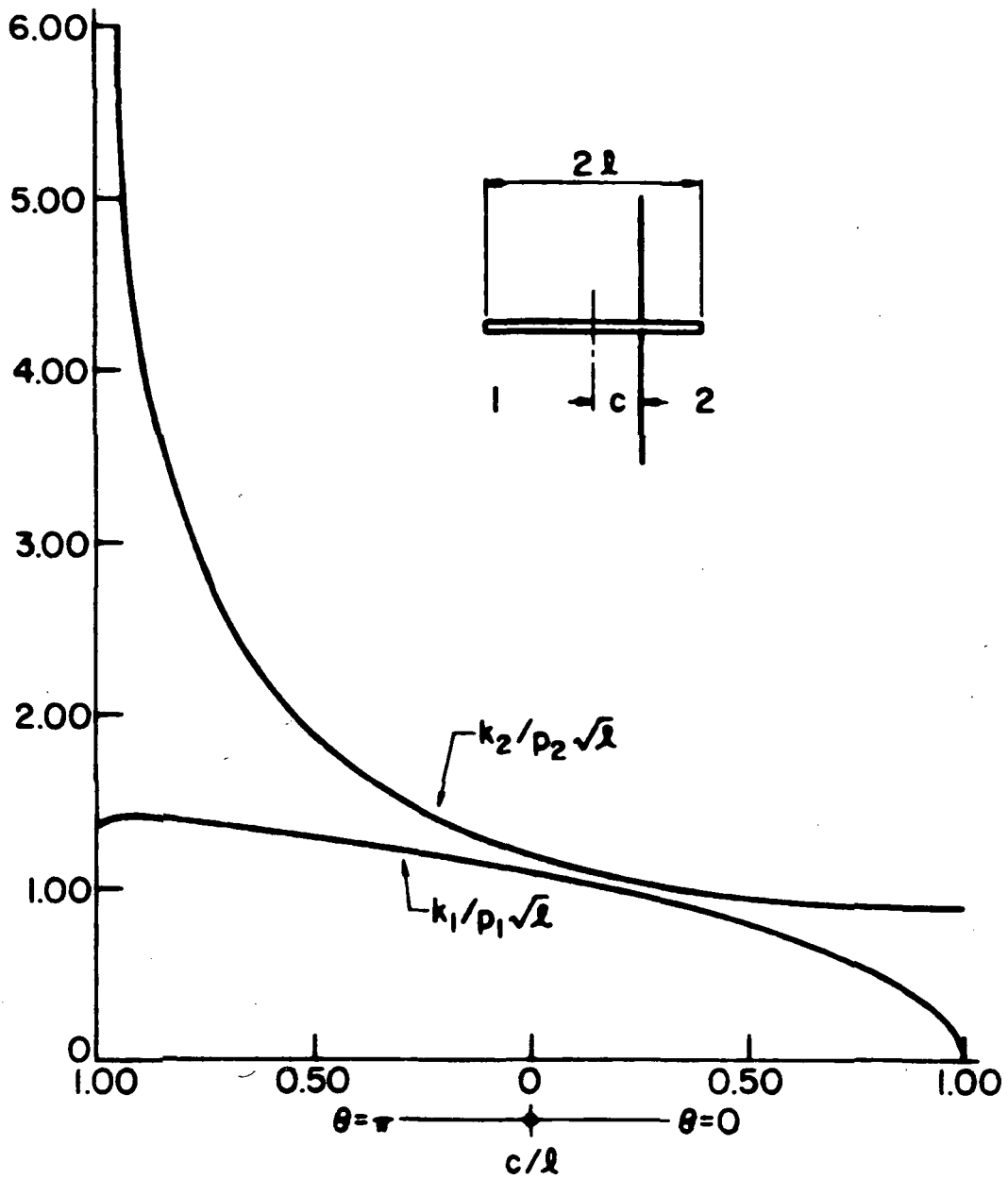
Figure 6.   Stress intensity factors for through crack
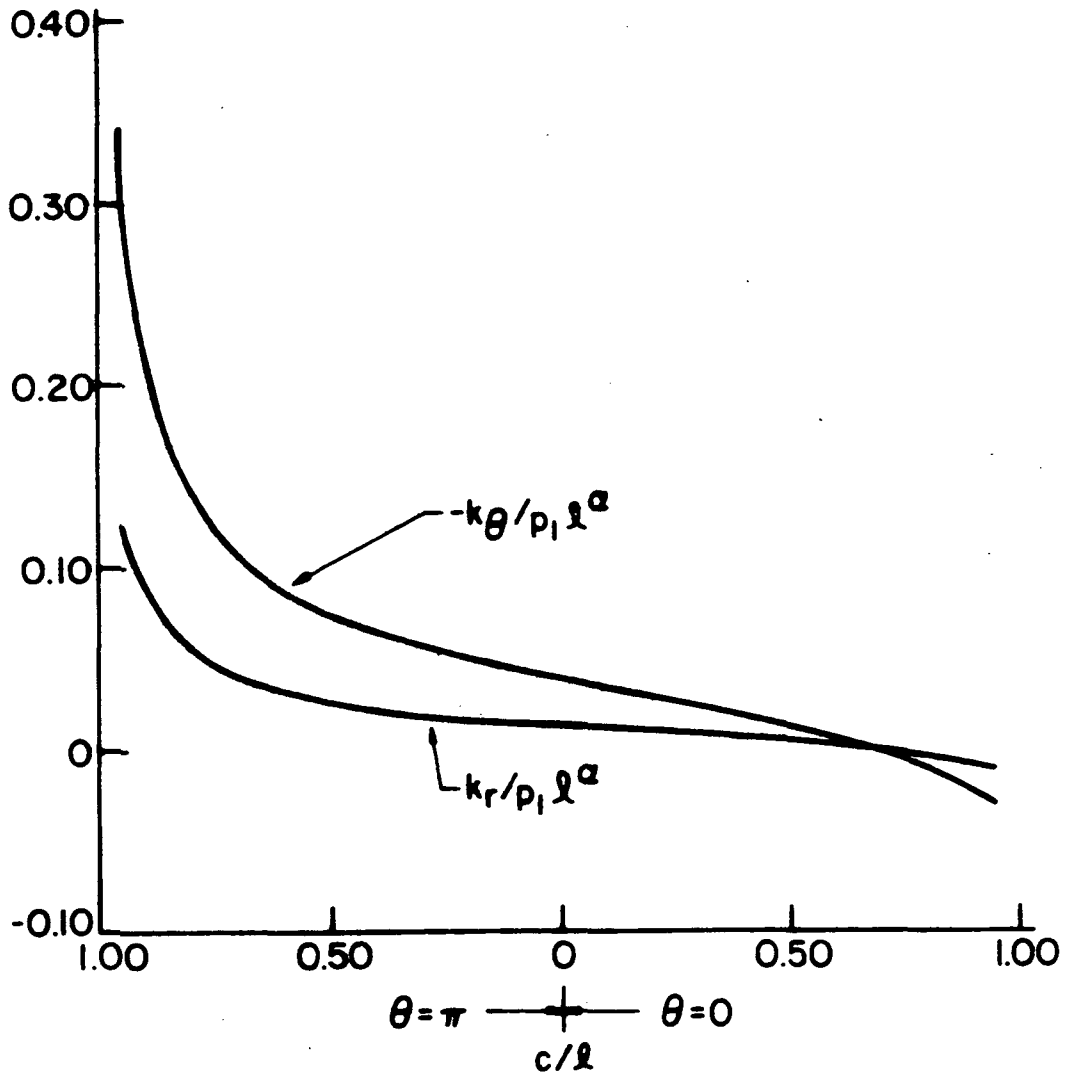(materials: 1 aluminum, 2 epoxy; $2\ell = 2 =$
constant, c variable).

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Figure 7. Stress intensity factors for normal and shear stresses on the interface for through crack (materials: 1 aluminum, 2 epoxy; $2\ell = 2 =$ constant, c variable).
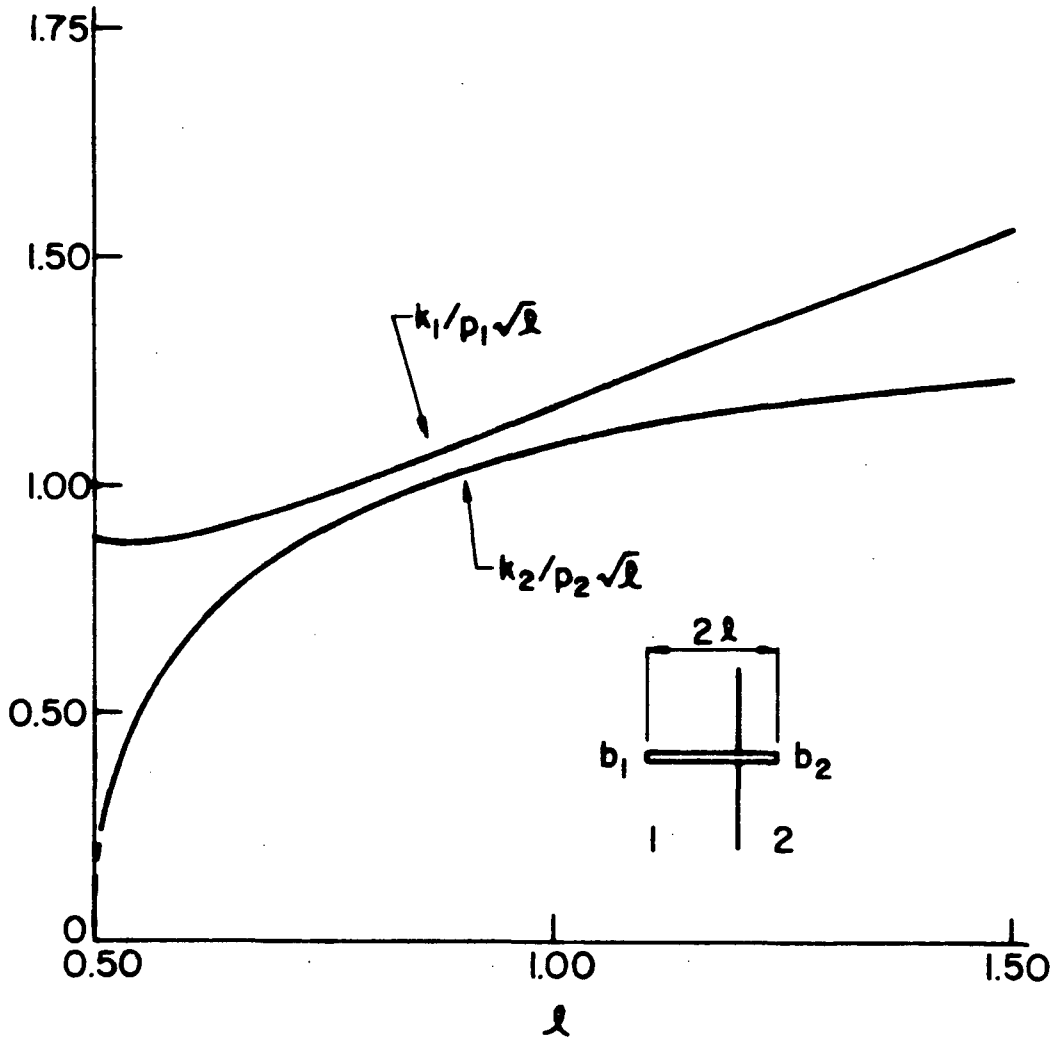
Figure 8. Stress intensity factors for through crack (materials: 1 epoxy, 2 aluminum; $b_1 = 1 =$ constant, ℓ variable).
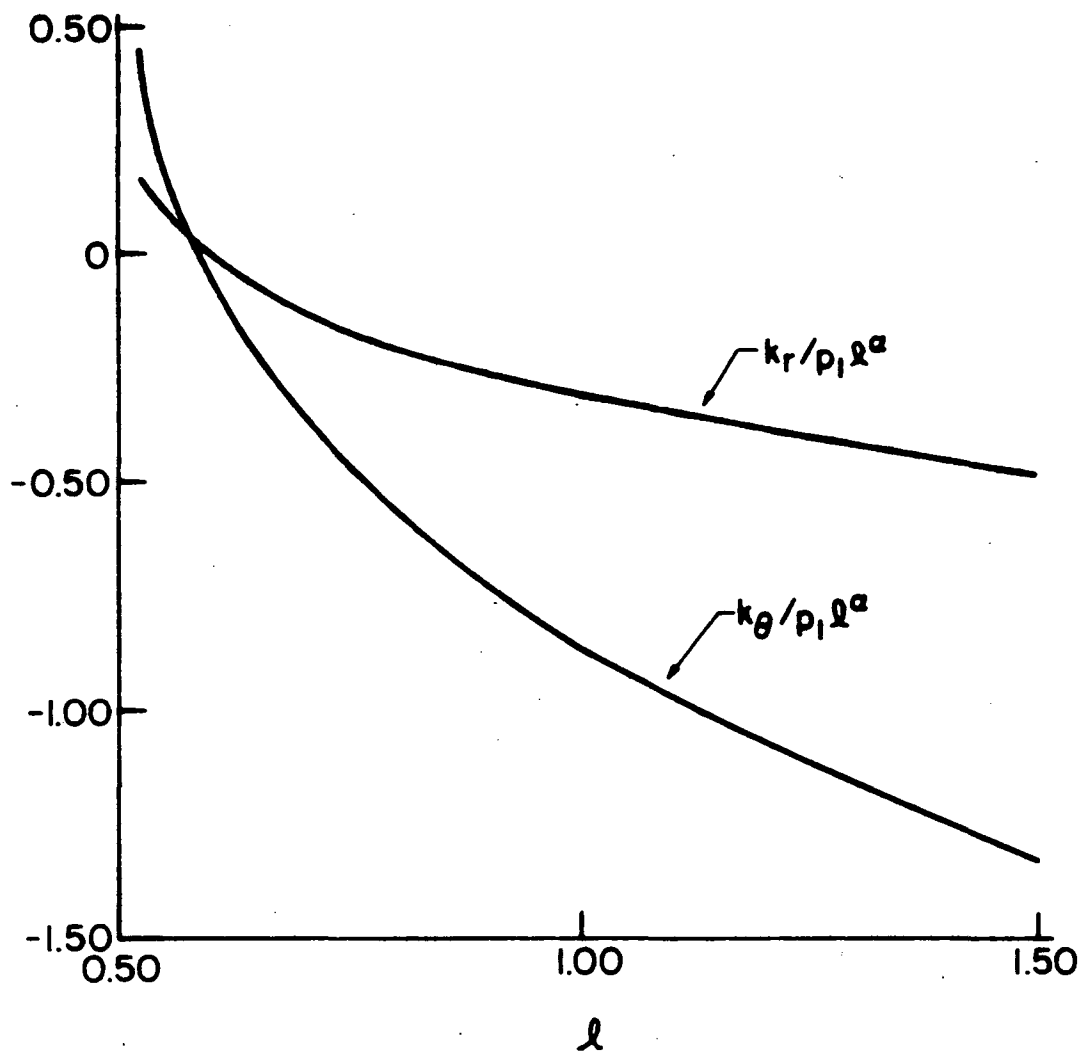
Figure 9.   Stress intensity factors for normal and shear stresses on the interface for through crack (materials: 1 epoxy, 2 aluminum; $b_1 = 1 =$ constant, ℓ variable).

shown in Tables 4 and 5.

From Tables 3-5 and Figures 5, 7, 9 it is seen that the normal component of the stress intensity factor, k_θ, (hence, the normal component of the bonding stress along the interface in the neighborhood of $r = 0$) may be positive or negative depending on the relative penetration depths b_1 and b_2 of the crack into the adjacent materials. As the ratio b_2/b_1 increases, k_θ increases if $\mu_1 > \mu_2$ (Figures 5, 7), and decreases if $\mu_1 < \mu_2$ (Figure 9).

Figure 10 shows a sample distribution of the density functions f_1 and f_2 defined by

$$f_1(s) = F_1(\xi_1) , \qquad (0 < s < b_1, \quad -1 < \xi_1 < 1),$$

$$f_2(s) = F_2(\xi_2) , \qquad (0 < s < b_2, \quad -1 < \xi_2 < 1). \quad (48.a,b)$$

The calculated values shown in the figure correspond to the example given in Table 3 with $b_2/b_1 = 0.05$ (i.e., material 1 is aluminum, material 2 is epoxy). The important feature of the distribution of the density functions f_1 and f_2 is that at $r = 0$, because of the common $r^{-\alpha}$ type singularity, they become infinite, $-f_1$ and f_2 having the same sign (i.e., at $r = 0$ the derivative of the crack surface displacement is discontinuous and unbounded).

Sample results for the crack surface displacement calculated from (39) are shown in Figures 11-13, where the variables are again the normalized quantities ξ_1 and ξ_2 defined by (41).
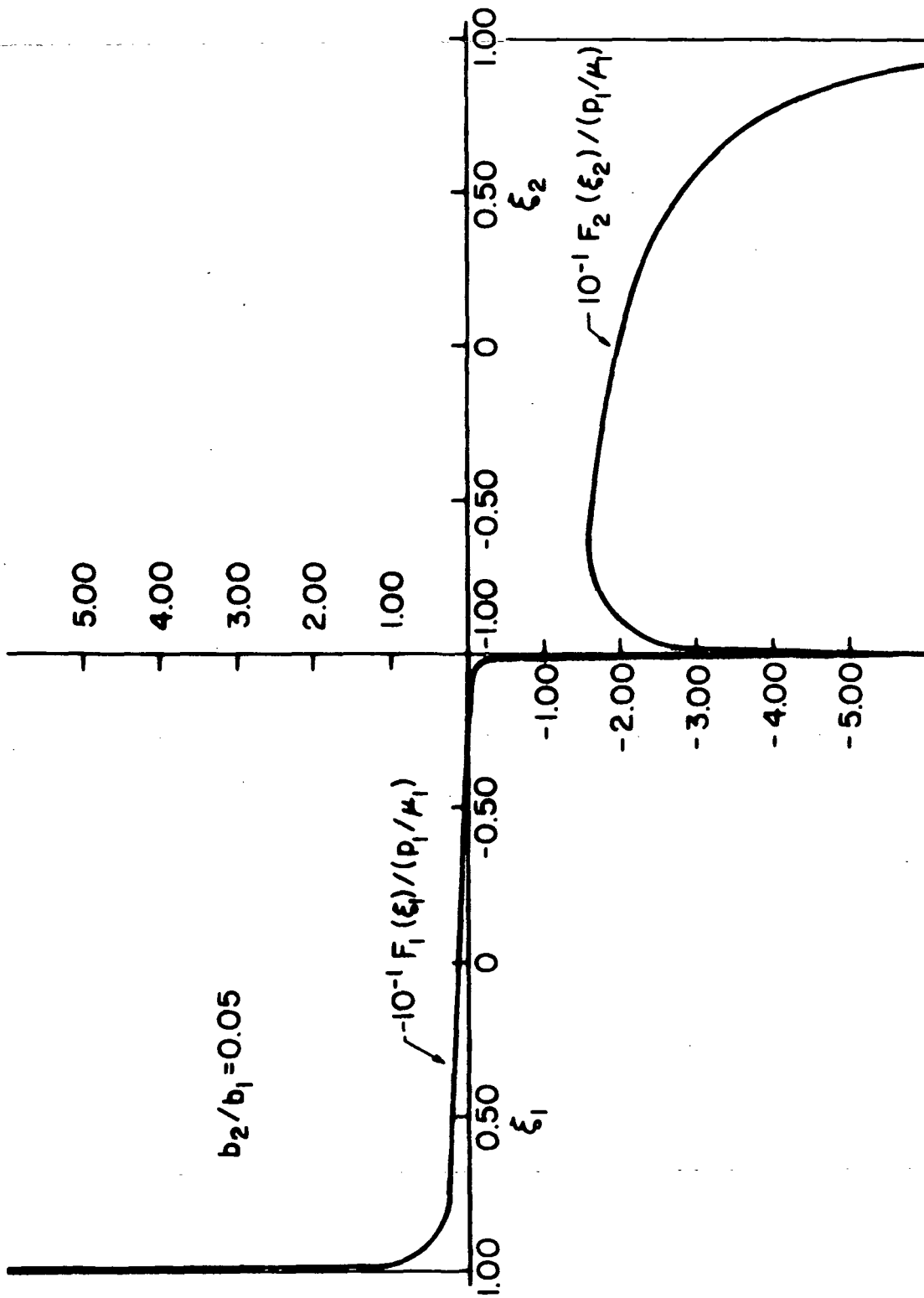
Figure 10. Distribution of the density functions $f_1(r) = F_1(\xi_1)$, $f_2(r) = F_2(\xi_2)$ (materials: 1 aluminum, 2 epoxy; $b_2/b_1 = 0.05$; $\xi_j = (2r-b_j-a_j)/(b_j-a_j)$, $j = 1,2$).

The results shown in Figures 11, 12, and 13 correspond to the examples given in the Tables 3, 4, and 5, respectively. In Figures 11 and 13 the length unit for displacements is b_1 and in Figure 12 it is $\ell = (b_1+b_2)/2$. A careful examination of the crack surface displacements in the adjacent materials around $r \approx 0$ would indicate that the crack opening in the medium with the higher modulus is greater than that with the lower modulus, both having an infinite slope.

The results given in this paper may be used in the analysis of fracture propagation in the composite medium by using any of the standard fracture criteria if the crack extension takes place at the singular point $r = b_1$ or $r = b_2$ where the singularity is of the form $(r-b_j)^{-1/2}$. However, in studying a possible fracture initiation at $r = 0$ (where the singularity is of the form $r^{-\alpha}$), i.e., initiation of a debonding crack along the interface or a cleavage crack in one of the adjacent materials, a criterion similar to that outlined in [8] may have to be used.

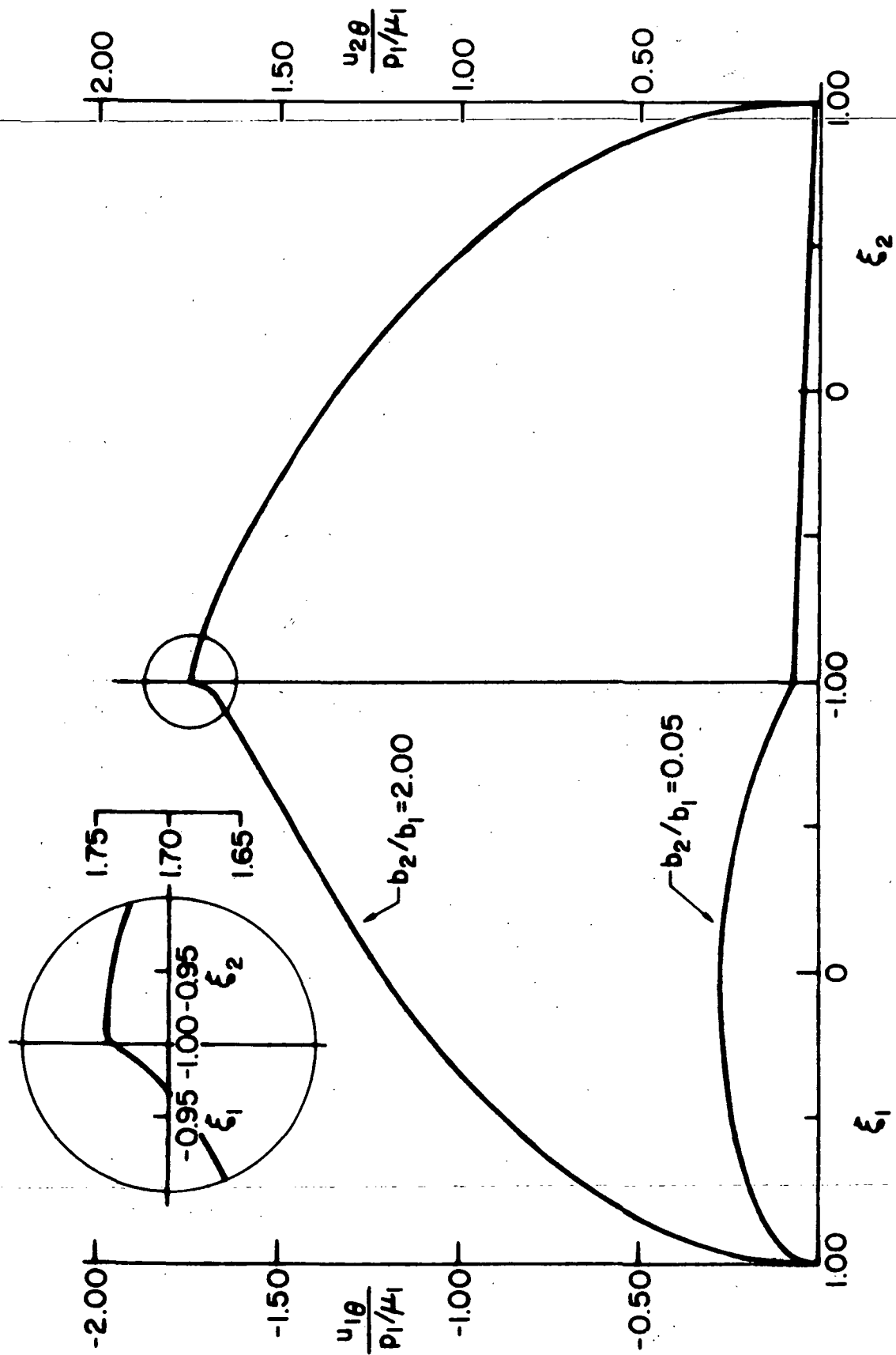**Page Intentionally Left Blank**

Figure 13. Crack surface displacements for through crack (materials: 1 epoxy, 2 aluminum; $b_1 = 1 = $ constant).

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REFERENCES

1. F. Erdogan and G. D. Gupta, "Stress Analysis of Multi-Layered Composites with a Flaw", Int. J. Solids, Structures, Vol. 7, p. 39 (1971).

2. F. Erdogan and G. D. Gupta, "Layered Composites with an Interface Flaw", Int. J. Solids, Structures, Vol. 7, p. 1089 (1971).

3. K. Arin and F. Erdogan, "Penny-Shaped Crack in an Elastic Layer Bonded to Dissimilar Half Spaces", Int. J. Engng. Sci., Vol. 9, p. 213 (1971).

4. F. Erdogan and K. Arin, "Penny-Shaped Interface Crack Between an Elastic Layer and a Half Space", Int. J. Engng. Sci. Vol. 10, p. 115 (1972).

5. F. Erdogan and T. Ozbek, "Stresses in Fiber-Reinforced Composites with Imperfect Bonding", J. Appl. Mech., Vol. 36, p. 865, Trans. ASME, Series E (1969).

6. F. Erdogan and G. D. Gupta, "Stresses Near a Flat Inclusion in Bonded Dissimilar Materials", Int. J. Solids, Structures, Vol. 8, p. 533 (1972).

7. F. Erdogan, "Fracture Problems in Composite Materials", J. Engineering Fracture Mechanics (to appear, 1972).

8. T. S. Cook and F. Erdogan, "Stresses in Bonded Materials with a Crack Perpendicular to the Interface", Int. J. Engng. Sci. (to appear, 1972).

9. F. Erdogan and G. D. Gupta, "On the Numerical Solution of Singular Integral Equations", Quarterly of Applied Mathematics, p. 525 (1972).

10. N. I. Muskhelishvili, Singular Integral Equations, P. Noordhoff, Groningen, The Netherlands (1953).

11. D. B. Bogy, "Edge-Bonded Dissimilar Orthogonal Elastic Wedges Under Normal and Shear Loading", J. Appl. Mech., Vol. 35, Trans. ASME, Series E, p. 460 (1968).

12. V. L. Hein and F. Erdogan, "Stress Singularities in a Two-Material Wedge", Int. J. Fracture Mechanics, Vol. 7, p. 317 (1971).

13. H. F. Bueckner, "On a Class of Singular Integral Equations", J. Math. Analysis and Applications, Vol. 14, p. 392 (1966).

14. G. T. Bierman, "A Particular Class of Singular Integral Equations", SIAM J. Appl. Math., Vol. 20, p. 99 (1971).

15. F. Erdogan, "Complex Function Technique", in Treatise on Continuum Physics, A. C. Eringen, Editor, Vol. II, Part III, Chapter 2, (1972).

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