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How I stumbled upon a new (to me) construction of the inverse of a point

Anna Baccaglini-Frank

Abstract

While explaining to a friend analyst that a theorem about circle inversion that he used could be proved with synthetic geometry, I stumbled upon a new to me construction of the inverse of a point with respect to a circle. In this snapshot I describe episodes from this discovery process, faithfully to how they played out in time, highlighting the main ways in which I used dynamic geometry as a research tool.

Keywords

circle inversion, dynamic geometry software, power of a point, synthetic geometry

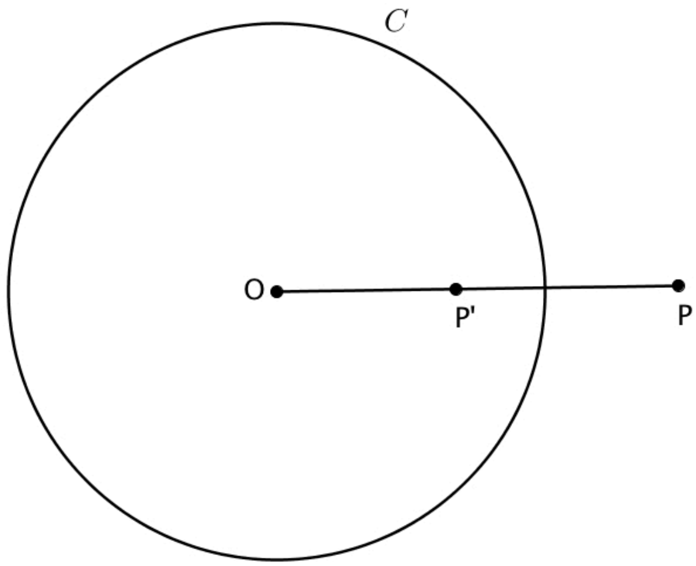
Introduction

In this paper I will be talking about a process of mathematical discovery in the field of "advanced Euclidean Geometry" (Coxeter & Greitzer, 1967). The name of this branch of geometry suggests a distinction from "elementary Euclidean geometry", the geometry that is contained in Euclid's writings. In many countries high school geometry includes topics from elementary Euclidean geometry, so the reader will probably be familiar with them. Advanced Euclidean geometry is the geometry discovered after Euclid's death by mathematicians who have continued to develop Euclidean geometry, discovering many new interesting relationships. Some people also refer to this branch as "synthetic geometry" (Scimemi, 2012), to highlight that they do not use analytic geometry to prove the theorems.

In particular, notions that I will be referring to are circle inversion and the power of a point with respect to a circle. For the reader's reference I define them here.

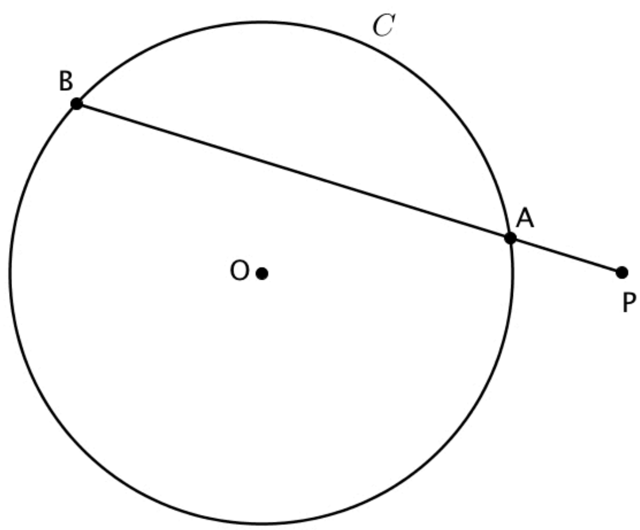
The *inverse of a point* P *with respect to a circle* C with radius r and center O is a point P' on the ray OP such that $OP \cdot OP' = r^2$ (P is assumed different from O)[1].

[1] In the proofs presented in this paper the notation AB indicates the (positive) length of the segment with endpoints in A and B.



caption: P' is the inverse of P with respect to the circle C.

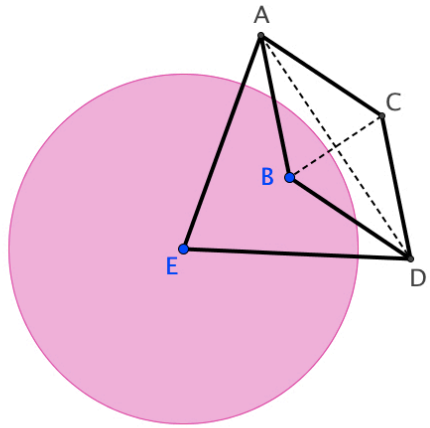
The *power of a point* P *with respect to a circle* $C, \prod_C(P)$, is the product $PA \cdot PB$ (where A and B are the intersection points of any line through P and through the circle C).



caption: The power of P with respect to C is the product of the lengths PA and PB.

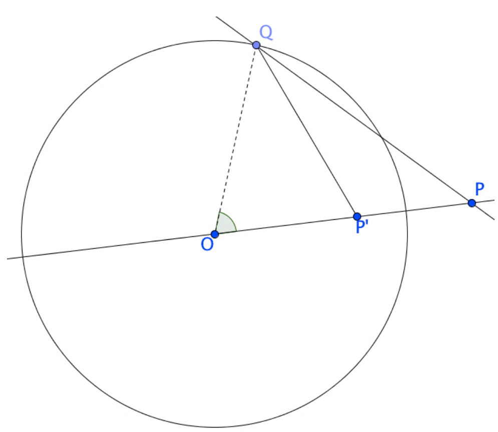
Episode 0

I am a mathematics educator working in a mathematics department; many of my neighbors on the ground floor hallway, where my office is, are analysts with whom I enjoy numerous discussions over coffee. One morning one of these neighbors came in while I was preparing for a new (for me) course I would be teaching to graduate students the following semester; a substantial part of the course includes topics in advanced Euclidean geometry. That morning I was refreshing my knowledge about circle inversion: I had sketched different constructions for an inverse of a point with respect to a circle with chalk on my blackboard, and open on my iPad was a reconstruction of Peaucellier's inversion machine that I had constructed using an app for dynamic geometry.



caption: The reconstruction of Peaucellier's inversion machine that I had made on my iPad using dynamic geometry.

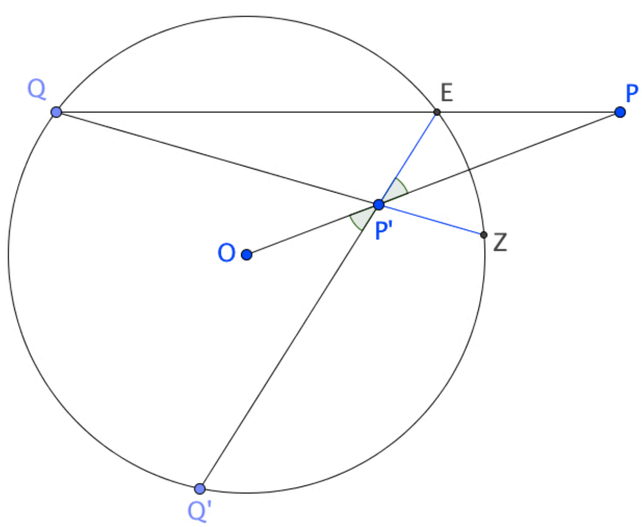
The inversion machine is a linkage made of six bars with joints at points A, B, C, D. Point E is fixed and is the center of the circle of inversion. Point B can be moved (inside or outside of the circle of inversion) and C is the inverse of B. The circle of inversion is not part of the machine, but I had represented it to visualize properties of the inversion (its radius is $\sqrt{AE^2 - AB^2}$).
My friend analyst wanted to see how the dynamic construction worked and to play around with it a bit. Then he said: "I don't know much about this kind of geometry, but I do use circle inversion for some theorems I teach. We use a "magic" fact about circle inversion, that is that the ratio between PQ and $P'Q$, where P' is the inverse of P with respect to the circle C, on which Q lies (Fig. 1), is constant." (we will refer to this as *Theorem 1* in this paper).



caption: The ratio between PQ and $P'Q$, where P' is the inverse of P with respect to the circle C, on which Q lies, is constant.

Then he proceeded to write down on the blackboard an analytical proof in an n-dimensional space for the theorem. I gazed at the figure while he was writing down the proof and mumbled (too loudly): “I think this can be done in synthetic geometry.” He heard and at that moment I felt “condemned” to finding a way to prove my conjecture.
During the next few days I proved not only the conjecture I was after, but I also stumbled upon some lovely invariant properties of circle inversion that I had not found in the books I had been studying from, and that led me to a new (to me) way of constructing the inverse of a point.
In the rest of this snapshot I will describe episodes from this discovery process, faithfully to how they played out in time (so results will not be presented in the formal way used in usual mathematical publications), highlighting the main ways in which I used dynamic geometry as a research tool.

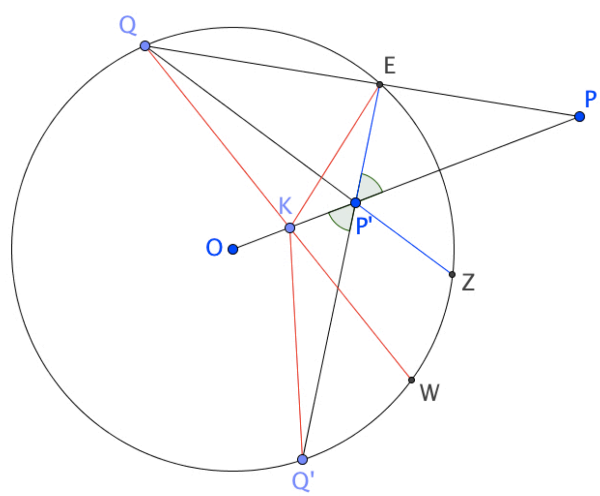
Episode 1: a (failed) search for similar triangles
I had a hunch I could work out the problem using similar triangles, and, at most, the notion of power of a point with respect to a circle, since I was interested in ratios (or products). I started looking for similar triangles, adding points and lines to my initial figure (Fig. 4) constructed in dynamic geometry. The first segment I drew was P’E, since the statement must hold both for Q and for E (that is, $\frac{QP'}{QP} = constant = \frac{EP'}{EP}$), because, more in general, the statement holds for any point X on the circle of inversion. But why look only at triangles “inside” P’QP? I also thought that definitely, if I were to use the notion of power of a point (I was thinking of the power of P’, in particular), I was going to need to consider lines through P’ and their intersections with the circle. So I thought of chords through P’ and drew QZ (I constructed Z as the second intersection of QP’ with the circle), and then constructed Q’ as the image of Q through reflectional symmetry across OP (Fig. 5).



Caption: Z is the second intersection of QP’ with C and Q’ as the image of Q through reflection symmetry over OP.

I tried dragging Q a bit (along the circle because it was constructed as a "point on the circle") and what I saw were "chords" QZ and Q'E "folding onto" one another when Q and Q' switched sides across OP; Z seemed to be symmetric (again across OP) to E, and Q', P', E seemed to *always* be lined up. I even checked the measures of angles $\angle OP'Q'$ and $\angle PP'E$: sure enough, they *always* seemed to be the same.
I hadn't found the similar triangles I was looking for, but now I was intrigued by this new regularity and I wondered if it happened all so "magically" just because of the position of P' given by its being the inverse of P over C.

Episode 2: stumbling upon a new construction for the inverse of a point
I wanted to make sure it was really P', the inverse of P, that was "special"; so I constructed a point K on OP (I was loosely thinking of point K as a possibly "less constrained" representation of the inverse of P), joined it with segments to Q, to Q' and to E, and found the second intersection, W, of QK with the circle (Fig. 6). As K moved along OP, and Q around the circle, never were Q', K and E collinear nor was W symmetric with respect to E, except for when K coincided with P' (in this case both invariants were present for any position of Q). I found this part of the exploration to be very convincing as to the fact that only for K coincident with P' did those interesting invariants seem true.

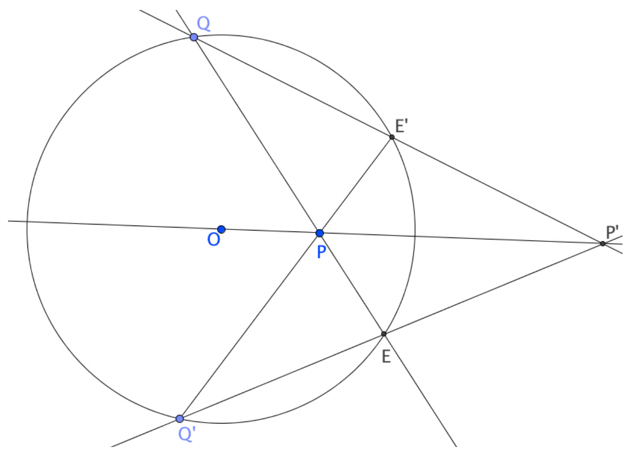


caption: Only when K = P', Q', K, E are collinear, and W is the image of E through reflectional symmetry over OP.

So I went a step further, and decided to assume these invariants true if and only if K = P', and used them to construct K so that it would have the desired properties, thus finding an alternative way of constructing P'…if eventually I had been able to prove that

Conjecture 1: Q', K, E are collinear if and only if K is the inverse of P with respect to a circle C with center O and radius r, that is, if O, P and P' are collinear and if $OP \cdot OP' = r^2$.

The construction is as follows (both in the case of P inside or outside of the circle): Construct a point Q on the circle and connect it with P; on the line P and Q construct the second intersection, E, with the circle *C.* Construct the segment joining the center O of the circle with P, and reflect QE on OP; mark the symmetric points Q' and E' to Q and E respectively. Construct lines through Q and E' and through E and Q'. The two lines meet at a point on the line through O and P, that is the inverse of P with respect to the circle (Fig. 7).

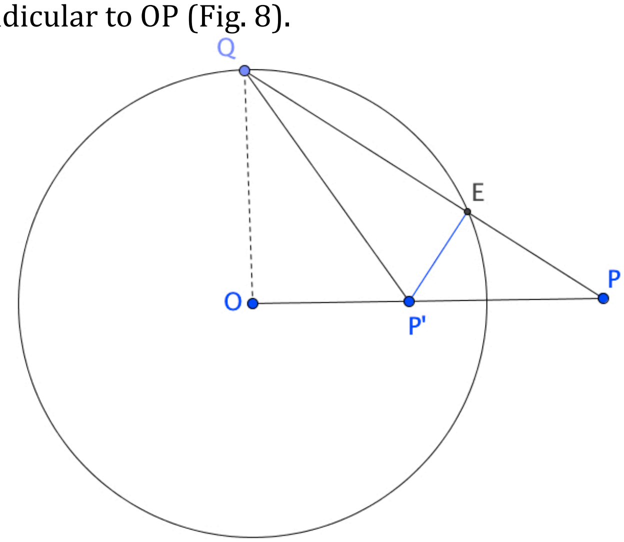


Caption: New construction of the inverse of a point P with respect to a circle C

So now instead of proving what I initially wanted, I had generated a new conjecture that I wanted to prove even more than the first one. However, diligently, I went back to my original problem.

Episode 3: back to the analyst's problem

I continued my search for similar triangles, and the third morning, walking to the department, I tried working with a particular configuration in my head: one in which QO is perpendicular to OP (Fig. 8).



Caption: Simpler configuration I imagined in my head.

Sure enough, the choice of the particular case to analyze mentally was a good one: I was able to see to very promising similar triangles: QOP' and P'EP, both similar to QOP. I worked out the proportions in my mind and saw that indeed these triangles should be similar in general. As soon as I got to the office I pulled out my iPad and dragged around Q to empirically check the similarities. The dragging confirmed my conjecture, and I wrote a proof for my friend analyst.

Proof (of Theorem 1):
We assume that $OP \cdot OP' = r^2$ (where r is the radius of C, on which Q is chosen), because P' is the inverse of P.
This is equivalent to $OP'/r = r/OP$ (as long as r is not 0 and P does not coincide with O), and also to $OP'/QO = QO/OP$.
Now, since triangles QOP and QOP' have $\angle QOP'$ in common, and P and P' are different points, they are similar because of the last proportion written.
So, we can write a proportion involving the third sides of the similar triangles QOP and QOP', and get:
$\frac{QP'}{OP'} = \frac{QP}{QO}$, that is $\frac{QP'}{\frac{r^2}{OP}} = \frac{QP}{r}$, that is $\frac{QP'}{QP} = \frac{r}{OP}$.
This proved the analyst's theorem (*Theorem 1*) with synthetic geometry, including the exact constant as a function of r and OP.

Episode 4: proving collinearity
Now I was determined to also prove that the new construction of the inverse of a point that I had stumbled upon was indeed a proper one, so I worked on this a bit longer and finally was able to prove the property that I was heavily using in it: the collinearity of Q', P' and E (refer to Fig. 5 or Fig. 9).
Theorem 2:
Given a circle C with center O and radius r, with Q a point of the circle and P any point, the points Q', P', E are collinear if and only if $OP \cdot OP' = r^2$.

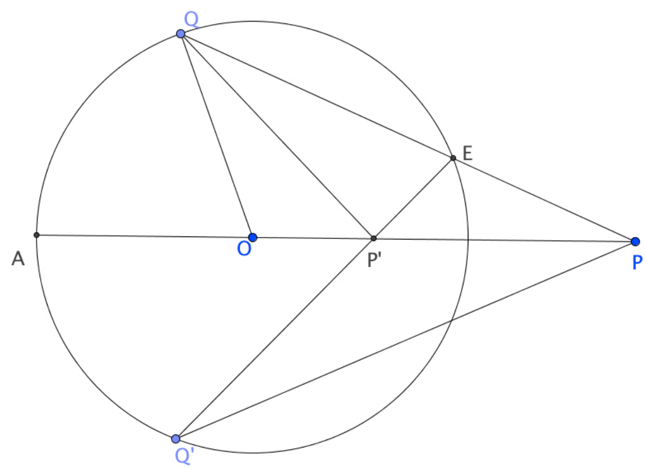
Proof (of Theorem 2):

Assuming that P' is the inverse of P, we have that triangle OQP' is similar to triangle OQP (see proof of Theorem 1), so $\angle OQP' = \angle QPO$.
Moreover, triangles EP'P and OQP are similar, because $PE/PP' = OP/PQ$ (which I prove below), and angle ∠QPO is common to both triangles.
Knowing that $OP \cdot OP' = r^2$, we can write $OP^2 - OP \cdot OP' = OP^2 - r^2$ (change sign to the initial terms and add OP^2 to both sides). Both terms now can be seen as the power[2] of P with respect to C. Substituting $OP - OP'$ with $P'P$, we get the following equality between products: $OP \cdot P'P = OP^2 - r^2 = PE \cdot PQ$, which is $PE/P'P =$

[2] If E and Q are points of intersection of a line through P with circle C, then the power of the point $PE \cdot PQ$ can also be written as $|OP^2 - r^2|$.

OP/PQ. Therefore $\angle OP'Q = \angle PQO = \angle EP'P$, and by construction $\angle Q'P'O = \angle OP'Q$, so $\angle Q'P'O = \angle EP'P$, which proves that points Q', P' and E are collinear.

Vice versa, if points Q', P' and E are collinear, triangles OQP' and EP'P are similar because two of their angles are respectively congruent: $\angle QP'O = \angle PP'E$ and $\angle P'QO = \angle P'PE$, as we prove below.
If A is the intersection of line P'O with C, on the side opposite to P (see Fig. 9), then $\angle QOA = \frac{1}{2}\angle QOQ' = \angle QEQ' = \angle QEP'$ (because $\angle QOQ'$ and $\angle QEQ'$ are respectively a central angle and an inscribed angle insisting on the same arc) and $\angle P'QO = \angle QOA - \angle QP'O = \angle QEP' - \angle PP'E = \angle P'PE$.



caption: A is the intersection of line P'O with C, on the side opposite to P; $\angle QOQ'$ and $\angle QEQ'$ are, respectively, a central angle and an inscribed angle insisting on the same arc.

From the similarity of triangles OQP' and EP'P, we have, in particular,

$$OP'/QP' = P'E/P'P$$

which is
$OP' \cdot PP' = QP' \cdot P'E$, and since $QP' = Q'P'$,
$OP' \cdot PP' = Q'P' \cdot P'E = \prod_C(P') = r^2 - (OP')^2$.
Substituting PP' with $PO - P'O$, we get the following equality $OP' \cdot (PO - P'O) = r^2 - OP'^2$, which is

$$OP' \cdot PO - (OP')^2 = r^2 - (OP')^2.$$

So $OP \cdot OP' = r^2$, as we wanted to prove.

The role of dynamic geometry in the episodes
Studies have shown that figures constructed in dynamic geometry environments can be interpreted in fundamentally two different ways, as Sinclair and Robutti describe (2013): in a first way, according to which the dynamic figure constitutes a discrete (and very large) set of "static" examples (Marrades & Gutierrez, 2000); or in a second way, according to which the dynamic figure is seen as an entity whose behavior is perceived and explored as a whole (Laborde, 1992; Battista, 2008).

Indeed, the authors state: "It is still unclear whether learners somehow naturally see the draggable diagrams as a series of examples or as one continuously changing object, and whether this depends on their previous exposure to the static geometric discourse of the typical classroom […]" (Sinclair & Robutti, 2013, p. 574).
In my personal experience, I have come to "see" dynamic figures (even the same figure) both as very large sets of discrete static examples, and as a sort of "continuous whole" to be explored and played with. Of course this way of seeing figures comes from my educational experiences with the software: initially (in 2003, as an undergraduate) I was introduced to Cabri by a mathematician, who clearly saw the software as a powerful generator of static examples, so of course I took on this perspective; later, when I studied dynamic geometry in depth as a graduate student (see Baccaglini-Frank & Mariotti, 2010), I also experienced the power of the "continuous whole" interpretation. So now I find myself happily switching from one way of interpreting the dynamic figure to the other, and back, without even noticing. Thinking back to how I used (and interpreted feedback from) the dynamic geometry software during the episodes from my mathematical explorations on circle inversion, I noticed that I jumped back and forth between these two interpretations.

In Episode 1 I used the software to generate static configurations as I searched for similar triangles, changing the figure only to try and notice examples or counter-examples when I thought I had found similar triangles. However, at the end of the episode, I perceived the dynamic geometry construction as a whole moving continuously: I saw the chords through P' "collapsing" and "switching places" as I dragged Q along the circle. This butterfly-like motion made me think of reflectional symmetry along the line through OP, leading to *Conjecture 1*.

In Episode 2 I used the software to test *Conjecture 1*, though a dragging test (Baccaglini-Frank & Mariotti, 2010). I empirically verified that K "worked" only when it coincided with P', and I saw this during dragging, interpreting the feedback as a very large number of different sketches of which only one was a "good case". This convinced me that the collinearity of Q', K and E was a necessary and sufficient condition for K to be P', the inverse of P.

In Episode 3, I again used the software to perform a dragging test for the similar triangles I had thought of. In this case I could not find a counter example in the many many examples generated through dragging, which confirmed my belief in the similarity of the triangles considered, the key idea for the proof of *Theorem 1*.

Finally, in Episode 4, I left a static figure on the screen and worked on the equations with paper and pen, so the software played the role of simple pen and paper (with the difference that the figure was more precise).

For the classroom
Generating conjectures and proving them are important *mathematical habits of mind* (Cuoco, Goldenberg & Mark, 1996), that should be promoted in the mathematics classroom. Geometry is a perfect setting to develop and practice such

habits. The specific topic in this paper is circle inversion, which is not usually treated in high school (at least in Italy), but in undergraduate and graduate geometry courses at some mathematics departments, such as the one where I teach. The rationale for teaching circle inversion and some of its properties as part of the "synthetic geometry" topics is fostering deeper knowledge of Euclidean geometry, but also better understanding how Poincare's disk model for hyperbolic geometry works. Although this year in the course I never made it to the particular findings in this paper, I have constantly asked the students to use dynamic geometry as a tool of exploration for generating and testing conjectures, and searching for key properties that might be useful in the proofs of the conjectures. Over the years I have noticed that undergraduate and graduate students seem to quickly come to appreciate the power of dynamic geometry, and are able to (more or less spontaneously) interpret dynamic figures in the two ways described above, using them to carry out various "mathematical habits of mind".

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