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APPLICATION OF MODERN CONTROL THEORY TO THE DESIGN OF OPTIMUM AIRCRAFT CONTROLLERS

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INTRODUCTION

One area of research which offers a great deal of hope that an inexpensive, systematic, control system synthesis procedure applicable to a large class of problems can be developed is the application of optimal control theory to time-invariant linear systems with quadratic performance criteria. For this class of problems, a library of subroutines has been developed at Ames Research Center[1] which can quickly perform all of the required computations.

The synthesis procedure presented in this paper is based on the solution of the output regulator problem of linear optimal control theory for time-invariant systems.[2] By this technique, solution of the matrix Riccati equation leads to a constant linear feedback control law for an output regulator which will maintain a plant in a particular equilibrium condition in the presence of impulse disturbances.

The use of this technique as a basis for flight-control system design has many very appealing advantages over root locus and frequency domain techniques. Some of these advantages are that this technique is completely automated after selection of a few parameters in a performance criterion; can easily handle multiple input-output systems; and is generally applicable to any controllable and observable system whose equations can be put into linear, constant coefficient state space form. Further advantages are that the resulting feedback control law guarantees system stability;[3] requires no dynamic compensation; and is optimal in the sense that it minimizes a quadratic cost function.

On the other hand, some fundamental difficulties must be overcome. One of these is that the regulator tends to minimize the effects of any disturbance, including desired state changes attempted by the pilot. To avoid the necessity of disengaging the regulator while performing a maneuver, thereby losing the advantage of the augmented stability, a feedforward control law must be computed which can be superimposed upon the regulator and which will result in a desired maneuver on command. A second difficulty is that the control law resulting from the solution of the matrix Riccati equation requires measurement of the entire state vector. If this is not possible or desirable, then either the approach must be abandoned or an adequate estimate of the unmeasurable states must be constructed. This can be accomplished satisfactorily, and with a minimum order dynamic compensator, with a Luenberger observer.[4,5] However, construction of an observer for high order systems can be a very tedious exercise that has, in the past, been carried out largely by hand. Before observer theory can be included in a general systematic synthesis procedure, an algorithm must be found to construct observers that can be carried out in a straightforward manner by a digital computer.

This paper presents two simple algorithms that can be used in an automatic synthesis procedure for the design of maneuverable output regulators requiring only selected state variables for feedback.

The first algorithm is for the construction of optimal feedforward control laws that can be superimposed upon a Kalman output regulator and that will drive the output of a plant to a desired constant value on command.

The second algorithm is for the construction of optimal Luenberger observers that can be used to obtain feedback control laws for the output regulator requiring measurement of only part of the state vector. This algorithm constructs observers which have minimum response time under the constraint that the magnitude of the gains in the observer filter be less than some arbitrary limit.

The algorithms can be useful for the design of flight controllers that are required to track a step command input using only selected state variables for feedback, but where zero steady-state error (i.e., integral feedback) is not a requirement. In many practical cases, the pilot can be expected to perform this function.

Illustrative examples of the use of both algorithms for the design of flight controllers for the Augmentor Wing Jet STOL Research Aircraft are presented.

1. SOLUTION OF THE CONSTANT OUTPUT TRACKING PROBLEM

Let a controllable and observable plant be represented by

$$\dot{x}(t) = Fx(t) + Gu(t) \; ; \quad x(t_o) = x_o \tag{1.1}$$

$$y(t) = Hx(t) \tag{1.2}$$

where x is an n-dimensional column vector defined as the state; y, a p-dimensional column vector defined as the output; and u, an m-dimensional column vector defined as the control or input to the plant. The matrices F, G, and H are constant, rectangular, and have appropriate dimension.

We would like to derive a control law with the properties of an optimal output regulator and, in addition, with the capability of driving the output $y(t)$ to some desired constant value y_d on command. Generally, this can only be accomplished by driving the control vector $u(t)$ to some constant value u_r which is required to satisfy the steady-state solution of (1.1-2), i.e.,

$$Fx_r + Gu_r = 0 \tag{1.3}$$

$$y_d = Hx_r \tag{1.4}$$

where the equilibrium point represented by the pair $[x_r, u_r]$ is unknown. A control law that will simultaneously drive the output $y(t)$ *toward* y_d and the control $u(t)$ *toward* u_r for an *arbitrary* choice of both y_d and u_r may be obtained by minimizing a scalar cost functional of the form

$$V[x(t_o),t_o,T;u(\cdot)] = \frac{1}{2}\int_{t_o}^{T} [\|y(t) - y_d\|_Q^2 + \|u(t) - u_r\|_R^2]dt \tag{1.5}$$

where the matrices Q and R are symmetric, positive definite, constant, and of appropriate dimension. The symbol $\|y\|_Q^2$ represents $y'Qy$ where the prime indicates transpose.

The cost functional (1.5) contains an unknown constant vector u_r. In addition to the control law that minimizes (1.5), the vector u_r must be determined so that the optimal scalar cost is finite as the final time T approaches infinity. This will ensure that the resulting optimal control law will drive the output $y(t)$ *to* y_d.

This tracking problem can be put in the form of an output regulator problem by defining new variables:

$$\bar{u}(t) = u(t) - u_r \; ; \quad \bar{y}(t) = y(t) - y_d \tag{1.6}$$

$$\bar{x}'(t) = [x(t) \quad u_r \quad y_d] \tag{1.7}$$

and new matrices:

$$\bar{F} = \begin{bmatrix} F & G & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad \bar{G} = \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix} ; \quad \bar{H} = [H \quad 0 \quad -I] \; . \tag{1.8}$$

Then for a plant represented by

$$\dot{\bar{x}}(t) = \bar{F}\bar{x}(t) + \bar{G}\bar{u}(t) \; ; \quad \bar{x}'(t_o) = [x_o \quad u_r \quad y_d] \tag{1.9}$$

$$\bar{y}(t) = \bar{H}\bar{x}(t) \tag{1.10}$$

the control law that minimizes a quadratic cost functional

$$V(\bar{x}(t_o),t_o,T;\bar{u}(\cdot)) = \frac{1}{2}\int_{t_o}^{T} [\|\bar{y}(t)\|_Q^2 + \|\bar{u}(t)\|_R^2]dt \qquad (1.11)$$

is given by

$$\bar{u}^*(t) = -R^{-1}\bar{G}'\bar{P}(t,T)\bar{x}(t) \qquad (1.12)$$

where $\bar{P}(t,T)$ is the solution of the matrix Riccati equation

$$\dot{\bar{P}}(t) = -\bar{P}(t)\bar{F} - \bar{F}'\bar{P}(t) + \bar{P}(t)\bar{G}R^{-1}\bar{G}'\bar{P}(t) - \bar{H}'Q\bar{H} \ ; \quad \bar{P}(T) = 0 \ . \qquad (1.13)$$

The optimal value of the scalar cost functional (1.11) using the optimal control law (1.12) is given by

$$V^*(\bar{x}(t_o),t_o,T) = \frac{1}{2}\|\bar{x}(t_o)\|^2_{\bar{P}(t_o,T)} \ . \qquad (1.14)$$

For all T, the solution $\bar{P}(t,T)$ exists, is unique, is nonnegative definite symmetric, and is defined for all $t \le T$.[3]

If $\bar{P}(t,T)$ is partitioned as

$$\bar{P}(t,T) = \begin{bmatrix} P(t,T) & P_{12}(t,T) & P_{13}(t,T) \\ P'_{12}(t,T) & P_{22}(t,T) & P_{23}(t,T) \\ P'_{13}(t,T) & P'_{23}(t,T) & P_{33}(t,T) \end{bmatrix} \qquad (1.15)$$

the optimal control law (1.12) can be written as

$$\bar{u}^*(t) = -R^{-1}G'P(t,T)x(t) - R^{-1}G'[P_{12}(t,T)u_r + P_{13}(t,T)y_d] \ . \qquad (1.16)$$

The matrix Riccati equation (1.13) becomes

$$\dot{P}(t) = -P(t)F - F'P(t) + P(t)GR^{-1}G'P(t) - H'QH \ ; \quad P(T) = 0 \qquad (1.17)$$

$$\dot{P}_{12}(t) = -(F - GR^{-1}G'P(t))'P_{12}(t) - P(t)G \ ; \quad P_{12}(T) = 0 \qquad (1.18)$$

$$\dot{P}_{13}(t) = -(F - GR^{-1}G'P(t))'P_{13}(t) + H'Q \ ; \quad P_{13}(T) = 0 \qquad (1.19)$$

$$\dot{P}_{22}(t) = P'_{12}(t)GR^{-1}G'P_{12}(t) - P'_{12}(t)G - G'P_{12}(t) \ ; \quad P_{22}(T) = 0 \qquad (1.20)$$

$$\dot{P}_{23}(t) = -G'P_{13}(t) + P'_{12}(t)GR^{-1}G'P_{13}(t) \ ; \quad P_{23}(T) = 0 \qquad (1.21)$$

$$\dot{P}_{33}(t) = P'_{13}(t)GR^{-1}G'P_{13}(t) - Q \ ; \quad P_{33}(T) = 0 \qquad (1.22)$$

and the optimal scalar cost (1.14) becomes

$$V^*(\bar{x}(t_o),t_o,T) = \frac{1}{2}\Big\{ \|x_o\|^2_{P(t_o,T)} + 2x'_o[P_{12}(t_o,T)u_r + P_{13}(t_o,T)y_d] + u'_rP_{22}(t_o,T)u_r$$

$$+u'_rP_{23}(t_o,T)y_d + y'_dP'_{23}(t_o,T)u_r + y'_dP_{33}(t_o,T)y_d\Big\} \ . \qquad (1.23)$$

If we define

$$b(t) = -P_{12}(t)u_r - P_{13}(t)y_d \qquad (1.24)$$

$$c(t) = u'_rP_{22}(t)u_r + u'_rP_{23}(t)y_d + y'_dP'_{23}(t)u_r + y'_dP_{33}(t)y_d \qquad (1.25)$$

equations (1.18-19) can be combined to form

$$\dot{b}(t) = -(F - GR^{-1}G'P(t))'b(t) - (H'Qy_d - P(t)Gu_r) \ ; \quad b(T) = 0 \qquad (1.26)$$

and (1.20-22) can be combined to form

$$\dot{c}(t) = b'(t)GR^{-1}G'b(t) - y'_dQy_d + b'(t)Gu_r + u'_rG'b(t) \ ; \quad c(T) = 0 \ . \qquad (1.27)$$

Thus, solution of the matrix Riccati equation (1.13) is equivalent to simultaneous solution of (1.17), (1.26), and (1.27). Since $\bar{u}^*(t) = u^*(t) - u_r$, the optimal control law for the plant (1.1-2) is given by

$$u^*(t) = -R^{-1}G'P(t,T)x(t) + R^{-1}G'b(t,T) + u_r \qquad (1.28)$$

where $b(t,T)$ is the solution of (1.26). The optimal scalar cost using (1.28) is given by

$$V^*(\bar{x}(t_o),t_o,T) = \frac{1}{2}\,[\|x_o\|^2_{P(t_o,T)} - 2x_o'b(t_o,T) + c(t_o,T)] \tag{1.29}$$

where $c(t,T)$ is the solution of (1.27). Equation (1.17) is the matrix Riccati equation for the output regulator problem for the plant (1.1-2).

The optimal control law (1.28) and the resulting optimal scalar cost (1.29) are functions of the final time T and the arbitrary vector u_r. The limiting solution as $T \to \infty$ can be shown to have the following properties:

$$P = \lim_{T\to\infty} P(t,T) = \lim_{t\to-\infty} P(t,T) \tag{1.30}$$

exists, is constant, and can be found by obtaining the steady-state solution of (1.17).

$$b = \lim_{T\to\infty} b(t,T) = \lim_{t\to-\infty} b(t,T) \tag{1.31}$$

exists, is constant, and can be found by obtaining the steady-state solution of the linear, first-order differential equation

$$\dot{b}(t) = (F - GR^{-1}G'P)'b(t) + (H'Qy_d - PGu_r)\ ; \quad b(t_o) = 0\ . \tag{1.32}$$

For arbitrary u_r, solution of (1.32) yields

$$b = Px_r \tag{1.33}$$

such that x_r satisfies

$$P(Fx_r + Gu_r) + H'Q(Hx_r - y_d) = 0\ . \tag{1.34}$$

A bounded optimal scalar cost (1.29) as $T \to \infty$ requires that $\dot{c}(t) = 0$ or $c(t) = c$ a constant, yielding the optimality condition

$$b'GR^{-1}G'b - y_d'Qy_d + b'Gu_r + u_r'G'b = 0 \tag{1.35}$$

i.e., if u_r satisfies (1.35) where b is obtained from (1.32), then the control law

$$u^*(t) = -R^{-1}G'Px(t) + R^{-1}G'b + u_r \tag{1.36}$$

is optimal, the controlled plant

$$\dot{x}(t) = (F - GR^{-1}G'P)x(t) + Gu_c^* \tag{1.37}$$

$$u_c^* = R^{-1}G'b + u_r \tag{1.38}$$

is asymptotically stable for all t, and the optimal feedforward control law u_c^* will drive the output $y(t)$ *to* y_d. The vectors x_r, u_r, and y_d satisfy (1.3-4) where $x_r = P^{-1}b$, the constant c is given by

$$c = x_r'Px_r \tag{1.39}$$

and the optimal scalar cost as $T \to \infty$ is given by

$$V^*(x(t_o),t_o,\infty) = \frac{1}{2}\,[\|x_o\|^2_P - 2x_o'b + c] = \frac{1}{2}\,\|x_o - x_r\|^2_P\ . \tag{1.40}$$

The problem now is: given a desired constant output vector y_d, find the corresponding constant required control vector u_r that satisfies (1.35) so that the optimal feedforward control law (1.38) can be computed. The problem has been solved by developing the iterative algorithm given in the following steps.

A. Solve (1.17) for the steady-state value P.
B. Set $u_r = 0$.
C. Solve (1.32) for the steady-state value b.
D. Substitute u_r and b into (1.37-38) and solve for the steady-state value x.
E. Substitute x into (1.2) and solve for y.

F. Compare this value of $-y$ with y_d. If a significant difference exists, compute u^* by substituting x, b, and u_r into (1.36), set $u_r = u^*$, and return to step *C*.

The iteration is terminated when step *F* shows y to be sufficiently close to y_d. The feedforward control law (1.38) used to achieve this result is considered optimal.

A sufficient condition for convergence of the algorithm is that the magnitude of the eigenvalues of the matrix $I - H\Gamma H'Q$ are less than unity where

$$\Gamma = \Sigma' P G R^{-1} G' P \Sigma \tag{1.41}$$

and

$$\Sigma = -[P(F - GR^{-1}G'P)]^{-1} . \tag{1.42}$$

2. SOLUTION OF THE OBSERVER EQUATION: TF - AT = B

Let r components of the n-dimensional state vector $x(t)$ of the plant represented by (1.1) be considered unmeasurable, where

$$1 \le r \le n-1 . \tag{2.1}$$

Let $z(t)$ be an r-dimensional column vector defined as the output of a filter of the form

$$\dot{z}(t) = Az(t) + Bx(t) + Cu(t) \ ; \quad z(t_o) = z_o . \tag{2.2}$$

The matrices A, B, and C are constant, rectangular, and of appropriate dimension.

It has been shown[4] that if a constant, rectangular transformation matrix T can be found satisfying the matrix equation

$$TF - AT = B \tag{2.3}$$

where A and F have no common eigenvalues, then, after a time associated with the eigenvalues of the matrix A, the state $x(t)$ of the plant (1.1) is related to the output $z(t)$ of the filter (2.2) by the transformation

$$z(t) = Tx(t) \tag{2.4}$$

and, in addition, the matrix C in (2.2) is given by

$$C = TG . \tag{2.5}$$

Therefore, if we could determine the matrices T, A, and B that satisfy (2.3), we could solve (2.4) for an estimate of those states which are considered unmeasurable in terms of the measurable states and the filter output. We could then construct a state vector from the measurable states and the estimates of the unmeasurable states which could be used in the feedback control law for the output regulator.

The observer filter (2.2) cannot use any of the unmeasurable states as input. Therefore, the columns of the matrix B associated with the unmeasurable states must have all zero elements. It would also be desirable if the solutions of (2.3) were such that the eigenvalues of A were large and negative to ensure quick response, and that the absolute value of the elements of T, B, and C were small enough to be practical.

Since we know nothing about the elements of T, A, and B, except that certain columns of B must be zero, there are many more unknowns than equations. Certain constraints are therefore imposed to facilitate the computational task.

Restrict the matrix A to be diagonal. This restriction greatly simplifies the filter equations for multidimensional filters by eliminating cross-coupling between the elements of the filter vector. Temporarily assume that all eigenvalues of the matrix A are equal and denoted by a. Then

$$A = aI \tag{2.6}$$

and (2.3) becomes

$$T[aI - F] + B = 0 . \tag{2.7}$$

The equation for the jth element of the filter becomes

$$\dot{z}_j(t) = az_j(t) + b_jx(t) + c_ju(t) \tag{2.8}$$

where b_j and c_j are the jth rows of B and C, respectively.

The problem therefore has been reduced to that of solving a matrix equation of the form (2.7) to obtain r simple, uncoupled, first-order filters of the form (2.8).

Assume that the kth element of the state vector x(t) is to be estimated primarily by $z_j(t)$. We would thus like t_{jk} (the kth component of t_j, which is in turn the jth row of T) to take on the value unity, and all other components of t_j and all components of b_j and c_j to be small in magnitude.

To achieve a unique solution of (2.7), r cost functionals of the following form will be minimized subject to the constraint of (2.7):

$$L_j(t_j,b_j) = \frac{1}{2}\left[(t_j - t_{o_j})(t_j' - t_{o_j}') + (b_j)R_o(b_j')\right] \tag{2.9}$$

where R_o is a constant diagonal matrix and t_{o_j} is zero except for the kth component which is unity.

A standard LaGrange multiplier minimization yields a solution t_j and b_j. To assure the desired zero columns in B, it will be assumed that the inverse R_o^{-1} has zeros at the appropriate diagonal components; i.e., R_o itself assigns infinite weight to the columns that must be zero. After the minimization procedure, the rows t_j, b_j, and c_j are normalized to assure that t_{jk} is unity. The normalized matrices are denoted by (~).

The preceding is implemented with the following algorithm. The algorithm incorporates the proper choice of the eigenvalues of the matrix A and the construction of all r filters simultaneously.

A. Given a plant represented by (1.1), select r elements of the state vector which are to be assumed unmeasurable, where

$$1 \le r \le n-1 \ .$$

B. Set the corresponding elements of the diagonal matrix R_o^{-1} to zero and the remaining elements to unity.

C. Construct a projection matrix T_o such that if z(t) were defined as

$$z(t) = T_ox(t) \tag{2.10}$$

then the elements of the vector z(t) would be the unmeasurable states. (T_o has one unity element in each row and zeros otherwise.)

D. Select a positive gain limit such that if the absolute value of any element of the solution is greater than this gain limit the solution is to be considered impractical.

E. Choose the parameter a to be some small negative value whose magnitude is approximately equal to the magnitude of the smallest eigenvalue of the matrix F.

F. Construct the matrix

$$M = [aI - F] \tag{2.11}$$

G. Carry out the minimization. Construct and solve the following matrix equations:

$$\begin{bmatrix} T' \\ \Lambda \end{bmatrix} = \begin{bmatrix} M' & R_o^{-1} \\ I & -M \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ T_o' \end{bmatrix} \tag{2.12}$$

$$B' = R_o^{-1}\Lambda \tag{2.13}$$

$$C = TG \tag{2.14}$$

$$A = Ia \tag{2.15}$$

which yields the following observer

$$z(t) = Tx(t) \tag{2.16}$$

$$\dot{z}(t) = Az(t) + Bx(t) + Cu(t) \tag{2.17}$$

H. Normalize the observer. For each row of T_o, find the unity element and then select the corresponding element of the same row of T (i.e., t_{jk}). Multiply this same row of T, B, and C by the inverse of this element (i.e., $1/t_{jk}$), yielding the normalized observer

$$z_n(t) = \tilde{T}x(t) \tag{2.18}$$

$$\dot{z}_n(t) = Az_n(t) + \tilde{B}x(t) + \tilde{C}u(t) \tag{2.19}$$

where $z_n(t)$ is used for the normalized variable. The matrices $\tilde{T}$, $\tilde{B}$, and $\tilde{C}$ satisfy

$$\tilde{T}F - A\tilde{T} = \tilde{B} \tag{2.20}$$

and

$$\tilde{C} = \tilde{T}G \tag{2.21}$$

I. Consider the rows of the matrices A, $\tilde{T}$, $\tilde{B}$, and $\tilde{C}$ to be solutions. Compare the absolute value of the elements of each solution with the gain limit. If the absolute value of each element of a solution is less than or equal to the gain limit, store that solution. If the absolute value of any element of a solution is greater than the gain limit, do not store that solution. If the absolute value of at least one element in all solutions is greater than the gain limit, print the stored solutions and stop. If not, increment the parameter a to some larger negative value and return to step *F*.

If the matrix inverse indicated in (2.12) exists, each iteration will produce r *independent* solutions as defined in step *I* of the algorithm. The iterative procedure will cause the r solutions with the quickest response time, whose elements do not exceed the gain limit in magnitude, to be stored. All other solutions are discarded. The diagonal elements of the resulting stored matrix A are *not* generally equal.

An illustrative example of the use of the algorithms will now be presented.

3. ILLUSTRATIVE EXAMPLE

The aircraft chosen to illustrate the procedure presented here is the Augmentor Wing Jet STOL Research Aircraft. For longitudinal flight, in addition to the usual elevator and throttle, this aircraft has vectored thrust provided by four Pegasus nozzles that direct the hot gas exhaust from the engines. It also has augmentor blowing ejector flaps through which the cold airflow from the engines is ducted.

For a 60-knot approach on a -7.5° flight-path angle, linearized longitudinal perturbation equations were obtained of the form (1.1-2) where

$$x'(t) = [v(t) \quad \alpha(t) \quad \theta(t) \quad q(t)] \tag{3.1}$$

$$u'(t) = [\delta_E(t) \quad \delta_N(t) \quad \delta_T(t) \quad \delta_F(t)] \tag{3.2}$$

$$y'(t) = [v_K(t) \quad \alpha(t) \quad \theta(t) \quad \gamma(t)] \tag{3.3}$$

and

$v(t)$ = variation in the forward component of the airspeed, ft/sec

$\alpha(t)$ = variation in angle of attack, deg

$\theta(t)$ = variation in pitch attitude, deg

$q(t)$ = variation in pitch rate, deg/sec

$\delta_E(t)$ = variation in elevator angle, deg

$\delta_N(t)$ = variation in nozzle angle, deg

$\delta_T(t)$ = variation in throttle angle, deg

$\delta_F(t)$ = variation in flap angle, deg

$v_K(t)$ = variation in the forward component of the airspeed, knots

$\gamma(t)$ = variation in the flight-path angle, deg

The matrices F, G, and H are

$$[F] = \begin{bmatrix} -0.05479 & 0.2606 & -0.5598 & -0.0880 \\ -0.1453 & -0.5078 & 0.02461 & 0.9521 \\ 0 & 0 & 0 & 1.0 \\ 0.08273 & -0.07091 & -0.00965 & -1.352 \end{bmatrix} \tag{3.4}$$

$$[G] = \begin{bmatrix} 0.004216 & -0.08378 & -0.01867 & -0.1031 \\ -0.04573 & 0 & -0.5028 & -0.09994 \\ 0 & 0 & 0 & 0 \\ -1.302 & -0.08190 & 0.2676 & 0.2167 \end{bmatrix} \tag{3.5}$$

$$[H] = \begin{bmatrix} 0.5925 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & -1.0 & 1.0 & 0 \end{bmatrix} . \tag{3.6}$$

The matrices Q and R^{-1} were chosen to be

$$[Q] = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0 \\ 0 & 0 & 1.6 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} ; \quad [R^{-1}] = \begin{bmatrix} 0.625 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{3.7}$$

For computational convenience R^{-1} was taken as a 4×4 matrix with a zero row and a zero column since the flap was not used for feedback control. The matrix Riccati equation was solved. The control law $u(t) = -R^{-1}G'Px(t)$ where

$$[R^{-1}G'P] = \begin{bmatrix} 0.1009 & -0.2650 & -0.9779 & -0.6761 \\ -0.6807 & -0.03284 & -1.049 & -0.5776 \\ 0.07422 & -4.392 & 4.953 & 0.3067 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

gave reasonably good response to initial condition errors and so was taken as an acceptable basis for the design of optimal feedforward control laws.

A number of optimal feedforward control laws were computed for various desired steady-state outputs y_d. The desired output criterion for each case is shown in table 1.

Table 1.- Steady-State Output Design Criteria

y_d \ Case	1	2	3	4	5	6	7	8	9	10	11
v_{K_d}	0	0	0	0	Free	Free	Free	Free	10.0	10.0	10.0
α_d	0	-1.0	Free	Free	0	-1.0	Free	Free	0	Free	Free
θ_d	1.0	0	Free	1.0	1.0	0	Free	1.0	0	Free	Free
γ_d	1.0	1.0	1.0	Free	1.0	1.0	1.0	Free	0	0	Free

Note that, in some cases, some of the elements of the output vector were allowed to remain free while others were to be driven to a specific value. This was achieved by starting with the free elements of y_d equal to zero in step *B* of the algorithm and by replacing the free elements of y_d with the corresponding elements of y in step *F*.

The desired output vector y_d and the required control vector u_r were computed for each case. The resulting steady-state output and control vector pairs for each case are shown in tables 2 and 3, respectively. The corresponding optimal feedforward control law u_c^* for each case is shown in table 4.

Table 2.- Desired Steady-State Output Vectors

y_d \ Case	1	2	3	4	5	6	7	8	9	10	11
v_{k_d}	0	0	0	0	-1.178	-.3120	-.5275	-.8907	10.0	10.0	10.0
α_d	0	-1.0	-.8095	.9332	0	-1.0	-.7513	.9210	0	-1.103	-.9284
θ_d	1.0	0	.1905	1.0	1.0	0	.2487	1.0	0	-1.103	-1.144
γ_d	1.0	1.0	1.0	.06676	1.0	1.0	1.0	.07916	0	0	-.2157

Table 3.- Required Steady-State Control Vectors

u_r \ Case	1	2	3	4	5	6	7	8	9	10	11
δ_{E_r}	0.414	0.461	0.452	-.016	.334	.440	.413	-.071	.680	.732	.635
δ_{N_r}	-6.663	-3.303	-3.943	-3.581	-5.497	-2.994	-3.616	-2.740	-9.907	-6.201	-5.351
δ_{T_r}	.011	.968	.786	-.892	.593	1.122	.991	-.440	-4.941	-3.886	-4.056
δ_{F_r}	0	0	0	0	0	0	0	0	0	0	0

Table 4.- Optimal Feedforward Control Laws

u_c^* \ Case	1	2	3	4	5	6	7	8	9	10	11
$\delta_{E_c}^*$	-.564	.726	.480	-1.242	-.845	.652	.279	-1.445	2.38	3.80	3.70
$\delta_{N_c}^*$	-7.712	-3.270	-4.117	-4.660	-5.191	-2.603	-3.246	-2.796	-21.40	-16.50	-15.61
$\delta_{T_c}^*$	4.964	5.360	5.285	-.038	5.399	5.475	5.456	.357	-3.68	-3.25	-4.39
$\delta_{F_c}^*$	0	0	0	0	0	0	0	0	0	0	0

For each case, the optimal feedforward commands (table 4) will drive the output $y(t)$ of the controlled plant (1.37) to the desired steady-state output (table 2) with the resulting steady-state control deflections (table 3). Transient responses for case 9, case 2, and case 1 are shown in figures 1, 2, and 3, respectively, where the elevator, nozzle, and throttle were assumed to have first-order actuator dynamics with time constants of 0.1, 0.0667, and 2.5 seconds, respectively. These figures show that the steady-state responses are exactly as desired, with quite reasonable dynamics as well.

Let us assume that $v(t)$ and $\alpha(t)$ are to be considered unmeasurable. Then if we construct

$$R_o^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} ; \quad T_o = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

and use a gain limit of 10, an observer of the form (2.18-19) is obtained where

$$[\tilde{T}] = \begin{bmatrix} 1.0 & 4.853 & -0.7202 & 3.141 \\ -0.4044 & 1.0 & 0.1679 & 6.865 \end{bmatrix} \tag{3.10}$$

$$[A] = \begin{bmatrix} -0.5 & 0 \\ 0 & -1.1 \end{bmatrix} \tag{3.11}$$

$$[\tilde{B}] = \begin{bmatrix} 0 & 0 & -0.8308 & 1.136 \\ 0 & 0 & 0.3695 & -0.5744 \end{bmatrix} \tag{3.12}$$

$$[\tilde{C}] = \begin{bmatrix} -4.308 & -0.3411 & -1.618 & 0.09264 \\ -8.986 & -0.5284 & 1.342 & 1.429 \end{bmatrix} \tag{3.13}$$

Figure 4 shows the transient response for case 1 using this observer as dynamic compensation, and using only pitch attitude and pitch rate for feedback. The same actuator dynamics were assumed. Comparison of figure 4 with figure 3 shows an almost identical response.

4. CONCLUSIONS

Simple algorithms for the construction of optimal feedforward control laws and optimal observers have been presented. These algorithms can be used to obtain control laws that behave as stable regulators for the plant in the absence of command inputs, that will drive the output of the plant to a desired constant value in response to a step command input, and that require only selected elements of the state vector for feedback.

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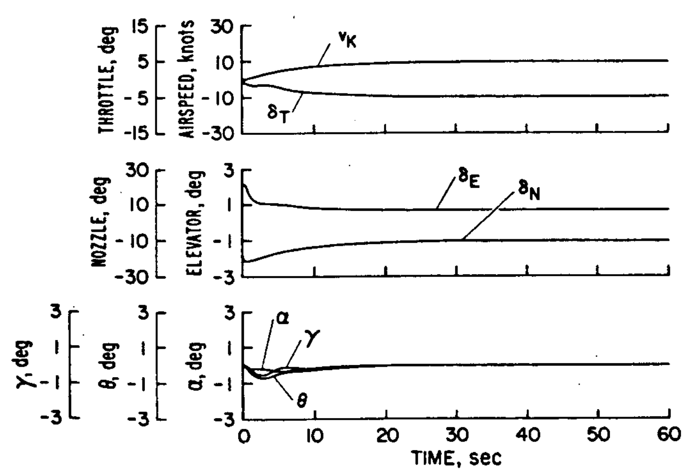

Figure 1.- Airspeed command input with entire state measurable.

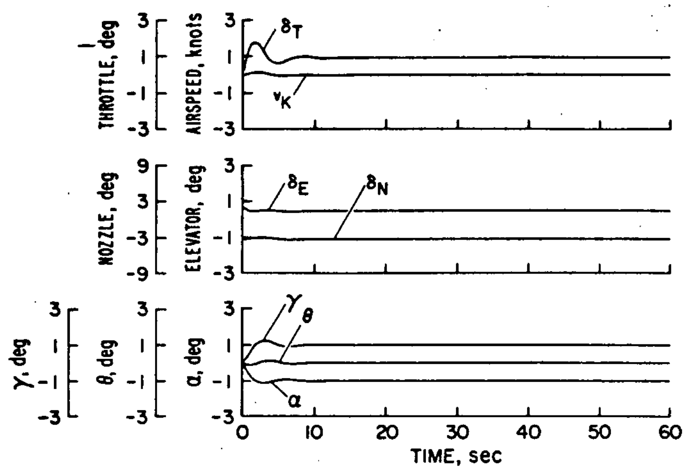

Figure 2.- Direct lift command input with entire state measurable.

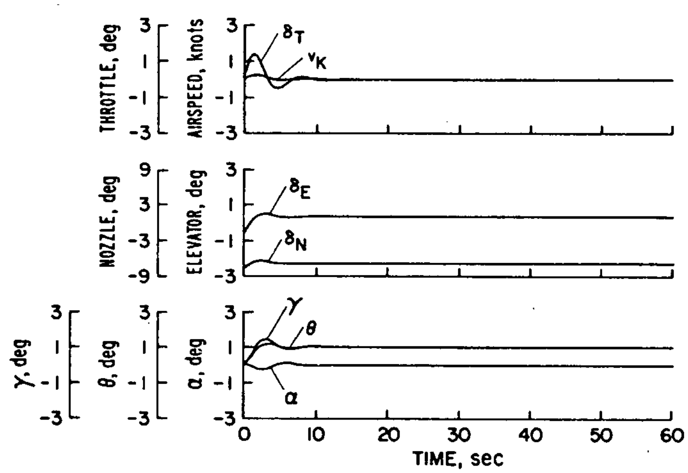

Figure 3.- Flight-path command input with entire state measurable.

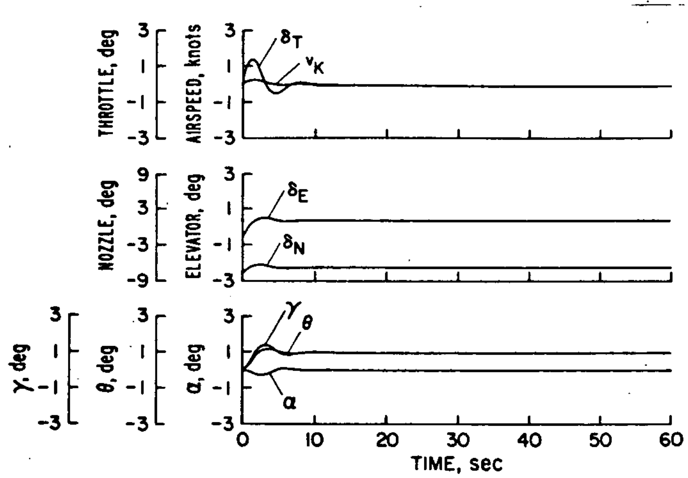

Figure 4.- Flight-path command input with second-order observer.