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## THE TURBULENT MEAN-FLOW, REYNOLDS-STRESS. AND HEAT-FLUX EQUATIONS IN MASS-AVERAGED DEPENDENT VARIABLES

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## SUMMARY

This note presents the time-dependent, turbulent mean-flow, Reynolds stress, and heat flux equations in mass-averaged dependent variables. These equations are given in conservative form for both generalized orthogonal and axisymmetric coordinates. For the case of small viscosity and thermal conductivity fluctuations, these equations are considerably simpler than the general Reynolds system of dependent variables for a compressible fluid and permit a more direct extension of low speed turbulence modeling to computer codes describing high speed turbulence fields.

# INTRODUCTION

In recent years, significant advancements in both computation techniques and computer hardware have led to sophisticated prediction techniques for describing complex compressible fluid flows. The most general techniques avoid boundary-layer approximations and solve the complete time-dependent, conservation equations of mass, momentum and thermal energy, and possibly species. (See ref (1) and (2).) The transport mechanisms that have been employed in these techniques are consistent with molecular or laminar processes. Methods such as those presented by MacCormack or Carter have been shown to apply to a flow as complex as a shock-wave boundary-layer interaction, even when the adverse pressure gradient generated by the shock wave is strong enough to cause separation of the boundary-layer flow.

The successful methods for laminar flows have not, however, been carried over to the computation of turbulent flows — particularly strongly interacting flows. Turbulent-flow solutions, which are presumably contained in the instantaneous viscous flow conservation equations, cannot be obtained at present because the small grid size required for accurately resolving the decay processes of turbulence cannot be accommodated in present computers. Therefore, these instantaneous equations must be time averaged to obtain a system of equations with spatial characteristics consistent with present computers. In the time averaging process there is a loss of phase and wave-length information of the turbulent motions. Although this information may be critical to the ultimate understanding of turbulence, time averaging has yielded most of the significant results related to the mean flow in boundary layers and may prove equally valuable for the more complex flow fields of current interest.

The first step i  establishing time-averaged turbulent flow equations is
the expansion of all the dependent variables into the sum of two terms, one
slowly varying and the other randomly varying at the high frequencies charac-
teristic of turbulence.  With compressible flows there is some freedom in the
choice of how this expansion is accomplished.  For example, instantaneous
velocity components can be expanded either as the sum of two velocities. as in
the classical Reynolds expansion, or as the sum of two mass fluxes divided by
the mean density, the mass averaged dependent variable expansion, first suggested
by Favre (ref. 3).  As an example of Reynolds· expansion, Donaldson (ref. 4) has
written the complete system of equations for compressible flow in generalized
tensor form so that they can be particularized to any desired coordinate system.
Donaldson's equations explicitly contain density fluctuations as well as velocity
fluctuations, thus, adding considerable complexity as compared with the constant
property case.  This added complexity can be reduced, as shown in ref. 5 and 6,
if the expansion of the dependent variables is carried out using mass averaged
expansions.  These equations essentially exhibit a term by term correspondence
with the constant property case.  This correspondence can be important if the
basic turbulence mechanisms in compressible flows remain similar to those in
constant property flows (the so-called Morkovin hypothesis), thereby permitting
convenient extension of the large body of experience existing with incompres-
sible flows.

The work of refs. 5 and 6 is limited, however, in that the Reynolds stress
and turbulent heat flux equations were not presented and, further, the equations
were confined to the steady state and expressed in cartesian coordinates.  The

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general objective of this note is to remove these limitations by

1. adding the time-dependent terms to all the equations

2. presenting the Reynolds stress and heat flux equations

3. employing a generalized orthogonal coordinate system

4. casting the equations in conservative form, consistent with existing computer codes (refs. 1 and 2)

and

5. particularizing the equations to cylindrical coordinates for comparison with axisymmetric flow data (e.g. ref. 7).

## SYMBOLS

e    total internal energy per unit of volume, $\rho h - p + \rho \dfrac{u_j u_j}{2}$

h    specific enthalpy

H    total specific enthalpy, $h + \dfrac{u_j u_j}{2}$

p    static pressure

P    period of integration for turbulence quantities

Pr    molecular Prandtl number

$q_j$    heat flux in j th direction, $- \lambda \dfrac{\partial T}{\partial x_j}$ or $- \dfrac{\mu}{Pr} \dfrac{\partial h}{\partial x_i}$

s    dummy variable of integration

t    time

T    temperature

$u_i$    velocity component in i th direction

$\lambda$    molecular thermal conductivity

$\mu$    molecular viscosity

$\rho$    density

$\tau_{ij}$    instantaneous shear stress in i th direction on a surface normal to the

j th direction,   $\mu \left[ \left( \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) - \dfrac{2}{3} \delta_{ij} \dfrac{\partial u_k}{\partial x_k} \right]$

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# TURBULENCE EQUATIONS IN ORTHOGONAL TENSOR FORM

In the derivation of the mean flow and turbulence equations, orthogonal tensor notation will be employed. The usual conventions of a repeated subscript indicating summation over the entire range of the indices and a comma representing partial differentiation will be used to make the equations compact. Orthogonal tensors were adopted to permit working directly with the physical components, thereby avoiding the need for identifying specific tensors as being covariant or contravariant. The use of orthogonal coordinate systems does not cause significant loss of generality in most fluid mechanical applications.

The time-dependent, conservation equations for a compressible fluid with zero external body forces are

## Continuity

$$\rho_{,t} + (\rho u_j)_{,j} = 0 \tag{1}$$

## Momentum in the ith direction (i = 1, 2, or 3)

$$(\rho u_i)_{,t} + (\rho u_i u_j + \delta_{ij} p - \tau_{ij})_{,j} = 0 \tag{2}$$

## Thermal Energy

$$(\rho H - p)_{,t} + (\rho u_j H + q_j - u_i \tau_{ij})_{,j} = 0 \tag{3}$$

to convert these equations to time averaged equations, the following expansions of the dependent variables are employed:

$$
\begin{aligned}
\rho &= \bar{\rho} + \rho' \\
p &= \bar{p} + p' \\
\tau_{ij} &= \bar{\tau}_{ij} + \tau'_{ij} \\
q_j &= \bar{q}_j + q'_j
\end{aligned}
\tag{4}
$$

and the mass averaged dependent variables

$$u_i = \tilde{u}_i + u_i''$$

$$H = \tilde{H} + H''$$ (5)

$$h = \tilde{h} + h''$$

The superscripts - , ~ represent time averaging over a period P that is long compared to the fluctuation period of the turbulence but short compared to the time constant for changes within the flow field as a whole. The superscript ' or " represents the instantaneous fluctuation about the corresponding time averaged quantity. Thus, we set

$$\bar{A}(t_o) = \frac{1}{P} \int_{t_o-(P/2)}^{t_o+(P/2)} A(t)dt$$ (6)

then,

$$\overline{\frac{\partial A(t_o)}{\partial t}} = \frac{1}{P} \int_{t_o-(P/2)}^{t_o+(P/2)} \frac{\partial A(t)}{\partial t} \, dt = \frac{\partial}{\partial t} \overline{A(t_o)}$$ (7)

For example, from Eqs. (4) and (6)

$$\bar{\rho}(t_o) = \frac{1}{P} \int_{t_o-(P/2)}^{t_o+(P/2)} (\bar{\rho}(t_o) + \rho'(t))dt$$ (8)

$$= \bar{\rho}(t_o) + \bar{\rho}'(t_o)$$ (9)

From Eq. (9),

$$\bar{\rho}' = 0$$ (10)

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Thus, in general

$$\bar{A}' = \frac{1}{P} \int_{t_o-(P/2)}^{t_o+(P/2)} A'(t)dt = 0 \qquad (11)$$

a single fluctuating quantity raised to the first power averages to zero.

The mass-averaged dependent variables are defined (see Appendix B) such that the mean velocity is

$$\tilde{u}_i = \bar{\tilde{u}}_i = \frac{\overline{\rho u_i}}{\bar{\rho}} \qquad (12)$$

and the fluctuating velocity averages to

$$\overline{\rho u_i''} = 0 \qquad (13)$$

From Eqs. (4), (5) and (13)

$$\overline{u_i''} = - \frac{\overline{\rho' u_i''}}{\bar{\rho}} \qquad (14)$$

so that $\overline{u_i''}$ is not zero in contrast to Eq. (11).

Similarly, the mass averaged static enthalpy is

$$\tilde{h} = \frac{\overline{\rho h}}{\bar{\rho}} \qquad (15)$$

and

$$\overline{\rho h''} = 0 \qquad (16)$$

Again $\overline{h''}$ is not zero, but rather

$$\overline{h''} = - \frac{\overline{\rho' h''}}{\bar{\rho}}$$

## Mean Flow Equations

When the instantaneous density and velocity, express as Eqs. (4) and (5), are inserted into the continuity equation (1) and the resulting equation is time averaged according to Eq. (6), there results

$$\bar{\rho}_{,t} + (\bar{\rho}\tilde{u}_j)_{,j} = 0 \tag{17}$$

This time-averaged continuity equation has identical form to Eq. (1) except that the instantaneous quantities $\rho$ and $u$ are replaced by the time averaged quantities $\bar{\rho}$ and $\tilde{u}$.

Similar operations of expanding the dependent variables according to Eqs. (4) and (5) and then Reynolds averaging converts the momentum equations, Eq. (2), and the energy equation, Eq. (3), to

$$(\bar{\rho}\tilde{u}_i)_{,t} + \left[\bar{\rho}\tilde{u}_j\tilde{u}_i + \delta_{ij}\bar{p} - \left(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right)\right]_{,j} = 0 \tag{18}$$

and

$$(\bar{\rho}\tilde{H} - \bar{p})_{,t} + \left[\bar{\rho}\tilde{u}_j\tilde{H} + \bar{q}_j + \overline{\rho u_j'' h''} - \tilde{u}_i\left(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right) - \overline{u_i''\left(\tau_{ij} - \frac{\overline{\rho u_i'' u_j''}}{2}\right)}\right]_{,j} = 0 \tag{19}$$

Again, their form is generally identical with their laminar flow counterparts except for the addition of the Reynolds stress, heat flux, and the additional mean energy dissipation terms given below:

$$\text{Reynolds stress} = \overline{\tau_{ij}^t} = -\overline{\rho u_i'' u_j''} \tag{20}$$

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$$\text{Reynolds heat flux} = \overline{q_j^t} = \overline{\rho u_j'' h''} \tag{21}$$

$$\text{Mean energy dissipation} = \overline{u_i'' \left( \tau_{ij} - \frac{\rho u_i'' u_j''}{2} \right)} \tag{22}$$

## Turbulence Growth Equations

The growth of the Reynolds stresses are found by manipulation of Eqs. (1) and (2) into a moment of momentum equation

$$(\rho u_i u_k)_{,t} + (\rho u_i u_k u_j)_{,j} + u_k p_{,i} + u_i p_{,k} - u_k \tau_{ij,j} - u_i \tau_{kj,j} = 0 \tag{23}$$

This equation is then expanded according to Eqs. (4) and (5) and time averaged according to Eq. (6) to yield the <u>sum</u> of the mean flow terms

$$(\bar{\rho} \bar{u}_i \bar{u}_k)_{,t} + (\bar{\rho} \bar{u}_j \bar{u}_i \bar{u}_k)_{,j}$$

$$= -\bar{u}_k \bar{p}_{,i} - \bar{u}_i \bar{p}_{,k} + \bar{u}_k \left( \bar{\tau}_{ij} - \overline{\rho u_i'' u_j''} \right)_{,j} + \bar{u}_i \left( \bar{\tau}_{kj} - \overline{\rho u_k'' u_j''} \right)_{,j} \tag{24}$$

and the turbulence quantities (growth of Reynolds stresses)

$$\left( \overline{\rho u_i'' u_k''} \right)_{,t} + \left( \bar{u}_j \overline{\rho u_i'' u_k''} \right)_{,j} = -\left( \overline{\rho u_i'' u_j''} \right) \bar{u}_{k,j} - \left( \overline{\rho u_k'' u_j''} \right) \bar{u}_{i,j} - \left( \overline{\rho u_i'' u_k'' u_j''} \right)_{,j}$$

$$- \left( \overline{u_k'' p} \right)_{,i} - \left( \overline{u_i'' p} \right)_{,k} + \overline{p \left( u_{k,i}'' + u_{i,k}'' \right)}$$

$$+ \left( \overline{u_k'' \tau_{ij}} \right)_{,j} + \left( \overline{u_i'' \tau_{kj}} \right)_{,j} - \overline{\tau_{ij} u_{k,j}''} - \overline{\tau_{kj} u_{i,j}''} \tag{25}$$

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Eq. (24) follows directly from Eqs. (17) and (18); so that the separation of
Eq. (24) and Eq. (25) as indicated is not arbitrary. Eq. (25) corresponds term
by term to the constant property equation in Ref. 8, except with $\rho$ occurring
inside of the operator signs of time-averaging and differentiation.

The turbulence kinetic energy equation, with

$$k = \frac{1}{2\bar{\rho}} \, \rho u_i'' u_i''$$

(26)

and

$$\bar{k} = \frac{1}{2\bar{\rho}} \left( \overline{\rho u_i'' u_i''} \right)$$

(27)

is

$$(\bar{\rho}\bar{k})_{,t} + (\bar{\rho}\tilde{u}_j \bar{k})_{,j} = -\left( \overline{\rho u_i'' u_j''} \right) \tilde{u}_{i,j} - \left( \overline{\rho u_j'' k} \right)_{,j} - \left( \overline{u_i'' p} \right)_{,i} + \overline{p u_{i,i}''} + \left( \overline{u_i'' \tau_{ij}} \right)_{,j} - \overline{\tau_{ij} u_{i,j}''}$$

(28)

The terms on the right are, sequentially, the turbulence production, the
turbulent diffusion, the pressure diffusion, pressure work, molecular diffu-
sion, and dissipation of the turbulence kinetic energy. Eq. (28) is consistent
with Rotta's and Townsend's (Ref. 9) constant property equations.

The equation for the growth of Reynolds heat flux, Eq. (21), follows from
the energy equation, Eq. (3), combined with Eq. (23) with $i = k$ and the
equation resulting from h times Eq. (2)

$$(\rho u_i h)_{,t} + (\rho u_i u_j h)_{,j} - u_i p_{,t} - u_i u_k p_{,k} + h p_{,i}$$

$$+ u_i q_{j,j} - h\tau_{ij,j} - u_i \tau_{kj} u_{k,j} = 0$$

(29)

Again, when this equation is expanded according to Eqs. (4) and (5), and the resulting equation is time averaged according to Eq. (6), there is obtained the sum of the mean quantities

$$(\bar{\rho}\tilde{h}\tilde{u}_i)_{,t} + (\bar{\rho}\tilde{u}_j\tilde{u}_i\tilde{h})_{,j} - \tilde{u}_i\bar{p}_{,t} + \tilde{u}_i\left[\bar{q}_j + \overline{\rho v_j'' h''} - \tilde{u}_k\bar{\tau}_{kj} - \overline{u_k''\tau_{kj}}\right]_{,j}$$

$$- \tilde{u}_i\tilde{u}_k\bar{p}_{,k} - \tilde{u}_i\overline{u_k''p}_{,k} + \tilde{u}_i\tilde{u}_k\bar{\tau}_{kj,j} + \tilde{u}_i\overline{u_k''\tau_{kj,j}} + \tilde{h}\bar{p}_{,i} - \tilde{h}\left(\bar{\tau}_{ij} - \overline{\rho u_i''u_j''}\right)_{,j} = 0$$

$$(30)$$

and Reynolds heat flux terms

$$\left(\overline{\rho u_i''h''}\right)_{,t} + \left(\tilde{u}_j\overline{\rho u_i''h''}\right)_{,j} + \left(\overline{\rho u_j''h''}\right)\tilde{u}_{i,j} + \left(\overline{\rho u_i''u_j''}\right)\tilde{h}_{,j} + \left(\overline{\rho u_j''u_i''h''}\right)_{,j} - \overline{u_i''p}_{,t}$$

$$- \tilde{u}_k\overline{u_i''p}_{,k} - \overline{u_i''u_k''p}_{,k} + \overline{h''p}_{,i} - \overline{h''\tau_{ij,j}} + \overline{u_i''q_{j,j}} - \tilde{u}_{k,j}\overline{u_i''\tau_{ki}} - \overline{u_{k,j}''u_i''\tau_{kj}} = 0$$

$$(31)$$

The proper separation of Eqs. (30) and (31) follows from the use of moments of Eqs. (18) and (19) to yield Eq. (30).

## THE AXISYMMETRIC-FLOW EQUATIONS

The equations given above in orthogonal tensor form are particularized in this section to an axisymmetric system (i.e. a cylindrical coordinate system with $\frac{\partial(\bar{\,})}{\partial\theta} = 0$ and $\tilde{w} = 0$). These equations are obtained using the particularized expansions and coordinate nomenclature given in the Appendix.

### Mean Flow Equations

The mean flow equations Eq. (17) through (19) particularize to

Continuity:

$$\frac{\partial(r\bar{\rho})}{\partial t} + \frac{\partial(r\bar{\rho}\tilde{u})}{\partial x} + \frac{\partial(r\bar{\rho}v)}{\partial r} = 0 \tag{32}$$

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**x-momentum**

$$\frac{\partial}{\partial t}(r\bar{\rho}\tilde{u}) + \frac{\partial}{\partial x}\left[r\left(\bar{\rho}\tilde{u}\tilde{u} + \bar{p} - \bar{\tau}_{xx} + \overline{\rho u''u''}\right)\right] + \frac{\partial}{\partial r}\left[r\left(\bar{\rho}\tilde{v}\tilde{u} - \bar{\tau}_{xr} + \overline{\rho u''v''}\right)\right] = 0 \quad (33)$$

**r-momentum**

$$\frac{\partial}{\partial t}(r\bar{\rho}\tilde{v}) + \frac{\partial}{\partial x}\left[r\left(\bar{\rho}\tilde{u}\tilde{v} - \bar{\tau}_{rx} + \overline{\rho u''v''}\right)\right] + \frac{\partial}{\partial r}\left[r\left(\bar{\rho}\tilde{v}\tilde{v} + \bar{p} - \bar{\tau}_{rr} + \overline{\rho v''v''}\right)\right]$$

$$+ \bar{\tau}_{\theta\theta} - \bar{p} - \overline{\rho w''w''} = 0 \quad (34)$$

**Total thermal energy**

$$\frac{\partial}{\partial t}(r\tilde{e}) + \frac{\partial}{\partial x}\, r\left[\tilde{u}(\tilde{e} + \bar{p}) + \bar{q}_x + \overline{\rho u''h''} - \tilde{u}(\bar{\tau}_{xx} - \overline{\rho u''u''}) - \tilde{v}(\bar{\tau}_{rx} - \overline{\rho v''u''}) - \overline{u''\left(\tau_{xx} - \frac{\rho u''u''}{2}\right)}\right.$$

$$\left. - \overline{v''\left(\tau_{rx} - \frac{\rho v''u''}{2}\right)} - \overline{w''\left(\tau_{\theta x} - \frac{\rho w''u''}{2}\right)}\right] + \frac{\partial}{\partial r}r\left[\tilde{v}(\tilde{e} + \bar{p}) + \bar{q}_r + \overline{\rho v''h''}\right.$$

$$- \tilde{u}(\bar{\tau}_{xr} - \overline{\rho u''v''}) - \tilde{v}(\bar{\tau}_{rr} - \overline{\rho v''v''}) - \overline{u''\left(\tau_{xr} - \frac{\rho u''v''}{2}\right)} - \overline{v''\left(\tau_{rr} - \frac{\rho v''v''}{2}\right)}$$

$$\left. - \overline{w''\left(\tau_{\theta r} - \frac{\rho w''v''}{2}\right)}\right] = 0 \quad (35)$$

These equations may be written in a condensed vector form convenient for numerical computations (ref. 1 or 2) as

$$\frac{\partial r\vec{U}}{\partial t} + \frac{\partial r\vec{F}}{\partial x} + \frac{\partial r\vec{G}}{\partial r} + \vec{A} = 0 \quad (36)$$

where, for example,

$$\vec{U} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\tilde{u} \\ \bar{\rho}\tilde{v} \\ \tilde{e} \end{pmatrix} \quad \text{and} \quad \vec{A} = \begin{pmatrix} 0 \\ 0 \\ \bar{\tau}_{\theta\theta} - \bar{p} - \overline{\rho w''w''} \\ 0 \end{pmatrix} \quad (37)$$

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These equations may also be written in an alternative form by dividing through by r to give

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial r} + \frac{1}{r}(\vec{G} + \vec{A}) = 0 \tag{38}$$

We may combine the two inhomogeneous terms as

$$\vec{B} = \frac{\vec{G} + \vec{A}}{r}$$

In this form $\vec{B}$ becomes the following vector:

$$\vec{B} = \begin{pmatrix} \dfrac{\bar{\rho}\tilde{v}}{r} \\[2ex] \dfrac{1}{r}\left[\bar{\rho}\tilde{u}\tilde{v} - \bar{\tau}_{xr} + \overline{\rho u''v''}\right] \\[2ex] \dfrac{1}{r}\left[\bar{\rho}\tilde{v}\tilde{v} - \bar{\tau}_{rr} + \overline{\rho v''v''} + \bar{\tau}_{\theta\theta} - \overline{\rho w''w''}\right] \\[2ex] \dfrac{1}{r}\left[\tilde{v}(\tilde{e} + \bar{p}) + \bar{q}_r + \overline{\rho v''h''} - \tilde{u}\left(\bar{\tau}_{xr} - \overline{\rho u''v''}\right) - \tilde{v}\left(\bar{\tau}_{rr} - \overline{\rho v''v''}\right)\right. \\[2ex] \left. - \overline{u''\left(\tau_{xr} - \dfrac{\rho u''v''}{2}\right)} - \overline{v''\left(\tau_{rr} - \dfrac{\rho v''v''}{2}\right)} - \overline{w''\left(\tau_{\theta r} - \dfrac{\rho w''v''}{2}\right)}\right] \end{pmatrix} \tag{39}$$

In the alternative form of the equations, the transformation from the axisymmetric case to the planar flow case is simple in that the vector B vanishes and the remainder of the equation is unchanged, i.e., we have for the planar case

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial r} = 0 \tag{40}$$

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## Turbulence Growth Equations

To affect a closure of the above equations (in either of the two forms presented) we require equations for $\overline{\rho u''^2}$, $\overline{\rho u''v''}$, $\overline{\rho v''^2}$, $\overline{\rho w''^2}$, $\overline{\rho u''h''}$, $\overline{\rho v''h''}$ as well as the terms in the thermal energy equation which have the form:

$$\overline{u''_i\left(\tau_{ij} - \frac{1}{2}\,\rho u''_i u''_j\right)}$$

Relative to these latter terms, Laufer (ref. 5) points out that, although all previous analyses have neglected them, they can be of the same order as the turbulence production terms.

The Reynolds stress equations in the axisymmetric form are

$$\frac{\partial \overline{\rho u_i'' u_k''}}{\partial t} + \frac{\partial}{\partial x} r \left[ \overline{\rho u_i'' u_k''} \, \tilde{u} - \overline{\tau_{ix} u_k''} - \overline{\tau_{kx} u_i''} + \overline{\rho u'' u_i'' u_k''} + \overline{u_k'' p} \, \delta_{i1} + \overline{u_i'' p} \, \delta_{k1} \right]$$

$$+ \frac{\partial}{\partial r} r \left[ \overline{\rho u_i'' u_k''} \, \tilde{v} - \overline{\tau_{ir} u_k''} - \overline{\tau_{kr} u_i''} + \overline{\rho v'' u_i'' u_k''} + \overline{u_k'' p} \, \delta_{i2} + \overline{u_i'' p} \, \delta_{k2} \right]$$

$$+ \delta_{k3} \left( \overline{\rho u_i'' w'' v''} - \overline{\tau_{\theta\theta} u_i''} + \overline{u_i'' p} \right) - \delta_{k2} \left( \overline{\rho u_i'' w'' w''} - \overline{\tau_{r\theta} u_i''} + \overline{u_i'' p} \right)$$

$$+ \delta_{i3} \left( \overline{\rho u_k'' v'' w''} - \overline{\tau_{r\theta} u_k''} \right) - \delta_{i2} \left( \overline{\rho u_k'' v'' w''} - \overline{\tau_{\theta\theta} u_k''} + \overline{u_k'' p} \right)$$

$$= -r \left[ \overline{\rho u_i'' u_k''} \frac{\partial \tilde{u}_i}{\partial x} + \overline{\rho u_k'' v''} \frac{\partial \tilde{u}_i}{\partial r} + \overline{\rho u_i'' u''} \frac{\partial \tilde{u}_k}{\partial x} + \overline{\rho u_i'' v''} \frac{\partial \tilde{u}_k}{\partial r} - \delta_{i3} \overline{\rho u_k'' w''} \, \tilde{v} - \delta_{k3} \overline{\rho u_i'' w''} \right]$$

$$- r \left[ \tau_{ix} \overline{\frac{\partial u_k''}{\partial x}} + \tau_{ir} \overline{\frac{\partial u_k''}{\partial r}} + \tau_{i\theta} \frac{\overline{u_k''}}{r} \overline{\frac{\partial \theta}{\theta}} + \tau_{kx} \overline{\frac{\partial u_i''}{\partial x}} + \tau_{kr} \overline{\frac{\partial u_i''}{\partial r}} + \tau_{k\theta} \frac{\overline{\theta}}{r} \overline{\frac{\partial u_i''}{\theta}} \right]$$

$$+ r \left[ \delta_{i1} \overline{p \frac{\partial u_k''}{\partial x}} + \delta_{i2} \overline{p \frac{\partial u_k''}{\partial r}} + \delta_{i3} \frac{p}{r} \overline{\frac{\partial u_k''}{\partial \theta}} + \delta_{k1} \overline{p \frac{\partial u_i''}{\partial x}} + \delta_{k2} \overline{p \frac{\partial u_i''}{\partial r}} + \delta_{k3} \frac{p}{r} \overline{\frac{\partial u_i''}{\theta}} \right]$$

(41)

where i and k take on values from 1 to 3.

The equation for the turbulence kinetic energy in axisymmetric form is

$$
\frac{\partial \bar{\rho} k r}{\partial t} + \frac{\partial}{\partial x} r\left[\bar{\rho}\bar{u}k - \overline{u''\tau_{xx}} - \overline{v''\tau_{rx}} - \overline{w''\tau_{\theta x}} + \bar{\rho}\,\overline{u''k} + \overline{u''p}\right]
$$

$$
+ \frac{\partial}{\partial r} r\left[\bar{\rho}\bar{v}k - \overline{u''\tau_{rx}} - \overline{v''\tau_{rr}} - \overline{w''\tau_{\theta r}} + \bar{\rho}\,\overline{v''k} + \overline{v''p}\right] - \overline{v''p}
$$

$$
= -r\left[\overline{\rho u''^2}\,\frac{\partial \bar{u}}{\partial x} + \overline{\rho u''v''}\left(\frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial x}\right) + \overline{\rho v''^2}\,\frac{\partial \bar{v}}{\partial r} + \overline{\rho w''^2}\,\frac{\bar{v}}{r}\right]
$$

$$
- r\left[\overline{\tau_{xx}\,\frac{\partial u''}{\partial x}} + \overline{\tau_{rx}\left(\frac{\partial u''}{\partial r} + \frac{\partial v''}{\partial x}\right)}\right]
$$

$$
+ \overline{\tau_{rr}\,\frac{\partial v''}{\partial r}} + \overline{\tau_{\theta r}\left(\frac{\partial w''}{\partial r} + \frac{1}{r}\frac{\partial v''}{\partial \theta} - \frac{w''}{r}\right)} + \overline{\tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial w''}{\partial \theta} + \frac{v''}{r}\right)} + \overline{\tau_{\theta x}\left(\frac{\partial w''}{\partial x} + \frac{1}{r}\frac{\partial u''}{\partial \theta}\right)}
$$

$$
+ r\left[\overline{p\,\frac{\partial u''}{\partial x}} + \overline{p\,\frac{\partial v''}{\partial r}} + \overline{\frac{p}{r}\frac{\partial w''}{\partial \theta}} + \frac{\overline{v''p}}{r}\right]
$$

(42)

The equations for the turbulent heat flux in axisymmetric form are

$$
\frac{\partial \overline{r\rho u_i'' h''}}{\partial t} + \frac{\partial}{\partial x} r\left(\overline{\rho u_i h''} - \overline{h'' \tau_{ix}} + \overline{u_i'' q_x} + \overline{\rho u'' u_i'' h''} + \overline{h'' p}\,\delta_{i1}\right) + \frac{\partial}{\partial r} r\left(\overline{\rho v u_i'' h''} - \overline{h'' \tau_{ir}} + \overline{u_i'' q_r}\right)
$$

$$
+ \overline{\rho v'' u_i'' h''} + \overline{h'' p}\,\delta_{i2}
$$

$$
= \delta_{i2}\left(\overline{h'' p} - \overline{h'' \tau_{\theta\theta}} + \overline{\rho w''^2 h''}\right) - \delta_{i3}\left(\overline{h'' \tau_{r\theta}} - \overline{\rho w'' v'' h''}\right) - r\left\{ \overline{\rho u'' h''}\,\frac{\partial \tilde{u}_i}{\partial x} + \overline{\rho v'' h''}\,\frac{\partial \tilde{u}_i}{\partial r} + \delta_{i3}\,\overline{\rho w'' h''}\,\frac{\tilde{v}}{r} \right.
$$

$$
+ \overline{\rho u_i'' u''}\,\frac{\partial \tilde{h}}{\partial x} + \overline{\rho u_i'' v''}\,\frac{\partial \tilde{h}}{\partial r} - \overline{u_i'' \tau_{xx}}\,\frac{\partial \tilde{u}}{\partial x} - u_i'' \tau_{rx}\,\frac{\partial \tilde{u}}{\partial r} - \overline{u_i'' \tau_{rr}}\,\frac{\partial \tilde{v}}{\partial r} + \tau_{ix}\,\frac{\partial h''}{\partial x}
$$

$$
+ \overline{\tau_{ir}\,\frac{\partial h''}{\partial r}} + \frac{\overline{\tau_{i\theta}}}{r}\frac{\partial h''}{\partial \theta} - q_x\,\frac{\partial u_i''}{\partial x} + \overline{u_i'' \tau_{xx}}\,\frac{\partial u''}{\partial x} + \overline{u_i'' \tau_{xr}}\,\frac{\partial v''}{\partial x} + \overline{u_i'' \tau_{rx}}\,\frac{\partial u''}{\partial r} + \overline{u_i'' \tau_{rr}}\,\frac{\partial v''}{\partial r}
$$

$$
+ \frac{\overline{u_i'' \tau_{x\theta}}}{r}\frac{\partial w''}{\partial x} + \overline{u_i'' \tau_{r\theta}}\,\frac{\partial w''}{\partial r} + u_i''\,\frac{\tau_{x\theta}}{r}\frac{\partial u''}{\partial \theta} + u_i''\,\frac{\tau_{r\theta}}{r}\frac{\partial v''}{\partial \theta} + u_i''\,\frac{\tau_{\theta\theta}}{r}\frac{\partial w''}{\partial \theta} - \frac{\overline{u_i'' \tau_{r\theta} w''}}{r}
$$

$$
+ \frac{\overline{u_i'' \tau_{\theta\theta} v''}}{r} + \delta_{i1}\, p\,\frac{\partial h''}{\partial x} + \delta_{i2}\, p\,\frac{\partial h''}{\partial r} + \delta_{i3}\, p\,\frac{\partial h''}{\partial \theta} + \overline{u u_i''}\,\frac{\partial p}{\partial x} + \overline{u_i'' u''}\,\frac{\partial p}{\partial r} + \overline{u_i'' v''}\,\frac{\partial p}{\partial \theta} + \frac{\overline{u_i'' w''}}{r}\frac{\partial p}{\partial \theta} + \overline{u_i''\,\frac{\partial p}{\partial t}} \left. \right\}
$$

$$
\tag{43}
$$

where i takes on values 1 and 2, for heat flux in axial and radial directions, respectively.

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COMMENTS ON THE USE OF MASS AVERAGED VARIABLES

The use of the mass-averaged variables results in mean flow turbulence
equations that contain a minimum number of terms, since terms involving $\rho'$ do
not appear explicitly, and possess almost a term by term correspondence with
their incompressible counterparts. On the other hand, the complete expressions
for the molecular shear terms $\tau_{ij}$ contain extra terms. It will be shown that
this latter condition does not seriously complicate matters.

Instantaneously,

$$\tau_{ij} = \mu\left[\left(u_{i,j} + u_{j,i}\right) - \frac{2}{3}\delta_{ij}u_{k,k}\right] \tag{44}$$

when the Stokes criterion for the second viscosity coefficient is invoked. For
compactness, we write the quantity in the braces as $S_{ij}$ so that

$$\tau_{ij} = \mu S_{ij} \tag{45}$$

When Eq. (45) is expanded in the Favre variables, there results

$$\tau_{ij} = \bar{\tau}_{ij} + \tau'_{ij} = (\tilde{\mu} + \mu'')(\tilde{S}_{ij} + S''_{ij}) \tag{46}$$

Here the viscosity is identified with the mass averaged enthalpy of Eq. (15) or
the equivalent mass averaged temperature. (See Appendix B for the interrela-
tionship of the Favre variable with the usual Reynolds expanded quantities.)
From Eq. (46)

$$\tau_{ij} = \bar{\tau}_{ij} + \tau'_{ij} = \tilde{\mu}\tilde{S}_{ij} + \tilde{\mu}S''_{ij} + \mu''\tilde{S}_{ij} + \mu''S''_{ij} \tag{47}$$

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Eq. (47), time averaged, yields

$$\bar{\tau}_{ij} = \widetilde{\tilde{\mu}\tilde{S}_{ij}} + \overline{\tilde{\mu}S''_{ij}} + \overline{\mu''\tilde{S}_{ij}} + \overline{\mu''S''_{ij}} \tag{48}$$

and

$$\tau'_{ij} = \tilde{S}_{ij}(\mu'' - \overline{\mu''}) + \tilde{\mu}(S''_{ij} - \overline{S''_{ij}}) + \mu''S''_{ij} - \overline{\mu''S''_{ij}} \tag{49}$$

In customary Reynolds expansions, not the mass-averaged coordinates, time averages of first order turbulent quantities vanish, which would have eliminated half the terms in Eqs. (48) and (49). Thus the mass-averaged notation appears to complicate the expressions involving the molecular viscosity. This additional complexity is mitigated through arguments of the relative magnitude of the individual terms of Eqs. (48) and (49) when equations as complex as Eqs. (41) through (43) are considered for solution.

# APPENDIX A

## Orthogonal Tensor Relationships

The following definitions apply for an orthogonal curvalinear coordinate system (Ref. 10). The <u>metric</u>, or length of a line element is

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 \tag{A-1}$$

where $h_1$, $h_2$, and $h_3$ are the metric elements of the orthogonal coordinates $x_1$, $x_2$ and $x_3$.*

The <u>gradient of a scalar</u>, a vector quantity, is given by

$$\phi_{,i} = \frac{1}{h_i} \frac{\partial \phi}{\partial x_i} \tag{A-2}$$

The <u>gradient of a vector</u>, a tensor quantity, is

$$A_{i,j} = \frac{1}{h_j} \frac{\partial A_i}{\partial x_j} + \delta_{ij} \sum_k \frac{A_k}{h_i h_k} \frac{\partial h_i}{\partial x_k} - \frac{A_j}{h_i h_j} \frac{\partial h_j}{\partial x_i} \tag{A-3}$$

where

$$\delta_{ij} (\text{Kronecker } \delta) = 1 \qquad i = j$$
$$= 0 \qquad i \neq j$$

Eq. (A-3) can be applied to define the <u>directional derivative of a vector</u>, namely

$$\sum_j B_j A_{i,j} = \sum_j \frac{B_j}{h_j} \frac{\partial A_i}{\partial x_j} + \sum_j \frac{B_i A_j}{h_i h_j} \frac{\partial h_i}{\partial x_j} - \sum_j \frac{B_j A_j}{h_i h_j} \frac{\partial h_j}{\partial x_i} \tag{A-4}$$

---

*Note that the summation convention of a repeated index is not used in this appendix.

This disagrees with Ref. (10) in that the subscripts of A and B in the second term on the right are interchanged there. No significant consequence occurs, however, since in Ref. (10) both B and A ultimately represent the same quantity.

Again, Eq. (A-3) can be summed to yield the divergence of a vector $A_j$.

$$\sum_j A_{j,j} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1}(h_2 h_3 A_1) + \frac{\partial}{\partial x_2}(h_1 h_3 A_2) + \frac{\partial}{\partial x_3}(h_1 h_2 A_3) \right] \tag{A-5}$$

The divergence of a tensor can be expressed from Eqs. (A-4) and (A-5) as

$$\sum_j \tau_{ij,j} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1}(h_2 h_3 \tau_{i1}) + \frac{\partial}{\partial x_2}(h_1 h_3 \tau_{i2}) + \frac{\partial}{\partial x_3}(h_1 h_2 \tau_{i3}) \right]$$

$$+ \tau_{1i} \frac{1}{h_i h_1} \frac{\partial h_i}{\partial x_1} + \tau_{2i} \frac{1}{h_i h_2} \frac{\partial h_i}{\partial x_2} + \tau_{3i} \frac{1}{h_i h_3} \frac{\partial h_i}{\partial x_3}$$

$$- \tau_{11} \frac{1}{h_1 h_i} \frac{\partial h_1}{\partial x_i} - \tau_{22} \frac{1}{h_2 h_i} \frac{\partial h_2}{\partial x_i} - \tau_{33} \frac{1}{h_3 h_i} \frac{\partial h_3}{\partial x_i} \tag{A-6}$$

Also required in the orthogonal tensor form of the equations is a tensor directional derivative analogous to the directional derivative of a vector which can be written as

$$\sum_j B_{kj} A_{i,j} = \sum_j \left[ \frac{B_{kj}}{h_j} \frac{\partial A_i}{\partial x_j} + \frac{B_{ki}}{h_i} \frac{A_j}{h_j} \frac{\partial h_i}{\partial x_j} - \frac{B_{kj}}{h_i} \frac{A_j}{h_j} \frac{\partial h_j}{\partial x_i} \right] \tag{A-7}$$

In addition, the <u>divergence of a third order tensor</u> is required. This form may be written as

$$
\sum_j E_{ijk,j} = \sum_j \left[ \frac{1}{h_j} \frac{\partial E_{ijk}}{\partial x_j} + \frac{E_{ikj}}{h_k h_j} \frac{\partial h_k}{\partial x_j} - \frac{E_{ijj}}{h_k h_j} \frac{\partial h_j}{\partial x_k} + \frac{E_{jik}}{h_i h_j} \frac{\partial h_i}{\partial x_j} - \frac{E_{jjk}}{h_i h_j} \frac{\partial h_j}{\partial x_i} \right.
$$
$$
\left. + \frac{E_{i1k}}{h_j h_1} \frac{\partial h_j}{\partial x_1} + \frac{E_{i2k}}{h_j h_2} \frac{\partial h_j}{\partial x_2} + \frac{E_{i3k}}{h_j h_3} \frac{\partial h_j}{\partial x_3} - \frac{E_{ijk}}{h_j h_j} \frac{\partial h_j}{\partial x_j} \right] \tag{A-8}
$$

## Axisymmetric Case

When particularizing to the axisymmetric coordinate system of interest here, the above general forms are simplified considerably. For the axisymmetric case we take a cylindrical coordinate system defined by

$$
\begin{aligned}
x_1 &= x & h_1 &= 1 \\
x_2 &= r & h_2 &= 1 \\
x_3 &= \theta & h_3 &= r
\end{aligned} \tag{A-9}
$$

We may then write the forms discussed previously as follows:

<u>Gradient of a Scalar:</u>

$$
\phi_{,t} = \frac{\partial \phi}{\partial t} \qquad \phi_{,1} = \frac{\partial \phi}{\partial x} \qquad \phi_{,2} = \frac{\partial \phi}{\partial r} \qquad \phi_{,3} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \tag{A-10}
$$

<u>Gradient of a vector:</u>

$$
A_{i,j} = \frac{1}{h_j} \frac{\partial A_i}{\partial x_j} + \delta_{j3} \delta_{i3} \frac{A_2}{r} - \delta_{i2} \delta_{3j} \frac{A_3}{r} \tag{A-11}
$$

<u>Divergence of a vector</u>

$$
A_{j,j} = \sum_j \frac{1}{h_j} \frac{\partial A_j}{\partial x_j} + \frac{A_2}{r} \tag{A-12}
$$

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Directional derivative of a vector

$$B_j A_{i,j} = \sum_j \frac{B_j}{h_j} \frac{\partial A_i}{\partial x_j} + \delta_{i3} \frac{B_3 A_2}{r} - \delta_{i2} \frac{B_3 A_3}{r} \qquad (A-13)$$

"Tensor directional derivative"

$$B_{kj} A_{i,j} = \sum_j \frac{B_{kj}}{h_j} \frac{\partial A_i}{\partial x_j} + \delta_{i3} \frac{h}{\cdot} \qquad \delta_{i2} \frac{B_{k3} A_3}{r} \qquad (A-14)$$

Divergence of 2nd order tensor

$$B_{ij,j} = \sum_j \frac{1}{h_j} \frac{\partial B_{ij}}{\partial x_j} + \frac{1}{r} \left\{ B_{i2} + \delta_{i3} B_{23} - \delta_{i2} B_{33} \right\} \qquad (A-15)$$

Divergence of 3rd order tensor

$$E_{ijk,j} = \sum_j \frac{1}{h_j} \frac{\partial E_{ijk}}{\partial x_j} + \frac{1}{r} \left[ \delta_{k3} E_{i32} - \delta_{k2} E_{i33} + \delta_{i3} E_{23k} - \delta_{i2} E_{33k} + E_{i2k} \right] \qquad (A-16)$$

The above forms were used, in conjunction with the conditions, $\hat{W} \equiv 0$ and $\frac{\partial ()}{\partial \theta} = 0$ deriving the axisymmetric equations presented in the body of this note.

APPENDIX B

The Relationship Between The Reynolds And

The Mass-Averaged Dependent Variables

In the past, the Reynolds system of dependent variables has been used in both analytical investigations and interpretation of experimental data. Since the equations discussed in this report are cast in terms of the mass-averaged variables introduced by Favre, it is important in relating past work to this report to know how the variables, particularly the fluctuating variables, are related in the two systems.

We begin by summarizing the definition of the dependent variables in the two systems.

| Dependent Variable | Reynolds | Mass Averaged |
|:---:|:---:|:---:|
| u | $\bar{u} + u'$ | $\tilde{u} + u''$ |
| v | $\bar{v} + v'$ | $\tilde{v} + v''$ |
| w | $\bar{w} + w'$ | $\tilde{w} + w''$ |
| h | $\bar{h} + h'$ | $\tilde{h} + h''$ |
| T | $\bar{T} + T'$ | $\tilde{T} + T''$ |
| $\mu$ | $\bar{\mu} + \mu'$ | $\tilde{\mu} + \mu''$ |
| $\rho$ | $\bar{\rho} + \rho'$ | $\bar{\rho} + \rho'$ |
| p | $\bar{p} + p'$ | $\bar{p} + p'$ |

The tilde indicates a mass-average defined in terms of the conventional time average as

$$\tilde{\alpha} \equiv \frac{\overline{\rho\alpha}}{\bar{\rho}} \tag{B-1}$$

We also note that the variables $\rho$ and $p$ are defined identically in terms of mean and fluctuating components in both systems.

To reveal the relationship between the systems we examine the characteristics of an arbitrary variable $\alpha$, i.e.

$$\alpha = \bar{\alpha} + \alpha' = \tilde{\alpha} + \alpha'' \tag{B-2}$$

irst, we note that in terms of the mass-averaged quantities

$$\overline{\rho\alpha} = \hat{\rho}\tilde{\alpha} + \overline{\rho\alpha''} = \overline{\rho\alpha} + \overline{\rho\alpha''} \tag{B-3}$$

and, thus,

$$\overline{\rho\alpha''} = 0 \tag{B-4}$$

Further, we may evaluate $\overline{\alpha''}$ from the above, since

$$0 = \overline{\rho\alpha''} = \hat{\rho}\overline{\alpha''} + \overline{\rho'\alpha''} \tag{B-5}$$

and, thus,

$$\overline{\alpha''} = - \frac{\overline{\rho'\alpha''}}{\hat{\rho}} \tag{B-6}$$

Note that, although $\overline{\alpha'} = 0$ in the Reynolds system, $\overline{\alpha''} \neq 0$ for the mass-averaged system.

In addition, since

$$\alpha = \bar{\alpha} + \alpha' = \tilde{\alpha} + \alpha'' \tag{B-7}$$

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we can time average this to show

$$\bar{\alpha} - \tilde{\alpha} = \overline{\alpha''} \tag{B-8}$$

and then show that

$$\alpha'' = \bar{\alpha} + \alpha' - \tilde{\alpha}$$

$$= \alpha' + \overline{\alpha''}$$

$$= \alpha' - \frac{\overline{\rho'\alpha'}}{\bar{\rho}} \tag{B-9}$$

Multiplying this last equation by $\rho$ and time averaging, i.e.

$$0 = \overline{\rho\alpha''} = \overline{\rho\alpha'} - \frac{\overline{\rho'\alpha''\rho}}{\bar{\rho}} = \overline{\rho'\alpha'} - \overline{\rho'\alpha''}$$

we see that

$$\overline{\rho'\alpha''} = \overline{\phantom{.}} \phantom{.} \tag{B-10}$$

and, therefore,

$$\alpha'' = \alpha' - \frac{\overline{\rho'\alpha'}}{\rho} \tag{B-11}$$

With equation (B-11), we may immediately deduce the relationship between turbulent transport of momentum and enthalpy expressed in the two systems of dependent variables. For momentum transport in the mass-averaged system, we require terms of the form

$$\overline{\rho\alpha''\beta''}$$

which can be expressed in the Reynolds system as follows:

$$\overline{\rho\alpha''\beta''} = \overline{\rho\left(\alpha' - \frac{\overline{\rho'\alpha'}}{\bar{\rho}}\right)\left(\beta' - \frac{\overline{\rho'\beta'}}{\bar{\rho}}\right)}$$

$$= \overline{\rho\alpha'\beta'} - \frac{\overline{\rho'\alpha'}\,\overline{\rho'\beta'}}{\bar{\rho}} \tag{B-12}$$

We may now examine some specific examples. Consider the turbulent shear stress term $\overline{\rho u''v''}$:

$$\overline{\rho u''v''} = \overline{\rho u'v'} - \frac{\overline{\rho'u'}\,\overline{\rho'v'}}{\bar{\rho}} \tag{B-13}$$

similarly, the normal stress term $\overline{\rho u''^2}$ is just

$$\overline{\rho u''^2} = \overline{\rho u'^2} - \frac{\left(\overline{\rho'u'}\right)^2}{\bar{\rho}} \tag{B-14}$$

The turbulent heat transfer terms are as follows

$$\overline{\rho u''h''} = \overline{\rho u'h'} - \frac{\overline{\rho'u'}\,\overline{\rho'h'}}{\bar{\rho}} \tag{B-15}$$

and

$$\overline{\rho v''h''} = \overline{\rho v'h'} - \frac{\overline{\rho'v'}\,\overline{\rho'h'}}{\bar{\rho}} \tag{B-16}$$

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