

ON 3-BRAID KNOTS OF FINITE CONCORDANCE ORDER

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ABSTRACT. We study 3-braid knots of finite smooth concordance order. A corollary of our main result is that a chiral 3-braid knot of finite concordance order is ribbon.

1. INTRODUCTION

In this paper we investigate 3-braid knots of finite concordance order. We work in the smooth category, therefore the words ‘slice’ and ‘concordance’ will always mean, respectively, ‘smoothly slice’ and ‘smooth concordance’. The *reverse* of an oriented knot will be denoted $-K$, and K^m will denote the *mirror image* of K . In this notation, a connected sum of the form $K\#(-K^m)$ is always a *slice* knot, i.e. it bounds a properly embedded disk $D \subset B^4$. Recall that a knot is *ribbon* if it bounds a properly embedded disk $D \subset B^4$ such that the restriction of the radial function $B^4 \rightarrow [0, 1]$ has no local maxima in the interior of D . That each slice knot is ribbon is the content of the well-known slice-ribbon conjecture. A knot $K \subset S^3$ is *amphichiral* if K^m is isotopic to either K or $-K$. The knot K is *chiral* if it is not amphichiral. Recall that the classical concordance group \mathcal{C} is the set of equivalence classes $[K]$ of oriented knots $K \subset S^3$ with respect to the equivalence relation which declares K_1 and K_2 equivalent if $K_1\#(-K_2^m)$ is slice. The group operation is induced by connected sum and $0 \in \mathcal{C}$ is the concordance class of the unknot. Several facts are known about the structure of \mathcal{C} , but its torsion subgroup is not understood, see e.g. [17]. We will say that a knot K is of *finite concordance order* if $[K] \in \mathcal{C}$ is of finite order.

In [1] Baldwin obtained some information on 3-braid knots of finite concordance order by computing a certain correction term of the Heegaard Floer homology of the two-fold branched cover (see Section 2 for more details). We will use his result together with constraints obtained via Donaldson’s ‘Theorem A’ [6] to establish Theorem 1.1 below, which is our main result. Our approach here is similar to the one used in [14, 15], where the concordance orders of 2-bridge knots are determined. We point out that the idea of combining information coming from Heegaard Floer correction terms with Donaldson’s Theorem A was also used in [5, 9, 10, 12, 13].

Before we can state Theorem 1.1 we need to introduce some terminology. Recall that a *symmetric union* knot is a special kind of ribbon knot first introduced by Kinoshita and Terasaka [11]. It is unknown whether every ribbon knot is a symmetric union. A braid $\beta \in B_n$ is called *quasi-positive* if it can be written as a product of conjugates of the standard generators $\sigma_1, \dots, \sigma_{n-1} \in B_n$, and *quasi-negative* if β^{-1} is quasi-positive. A knot K is *quasi-positive* (respectively *quasi-negative*) if K is the closure of a quasi-positive (respectively quasi-negative) braid.

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We now recall the notion of ‘blowup’ from [16]. Let \mathbb{N} be the set of (positive) natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and let $k \in \mathbb{N}$. We say that $\hat{z} \in \mathbb{N}_0^{k+1}$ is a *blow-up* of $z = (n_1, \dots, n_k) \in \mathbb{N}_0^k$ if

$$\hat{z} = \begin{cases} (1, n_1 + 1, n_2, \dots, n_{k-1}, n_k + 1), \text{ or} \\ (n_1, \dots, n_i + 1, 1, n_{i+1} + 1, \dots, n_k), \text{ for some } 1 \leq i < k, \text{ or} \\ (n_1 + 1, n_2, \dots, n_{k-1}, n_k + 1, 1). \end{cases}$$

There is a well-known isomorphism between the 3-braid group and the mapping class group of the one-holed torus. Let T be an oriented, one-holed torus, and let $\text{Mod}(T)$ be the group of isotopy classes of orientation-preserving diffeomorphisms of T , where isotopies are required to fix the boundary pointwise. Let $x, y \in \text{Mod}(T)$ be right-handed Dehn twists along two simple closed curves in T intersecting transversely once. The group $\text{Mod}(T)$ is generated by x and y subject to the relation $xyx = yxy$, and there is an isomorphism from $\psi : B_3 \rightarrow \text{Mod}(T)$ sending the standard generators σ_1, σ_2 to x , respectively y . The isomorphism ψ can be realized geometrically by viewing T as a two-fold branched cover over the 2-disk with three branch points: elements of B_3 , viewed as automorphisms of the triply-pointed disk, lift uniquely to elements of $\text{Mod}(T)$. It is easy to check that an element $h \in \text{Mod}(T)$ is the image under ψ of a quasi-positive 3-braid if and only if h can be written as a product of right-handed Dehn twists.

Keeping the isomorphism ψ in mind, it is easy to check that by [16, Theorem 2.3], if (s_1, \dots, s_N) is obtained from $(0, 0)$ via a sequence of blowups and the string

$$(c_1, \dots, c_N) := (x_1 + 2, \overbrace{2, \dots, 2}^{y_1-1}, \dots, x_t + 2, \overbrace{2, \dots, 2}^{y_t-1})$$

satisfies $c_i \geq s_i$ for $i = 1, \dots, N$, then the 3-braid $(\sigma_1 \sigma_2)^3 \prod_{i=1}^t \sigma_1^{x_i} \sigma_2^{-y_i}$ is quasi-positive.

Observe that any string obtained from $(0, 0)$ via a sequence of blowups contains always at least two 1’s and, typically, more than two 1’s.

Examples. (1) According to Knotinfo [4], the knot 12_{n0721} is slice, chiral, and equal to the closure of the 3-braid $\alpha = \sigma_1 \sigma_2^2 \sigma_1^4 \sigma_2^{-5}$. Applying [19, Proposition 2.1] it is easy to check that α is conjugate to $(\sigma_1 \sigma_2)^3 \sigma_1^3 \sigma_2^{-7}$, whose associated string of integers is obtained by changing the two 1’s into 2’s in $(5, 1, 2, 2, 2, 2, 1)$. Moreover, $(5, 1, 2, 2, 2, 2, 1)$ is an iterated blowup of $(0, 0)$:

$$(5, 1, 2, 2, 2, 2, 1) \rightarrow (4, 1, 2, 2, 2, 1) \rightarrow (3, 1, 2, 2, 1) \rightarrow (2, 1, 2, 1) \rightarrow (1, 1, 1) \rightarrow (0, 0).$$

It follows from [16, Theorem 2.3] that α is quasi-positive.

(2) As another example one may consider the knot 12_{n0708} , which according to Knotinfo is slice, chiral and equal to the closure of the 3-braid $\beta = \sigma_1 \sigma_2^{-3} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^3$. Applying [19, Proposition 2.1] one can check that β is conjugate to $(\sigma_1 \sigma_2)^3 \sigma_1 \sigma_2^{-3} \sigma_1^2 \sigma_2^{-4}$, corresponding to the string obtained changing the two 1’s into 2’s in $(3, 1, 2, 4, 1, 2, 2)$, which is an iterated blowup of $(0, 0)$. Therefore 12_{n0708} is also quasi-positive. Previously, the quasi-positivity of 12_{n0708} and 12_{n0721} appear to have been unknown; compare [4]¹.

Before we can state our main result we need a little more terminology. Let $P \subset \mathbb{R}^2$ be a regular polygon with $t \geq 2$ vertices. Let $V_P = \{v_1, \dots, v_t\}$ and $E_P = \{e_1, \dots, e_t\}$ be the sets of vertices and edges of P , indexed so that, for each $i = 1, \dots, t-1$, the edge e_i is between v_i and

¹Although there exists an algorithm to establish the quasi-positivity of any 3-braid (but not the quasi-positivity of any 3-braid closure) [20].

v_{i+1} . A labelling of P will be a pair (X, Y) of maps $X : V_P \rightarrow \mathbb{N}$ and $Y : E_P \rightarrow \mathbb{N}$. We say that a $2t$ -uple $(x_1, y_1, x_2, y_2, \dots, x_t, y_t) \in \mathbb{N}^{2t}$ encodes a labelling (X, Y) of P if

$$(x_1, y_1, x_2, y_2, \dots, x_t, y_t) = (X(v_1), Y(e_1), \dots, X(v_t), Y(e_t)).$$

Given a labelling (X, Y) and a symmetry $\varphi : P \rightarrow P$ of P , one can define a new labelling of P by setting $(X, Y)^\varphi := (X \circ \varphi_V, Y \circ \varphi_E)$, where $\varphi_V : V_P \rightarrow V_P$ and $\varphi_E : E_P \rightarrow E_P$ are the 1-1 maps induced by φ . We are now ready to state our main result.

Theorem 1.1. *Let $K \subset S^3$ be a 3-braid knot of finite concordance order. Then, one of the following holds:*

- (1) either K or K^m is the closure of a 3-braid β of the form

$$\beta = (\sigma_1 \sigma_2)^3 \sigma_1^{x_1} \sigma_2^{-y_1} \dots \sigma_1^{x_t} \sigma_2^{-y_t}, \quad t, x_i, y_i \geq 1,$$

where $\sum_i y_i = \sum_i x_i + 4$ and the string of positive integers $(x_1 + 2, \overbrace{2, \dots, 2}^{y_1 - 1}, \dots, x_t + 2, \overbrace{2, \dots, 2}^{y_t - 1})$ is obtained from an iterated blowup of $(0, 0)$ by replacing two 1's with 2's. Moreover, β is quasi-positive and K is ribbon.

- (2) K is a symmetric union of the form L_a given in Figure 1, where $a \in B_3$;

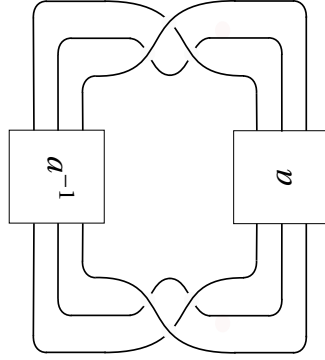


FIGURE 1. The link L_a

- (3) K is isotopic to the closure of a 3-braid β of the form

$$\beta = \sigma_1^{x_1} \sigma_2^{-y_1} \dots \sigma_1^{x_t} \sigma_2^{-y_t}, \quad t \geq 2, x_1, \dots, x_t \geq 1,$$

where $\sum_i y_i = \sum_i x_i$ and there exist a regular polygon P with t vertices and a symmetry $\varphi : P \rightarrow P$ such that $(X, Y) = (Y, X')^\varphi$, where (X, Y) is the labelling encoded by $(x_1, y_1, \dots, x_t, y_t)$ and (Y, X') is the labelling encoded by $(y_1, x_2, y_2, x_3, \dots, y_t, x_1)$. Moreover, K is amphichiral.

It is natural to wonder how the three families of knots appearing in Theorem 1.1 intersect each other. It is easy to check that a knot belonging to Family (1) is the closure of a 3-braid β satisfying $e(\beta) = \pm 2$, where $e : B_3 \rightarrow \mathbb{Z}$ denotes the abelianization homomorphism (exponent sum with respect to the standard generators σ_i). On the other hand, a knot K belonging to Family (2) or Family (3) is the closure of a 3-braid β with $e(\beta) = 0$. By the main result of [2], a link which can be represented as a 3-braid closure admits a unique conjugacy class of 3-braid representatives, with the exceptions of (i) the unknot, which can be represented only by the

conjugacy classes of $\sigma_1\sigma_2$, $\sigma_1\sigma_2^{-1}$ or $\sigma_1^{-1}\sigma_2^{-1}$, (ii) a type $(2, k)$ torus link with $k \neq \pm 1$ and (iii) a special class of links admitting at most two conjugacy classes of 3–braid representatives having the same exponent sum. By the results of [19] none of the 3–braids giving rise to the knots of Family (1) is conjugate to either $\sigma_1\sigma_2$ or $\sigma_1^{-1}\sigma_2^{-1}$, therefore the unknot does not belong to the first family. Moreover, a $(2, k)$ –torus knot with $k \neq \pm 1$ has non–vanishing signature, while clearly knots belonging to Family (2) have vanishing signature. By the computations of [7] (see the proof of Lemma 2.1) knots belonging to Family (3) also have vanishing signature. Therefore we can conclude that Family (1) is disjoint from the union of Families (2) and (3). On the other hand, there are knots belonging simultaneously to Families (2) and (3). The easiest example is the knot 8_9 , which coincides with $L_{\sigma_1\sigma_2^{-1}\sigma_1^2}$ and therefore belongs to Family (2), while according to Knotinfo [4] it is the closure of $\sigma^3\sigma_2^{-1}\sigma_1\sigma_2^{-3}$ and therefore belongs to Family (3) as well. As a final comment we point out that both Families (1) and (2) contain chiral knots. For instance, according to Knotinfo the 3–braid slice knot 8_{20} is chiral and equal to the closure of a quasi–negative 3–braid, therefore it belongs to Family (1). More examples of chiral knots belonging to Family (1) are 12_{n0708} and 12_{n0721} described in the examples before Theorem 1.1. Similarly, the 3–braid slice knots 10_{48} and 12_{a1011} are chiral and neither quasi–positive nor quasi–negative, therefore they belong to Family (2).

The following corollary follows immediately from Theorem 1.1.

Corollary 1.2. *A chiral 3–braid knot of finite concordance order is ribbon.* □

Remarks. (1) Corollary 1.2 holds for 2–bridge knots. Indeed, by [15, Corollary 1.3] a chiral 2–bridge knot of finite concordance order is slice, and by [14] a slice 2–bridge knot is ribbon.

(2) It is not difficult to find chiral knots of finite concordance order which do not satisfy the conclusion of Corollary 1.2 (and therefore cannot be closures of 3–braids). For instance, according to Knotinfo [4] the chiral 4–braid knots 9_{24} and 9_{37} have concordance order 2.

(3) There are 185 knots with crossing number at most twelve and braid index 3. Among these, 12 are ribbon (some of which quasi–positive or quasi–negative), 17 are amphichiral and have concordance order 2, while 151 have infinite concordance order. The remaining knots 10_{91} , 12_{a1199} , 12_{a1222} , 12_{a1231} and 12_{a1258} are chiral and non–slice. In view of Corollary 1.2, the five knots above have infinite concordance order. Previously, their concordance orders appear to have been unknown; compare [4].

Theorem 1.1 naturally leads to the following problem, to which we hope to return in the future.

Problem. Determine the concordance orders of the knots in Family (3) of Theorem 1.1.

The paper is organized as follows. In Section 2 we collect some preliminary results on knots of finite concordance order which are 3–braid closures. The purpose of the section is to prove Proposition 2.6, which uses Donaldson’s ‘Theorem A’ to show that if a 3–braid knot K has concordance order k , then the orthogonal sum of k copies of a certain negative definite integral lattice embeds isometrically in the standard negative definite lattice of the same rank. In Section 3 we draw the lattice–theoretical consequences of the existence of the isometric embedding given by Proposition 2.6. Section 3 contains the bulk of the technical work. In Section 4 we prove Theorem 1.1 using the results of Sections 2 and 3.

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2. PRELIMINARIES

The following simple lemma uses the slice–Bennequin inequality [22] to establish a basic property of braid closures having finite concordance order.

Lemma 2.1. *Let $K \subset S^3$ be a knot of finite concordance order which is the closure of a braid $\beta \in B_n$. Then,*

$$1 - n \leq e(\beta) \leq n - 1,$$

where $e : B_n \rightarrow \mathbb{Z}$ is the exponent sum homomorphism.

Proof. Recall that, if a knot $N \subset S^3$ is the closure of an n –braid $\gamma \in B_n$, the slice Bennequin inequality [22] reads

$$(2.1) \quad 1 - g_s(N) \leq n - e(\gamma),$$

where $g_s(N)$ is the slice genus of N . Now observe that, for each $m \geq 1$, the knot $mK := K \# \overset{(m)}{\dots}$ is the closure of the mn –braid η described in Figure 2. If K has concordance order m then

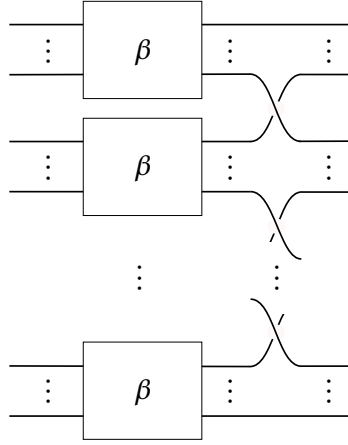


FIGURE 2. The mn –braid η with $\hat{\eta} = mK$

$g_s(mK) = 0$. Since $e(\eta) = me(\beta) + m - 1$, applying Inequality (2.1) to η we get the inequality $e(\beta) \leq n - 1$. Notice that the mirror image of K is a knot of concordance order m which is the closure of a 3–braid $\bar{\beta}$ such that $e(\bar{\beta}) = -e(\beta)$. Arguing as before we have $e(\bar{\beta}) \leq n - 1$, and the statement follows. \square

In [1] Baldwin combines Murasugi’s normal form for 3–braid closures [19] and the signature computations of [7] with computations of the concordance invariant δ defined by Manolescu–Owens [18] to establish the following proposition. Here we provide a short proof of Baldwin’s result based on [19, 7] and Lemma 2.1.

Proposition 2.2 ([1, Prop. 8.6]). *Let $K \subset S^3$ be a 3–braid knot of finite concordance order. Then, K is the closure of a 3–braid $\beta \in B_3$ of the form*

$$(2.2) \quad \beta = (\sigma_1 \sigma_2)^{3d} \sigma_1^{x_1} \sigma_2^{-y_1} \cdots \sigma_1^{x_t} \sigma_2^{-y_t}, \quad t, x_i, y_i \geq 1,$$

where $d \in \{-1, 0, 1\}$ and $\sum_{i=1}^t (x_i - y_i) = -4d$. Moreover, if $d = \pm 1$ then up to replacing K with K^m one can take $d = 1$ in Equation (2.2).

Proof. As observed in [1, Remark 8.4], the results of [19] immediately imply that a 3–braid knot of finite concordance order is either the unknot or is isotopic to the closure $\hat{\beta}$ of a 3–braid β of the form:

$$(2.3) \quad \beta = (\sigma_1 \sigma_2 \sigma_1)^{2d} \sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_n}, \quad a_i \geq 0, \text{ some } a_j > 0,$$

where $d \in \mathbb{Z}$. By [7] K has signature

$$(2.4) \quad \sigma(K) = -n - 4d + \sum_{i=1}^n a_i = 2d - e(\beta),$$

where $e : B_3 \rightarrow \mathbb{Z}$ is the exponent sum homomorphism. Since $\sigma : \mathcal{C} \rightarrow \mathbb{Z}$ is a homomorphism, the fact that K has finite concordance order implies $\sigma(K) = 0$, therefore $e(\beta) = 2d$. Moreover, Lemma 2.1 implies $d \in \{-1, 0, 1\}$. Since $(\sigma_2 \sigma_1 \sigma_2)^2 = (\sigma_1 \sigma_2 \sigma_1)^2 = (\sigma_1 \sigma_2)^3$ in B_3 , if $d = 1$ we are done. If $d = -1$ and K is the closure of $(\sigma_1 \sigma_2 \sigma_1)^{-2} \sigma_1^{x_1} \sigma_2^{-y_1} \cdots \sigma_1^{x_t} \sigma_2^{-y_t}$ then K^m is the closure of

$$(\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1})^{-2} \sigma_1^{-x_1} \sigma_2^{y_1} \cdots \sigma_1^{-x_t} \sigma_2^{y_t} = (\sigma_1 \sigma_2 \sigma_1)^2 \sigma_1^{-x_1} \sigma_2^{y_1} \cdots \sigma_1^{-x_t} \sigma_2^{y_t}.$$

If $f : B_3 \rightarrow B_3$ is the automorphism which sends σ_1 to σ_2 and σ_2 to σ_1 , it is easy to check that for each $\beta \in B_3$ the closure of β is isotopic to the closure of $f(\beta)$. Therefore, K^m is also the closure of $(\sigma_2 \sigma_1 \sigma_2)^2 \sigma_2^{-x_1} \sigma_1^{y_1} \cdots \sigma_2^{-x_t} \sigma_1^{y_t}$, which is conjugate to

$$(\sigma_2 \sigma_1 \sigma_2)^2 \sigma_1^{y_1} \sigma_2^{-x_2} \cdots \sigma_1^{-x_t} \sigma_1^{y_t} \sigma_2^{-x_1}$$

because the element $(\sigma_2 \sigma_1 \sigma_2)^2$ is central. This shows that up to replacing K with K^m we may assume $d \in \{0, 1\}$, therefore, the braid β of Equation (2.3) is of the form given in Equation (2.2). \square

Our next task is to show that a 2–fold cover of S^3 branched along a 3–braid knot of finite concordance order bounds a smooth 4–manifold with an intersection lattice Λ_Γ associated with a certain weighted graph Γ . Let Λ_Γ be the free abelian group generated by the vertices of the integrally weighted graph Γ of Figure 3, where $d \in \mathbb{Z}$, $t, x_i, y_i \geq 1$ and $\sum_i y_i \geq 2$. The graph Γ naturally determines a symmetric, bilinear form $\cdot : \Lambda_\Gamma \times \Lambda_\Gamma \rightarrow \mathbb{Z}$ such that, if v, w are two vertices of Γ then $v \cdot v$ equals the weight of v , $v \cdot w$ equals 1 if v, w are connected by an unlabelled edge, $(-1)^d$ if they are connected by the $(-1)^d$ –labelled edge, and 0 otherwise.

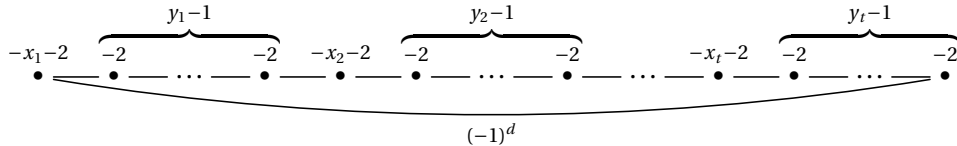


FIGURE 3. The weighted graph Γ

Lemma 2.3. *Let $K \subset S^3$ be the closure of a 3-braid $\beta \in B_3$ of the form given in Equation (2.2) for some $t, x_i, y_i \geq 1$ with $\sum_i y_i \geq 2$ and $d \in \{0, 1\}$. Let Y_K be the 2-fold cover of S^3 branched along K . Then, there is a smooth, oriented 4-manifold M whose intersection lattice $H_2(M; \mathbb{Z})/\text{Tor}$ is isometric to Λ_Γ , with $\partial M = -Y_K$.*

Proof. We prove the statement considering separately the two cases $d = 0$ and $d = 1$.

First case: $d = 0$. Let β^m denote the ‘mirror braid’ obtained from β by replacing σ_i with σ_i^{-1} , for $i = 1, 2$. Consider the regular projection $P \subset \mathbb{R}^2$ of the closure $K^m = \widehat{\beta}^m$ given in Figure 4. Colour the regions of $\mathbb{R}^2 \setminus P$ alternately black and white, so that the unbounded region is

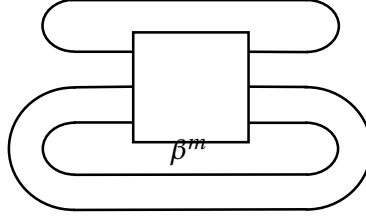


FIGURE 4. Regular projection of $\widehat{\beta}^m$

white. The black regions determine a spanning surface S for the knot K^m . Push the interior of S into the 4-ball and let $M \rightarrow B^4$ be the associated 2-fold branched cover. Gordon and Litherland [8, Theorem 3] give a recipe to compute the intersection form of M . Applying their recipe it is easy to check that the intersection lattice $H_2(M; \mathbb{Z})/\text{Tor}$ is isomorphic to Λ_Γ . Clearly $\partial M = -Y_K$, and the claim is proved when $d = 0$.

Second case: $d = 1$. Let T be an oriented, one-holed torus viewed as a 2-dimensional manifold with boundary and $\text{Diff}^+(T, \partial T)$ be the group of orientation-preserving diffeomorphisms of T which restrict to the identity on the boundary.

Let Y_K be the 2-fold cover of S^3 branched along K , which up to isotopy can be assumed to be positively transverse to the standard open book decomposition of S^3 with disk pages. Then, the pull-back of the standard open book on S^3 under the covering map $Y_K \rightarrow S^3$ has pages homeomorphic to a one-holed torus T and, there are right-handed Dehn twists $\gamma_1, \gamma_2 \in \text{Diff}^+(T, \partial T)$ along two simple closed curves in T intersecting transversely once, such that the monodromy $h : T \rightarrow T$ is

$$h = (\gamma_1 \gamma_2)^3 \gamma_1^{x_1} \gamma_2^{-y_1} \cdots \gamma_1^{x_t} \gamma_2^{-y_t} \in \text{Diff}^+(T, \partial T).$$

In [16, §4] it is shown, using the open book (T, h) , that Y_K bounds a smooth, oriented, positive definite 4-manifold N such that the intersection lattice of $M = -N$ is isometric to Λ_Γ . \square

Remark 2.4 (Pointed out by the referee). *It is possible to give another proof for the case $d = 1$ of Lemma 2.3 using the same method as in the case $d = 0$ by writing the knot K as the almost-alternating closure of*

$$\sigma_1^{x_1} \sigma_1^{-y_1} \sigma_2^{-x_2} \cdots \sigma_1^{x_t} \sigma_1^{-y_t+1} \sigma_2^{-x_1} \sigma_1^2 \sigma_2 \sigma_1^2,$$

using the relation $(\sigma_1 \sigma_2)^3 = (\sigma_2 \sigma_1^2)^2$. This also gives an alternative proof of the following Lemma 2.5: alternating diagrams have definite Goeritz matrices, almost-alternating diagrams have precisely one definite Goeritz, and one can easily check that the other Goeritz of the almost-alternating K is indefinite.

Lemma 2.5. *The lattice Λ_Γ is negative definite for any $t, x_i, y_i, \geq 1$ with $\sum_i y_i \geq 2$.*

Proof. Choose a ‘circular’ order on the set V of vertices of Γ and, for each $v \in V$, let v' (respectively v'') be the vertex coming immediately before (respectively after) v . Let $\xi = \sum_{v \in V} a_v v \in \Lambda_\Gamma$, with $a_v \in \mathbb{Z}$. Then,

$$\xi \cdot \xi = \left(\sum_{v \in V} a_v v \right) \cdot \left(\sum_{w \in V} a_w w \right) = \sum_{v \in V} (a_v a_{v'} v \cdot v' + a_v^2 v \cdot v + a_v a_{v''} v \cdot v'').$$

Since $\sum_{v \in V} a_v a_{v'} v \cdot v' = \sum_{v \in V} a_v a_{v''} v \cdot v''$ and $v \cdot v \leq -2$ for each $v \in V$, we have

$$\begin{aligned} \xi \cdot \xi &= \sum_{v \in V} (2a_v a_{v'} v \cdot v' + a_v^2 v \cdot v) \leq \sum_{v \in V} (2a_v a_{v'} v \cdot v' - 2a_v^2) = \\ &= \sum_{v \in V} (2a_v a_{v'} v \cdot v' - a_v^2 - a_{v'}^2) = - \sum_{v \in V} (a_v - a_{v'} v \cdot v')^2 \leq 0. \end{aligned}$$

Therefore, $\xi \cdot \xi = 0$ implies $a_v = a_{v'} v \cdot v'$ for each $v \in V$. In particular, $a_v^2 = a_{v'}^2$ for each $v \in V$. If we call a^2 this common value of the a_v^2 's, we have

$$0 = \xi \cdot \xi = \sum_{v \in V} (2a_v a_{v'} v \cdot v' + a_v^2 v \cdot v) = \sum_{v \in V} (2a_v^2 + a_v^2 v \cdot v) = a^2 \sum_{v \in V} (v \cdot v + 2).$$

Since $\sum_{v \in V} (v \cdot v + 2) = -\sum_i x_i < 0$, this implies $a = 0$, hence $a_v = 0$ for each $v \in V$ and $\xi = 0$. \square

Proposition 2.6. *Let $K \subset S^3$ be a knot of concordance order k which is the closure of a 3–braid of the form given by Equation (2.2), with $d \in \{0, 1\}$ and $\sum_{i=1}^t (x_i - y_i) = -4d$. Let Γ denote the weighted graph of Figure 3 determined by the given values of d, x_i and y_i . Then, the orthogonal sum Λ_Γ^k of k copies of Λ_Γ embeds isometrically in the standard negative definite lattice \mathbb{Z}^N , where $N = k \sum_i y_i$. Moreover, Λ^k has finite and odd index as a sublattice of \mathbb{Z}^N .*

Proof. Since $K = \hat{\beta}$ has concordance order k , $K \# \overset{(k)}{\cdot\cdot\cdot} \# K$ bounds a properly and smoothly embedded disk $D \subset B^4$. The 2–fold cover $N_D \rightarrow B^4$ branched along D is a rational homology ball [3, Lemma 2] and $\partial N_D = Y_K \# \overset{(k)}{\cdot\cdot\cdot} \# Y_K$. Let $M^k := \natural^k M$ be the boundary–connect sum of k copies of the 4–manifold M of Lemma 2.3. Then, M^k has intersection lattice isomorphic to $\Lambda_\Gamma^k := \Lambda_\Gamma \perp \overset{(k)}{\cdot\cdot\cdot} \perp \Lambda_\Gamma$. In view of Lemma 2.5 the smooth, closed, 4–manifold $X = N_D \cup M^k$ has negative definite intersection form, and by Donaldson’s ‘Theorem A’ [6, Theorem 1] the intersection lattice $H_2(X; \mathbb{Z})/\text{Tor}$ is isometric to the standard negative lattice \mathbb{Z}^N , where $N = b_2(X) = b_2(M^k) = k \sum_i y_i$. This gives an isometric embedding $\Lambda_\Gamma^k \subset \mathbb{Z}^N$. The fact that Λ_Γ^k has finite and odd index follows, via standard arguments, from the fact that, since K is a knot, the determinant of K is odd. \square

3. LATTICE ANALYSIS

3.1. Circular subsets. The standard negative lattice $(\mathbb{Z}^N, -I)$ admits a basis $\mathcal{E} = \{e_1, \dots, e_N\}$ such that the intersection product between e_i and e_j is $e_i \cdot e_j = -\delta_{ij}$. Such a basis is unique up to permutations and sign reversals of its elements. From now on, we shall call any such basis $\mathcal{E} \subset \mathbb{Z}^N$ a *canonical basis*, and denote the standard negative lattice simply \mathbb{Z}^N . Given a subset $\mathcal{V} \subset \mathbb{Z}^N$, we shall denote by $|\mathcal{V}|$ the cardinality of \mathcal{V} , and by $\Lambda_{\mathcal{V}}$ the intersection lattice consisting of the subgroup of \mathbb{Z}^N generated by \mathcal{V} , endowed with the restriction of the intersection form on \mathbb{Z}^N . Elements of the set \mathcal{V} will be called indifferently *elements* or *vectors*. We will say that a vector $u \in \mathbb{Z}^N$ *hits* a vector $v \in \mathbb{Z}^N$ if $u \cdot v \neq 0$. On a finite subset $\mathcal{V} \subset \mathbb{Z}^N$ is

defined the equivalence relation R generated by the reflexive and symmetric relation given by $u \sim_R v$ if and only if $u \cdot v \neq 0$. We call *connected components* of \mathcal{V} its R -equivalence classes, and we call *connected* a subset consisting of a single R -equivalence class.

Definition 3.1. A (not necessarily connected) subset $\mathcal{V} \subset \mathbb{Z}^N$ is a *circular subset* if:

- $|C| \geq 3$ for each connected component $C \subset \mathcal{V}$;
- $v \cdot w \in \{-1, 0, 1\}$ for any $v, w \in \mathcal{V}$ with $v \neq w$;
- for each $v \in \mathcal{V}$, the set $\{u \in \mathcal{V} \mid |u \cdot v| = 1\}$ has two elements.

Given a circular subset $\mathcal{V} \subset \mathbb{Z}^N$, the vector $W := \sum_{v \in \mathcal{V}} v \in \Lambda_{\mathcal{V}}$ will be called the *Wu element* of \mathcal{V} . For any subset $\mathcal{V} \subset \mathbb{Z}^N$, a canonical basis $\mathcal{E} \subset \mathbb{Z}^N$ such that $\sum_{v \in \mathcal{V}} v = -\sum_{e \in \mathcal{E}} e$ will be called *adapted to \mathcal{V}* .

Lemma 3.2. Let $\mathcal{V} \subset \mathbb{Z}^N$ be a circular subset such that $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite and odd index and whose Wu element W satisfies $W \cdot W = -N$. Then, there is a canonical basis $\mathcal{E} \subset \mathbb{Z}^N$ adapted to \mathcal{V} . In particular, for each $e \in \mathcal{E}$ we have

$$(3.1) \quad \sum_{v \in \mathcal{V}} v \cdot e = 1.$$

Proof. Since the index of $\Lambda_{\mathcal{V}}$ is finite and odd, given $v \in \mathbb{Z}^N$ there exists an odd integer d such that $dv \in \Lambda_{\mathcal{V}}$. It is easy to check that $W \in \Lambda_{\mathcal{V}}$ is characteristic in $\Lambda_{\mathcal{V}}$, therefore $W \cdot (dv)$ and $(dv) \cdot (dv)$ are congruent modulo 2. But, since d is odd, the first number is congruent to $W \cdot v$ and the second one to $v \cdot v$. This shows that W is a characteristic vector when viewed in \mathbb{Z}^N . In particular, given any canonical basis $\mathcal{E} \subset \mathbb{Z}^N$ we have $W \cdot e \neq 0$ for each $e \in \mathcal{E}$. Since $W \cdot W = -N$, we have $W \cdot e = \pm 1$ for each e , and up to reversing the signs of some of the e 's we have $W = -\sum_{e \in \mathcal{E}} e$. Equation (3.1) follows immediately using the definition of W . \square

Lemma 3.3. Let $\mathcal{V} \subset \mathbb{Z}^N$ be a circular subset and $\mathcal{E} \subset \mathbb{Z}^N$ a canonical basis adapted to \mathcal{V} . Then, for every $v \in \mathcal{V}$ and $e \in \mathcal{E}$ we have $v \cdot e \in \{-1, 0, 1, 2\}$, and there is a map

$$\mathcal{W} := \{v \in \mathcal{V} \mid W \cdot v = v \cdot v + 2\} \rightarrow \mathcal{E}$$

defined by sending each $v \in \mathcal{W}$ to the unique $e_v \in \mathcal{E}$ such that $v \cdot e_v \in \{-1, 2\}$.

Proof. We can write each $v \in \mathcal{V}$ as a linear combination

$$v = -\sum_{e \in \mathcal{E}} (v \cdot e)e.$$

For each $v \in \mathcal{V}$ we have the equality

$$-v \cdot v + v \cdot W = v \cdot (-v + W) = \sum_{e \in \mathcal{E}} v \cdot e(v \cdot e - 1).$$

By the second and third conditions in Definition 3.1, we have

$$-2 \leq -v \cdot v + v \cdot W \leq 2$$

for each $v \in \mathcal{V}$. The quantity $x(x-1)$ is always nonnegative when $x \in \mathbb{Z}$, so we conclude

$$(3.2) \quad \sum_{e \in \mathcal{E}} v \cdot e(v \cdot e - 1) = \begin{cases} 2 & \text{if } v \in \mathcal{W} \\ 0 & \text{if } v \notin \mathcal{W}. \end{cases}$$

By the second and third condition in Definition 3.1, Equation (3.2) implies $v \cdot e \in \{-1, 0, 1, 2\}$ for every e . Moreover, if $v \in \mathcal{W}$ there is exactly one $e_v \in \mathcal{E}$ such that $v \cdot e_v \in \{-1, 2\}$. This gives the statement and concludes the proof. \square

3.2. Semipositive circular subsets. A circular subset $\mathcal{V} \subset \mathbb{Z}^N$ will be called *semipositive* if each connected component $C \subset \mathcal{V}$ contains a single pair of vectors $u, v \in C$ such that $u \cdot v = -1$.

Lemma 3.4. *Let $\mathcal{V} \subset \mathbb{Z}^{|\mathcal{V}|}$ be a semipositive circular subset such that $v \cdot v \leq -2$ for each $v \in \mathcal{V}$ and $\sum_{v \in C} (v \cdot v + 2) < -1$ for each connected component $C \subset \mathcal{V}$. Let \mathcal{E} be a canonical basis adapted to \mathcal{V} . Then, for each $e \in \mathcal{E}$ not in the image of the map of Lemma 3.3, there exists $u \in \mathcal{V}$ such that $u \cdot u = -2$, $u \cdot e = 1$ and u is the only vector of \mathcal{V} which hits e .*

Proof. Let $e \in \mathcal{E}$ be an element not in the image of the map of Lemma 3.3. Then, we have $v \cdot e \in \{0, 1\}$ for each $v \in \mathcal{V}$. In view of Equation (3.1) there exists a unique $u \in \mathcal{V}$ such that $u \cdot e = 1$. The subset $\mathcal{V}' = \mathcal{V} \setminus \{u\} \cup \{u + e\}$ is semipositive circular, contained in the span of $\mathcal{E} \setminus \{e\}$, and satisfies $\sum_{v \in C} (v \cdot v + 2) < 0$ for each connected component $C \subset \mathcal{V}'$. If $v \cdot v \leq -2$ for each $v \in \mathcal{V}'$, then $\Lambda_{\mathcal{V}'}$ is isometric to Λ_{Γ} for some $t, x_i, y_i \geq 1$. By Lemma 2.5 this would imply $\text{rk}(\Lambda_{\mathcal{V}'}) = |\mathcal{V}'|$. Since \mathcal{V}' can be viewed as a subset of $\mathbb{Z}^{|\mathcal{V}'|-1}$ and $|\mathcal{V}'| = |\mathcal{V}|$, we must have $(u + e) \cdot (u + e) = -1$, which implies $u \cdot u = -2$. \square

Lemma 3.5. *Let $\mathcal{V} \subset \mathbb{Z}^N$ be a semipositive circular subset, and let $\mathcal{E} \subset \mathbb{Z}^N$ be a canonical basis adapted to \mathcal{V} . Let $u \in \mathcal{V}$ with $u \cdot u = -1$ and suppose that the connected component containing u has cardinality at least 4. Let $u', u'' \in \mathcal{V}$ be the two distinct vectors such that $|u' \cdot u| = |u'' \cdot u| = 1$. Then, the subset*

$$\mathcal{V}' = \mathcal{V} \setminus \{u\} \cup \{u' + (u' \cdot u)u, u'' + (u'' \cdot u)u\}$$

is a semipositive, circular subset contained in the span of $\mathcal{E}' = \mathcal{E} \setminus \{e\}$, where $e \in \mathcal{E}$ is the only element such that $e \in \{u, -u\}$, and \mathcal{E}' is adapted to \mathcal{V}' .

Proof. The proof is an easy exercise left to the reader. \square

Proposition 3.6. *Let $\mathcal{V} \subset \mathbb{Z}^N$ be a semipositive circular subset such that:*

- $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite and odd index;
- its Wu element $W = \sum_{v \in \mathcal{V}} v$ satisfies $W \cdot W = -N$;
- $v \cdot v \leq -2$ for each $v \in \mathcal{V}$;
- $\sum_{v \in D} (v \cdot v + 2) = 4 - |D| < -1$ for each connected component $D \subset \mathcal{V}$.

Let $C = \{v_1, \dots, v_{|C|}\} \subset \mathcal{V}$ be any connected component of \mathcal{V} , and $(-c_1, \dots, -c_{|C|})$ the corresponding string of self-intersections. Then, there is an iterated blowup $(s_1, \dots, s_{|C|})$ of $(0, 0)$ containing exactly two 1's, and such that $c_i \geq s_i$ for each $i = 1, \dots, |C|$.

Proof. By Lemma 3.2 there is a canonical basis $\mathcal{E} \subset \mathbb{Z}^N$ adapted to \mathcal{V} , and Lemma 3.3 applies. We observe that the subset $\mathcal{W} \subset \mathcal{V}$ defined in Lemma 3.3 does not coincide with \mathcal{V} . In fact, it is easy to check that if k is the number of connected components of \mathcal{V} then $|\mathcal{W}| = |\mathcal{V}| - 2k$. Since $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite index we have $|\mathcal{V}| = |\mathcal{E}|$, therefore there are at least $2k$ distinct elements in the complement of the image of the map $\mathcal{W} \rightarrow \mathcal{E}$ of Lemma 3.3. As a result, there are at least $2k$ distinct vectors $u \in \mathcal{V}$ as in Lemma 3.4. We modify all of those vectors by replacing each u with $u + e$, where e is the associated vector of \mathcal{E} . The result is a new subset \mathcal{V}' with $|\mathcal{V}'| = |\mathcal{V}|$, and at least $2k$ vectors of square -1 . Let $\mathcal{E}' \subset \mathcal{E}$ be the subset obtained by erasing from \mathcal{E} all the vectors e corresponding to the u 's that were modified. Then, \mathcal{V}' is contained in the span of \mathcal{E}' and \mathcal{E}' is adapted to \mathcal{V}' , i.e. $\sum_{v \in \mathcal{V}'} v = -\sum_{e \in \mathcal{E}'} e$. Lemma 3.5 can then be applied several times. We apply the lemma as many times as possible, i.e. until the resulting subset \mathcal{V}'' has no vectors of square -1 belonging to a connected component with at least four vectors. We claim that every connected component of \mathcal{V}'' contains three vectors. In fact, a

connected component $C'' \subset \mathcal{V}''$ with no (-1) -vectors would contain two vectors v such that $W \cdot v = v \cdot v$, and Equations (3.1) and (3.2) would imply that each such v would be the unique vector of \mathcal{V}'' hitting some $e \in \mathcal{E}$. But we had eliminated from \mathcal{V} all e 's hitting a single vector at the beginning, and the construction of \mathcal{V}'' from \mathcal{V} implies that a vector of \mathcal{V}'' can have this property only if it has square -1 . Therefore C'' must consist of three vectors, and the claim holds. Next, we claim that each connected component of \mathcal{V}'' consists of (-1) -vectors. To see this it suffices to show that $\sum_{v \in \mathcal{V}''} (v \cdot v + 2) = 3k$, since each of the k components can contribute at most 3 to this quantity. By our assumptions on \mathcal{V} we have

$$\sum_{v \in \mathcal{V}} (v \cdot v + 2) = \sum_{C \subset \mathcal{V}} (4 - |C|) = 4k - |\mathcal{V}|.$$

In the first step of the above construction we turn a certain number $m \geq 2k$ of (-2) -vectors into (-1) -vectors, therefore $\sum_{v \in \mathcal{V}'} = 4k - |\mathcal{V}| + m$. Then we apply Lemma 3.5 $\sum_{C \subset \mathcal{V}'} (|C| - 3) = |\mathcal{V}'| - 3k$ times, each time increasing the quantity by 1. The result is $k + m$. This number is at most $3k$ because each component can contribute at most 3. On the other hand, $m \geq 2k$ by construction. This forces $m = 2k$, and the last claim is proved. If we now look at how \mathcal{V}' is obtained from \mathcal{V}'' , we see that, in view of Equations (3.1) and (3.2), each component of \mathcal{V}' must contain at least two (-1) -vectors. Since $m = 2k$, each component of \mathcal{V}' contains exactly two (-1) -vectors. It is now easy to check that this implies the statement. \square

Theorem 3.7. *Let Γ be a weighted graph as in Figure 3 with $d = 1$ and $\sum_i y_i = \sum_i x_i + 4 \geq 6$. Let $N = k \sum_i y_i$ for some $k \geq 1$ and suppose that there is an isometric embedding $\Lambda_\Gamma^k \subset \mathbb{Z}^N$ of finite*

and odd index. Then, the string of positive integers $(x_1 + 2, \overbrace{2, \dots, 2}^{y_1-1}, \dots, x_t + 2, \overbrace{2, \dots, 2}^{y_t-1})$ is obtained from an iterated blowup of $(0, 0)$ by replacing two 1's with 2's.

Proof. The lattice Λ_Γ^k has a natural basis, whose elements are in 1-1 correspondence with the vertices of the disjoint union of k copies of Γ , and intersect as prescribed by edges and weights. Since $d = 1$, the image of such a basis is a semipositive circular subset $\mathcal{V} \subset \mathbb{Z}^N$. The Wu element $W = \sum_{v \in \mathcal{V}} v$ satisfies

$$W \cdot W = k \left(\sum_i (-2 - x_i) - 2 \sum_i (y_i - 1) + 2 \sum_i y_i - 4 \right) = -k \sum_i y_i = -N$$

and since $\Lambda_{\mathcal{V}} = \Lambda_\Gamma^k$, the embedding $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite and odd index. Moreover, each connected component $D \subset \mathcal{V}$ satisfies

$$\sum_{v \in D} (v \cdot v + 2) = - \sum_i x_i = 4 - \sum_i y_i = 4 - |D| \leq -2.$$

Therefore we can apply Proposition 3.6 to conclude that for any connected component $C \subset \mathcal{V}$ there is an iterated blowup $(s_1, \dots, s_{|C|})$ of $(0, 0)$ containing exactly two 1's such that

the elements of the string $(c_1, \dots, c_{|C|}) := (x_1 + 2, \overbrace{2, \dots, 2}^{y_1-1}, \dots, x_t + 2, \overbrace{2, \dots, 2}^{y_t-1})$ satisfy $c_i \geq s_i$ for $i = 1, \dots, |C|$. It only remains to check that $c_i = s_i + 1$ if and only if $s_i = 1$. This follows immediately from the fact that

$$\sum_i c_i = - \sum_{v \in C} v \cdot v = 3|C| - 4 = \sum_i s_i + 2,$$

where the last equality can be easily established by induction on the number of blowups. \square

3.3. Positive circular subsets. A circular subset $\mathcal{V} \subset \mathbb{Z}^N$ is called *positive* if $u \cdot v \geq 0$ for any $u, v \in \mathcal{V}$ with $u \neq v$. Observe that for a positive circular subset \mathcal{V} the subset $\mathcal{W} \subset \mathcal{V}$ defined in Lemma 3.3 coincides with \mathcal{V} . Moreover, two canonical bases adapted to \mathcal{V} differ by a permutation of their elements. This easily implies that, whether the map $\mathcal{W} \rightarrow \mathcal{E}$ of Lemma 3.3 is injective or not, does not depend on the choice of \mathcal{E} . We will assume these facts as understood throughout this subsection.

The analysis of positive circular subsets admitting an adapted canonical basis splits naturally into two subcases, according to whether the map of Lemma 3.3 is injective or not. We first deal with the simplest case, when the map is not injective.

First subcase. Let $\mathcal{V} \subset \mathbb{Z}^N$ be a positive circular subset admitting an adapted canonical basis. From now on, and until further notice, we will assume that the map of Lemma 3.3 is not injective.

Lemma 3.8. *Let $\mathcal{V} \subset \mathbb{Z}^N$ be a positive circular subset and $\mathcal{E} \subset \mathbb{Z}^N$ a canonical basis adapted to \mathcal{V} . Suppose that:*

- $|\mathcal{V}| = N$;
- *the map of Lemma 3.3 is not injective;*
- $v \cdot v \leq -2$ for each $v \in \mathcal{V}$;
- $\sum_{v \in D} (v \cdot v + 2) = -|D|$ for each connected component $D \subset \mathcal{V}$.

Then, for each $e \in \mathcal{E}$ not in the image of the map of Lemma 3.3 there exists $u \in \mathcal{V}$ such that $u = e_u - e$ and $\{v \in \mathcal{V} \mid v \cdot e = 1\} = \{u\}$. Moreover, let $C \subset \mathcal{V}$ be the connected component containing u . Then,

- *if $|C| > 3$ there is $v \in C$ such that $v \cdot e_u = 1$ and v hits ≥ 3 distinct vectors of \mathcal{E} ;*
- *if $|C| = 3$ we have $C = \{e_1 - e_2, e_3 - e_1, -2e_3 - e_1\}$ for some $e_1, e_2, e_3 \in \mathcal{E}$.*

Proof. By Lemma 2.5 and the first assumption we have $\text{rk } \Lambda_{\mathcal{V}} = |\mathcal{V}| = N$. Therefore, since the map of Lemma 3.3 is not injective, it cannot be surjective. For each $e \in \mathcal{E}$ not in the image of the map we have $v \cdot e \in \{0, 1\}$ for each $v \in \mathcal{V}$. By Equation (3.1) there is a unique $u \in \mathcal{V}$ such that $u \cdot e = 1$. We claim that the set $\{e' \in \mathcal{E} \mid u \cdot e' \neq 0\}$ contains only e and e_u and moreover $u \cdot e_u = -1$. Otherwise, we would have $u \cdot u < -2$, and by replacing u in \mathcal{V} with $u + (u \cdot e)e$ we would obtain a new set $\mathcal{V}' \subset \mathbb{Z}^N$ of vectors contained in the span of $\mathcal{E} \setminus \{e\}$, such that $\Lambda_{\mathcal{V}'}$ would be isometric to Λ_{Γ} for some $t, x_i, y_i \geq 1$. Then, by Lemma 2.5 we would have $\text{rk}(\Lambda_{\mathcal{V}'}) = |\mathcal{V}'|$. This would contradict the fact that $|\mathcal{V}'| = |\mathcal{V}| = |\mathcal{E}| > |\mathcal{E} \setminus \{e\}|$, therefore the claim is proved, and we have $u = e_u - e$. Let $v, w \in \mathcal{V}$ be the two vectors satisfying $v \cdot u = w \cdot u = 1$. Then, $v \cdot e_u = w \cdot e_u = 1$, and we may write

$$(3.3) \quad v = a_v e_v - e_u - \cdots, \quad w = a_w e_w - e_u - \cdots \quad \text{for some } a_v, a_w \in \{-2, 1\}.$$

Let $C \subset \mathcal{V}$ be the connected component containing u . By contradiction, suppose that $|C| > 3$ and both v and w hit only two vectors of \mathcal{E} . Then,

$$0 = v \cdot w = (a_v e_v - e_u) \cdot (a_w e_w - e_u) = a_v a_w (e_v \cdot e_w) - 1,$$

which is impossible because $e_v \cdot e_w \in \{-1, 0\}$. Therefore, up to renaming v and w we may assume that v hits at least three distinct vectors of \mathcal{E} . If $|C| = 3$ we also have $v \cdot w = 1$, therefore by Equation (3.3)

$$1 = v \cdot w = (a_v e_v - e_u - \cdots) \cdot (a_w e_w - e_u - \cdots) \leq (a_v e_v - e_u) \cdot (a_w e_w - e_u) = a_v a_w (e_v \cdot e_w) - 1,$$

which implies $e_v = e_w$ and $a_v a_w = -2$. Up to swapping v and w we may assume $a_v = 1$ and $a_w = -2$. This implies $v \cdot v \leq -2$ and $w \cdot w \leq -5$, and since $-3 = \sum_{v \in C} (v \cdot v + 2)$ we must have $v \cdot v = -2$ and $w \cdot w = -5$, and the statement follows immediately. \square

Definition 3.9. A string of negative integers $(-n_1, \dots, -n_{k+1})$ is obtained from $(-m_1, \dots, -m_k)$ by a *(-2)-expansion* if $m_1 = 2$ and

$$(-n_1, \dots, -n_{k+1}) = \begin{cases} (-2, -m_1, \dots, -m_k - 1) & \text{or} \\ (-2, -m_2 - 1, \dots, -m_k, -m_1). \end{cases}$$

Proposition 3.10. Let $\mathcal{V} \subset \mathbb{Z}^N$ be a positive circular subset such that:

- $v \cdot v \leq -2$ for each $v \in \mathcal{V}$;
- $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite and odd index;
- the Wu element $W = \sum_{v \in \mathcal{V}} v$ satisfies $W \cdot W = -N$;
- the map of Lemma 3.3 is not injective;
- $\sum_{v \in D} (v \cdot v + 2) = -|D|$ for each connected component $D \subset \mathcal{V}$.

Then, there is a connected component $C \subset \mathcal{V}$ whose string of self-intersections is obtained, up to circular symmetry, by a sequence of *(-2)-expansions* from $(-2, -2, -5)$.

Proof. Observe that $\Lambda_{\mathcal{V}}$ is isometric to an intersection lattice Λ_{Γ} for some $t, x_i, y_i \geq 1$. By Lemma 2.5 and the second assumption we have $\text{rk } \Lambda_{\mathcal{V}} = |\mathcal{V}| = N$. Therefore, since the map of Lemma 3.3 is not injective, it cannot be surjective. If a connected component of \mathcal{V} has exactly three elements the statement follows immediately from Lemma 3.8. Hence, suppose that each connected component of \mathcal{V} has at least four elements. By Lemma 3.2, the first two assumptions imply the existence of a canonical basis \mathcal{E} adapted to \mathcal{V} . This, together with the remaining assumptions imply that Lemma 3.8 can be applied to \mathcal{V} . Let u and v be vectors as in Lemma 3.8 belonging to a connected component $C \subset \mathcal{V}$. We can change \mathcal{V} into a new subset \mathcal{V}' by a process we will call *contraction*. The subset \mathcal{V}' is obtained by replacing C with $C' := (C \setminus \{u, v\}) \cup \{v + e_u\}$, which is contained in the span of $\mathcal{E}' := \mathcal{E} \setminus \{e\}$. When regarded as a subset of \mathbb{Z}^{N-1} , \mathcal{V}' is still a positive circular subset, \mathcal{E}' is adapted to \mathcal{V}' , $|\mathcal{V}'| = N - 1$, $v' \cdot v' \leq -2$ for each $v' \in \mathcal{V}'$ and $\sum_{v' \in C'} (v' \cdot v' + 2) = -|C'|$. Moreover, e_u does not belong to the image of the map of Lemma 3.3 and $|C'| \geq 3$. If $|C'| > 3$ we can apply Lemma 3.8 again and contract \mathcal{V}' to a subset \mathcal{V}'' , possibly modifying a connected component different from C' . We can keep contracting as long as all connected components of the resulting subset have cardinality greater than 3. When one of the components reaches cardinality 3 we can apply the last part of Lemma 3.8. The statement is easily obtained combining the information from the lemma with the fact that the component is the result of a sequence of contractions. \square

Second subcase. In this subsection we study the positive circular subsets $\mathcal{V} \subset \mathbb{Z}^N$ with an adapted canonical basis such that the map of Lemma 3.3 is injective.

Lemma 3.11. Let $\mathcal{V} \subset \mathbb{Z}^{|\mathcal{V}|}$ be a positive circular subset such that:

- $v \cdot v \leq -2$ for each $v \in \mathcal{V}$;
- the Wu element $W = \sum_{v \in \mathcal{V}} v$ satisfies $W \cdot W = -|\mathcal{V}|$;
- there is a canonical basis $\mathcal{E} \subset \mathbb{Z}^{|\mathcal{V}|}$ adapted to \mathcal{V} and the map of Lemma 3.3 is injective.

Then, the following properties hold:

- (1) for each $v \in \mathcal{V}$ we have $v \cdot e_v = -1$;

- (2) for each $e \in \mathcal{E}$ there exist distinct elements $u_e, v_e, w_e \in \mathcal{V}$ such that $u_e \cdot e = -1$ and $v_e \cdot e = w_e \cdot e = 1$. Moreover, $x \cdot e = 0$ for each $x \in \mathcal{V} \setminus \{u_e, v_e, w_e\}$;
- (3) for each $u \in \mathcal{V}$ with $u \cdot u = -2$, there exist $f, g \in \mathcal{E}$ and $v, w, z \in \mathcal{V}$ such that:
- $u = e_u - f$;
 - $v \cdot u = 1, v \cdot f = 0$ and $v = e_v - e_u - \dots$;
 - $w \cdot u = 1, w \cdot e_u = 0$ and $w = f - \dots$;
 - $z \cdot u = 0$ and $z = e_z - e_u - f - g - \dots$.

Proof. (1) By the injectivity of the map each $v \in \mathcal{V}$ is characterized as the unique element $u \in \mathcal{V}$ such that $u \cdot e_v \in \{-1, 2\}$. It follows that for each $u \in \mathcal{V} \setminus \{v\}$ we have $u \cdot e_v \in \{0, 1\}$. Clearly, Equation (3.1) can be satisfied only if $v \cdot e_v = -1$.

(2) Since $|\mathcal{V}| = |\mathcal{E}|$, the map $v \mapsto e_v$ is a bijection. Denote by $e \mapsto u_e$ the inverse map. By (1) $u_e \cdot e = -1$, and by Equation (3.1) there exist distinct elements $u_e, v_e, w_e \in \mathcal{V}$ such that $u_e \cdot e = -1$ and $v_e \cdot e = w_e \cdot e = 1$, while $x \cdot e = 0$ for each $x \in \mathcal{V} \setminus \{u_e, v_e, w_e\}$.

(3) The fact that $u = e_u - f$ for some $f \in \mathcal{E}$ follows immediately from (1), the definition of e_u and the fact that $u \cdot u = -2$. Observe that for any $v \in \mathcal{V}$ with $v \cdot u = 1$, since $v \cdot e \in \{-1, 0, 1\}$ for each $e \in \mathcal{E}$, we have $(v \cdot e_u, v \cdot f) \in \{(1, 0), (0, -1)\}$. There are exactly two elements $v, w \in \mathcal{V}$ such that $u \cdot v = u \cdot w = 1$, and by the previous observation we have $v \cdot f, w \cdot f \in \{-1, 0\}$. On the other hand, $v \cdot f = w \cdot f = -1$ would imply $e_v = f = e_w$, which is impossible because we are assuming that the map $u \mapsto e_u$ is injective. Therefore, either $(v \cdot e_u, v \cdot f) = (1, 0)$ or $(w \cdot e_u, w \cdot f) = (1, 0)$, and up to renaming v and w we may assume $(v \cdot e_u, v \cdot f) = (1, 0)$. Therefore $v \cdot f = 0$ and $v = e_v - e_u - \dots$. This forces $(w \cdot e_u, w \cdot f) = (0, -1)$, because otherwise $w \cdot e_u = 1$, and there would be $w' \in \mathcal{V}$ distinct from v and w with $e_{w'} = f$. Since $w' \cdot u = 0$, this would imply $w' \cdot e_u = 1$, contradicting (2) because e_u would already appear in u, v and w . Therefore $w \cdot e_u = 0$ and $w = f - \dots$.

By (2) and the fact that $(v \cdot e_u, w \cdot e_u) = (-1, 0)$, there exist $z \in \mathcal{V}$ such that $z \cdot e_u = -1$. Since the vectors adjacent to u are v and w , we have $z \cdot u = 0$ and therefore $z \cdot f = -1$ as well. Thus, $z = e_z - e_u - f - \dots$ and $z \cdot z \leq -3$. We claim that $z \cdot z < -3$. In fact, suppose by contradiction that $z \cdot z = -3$, so that $z = e_z - e_u - f$. Since e_u and f already appear three times, e_z must appear in both the adjacent vectors of z , say z' and z'' . On the other hand, we have $1 \geq w \cdot z = (f - \dots) \cdot (e_z - e_u - f) \geq 1$, therefore $w \cdot z = 1$, which implies $w \in \{z', z''\}$. But this is not possible because e_z cannot appear in w . Therefore, there exist some $g \in \mathcal{E}$ with $z = e_z - e_u - f - g - \dots$. \square

Definition 3.12. Let $\mathcal{V} \subset \mathbb{Z}^{|\mathcal{V}|}$ be a positive circular subset satisfying the hypotheses of Lemma 3.11. If $u \in \mathcal{V}$ satisfies $u \cdot u = -2$ and the connected component of u contains more than 3 elements, by Lemma 3.11 there exist $f, g \in \mathcal{E}$ and $v, z \in \mathcal{V}$ such that $u = e_u - f$, $v = e_v - e_u - \dots$ and $z = e_z - e_u - f - g - \dots$. Then, we define $v' := u + v = e_u - f - \dots$, $z' := z + e_u = e_z - f - g - \dots$ and

$$\mathcal{V}' := \mathcal{V} \setminus \{u, v, z\} \cup \{v', z'\}.$$

We say that the set \mathcal{V}' is obtained from \mathcal{V} by a (-2)-contraction.

Remark 3.13. Observe that if every connected component C of the set \mathcal{V}' in Definition 3.12 satisfies $|C| \geq 3$, then \mathcal{V}' is circular when regarded as a subset of the intersection lattice $\mathbb{Z}^{|\mathcal{V}'|}$ spanned by $\mathcal{E}' = \mathcal{E} \setminus \{e_u\}$, and $v \cdot v \leq -2$ for each $v \in \mathcal{V}'$. A simple calculation shows that the Wu element $W' = \sum_{v \in \mathcal{V}'} v$ satisfies $W' \cdot W' = W \cdot W + 1 = -|\mathcal{V}'|$ and $W' = -\sum_{e \in \mathcal{E}'} e$. In particular, \mathcal{E}' is a canonical basis adapted to \mathcal{V}' . Moreover, for each $v \in \mathcal{V}'$ there is an $e \in \mathcal{E}'$ such that

$v \cdot e = -1$. Therefore the map $\mathcal{V}' \rightarrow \mathcal{E}'$ defined in Lemma 3.3 is surjective, hence also injective. This shows that \mathcal{V}' satisfies all the assumptions of Lemma 3.11.

If, after applying a (-2) -contraction to a positive circular subset \mathcal{V} satisfying the assumptions of Lemma 3.11, we obtain a new positive circular subset \mathcal{V}' which still contains a (-2) -vector whose connected component has more than 3 elements, by Remark 3.13 we can apply a (-2) -contraction again, obtaining a set \mathcal{V}'' , again satisfying the assumptions of Lemma 3.11, and so on. Furthermore, at each step a (-2) -vector is eliminated but, in view of Definition 3.12, no new (-2) -vector is created. Clearly, after a finite number of (-2) -contractions we end up with a circular subset $\mathcal{Z} \subset \mathbb{Z}^{|\mathcal{Z}|}$ satisfying the assumptions of Lemma 3.11 and having the property that each (-2) -vector of \mathcal{Z} belongs to a connected component with exactly three elements.

In order to understand the set \mathcal{V} we need more information about the set \mathcal{Z} . In the following, we shall denote by $\mathcal{F} \subset \mathbb{Z}^{|\mathcal{Z}|}$ a canonical basis adapted to \mathcal{Z} . For each $u \in \mathcal{Z}$, we define

$$\mathcal{F}_u := \{f \in \mathcal{F} \mid u \cdot f \neq 0\} \subset \mathcal{F}.$$

Following [14], we consider the equivalence relation on \mathcal{Z} generated by the relation given by $u \sim v$ if and only if $\mathcal{F}_u \cap \mathcal{F}_v \neq \emptyset$ and we call *irreducible components* of \mathcal{Z} the resulting equivalence classes. We shall now analyze \mathcal{Z} considering separately the two cases $t = 2$ and $t \geq 3$, where t is the parameter appearing in Equation (2.2) (when $t = 1$ the analysis is not necessary, as will be shown in the proof of Theorem 3.22).

Suppose $t = 2$. By the considerations following Remark 3.13, it is easy to see that \mathcal{Z} consists of some number n of connected components C_1, \dots, C_n , where each component C_i contains three vectors u_i, v_i and w_i satisfying $u_i \cdot v_i = v_i \cdot w_i = w_i \cdot u_i = 1$, $u_i \cdot u_i = -2$, $v_i \cdot v_i = -2 - a_i$ and $w_i \cdot w_i = -2 - b_i$, with $a_i, b_i \geq 1$.

Lemma 3.14. $\sum_{i=1}^n (a_i + b_i) = 3n$.

Proof. We refer to the notation of Figure 3. Each connected component of the original positive circular subset \mathcal{V} contains $y_1 + y_2 - 2 = x_1 + x_2 - 2$ vectors of square -2 . Therefore, the number of (-2) -contractions applied to each connected component of \mathcal{V} to obtain \mathcal{Z} is $x_1 + x_2 - 3$. Each time we apply a (-2) -contraction the self-intersection of some vector w with $w \cdot w \leq -4$ increases by 1. This shows that

$$\sum_{i=1}^n (a_i + b_i) = n(x_1 + x_2) - n(x_1 + x_2 - 3) = 3n.$$

□

Lemma 3.15. $a_i + b_i = 3$ for each $i = 1, \dots, n$.

Proof. In view of Lemma 3.14, it suffices to show that $a_i + b_i \geq 3$ for each $i = 1, \dots, n$. We do this by showing that the case $a_i + b_i = 2$ cannot occur. Arguing by contradiction, suppose that $a_i + b_i = 2$ for some i . Then, $a_i = b_i = 1$ and, up to renaming the elements of \mathcal{F} we must have $u_i = f_1 - f_2$, $v_i = f_2 - f_3 - f_4$ and $w_i = f_3 - f_1 - f_5$ for some $f_1, \dots, f_5 \in \mathcal{F}$. Let v be the unique vector of $\mathcal{F} \setminus C_i$ such that $v \cdot f_1 = 1$. Since $v \cdot u_i = 0$, we must have $v = e_v - f_1 - f_2 - \dots$. Since $v \cdot w_i = 0$, we have either (i) $v = f_5 - f_1 - f_2 - \dots$ or (ii) $v = e_v - f_1 - f_2 - f_3 - \dots$. In Case (ii), consider the unique vector $w \in \mathcal{Z} \setminus C_i$ with $e_w = f_4$. Since $w \cdot v_i = 0$, \mathcal{F}_w should contain either f_2 or f_3 . But f_2 already appears in u_i, v_i and v , and f_3 in v_i, w_i and v . The only possibility

is that $w = v$ and $v = f_4 - f_1 - f_2 - f_3 - \dots$, but this is incompatible with $v \cdot v_i = 0$, therefore Case (ii) cannot occur. In Case (i), since $v \cdot v_i = 0$ we must have $v = f_5 - f_1 - f_2 - f_4 - \dots$. Let $w \in \mathcal{Z} \setminus C_i$ be the unique vector with $e_w = f_4$. Then, since $w \cdot v_i = 0$ we must have $w = f_4 - f_3 - \dots$ and since $w \cdot w_i = 0$ we must have $w = f_4 - f_3 - f_5 - \dots$. Since $v \cdot w \leq 1$, we must have $v = f_5 - f_1 - f_2 - f_4 - f_6 - \dots$ and $w = f_4 - f_3 - f_5 - f_6 - \dots$ for some $f_6 \in \mathcal{F}$. Now consider the new set \mathcal{Z}' obtained from \mathcal{Z} by eliminating the vectors u_i, v_i and w_i and replacing v with $v' = v + f_1 + f_2 + f_4$ and w with $w' = w + f_3 - f_4$. By construction, the set \mathcal{Z}' is a positive circular subset of the span of $\mathcal{F} \setminus \{f_1, f_2, f_3, f_4\}$, and $|\mathcal{Z}'| = |\mathcal{Z}| - 3$. Moreover, each connected component of \mathcal{Z}' contains a vector with square ≤ -3 . This contradicts Lemma 2.5, showing that Case (i) cannot occur either. \square

Proposition 3.16. *When $t = 2$ the set \mathcal{Z} has an even number of connected components. Indeed, each irreducible component of \mathcal{Z} is the union of two connected components C_1 and C_2 , with*

$$C_1 = \{f_1 - f_2, f_2 - f_3 - f_4, f_4 - f_5 - f_6 - f_1\} \quad \text{and} \quad C_2 = \{f_3 - f_4 - f_6, f_6 - f_5, f_5 - f_1 - f_2 - f_3\}$$

for some $f_1, \dots, f_6 \in \mathcal{F}$.

Proof. Let $S \subset \mathcal{Z}$ be an irreducible component, and let $C_1 \subset S$ be one of its connected components. In view of Lemma 3.15, up to permuting the elements of \mathcal{F} we may assume that

$$C_1 = \{u_1 = f_1 - f_2, v_1 = f_2 - f_3 - f_4, w_1 = f_4 - f_5 - f_6 - f_1\}.$$

Let $w_2 \in S \setminus C_1$ be the unique vector with $w_2 \cdot f_1 = 1$. Since $w_2 \cdot u_1 = w_2 \cdot v_1 = 0$, we have $w_2 \cdot w_2 = -4$ and either (i) $w_2 = e_{w_2} - f_1 - f_2 - f_3$ or (ii) $w_2 = e_{w_2} - f_1 - f_2 - f_4$. Case (ii) cannot occur, because if it did f_2 and f_4 would already have been used three times each, therefore the unique vector $v_2 \in S \setminus C_1$ such that $e_{v_2} = f_3$ could not satisfy $v_2 \cdot v_1 = 0$.

In Case (i), since $w_2 \cdot w_1 = 0$, up to swapping f_5 and f_6 we have $w_2 = f_5 - f_1 - f_2 - f_3$. Let $v_2 \in S \setminus C_1$ be the unique vector with $e_{v_2} = f_3$. Since $v_2 \cdot v_1 = 0$, we have $v_2 = f_3 - f_4 - \dots$ and since $v_2 \cdot w_1 = 0$ we have either $v_2 = f_3 - f_4 - f_5 - \dots$ or $v_2 = f_3 - f_4 - f_6 - \dots$. The first case is not possible because it would imply $v_2 \cdot w_2 = 2$, therefore the second case occurs and $v_2 \cdot w_2 = 1$, which implies that v_2 belongs to the same connected component as w_2 , hence $v_2 = f_3 - f_4 - f_6$. The third vector u_2 of the connected component of S containing v_2 and w_2 must share a vector with both v_2 and w_2 . Since f_1, \dots, f_4 have already been used three times, this forces $u_2 = f_6 - f_5$. Therefore, $S = C_1 \cup C_2$ where $C_2 = \{u_2, v_2, w_2\}$ is a connected component of the stated form. \square

Suppose $t \geq 3$. By the considerations following Remark 3.13, it is easy to see that \mathcal{Z} consists of some number n of connected components C_1, \dots, C_n , where each component C_i consists of t vectors of square ≤ -3 .

Lemma 3.17. *Each $z \in \mathcal{Z}$ satisfies $z \cdot z = -3$.*

Proof. For each $z \in \mathcal{Z}$ we have $z = -\sum_{f \in \mathcal{F}} (z \cdot f) f$. By Lemma 3.11(2) we have

$$3|\mathcal{Z}| \leq -\sum_{z \in \mathcal{Z}} z \cdot z = \sum_{z \in \mathcal{Z}} z \cdot \sum_{f \in \mathcal{F}} (z \cdot f) f = \sum_{f \in \mathcal{F}} \sum_{z \in \mathcal{Z}} (z \cdot f)^2 = 3|\mathcal{F}| = 3|\mathcal{Z}|.$$

Therefore $3|\mathcal{Z}| = -\sum_{z \in \mathcal{Z}} z \cdot z$. Since $z \cdot z \leq -3$ for each $z \in \mathcal{Z}$, we must have $z \cdot z = -3$ for every $z \in \mathcal{Z}$. \square

Observe that, in view of Lemmas 3.11 and 3.17, for each $u \in \mathcal{Z}$ there are distinct elements $f_1, f_2, f_3 \in \mathcal{F}$ such that $u = f_1 - f_2 - f_3$. In particular, $\mathcal{F}_u = \{f_1, f_2, f_3\}$.

Lemma 3.18. *Let $u, v \in \mathcal{Z}$ with $u \cdot v = 1$. Then, up to swapping u and v , one of the following holds:*

- (1) $|\mathcal{F}_u \cap \mathcal{F}_v| = 1$ and there are five distinct elements $f_1, \dots, f_5 \in \mathcal{F}$ such that $u = f_1 - f_2 - f_3$ and $v = f_3 - f_4 - f_5$;
- (2) $|\mathcal{F}_u \cap \mathcal{F}_v| = 3$ and there are three distinct elements $f_1, f_2, f_3 \in \mathcal{F}$ such that $u = f_1 - f_2 - f_3$, $v = f_3 - f_1 - f_2$.

Moreover, if u and v belong to a connected component $C \subset \mathcal{F}$ with cardinality $|C| > 3$ then Case (1) holds.

Proof. Since $u \cdot f, v \cdot f \in \{-1, 0, 1\}$ for each $f \in \mathcal{F}$, $u \cdot v = 1$ implies that $|\mathcal{F}_u \cap \mathcal{F}_v|$ is either 1 or 3. In the latter case, up to swapping u and v we necessarily have $u = f_1 - f_2 - f_3$ and $v = f_3 - f_1 - f_2$ for some distinct $f_1, f_2, f_3 \in \mathcal{F}$, hence (2) holds. In the first case, up to swapping u and v we must have $\mathcal{F}_u \cap \mathcal{F}_v = \{f_3\}$, $u = f_1 - f_2 - f_3$ and $v = f_3 - f_4 - f_5$ for some distinct $f_1, \dots, f_5 \in \mathcal{F}$. Therefore, (1) holds. If u and v belong to a connected component C with $|C| > 3$, there exists an element $w \in C$ such that $w \cdot u = 1$ and $w \cdot v = 0$. Then, since $|\mathcal{F}_w \cap \mathcal{F}_u|$ is odd and $|\mathcal{F}_w \cap \mathcal{F}_v|$ is even, we must have $\mathcal{F}_u \neq \mathcal{F}_v$, and only Case (1) can occur. \square

Lemma 3.19. *Let $u, v \in \mathcal{Z}$ with $u \cdot v = 0$ and $\mathcal{F}_u \cap \mathcal{F}_v \neq \emptyset$. Then, up to swapping u and v the following hold:*

- (1) *there are four distinct elements $f_1, f_2, f_3, f_4 \in \mathcal{F}$ such that $u = f_1 - f_2 - f_3$ and $v = f_2 - f_3 - f_4$;*
- (2) *let C_u and C_v be the connected components of \mathcal{Z} containing u and v , and suppose $C_u \neq C_v$. If $|C_u| = 3$, then $|C_v| = 3$ and one of the following holds:*
 - (a) *there exist $f_1, \dots, f_6 \in \mathcal{F}$ such that*

$$C_u = \{f_1 - f_2 - f_3, f_3 - f_4 - f_5, f_5 - f_6 - f_1\}, \quad C_v = \{f_2 - f_3 - f_4, f_4 - f_5 - f_6, f_6 - f_1 - f_2\};$$

- (b) *there exist $f_1, \dots, f_8 \in \mathcal{F}$ such that*

$$C_u = \{f_1 - f_2 - f_3, f_5 - f_6 - f_1, f_6 - f_5 - f_1\}, \quad C_v = \{f_4 - f_7 - f_8, f_2 - f_3 - f_4, f_3 - f_2 - f_4\}.$$

Proof. (1) Clearly $\mathcal{F}_u \cap \mathcal{F}_v$ contains two elements, say f_2 and f_3 , one of which, say f_2 , must satisfy $(f_2 \cdot u, f_2 \cdot v) \in \{(-1, 1), (1, -1)\}$. Up to swapping u and v we may assume that $f_2 \cdot u = 1$ and $f_2 \cdot v = -1$. Then, $u = f_1 - f_2 - f_3$ for some $f_1, f_3 \in \mathcal{F}$, and necessarily $v = f_2 - f_3 - f_4$ for some $f_4 \in \mathcal{F}$, with f_1, f_2, f_3 and f_4 pairwise distinct.

(2) Let $u', u'' \in C_u$ be the two elements adjacent to u , i.e. such that $u' \cdot u = u'' \cdot u = 1$. Analogously, let $v', v'' \in C_v$ be the two elements adjacent to v . By Lemma 3.18 we have either $|\mathcal{F}_{u'} \cap \mathcal{F}_u| = 1$ or $|\mathcal{F}_{u'} \cap \mathcal{F}_u| = 3$. In the latter case we would have $u' \in \{f_3 - f_1 - f_2, f_2 - f_1 - f_3\}$, which would be incompatible with $u' \cdot v = 0$, therefore the first case occurs. The only possibilities are $\mathcal{F}_{u'} \cap \mathcal{F}_u = \{f_1\}$ and $\mathcal{F}_{u'} \cap \mathcal{F}_u = \{f_3\}$, which correspond, respectively, to $u' = f_5 - f_6 - f_1$ and $u' = f_3 - f_4 - f_5$, for some $f_5, f_6 \in \mathcal{F}$. If the first possibility is realized, then it is easy to check that either $u'' = f_3 - f_4 - f_5$ or $u'' = f_6 - f_5 - f_1$. We are going to analyze the various cases.

Suppose first that $u' = f_5 - f_6 - f_1$ and $u'' = f_3 - f_4 - f_5$. by Lemma 3.11(2) there must be some vector $w \neq u$ such that $w \cdot f_2 = 1$. Since $w \cdot u = 0$, we must have $w \cdot f_1 = 1$, and this forces $w = f_6 - f_1 - f_2$. Therefore w is either v' or v'' , say v'' . Then, it is easy to check that $|\mathcal{F}_v \cap \mathcal{F}_{v'}| = 3$ is impossible, therefore $|\mathcal{F}_v \cap \mathcal{F}_{v'}| = 1$, which forces $v' = f_4 - f_5 - f_6$. Thus, v' exhausts both C_v and the irreducible component, and (2)(a) holds.

Now suppose that $u' = f_5 - f_6 - f_1$ and $u'' = f_6 - f_5 - f_1$. Then, by Lemma 3.11(2) there is some $w \in \mathcal{Z}$ such that $w \cdot f_2 = 1$. We must have $w \cdot u = 0$, which implies $w \cdot f_1 = 1$ or $w \cdot f_3 = -1$. But $w \cdot f_1 = 1$ is incompatible with $w \cdot u' = w \cdot u'' = 0$, therefore we must have $w \cdot f_3 = -1$. Then, $w \cdot v > 0$ and therefore $w = f_3 - f_2 - f_4 \in \{v', v''\}$, so we may assume $w = v''$. If we now consider the only vector $w' \in \mathcal{Z}$ which, by Lemma 3.11(2), satisfies $w' \cdot f_4 = -1$ we can easily conclude that $w' = f_4 - f_7 - f_8 = v'$ for some $f_7, f_8 \in \mathcal{F}$, and $v' \cdot v'' = 1$. Therefore $|C_v| = 3$ and C_u, C_v are as in (2)(b).

There remains to examine the possibility $u' = f_3 - f_4 - f_5$. In this case, it is easy to check that $|\mathcal{F}_{u''} \cap \mathcal{F}_u| = 3$ is impossible, and $|\mathcal{F}_{u''} \cap \mathcal{F}_u| = 1$ forces $u'' = f_5 - f_6 - f_1$ for some $f_6 \in \mathcal{F}$. As before, this implies $\{v', v''\} = \{f_4 - f_5 - f_6, f_6 - f_1 - f_2\}$. Therefore $|C_v| = 3$ and (2)(a) holds. \square

Proposition 3.20. *Suppose that $u, v \in \mathcal{Z}$ belong to distinct connected components C_u , respectively C_v , and $\mathcal{F}_u \cap \mathcal{F}_v \neq \emptyset$. If $|C_u|, |C_v| > 3$ then $|C_u| = |C_v|$ and, up to swapping u and v there are distinct elements $f_1, \dots, f_{2|C_u|} \in \mathcal{F}$ such that*

$$C_u = \{f_{2i-1} - f_{2i} - f_{2i+1}\}, \quad C_v = \{f_{2i} - f_{2i+1} - f_{2i+2}\} \quad \text{for } i = 1, \dots, |C_u|,$$

where $f_{2|C_u|+1} = f_1$ and $f_{2|C_u|+2} = f_2$.

Proof. According to Lemma 3.19, up to swapping u and v we can write $u = f_1 - f_2 - f_3$ and $v = f_2 - f_3 - f_4$ for some distinct $f_1, f_2, f_3, f_4 \in \mathcal{F}$. Let $u', u'' \in C_u$ be the two elements adjacent to u , i.e. such that $u' \cdot u = u'' \cdot u = 1$. Analogously, let $v', v'' \in C_v$ be the two elements adjacent to v . By Lemma 3.18 we have

$$|\mathcal{F}_{u'} \cap \mathcal{F}_u| = |\mathcal{F}_{u''} \cap \mathcal{F}_u| = |\mathcal{F}_{v'} \cap \mathcal{F}_v| = |\mathcal{F}_{v''} \cap \mathcal{F}_v| = 1.$$

Moreover, we claim that either $u' \cdot f_1 = 0$ or $u'' \cdot f_1 = 0$. To prove the claim, suppose by contradiction that $u' \cdot f_1 \neq 0$ and $u'' \cdot f_1 \neq 0$. Then, $\mathcal{F}_{u'} \cap \mathcal{F}_u = \mathcal{F}_{u''} \cap \mathcal{F}_u = \{f_1\}$. By Lemma 3.11(1) and the surjectivity of the map of Lemma 3.3, there exists $w \in \mathcal{Z}$ with $w \cdot f_2 = -1$. This implies, in particular, that $w \notin \{u', u''\}$, and therefore $w \cdot u = 0$. By Lemma 3.19 we have $w = f_2 - f_3 - f_5$ for some $f_5 \in \mathcal{Z}$. But then $w \cdot v < 0$, which is impossible. Therefore, the claim is proved and without loss of generality we may assume $u' \cdot f_1 = 0$. Since $u' \cdot u = 1$, $u' \cdot v = 0$ and we already have $v \cdot f_2 = -1$, we must have $u' = f_3 - f_4 - f_5$ for some $f_5 \in \mathcal{F}$. If we apply the same argument with the pair (v, u') in place of (u, v) we see that we may assume without loss of generality that $v' = f_4 - f_5 - f_6$ for some $f_6 \in \mathcal{Z}$. Now let $u^1 := u$, $u^2 := u'$, $v^1 := v$ and $v^2 := v'$. The same argument applied to (v^1, u^1) yields an element $u^3 = f_5 - f_6 - f_7$, and so on. We end up with two sequences of the form

$$u^i = f_{2i-1} - f_{2i} - f_{2i+1} \in C_u, \quad v^i = f_{2i} - f_{2i+1} - f_{2i+2} \in C_v, \quad i = 1, 2, \dots$$

such that $u^i \cdot u^{i+1} = v^i \cdot v^{i+1} = 1$ for every i . Suppose without loss of generality $|C_u| \leq |C_v|$. It is easy to check that, as long as $i \leq |C_u| - 1$ the f_i^j 's are all distinct. On the other hand, $u^{|C_u|} \cdot u^1 = 1$ implies $u^{|C_u|} = f_{2|C_u|-1} - f_{2|C_u|} - f_1$ and $u^{|C_u|+1} = u^1$, while $v^{|C_u|} \cdot u^1 = 0$ implies $v^{|C_u|} = f_{2|C_u|} - f_1 - f_2$ and $v^{|C_u|+1} = v^1$. Therefore $|C_u| = |C_v|$ and the statement holds. \square

Proposition 3.21. *Let $C \subset \mathcal{Z}$ be a connected component of \mathcal{Z} such that, for each $u \in C$ and $v \in \mathcal{Z}$, $\mathcal{F}_u \cap \mathcal{F}_v \neq \emptyset$ implies $v \in C$. Then, $|C|$ is odd and there exist $f_i \in \mathcal{F}$, $i \in \mathbb{Z}/(2m+1)\mathbb{Z}$, such that*

$$C = \{f_{2i-1} - f_{2i} - f_{2i+1} \mid i \in \mathbb{Z}/(2m+1)\mathbb{Z}\}.$$

Proof. Without loss of generality we may assume $C = \mathcal{Z}$. We prove the statement by induction on $|\mathcal{Z}| \geq 3$, treating separately the two cases $|\mathcal{Z}|$ odd and $|\mathcal{Z}|$ even. For the basis of the induction in the odd case, we observe that if $|\mathcal{Z}| = 3$ it is easy to check that there exist $f_1, f_2, f_3 \in \mathcal{F}$ such that

$$\mathcal{Z} = \{f_1 - f_2 - f_3, f_3 - f_1 - f_2, f_2 - f_3 - f_1\},$$

therefore the statement holds. In the even case the basis is the case $|\mathcal{Z}| = 4$. Then, $|\mathcal{F}| = 4$ as well, which is impossible by Lemma 3.18 (Case (1) applies because $|\mathcal{Z}| > 3$). Therefore, there is no set \mathcal{Z} satisfying the assumptions of the proposition with $|\mathcal{Z}| = 4$. In the proof we will show that the same conclusion holds if $|\mathcal{Z}|$ is even.

Now suppose that $|\mathcal{Z}| \geq 5$, and let $u, v \in \mathcal{Z}$ with $u \cdot v = 1$. By Lemma 3.18, up to renaming u and v there exist five distinct elements $f_1, \dots, f_5 \in \mathcal{F}$ such that $u = f_1 - f_2 - f_3$ and $v = f_3 - f_4 - f_5$. Let w be the only element of \mathcal{Z} such that $w \neq u$ and $w \cdot f_3 = 1$. By Lemma 3.18 we must have $w \cdot u = 0$. We claim that $w \cdot v = 0$ as well. Suppose by contradiction that $w \cdot v = 1$. Then, let w' and w'' be the unique elements of \mathcal{Z} such that $w' \cdot f_4 = -1$ and $w'' \cdot f_5 = -1$. Since necessarily $w' \cdot v = w'' \cdot v = 0$ and f_3 already hits u, v and w , we must have $w' \cdot f_5 = w'' \cdot f_4 = 1$. But this would imply $w' \cdot w'' > 0$ and $|\mathcal{F}_{w'} \cap \mathcal{F}_{w''}| > 1$, contradicting Lemma 3.18. Therefore the claim is proved and $w \cdot v = 0$. Up to swapping f_4 and f_5 we have $w = f_2 - f_3 - f_4$. Now let $w' = f_2 - f_4$, and consider the new set

$$\mathcal{Z}' = \mathcal{Z} \setminus \{u, v, w\} \cup \{u + v, w'\} \subset \mathcal{Z}.$$

By construction, \mathcal{Z}' has clearly cardinality $|\mathcal{Z}| - 1$ and is contained in the span of $\mathcal{F}' = \mathcal{F} \setminus \{f_3\}$. Thus, it is a positive, circular subset in $\mathbb{Z}^{|\mathcal{Z}'|}$ admitting a canonical adapted basis, and the map of Lemma 3.3 is injective. Moreover, the Wu element W' of \mathcal{Z}' clearly satisfies $W' \cdot W' = -|\mathcal{Z}'|$. Hence, the assumptions of Lemma 3.11 are satisfied for \mathcal{Z}' . We can therefore apply a (-2)-contraction using the vector w' . Since $u + v = f_1 - f_2 - f_4 - f_5$, the (-2)-contraction gives a new positive circular subset \mathcal{Z}'' of cardinality $|\mathcal{Z}''| = |\mathcal{Z}'| - 1 = |\mathcal{Z}| - 2 \geq 3$ contained in the span of $\mathcal{F}'' = \mathcal{F} \setminus \{f_2, f_3\}$. By construction, all the elements of \mathcal{Z}'' have square -3 , and \mathcal{Z}'' has a single connected component. Moreover, it is easy to check that the assumptions of Lemma 3.11 are still satisfied by \mathcal{Z}'' . If $|\mathcal{Z}|$ and $|\mathcal{Z}''|$ were even, we could repeat the same construction – several times if necessary – to obtain a set $\tilde{\mathcal{Z}}$ with $|\tilde{\mathcal{Z}}| = 4$, satisfying the assumptions of the proposition, which we have already shown to be impossible. Therefore $|\mathcal{Z}|$ and $|\mathcal{Z}''|$ must be odd. By the inductive assumption there exist $f_i \in \mathcal{F}''$, $i \in \mathbb{Z}/(2m-1)\mathbb{Z}$, such that

$$\mathcal{Z}'' = \{f_{2i-1} - f_{2i} - f_{2i+1} \mid i \in \mathbb{Z}/(2m-1)\mathbb{Z}\}.$$

Observe that \mathcal{Z} is obtained from \mathcal{Z}'' by replacing, for some $j \in \mathbb{Z}/(2m-1)\mathbb{Z}$, the subset

$$\{f_{2j-1} - f_{2j} - f_{2j+1}, f_{2j+1} - f_{2j+2} - f_{2j+3}\}$$

with

$$\{f_{2j-1} - f_{2j} - f_{2m+3}, f_{2m+2} - f_{2m+3} - f_{2j+1}, f_{2j+1} - f_{2j+2} - f_{2j+3}\}$$

and the subset

$$\{f_{2j} - f_{2j+1} - f_{2j+2}\} \quad \text{with} \quad \{f_{2j} - f_{2m+2} - f_{2m+3}, f_{2m+3} - f_{2j+1} - f_{2j+2}\}.$$

It is now a simple matter to deduce the statement for \mathcal{Z} . □

Conclusions. We now have enough information about positive circular subsets of \mathbb{Z}^N , so we can draw the conclusions we are interested in. The following theorem will be used in the next section, together with Theorem 3.7, to prove Theorem 1.1.

Theorem 3.22. *Let Γ be a weighted graph as in Figure 3 with $d = 0$, $t \geq 2$, $x_i, y_i \geq 1$ and $\sum_i y_i = \sum_i x_i \geq 3$. Let $N = k \sum_i y_i$ for some $k \geq 1$, and suppose that there is an isometric embedding $\Lambda_\Gamma^k \subset \mathbb{Z}^N$ of finite and odd index. Then, the string of negative integers*

$$(3.4) \quad S := (-2 - x_1, \overbrace{-2, \dots, -2}^{y_1-1}, \dots, -2 - x_t, \overbrace{-2, \dots, -2}^{y_t-1})$$

satisfies one of the following:

- (1) S is obtained, up to circular symmetry, from $(-2, -2, -5)$ by iterated (-2) -expansions (in the sense of Definition 3.9);
- (2) there exist a regular polygon P with t vertices and a symmetry $\varphi : P \rightarrow P$ such that $(X, Y) = (Y, X')^\varphi$, where (X, Y) is the labelling of P encoded by $(x_1, y_1, \dots, x_t, y_t)$ and (Y, X') is the labelling encoded by $(y_1, x_2, y_2, x_3, \dots, y_t, x_1)$.

Proof. We proceed as in Theorem 3.7. The lattice Λ_Γ^k has a natural basis, whose elements are in 1–1 correspondence with the vertices of the disjoint union of k copies of Γ , and intersect as prescribed by edges and weights. Since $d = 0$, the image of such a basis is a positive circular subset $\mathcal{V} \subset \mathbb{Z}^N$. The Wu element $W = \sum_{v \in \mathcal{V}} v$ satisfies

$$W \cdot W = k \left(\sum_i (-2 - x_i) - 2 \sum_i (y_i - 1) + 2 \sum_i y_i \right) = -k \sum_i y_i = -N$$

and since $\Lambda_{\mathcal{V}} = \Lambda_\Gamma^k$, the embedding $\Lambda_{\mathcal{V}} \subset \mathbb{Z}^N$ has finite and odd index. By Lemma 3.2, there is a canonical basis $\mathcal{E} \subset \mathbb{Z}^N$ adapted to \mathcal{V} . Moreover, each connected component $D \subset \mathcal{V}$ satisfies

$$\sum_{v \in D} (v \cdot v + 2) = -\sum_i x_i = -\sum_i y_i = -|D|.$$

If the map of Lemma 3.3 is not injective then the assumptions of Proposition 3.10 are satisfied. Applying the proposition, Case (1) of the statement follows immediately.

If the map of Lemma 3.3 is injective, the considerations following Remark 3.13 show that applying a finite number of (-2) -contractions (in the sense of Definition 3.12) to the subset \mathcal{V} one obtains a circular subset $\mathcal{Z} \subset \mathbb{Z}^{|\mathcal{Z}|}$ satisfying all the assumptions of Lemma 3.11 and having the property that each (-2) -vector of \mathcal{Z} belongs to a connected component with exactly three elements. In particular, there is a canonical basis $\mathcal{F} \subset \mathbb{Z}^{|\mathcal{Z}|}$ adapted to \mathcal{Z} . Clearly, every irreducible component of \mathcal{Z} is a union of connected components. Since the string S given by (3.4) is associated to any connected component of \mathcal{V} , we can assume without loss of generality that \mathcal{Z} is irreducible.

When $t = 2$ the subset \mathcal{Z} satisfies the conclusions of Proposition 3.16, therefore it is the union of two connected components $C_1 = \{u_1, v_1, w_1\}$ and $C_2 = \{u_2, v_2, w_2\}$, where $u_1 = f_1 - f_2$, $v_1 = f_2 - f_3 - f_4$, $w_1 = f_4 - f_5 - f_6 - f_1$ and $u_2 = f_6 - f_5$, $v_2 = f_3 - f_4 - f_6$ and $w_2 = f_5 - f_1 - f_2 - f_3$ for some $f_1, \dots, f_6 \in \mathcal{F}$. Now we want to recover the self-intersections of the elements of \mathcal{V} using the fact that \mathcal{Z} is obtained from \mathcal{V} by (-2) -contractions. Of course, \mathcal{V} is obtained from \mathcal{Z} by (-2) -expansions. Suppose that, for $i = 1, 2$, \mathcal{V} is obtained by applying a_i (-2) -expansions between u_i and v_i , b_i (-2) -expansions between v_i and w_i , and c_i (-2) -expansions between w_i and u_i . Then, it is easy to check that $v_i \in \mathcal{Z}$ becomes a vector of \mathcal{V} of self-intersection

$-3 - b_{3-i}$ and w_i a vector of self-intersection $-4 - a_{3-i} - c_{3-i}$, for $i = 1, 2$. Therefore, up to a symmetry, the weights of the weighted graphs associated to the two connected components of \mathcal{V} are

$$(-4 - a_2 - c_2, \overbrace{-2, \dots, -2}^{a_1+c_1+1}, -3 - b_2, \overbrace{-2, \dots, -2}^{b_1})$$

and

$$(-4 - a_1 - c_1, \overbrace{-2, \dots, -2}^{a_2+c_2+1}, -3 - b_1, \overbrace{-2, \dots, -2}^{b_2}).$$

Since the two components of \mathcal{V} had the same self-intersections, in terms of the parameters x_i, y_i this easily translates into the condition that either (i) $x_1 = y_1$ and $x_2 = y_2$ or (ii) $x_1 = y_2$ and $x_2 = y_1$. In both cases, there exist a bigon P and a symmetry $\varphi : P \rightarrow P$ such that $(X, Y) = (Y, X')^\varphi$, where (X, Y) is the labelling of P encoded by (x_1, y_1, x_2, y_2) and (Y, X') is the labelling encoded by (y_1, x_2, y_2, x_1) . In fact, in the first case we can take the reflection of P which fixes the vertices and switches the edges, in the second case the reflection which switches the vertices and fixes the edges. Therefore Case (2) holds.

When $t \geq 3$ we first assume that \mathcal{Z} has more than one connected component. Then, by Lemma 3.19 and Proposition 3.20 all the connected components of \mathcal{Z} have the same cardinality. As in the case $t = 2$, we want to recover the self-intersections of the elements of \mathcal{V} from the structure of \mathcal{Z} . Let $C \subset \mathcal{Z}$ be a connected component. If $|C| > 3$ then Proposition 3.20 applies, and we shall now refer to the notation and terminology of that proposition. Starting from \mathcal{Z} , if we do, say, x_i (-2) -expansions between $f_{2i-1} - f_{2i} - f_{2i+1}$ and $f_{2i+1} - f_{2i+2} - f_{2i+3}$, the weight of the vertex corresponding to $f_{2i} - f_{2i+1} - f_{2i+2}$ becomes $-3 - x_i$. Similarly, doing y_i (-2) -expansions between $f_{2i} - f_{2i+1} - f_{2i+2}$ and $f_{2i+2} - f_{2i+3} - f_{2i+4}$ makes the vertex corresponding to $f_{2i-1} - f_{2i} - f_{2i+1}$ acquire weight $-3 - y_i$. This shows that the two connected components C_u and C_v of Proposition 3.20 come from two connected components $\tilde{C}_u, \tilde{C}_v \subset \mathcal{V}$ having self-intersections given, up to a symmetry of the corresponding weighted graphs, by

$$\left(\overbrace{2, \dots, 2}^{x_1}, -3 - y_1, \overbrace{2, \dots, 2}^{x_2}, \dots, \overbrace{2, \dots, 2}^{x_t}, -3 - y_t\right) \quad \text{and} \quad \left(-3 - x_1, \overbrace{2, \dots, 2}^{y_1}, -3 - x_2, \dots, -3 - x_t, \overbrace{2, \dots, 2}^{y_t}\right).$$

Since the weighted graphs associated to \tilde{C}_u and \tilde{C}_v are isomorphic, it follows easily that there is a regular t -gon P and a symmetry $\varphi : P \rightarrow P$ such that $(X, Y) = (Y, X')^\varphi$, where (X, Y) is the labelling of P encoded by $(x_1, y_1, \dots, x_t, y_t)$ and (Y, X') is the labelling encoded by $(y_1, x_2, y_2, x_3, \dots, y_t, x_1)$. Therefore Case (2) holds. If $|C| = 3$ then Lemma 3.19(2) applies. Conclusion (a) in the statement of the lemma is perfectly analogous to the statement of Proposition 3.20. Therefore, if Case (a) in the lemma holds, Case (2) of the statement holds. If the conclusion of Lemma 3.19(2)(b) holds, an analysis similar to the case $|C| > 3$ shows that the connected component C_u corresponds to a connected component $\tilde{C}_u \subset \mathcal{V}$ with string of self-intersections given, up to a symmetry of the corresponding weighted graph, by

$$(-3 - x_1, \overbrace{2, \dots, 2}^{x_3}, -3 - x_2, \overbrace{2, \dots, 2}^{x_1}, -3 - x_3, \overbrace{2, \dots, 2}^{x_2}).$$

In other words, up to symmetry, the string is of the type given by Equation (3.4) with $t = 3$ and $(y_1, y_2, y_3) = (x_3, x_1, x_2)$. In other words, there is an equilateral triangle P and a $2\pi/3$ -rotation $\varphi : P \rightarrow P$ such that $(x_1, y_1, x_2, y_2, x_3, y_3) = (y_1, x_2, y_2, x_3, y_3, x_1)^\varphi$. Therefore Case (2) holds again.

Now we assume that \mathcal{Z} is connected. Then, Proposition 3.21 applies. This case is quite similar to the case of Proposition 3.20. For each index i , we can only do x_i (-2) -expansions between $f_{2i-1} - f_{2i} - f_{2i+1}$ and $f_{2i+1} - f_{2i+2} - f_{2i+3}$. Then, the vertex corresponding to $f_{2i} - f_{2i+1} - f_{2i+2}$ acquires weight $-3 - x_i$. This implies that the weights of each component of \mathcal{V} are given by Equation (3.4), with $t = 2m + 1$ and $y_i = x_{i+m+1}$ for each $i \in \mathbb{Z}/(2m+1)\mathbb{Z}$. Arguing as when \mathcal{Z} is disconnected, we see again that Case (2) holds. This concludes the proof. \square

4. THE PROOF OF THEOREM 1.1

In this section we use the results of Sections 2 and 3 to prove Theorem 1.1. By Proposition 2.2, K is the closure of a 3-braid $\beta \in B_3$ of the form

$$\beta = (\sigma_1 \sigma_2)^{3d} \sigma_1^{x_1} \sigma_2^{-y_1} \cdots \sigma_1^{x_t} \sigma_2^{-y_t}, \quad t, x_i, y_i \geq 1,$$

for some $d \in \{-1, 0, 1\}$ with $\sum_{i=1}^t (x_i - y_i) = -4d$. We consider separately the two cases $d = \pm 1$ and $d = 0$.

First case: $d = \pm 1$. If $d = -1$ then K^m is the closure of an analogous 3-braid with $d = 1$. Therefore, we will assume without loss that $d = 1$. By Proposition 2.6, for some $k \geq 1$ the orthogonal sum Λ_Γ^k of k copies of Λ_Γ embeds isometrically, with finite and odd index, in the standard negative definite lattice \mathbb{Z}^N , where $N = k \sum_i y_i$. If $\sum_i x_i = 1$ then $t = 1$, $x_1 = 1$ and $y_1 = x_1 + 4 = 5$. Therefore $\beta = (\sigma_1 \sigma_2)^3 \sigma_1 \sigma_2^{-5}$, which belongs to Family (1) of Theorem 1.1.

Indeed, $(x_1 + 2, \overbrace{2, \dots, 2}^{y_1-1}) = (3, 2, 2, 2, 2)$ and it is easy to check that $(3, 1, 2, 2, 1)$ is an iterated blowup of $(0, 0)$ and β is quasi-positive because it is conjugate to $\sigma_2^3 \sigma_1 \sigma_2^{-3} \sigma_1$, or by [16, Theorem 2.3]. We have $e(\beta) = 2$, and we show below that a knot with vanishing signature which is the closure of a quasi-positive 3-braid having exponent sum equal to 2 is ribbon. Thus, we may assume $\sum_i y_i = \sum_i x_i + 4 \geq 6$. Theorem 3.7 implies that K belongs to Family (1). To conclude the proof in this case we need to argue that β is quasi-positive and K is ribbon. As observed in Section 1, the fact that β is quasi-positive follows from [16, Theorem 2.3]. Next, we observe that since K has finite concordance order its signature vanishes, therefore by Equation (2.4) we have $e(\beta) = 2d = 2$, where $e : B_3 \rightarrow \mathbb{Z}$ is the exponent sum homomorphism. Finally, the constructions described in [21, § 2] show that the closure of a quasi-positive 3-braid with exponent sum equal to 2 bounds an immersed surface $S \subset S^3$ with only ribbon singularities, constructed as the disjoint union of three embedded disks and two embedded 1-handles which intersect the disks in ribbon singularities. Since $K = \hat{\beta} = \partial S$ is a knot, the fact that $\chi(S) = 3 - 2 = 1$ implies that in our case we can find such an S which is an immersed ribbon disk. It is well-known that suitably pushing the interior of S into the 4-ball we obtain a ribbon disk for K .

Second case: $d = 0$. Let $n := \sum_i x_i = \sum_i y_i$. If $n = 1$ we have $t = x_1 = y_1 = 1$, therefore $\beta = \sigma_1 \sigma_2^{-1}$, $K = \hat{\beta}$ belongs to Family (3) and it is the unknot, which is amphichiral. If $n = 2$ there are two possible cases: either $t = 2$ and $x_1 = y_1 = x_2 = y_2 = 1$, or $t = 1$ and $x_1 = y_1 = 2$. In the first case $\beta = (\sigma_1 \sigma_2^{-1})^2$, K belongs to Family (3) and it is the figure-eight knot, which is well-known to be amphichiral. In the second case $\beta = \sigma_1^2 \sigma_2^{-2}$, and $K = \hat{\beta}$ is a 3-component link. Therefore, from now on we assume $n = \sum_i x_i = \sum_i y_i \geq 3$.

By Proposition 2.6, for some $k \geq 1$ the orthogonal sum Λ_Γ^k of k copies of Λ_Γ embeds isometrically, with finite and odd index, in the standard negative definite lattice \mathbb{Z}^N , where $N = k \sum_i y_i$. The embedding $\Lambda_\Gamma^k \subset \mathbb{Z}^N$ gives rise to a positive circular subset $\mathcal{V} \subset \mathbb{Z}^N$ such that

$v \cdot v \leq -2$ for each $v \in \mathcal{V}$. By Theorem 3.22, the string S of negative integers of Equation (3.4) falls under Case (1) or Case (2) of the statement of that theorem. We will now verify that in Case (1) the knot K belongs to Family (2) of Theorem 1.1. We shall argue by induction on the length of the string S . Observe first that $L_a = \widehat{\beta}_a$, where

$$\beta_a := \sigma_2 \sigma_1 \sigma_2^{-1} a \sigma_2 \sigma_1^{-1} \sigma_2^{-1} a^{-1} \in B_3.$$

The 3-braid β_a is conjugate to $\beta'_a = \sigma_2^{-2} a \sigma_2^2 \Delta^{-1} a^{-1} \Delta$, where $\Delta = \sigma_1 \sigma_2 \sigma_1 \in B_3$ is the Garside element. To prove the basis of the induction, note that the knot corresponding to $(-2, -2, -5)$ is the closure of $\sigma_2^{-3} \sigma_1^3 \in B_3$, and we may also write this braid as

$$\beta'_{\sigma_2^{-1} \sigma_1^2} = \sigma_2^{-2} (\sigma_2^{-1} \sigma_1^2) \sigma_2^2 \Delta^{-1} (\sigma_2^{-1} \sigma_1^2)^{-1} \Delta.$$

This shows that K belongs to Family (2). To prove the inductive step, we first consider the two (-2) -expansions of $(-2, -2, -5)$, which are $(-2, -2, -2, -6)$ and $(-2, -3, -5, -2)$. As one can easily check, the knot corresponding to the first expansion is the closure of the braid obtained from $\beta'_{\sigma_2^{-1} \sigma_1^2}$ by inserting $\sigma_1 \sigma_2^{-1}$ immediately before the factor σ_2^{-2} , while the knot corresponding to the second expansion is the closure of the braid obtained inserting $\sigma_2^{-1} \sigma_1$ immediately after the factor σ_2^{-2} . Observe that, in general, since $\Delta \sigma_1 = \sigma_2 \Delta$,

$$\sigma_1 \sigma_2^{-1} \beta'_a = \sigma_1 \sigma_2^{-2} (\sigma_2^{-1} a) \sigma_2^2 \Delta^{-1} a^{-1} \Delta \sim \sigma_2^{-2} (\sigma_2^{-1} a) \sigma_2^2 \Delta^{-1} (\sigma_2^{-1} a)^{-1} \Delta = \beta'_{\sigma_2^{-1} a},$$

where \sim denotes conjugation. Similarly,

$$\sigma_2^{-2} (\sigma_2^{-1} \sigma_1) a \sigma_2^2 \Delta^{-1} a^{-1} \Delta \sim \sigma_2^{-2} (\sigma_1 a) \sigma_2^2 \Delta^{-1} (\sigma_1 a)^{-1} \Delta = \beta'_{\sigma_1 a}.$$

This shows that the knots corresponding to the two (-2) -expansions of $(-2, -2, -5)$ are of the form $\widehat{\beta}_{\sigma_2^{-2} \sigma_1^2}$ and $\widehat{\beta}_{\sigma_1 \sigma_2^{-1} \sigma_1^2}$. In general, suppose that the string $(-m_1, \dots, -m_k)$ was obtained by a sequence of (-2) -expansions from $(-2, -2, -5)$, and that the corresponding knot was the closure of a braid of the form β'_a . Arguing as before we see that the knots corresponding to the two (-2) -expansions of $(-m_1, \dots, -m_k)$ are closures of braids obtained from β'_a by inserting either $\sigma_1 \sigma_2^{-1}$ immediately before the factor σ_2^{-2} or $\sigma_2^{-1} \sigma_1$ immediately after the factor σ_2^{-2} . Exactly as before, the results are $\beta'_{\sigma_2^{-1} a}$ and $\beta'_{\sigma_1 a}$. This proves the inductive step and shows that if Case (1) of Theorem 3.22 holds, then K belongs to Family (2) of Theorem 1.1. The fact that the link L_a is a symmetric union, hence ribbon, is evident by looking at Figure 1.

Before considering the next case, observe that if $t = 1$ then $x = y \geq 3$, and K is the closure of $\sigma_1^x \sigma_2^{-x}$. In this case, the string S given by Equation (3.4) is of the form given in Case (1) of the statement of Theorem 3.22. Therefore, by the above argument K belongs to Family (2) of Theorem 1.1. From now on we will assume that $t \geq 3$.

Now suppose that the string S given by Equation (3.4) falls under Case (2) of Theorem 3.22. This immediately implies that K belongs to Family (3) of Theorem 1.1, and we are only left to show that K is amphichiral. We start observing that the condition $(X, Y) = (Y, X')^\varphi$ translates, using the notations of Section 1, into

$$(4.1) \quad x_i = y_{\varphi_E(i)}, \quad y_i = x_{\varphi_V(i+1)}, \quad i = 1, \dots, t,$$

where from now on ' $i + 1$ ' will mean ' 1 ' when $i = t$. Moreover, it is easily checked that if φ is a rotation then

$$(4.2) \quad \varphi_V(i + 1) = \varphi_V(i) + 1 \quad \text{and} \quad \varphi_E(i) = \varphi_V(i),$$

while if φ is a reflection then

$$(4.3) \quad \varphi_V(i+1), = \varphi_V(i) - 1 \quad \text{and} \quad \varphi_E(i) = \varphi_V(i) - 1.$$

By Equation (4.1), the knot K is the closure of

$$\prod_{i=1}^t \sigma_1^{x_i} \sigma_2^{-y_i} = \prod_{i=1}^t \sigma_1^{x_i} \sigma_2^{-x_{\varphi_V(i+1)}},$$

therefore K^m is the closure of

$$\prod_{i=1}^t \sigma_1^{-x_i} \sigma_2^{x_{\varphi_V(i+1)}},$$

therefore (applying an isotopy) also the closure of

$$\prod_{i=1}^t \sigma_2^{-x_i} \sigma_1^{x_{\varphi_V(i+1)}} \sim \prod_{i=1}^t \sigma_1^{x_{\varphi_V(i+1)}} \sigma_2^{-x_{i+1}} \sim \prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-x_i} = \prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-y_{\varphi_E(i)}}.$$

where ‘ \sim ’ denotes conjugation and to obtain the equality we applied Equation (4.1). If φ is a rotation then $(\varphi_V(1), \dots, \varphi_V(t)) = (1, 2, \dots, t)$, hence by Equation (4.2) we have

$$\prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-y_{\varphi_E(i)}} = \prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-y_{\varphi_V(i)}} \sim \prod_{i=1}^t \sigma_1^{x_i} \sigma_2^{-y_i},$$

therefore K^m is isotopic to K . If φ is a reflection then $(\varphi_V(1), \dots, \varphi_V(t)) = (t, t-1, \dots, 1)$, hence by Equation (4.3) we have

$$\prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-y_{\varphi_E(i)}} = \prod_{i=1}^t \sigma_1^{x_{\varphi_V(i)}} \sigma_2^{-y_{\varphi_V(i)-1}} \sim \prod_{i=1}^{t-1} \sigma_1^{x_{t+1-i}} \sigma_2^{-y_{t-i}} \cdot \sigma_1^{x_1} \sigma_2^{-y_t} \sim \prod_{i=0}^{t-1} \sigma_2^{-y_{t-i}} \sigma_1^{x_{t-i}},$$

therefore K^m is isotopic to $-K$. This concludes the proof of Theorem 1.1.

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