Numerische Mathematik manuscript No.
(will be inserted by the editor)

Convergence Analysis for a Finite Element Approximation of a Steady Model for Electrorheological Fluids

Luigi C. Berselli · Dominic Breit · Lars Diening

the date of receipt and acceptance should be inserted later

Abstract In this paper we study the finite element approximation of systems of $p(\cdot)$-Stokes type, where $p(\cdot)$ is a (non constant) given function of the space variables. We derive –in some cases optimal– error estimates for finite element approximation of the velocity and of the pressure, in a suitable functional setting.

Keywords. Error analysis, inf-sup condition, velocity, pressure, conforming elements, variable exponents.

1 Introduction

The stationary flow of an incompressible homogeneous fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ is described by the set of equations

$$-\operatorname{div}\mathcal{S} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla q = \mathbf{f}\,, \qquad \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega. \tag{1.1}$$

Here $\mathbf{v} : \Omega \to \mathbb{R}^n$ and $q : \Omega \to \mathbb{R}$ are the unknown velocity field and pressure respectively, whereas $\mathbf{f} : \Omega \to \mathbb{R}^n$ is a given volume force. A popular model for Non-Newtonian (Newtonian if $p = 2$) fluids is the power-law model

$$\mathcal{S} = \mathcal{S}(\mathbf{D}\mathbf{v}) = \mu(\kappa + |\mathbf{D}\mathbf{v}|)^{p-2}\mathbf{D}\mathbf{v}, \tag{1.2}$$

with $\mu > 0$, $\kappa \in [0, 1]$, and $1 < p < \infty$. The extra stress tensor $\mathcal{S}(\mathbf{D}\mathbf{v})$ depends on $\mathbf{D}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^\top)$, the symmetric part of the velocity gradient $\nabla\mathbf{v}$. Physical interpretation and discussion of some non-Newtonian fluid models can be found, e.g., in [8,28]

In this paper we consider a further generalization of (1.2), which is motivated by a model introduced in [32,33] to describe motions of electrorheological fluids, further studied in [34]. Electrorheological fluids are special smart fluids, which change their material properties due to the application of an electric field; especially the viscosity can locally change by a factor of 10^3 in 1ms. Electrorheological fluids can be used in the construction of clutches and shock absorbers. In the model introduced in [33] the exponent p is not a fixed constant, but a function of the electric field $\mathbf{E}$, in particular $p := p(|\mathbf{E}|^2)$. The electric field itself is a solution to the quasi–static Maxwell equations and is not influenced by the motion of the fluid. In a first preliminary step, it is then justified to separate the Maxwell equation from (1.1) and to study, for a given function $p : \Omega \to (1, \infty)$, the system (1.1) with $\mathbf{S} : \Omega \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}^{n\times n}_{\text{sym}}$ satisfying for all $x \in \Omega$ and for all $\boldsymbol{\eta} \in \mathbb{R}^{n\times n}_{\text{sym}}$

$$\mathbf{S}(x, \boldsymbol{\eta}) = \mu(\kappa + |\boldsymbol{\eta}|)^{p(x)-2}\boldsymbol{\eta}. \tag{1.3}$$

Luigi C. Berselli
Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 1/c, I-56127 Pisa, ITALY.
E-mail: berselli@dma.unipi.it

Dominic Breit · Lars Diening
Institute of Mathematics, LMU Munich Theresienstr., 39 D-80333 Munich, Germany
E-mail: diening@math.lmu.de, E-mail: breit@math.lmu.de

This model comprises all the mathematical difficulties of the full system for electrorheological fluids (as in [34]) and the results below can be directly extended to the general case. In this first study we consider the case of a slow flow and therefore neglect the convective term $\operatorname{div}(\mathbf{v}\otimes\mathbf{v}) = (\nabla\mathbf{v})\mathbf{v}$. A reintroduction of this term causes the usual difficulties as for instance the possible non-uniqueness of the solution. This problem also arises for the continuous problem. For small data and large exponents p one can recover uniqueness. In this situation it should be possible to generalize the results of this paper to the presence of the convection term. However, a numerical analysis for small exponents will be complicated, but this difficulty also appears for constant exponents.

Therefore, as a first step (to focus on peculiar difficulties of variable exponents) in this paper we study the numerical approximation of steady systems of the $p(\cdot)$-Stokes type

$$\begin{aligned} -\operatorname{div}\mathbf{S}(\cdot,\mathbf{Dv}) + \nabla q &= \mathbf{f} && \text{in } \Omega,\\ -\operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega,\\ \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

with $\mathbf{S}$ with variable exponent given by (1.3). Our approach is based on conforming finite element spaces satisfying the classical discrete inf-sup condition. We assume that $\Omega \subset \mathbb{R}^n$ is a polyhedral, bounded domain.

The mathematical investigation of fluids with shear-dependent viscosities (p =const.) started with the celebrated works of O.A. Ladyzhenskaya [24] and J.-L. Lions [25] in the late sixties. In recent years there has been an enormous progress in the understanding of this problem and we refer the reader to [22,27,36] and the references therein for a detailed discussion.

The first results regarding the numerical analysis date back to [37], with improvements in [4]) where the error estimates are presented in the setting of quasi-norms. The notion of quasi-norm is the natural one for this type of problem, cf. [2,3,4,21,26], since the quasi-norm is equivalent to the distance naturally defined by the monotone operator $-\operatorname{div}\mathcal{S}(\mathbf{Dv})$. Given the discrete solution $\mathbf{v}_h$ and the continuous solution $\mathbf{v}$, the error is measured as the L^2-difference of $\mathcal{F}(\mathbf{Dv})$ and $\mathcal{F}(\mathbf{Dv}_h)$, where $\mathcal{F}(\boldsymbol{\eta}) = (\kappa + |\boldsymbol{\eta}|)^{\frac{p-2}{2}}\boldsymbol{\eta}$. For the system (1.1)–(1.2), (without convective term) the following error estimates are shown for constant $p > 1$ in [5]:

$$\|\mathcal{F}(\mathbf{Dv}) - \mathcal{F}(\mathbf{Dv}_h)\|_2 \le c\, h^{\min\{1,\frac{p'}{2}\}}, \tag{1.5}$$

$$\|q - q_h\|_{p'} \le c\, h^{\min\left\{\frac{p'}{2},\frac{2}{p'}\right\}}. \tag{1.6}$$

For the validity of the above estimates the natural assumptions that $\mathcal{F}(\mathbf{Dv}) \in (W^{1,2}(\Omega))^{n\times n}$ and also that $q \in W^{1,p'}(\Omega)$ are made. The convergences rates in (1.5) and (1.6) are the best known ones.

The purpose of the present paper is to extend the estimates (1.5)-(1.6) to the setting of variable exponents for the $p(\cdot)$-Stokes system. Since the development of the model for electrorheological fluids in [32,33,35] there has been a huge progress regarding its mathematical analysis [20,1,6,18] and especially the precise characterization of the corresponding functional setting [19]. However there are only very few results about the numerical analysis, see for instance results for the time-discretization in [16].

We observe that in [14] a time-dependent system with the same stress-tensor (1.3) and (smoothed) convective terms is studied and the convergence of the finite element approximation is shown without convergence rate. To our knowledge no quantitative estimate on the convergence rate for problems with variable exponents is known, while recent results for the $p(\cdot)$-Laplacian (i.e. the scalar system without pressure) can be found in [12]

The main purpose of the present paper is to obtain precise convergence estimates for the system (1.4), generalizing to the variable exponent the estimates (1.5)-(1.6). By assuming Hölder regularity on the exponent $p(\cdot)$, the main results we will prove (see Theorem 3.6) are the following estimates

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\|_2 &\le c\left(h^{\min\{1,\frac{(p^+)'}{2}\}} + h^{\alpha}\right),\\ \|q - q_h\|_{p'(\cdot)} &\le c\left(h^{\frac{\min\{((p^+)')^2,4\}}{2(p^-)'}} + h^{\alpha}\right). \end{aligned}$$

Here $\mathbf{F}_{\mathcal{T}}$ is a locally constant approximation to $\mathbf{F}(x,\boldsymbol{\eta}) = (\kappa+|\boldsymbol{\eta}|)^{\frac{p(x)-2}{2}}\boldsymbol{\eta}$, see (3.4), the number $\alpha \in (0,1]$ is the Hölder exponent of $p(\cdot)$ and p^+ and p^- supremum and infimum value of $p(\cdot)$, respectively. As usual $(p^+)'$ and $(p^-)'$ are their conjugate exponents. Our analysis is based on the recent studies in [12] about the finite element approximation of the $p(\cdot)$-Laplacian and on the numerical analysis in [5] for the p-Stokes system.

Plan of the paper: In Sec. 2 we recall the basic results on variable exponent space, we will use. In Sec. 3 we recall the basic existence results for the $p(\cdot)$-Stokes system, the finite element setting, and we state the main results of the paper. The convergence analysis of the velocity is presented in Sec. 4 and the convergence analysis of the pressure in Sec. 5. An appendix with some technical results on Orlicz spaces is also added

2 Variable exponent spaces

For a measurable set $E \subset \mathbb{R}^n$ let $|E|$ be the Lebesgue measure of E and χ_E its characteristic function. For $0 < |E| < \infty$ and $f \in L^1(E)$ we define the mean value of f over E by

$$\langle f\rangle_E := \fint_E f\,dx := \frac{1}{|E|}\int_E f\,dx.$$

For an open set $\Omega \subset \mathbb{R}^n$ let $L^0(\Omega)$ denote the set of measurable functions.

Let us introduce the spaces of variable exponents $L^{p(\cdot)}$. We use the same notation used in the recent book [19]. We define $\mathcal{P}$ to consist of all $p \in L^0(\mathbb{R}^n)$ with $p : \mathbb{R}^n \to [1,\infty]$ (called variable exponents). For $p \in \mathcal{P}$ we define $p^-_\Omega := \operatorname{essinf}_\Omega p$ and $p^+_\Omega := \operatorname{esssup}_\Omega p$. Moreover, let $p^+ := p^+_{\mathbb{R}^n}$ and $p^- := p^-_{\mathbb{R}^n}$.

For $p \in \mathcal{P}$ the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) := \Big\{ f \in L^0(\Omega) \,:\, \|f\|_{L^{p(\cdot)}(\Omega)} < \infty \Big\},$$

where

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\lambda > 0 \,:\, \int_{\mathbb{R}^n} \Big|\frac{f(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\}.$$

By using a standard notation, by $\|\cdot\|_p$ we mean the usual Lebesgue L^p-norm, for a fixed $p \ge 1$. We say that a function $g\colon \mathbb{R}^n \to \mathbb{R}$ is log-*Hölder continuous* on Ω if there exist constants $c \ge 0$ and $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/|x-y|)} \qquad \text{and} \qquad |g(x) - g_\infty| \le \frac{c}{\log(e + |x|)},$$

for all $x \neq y \in \mathbb{R}^n$. The first condition describes the so called local log-Hölder continuity and the second the decay condition. The smallest such constant c is the log-Hölder constant of g. We define $\mathcal{P}^{\log}$ to consist of those exponents $p \in \mathcal{P}$ for which $\frac{1}{p} : \mathbb{R}^n \to [0,1]$ is log-Hölder continuous. By p_∞ we denote the limit of p at infinity, which exists for $p \in \mathcal{P}^{\log}$. If $p \in \mathcal{P}$ is bounded, then $p \in \mathcal{P}^{\log}$ is equivalent to the log-Hölder continuity of p. However, working with $\frac{1}{p}$ gives better control of the constants especially in the context of averages and maximal functions. Therefore, we define $c_{\log}(p)$ as the log-Hölder constant of $1/p$. Expressed in p we have for all $x, y \in \mathbb{R}^n$

$$|p(x) - p(y)| \le \frac{(p^+)^2 c_{\log}(p)}{\log(e + 1/|x-y|)} \qquad \text{and} \qquad |p(x) - p_\infty| \le \frac{(p^+)^2 c_{\log}(p)}{\log(e + |x|)}.$$

For a cube $Q \subset \mathbb{R}^n$ we denote by $\ell(Q)$ its side length and we have the following results.

Lemma 2.1 (Lemma 2.1 in [12]) *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$ and $m > 0$. Then for every cube $Q \subset \mathbb{R}^n$ with $\ell(Q) \le 1$, $\kappa \in [0,1]$, and $t \ge 0$ such that $|Q|^m \le t \le |Q|^{-m}$, then*

$$(\kappa + t)^{p(x)-p(y)} \le c,$$

for all $x, y \in Q$. The constant depends on $c_{\log}(p)$, m, and p^+.

For every convex function ψ and every cube Q we have by Jensen's inequality

$$\psi\bigg(\fint_Q |f(y)|\,dy\bigg) \le \fint_Q \psi(|f(y)|)\,dy. \tag{2.1}$$

This simple but crucial estimate allows for example to transfer the L^1-L^∞ estimates for the interpolation operators to the setting of Orlicz spaces, see [21]. A suitable analogue for variable exponent spaces bounds $(\fint_Q |f(y)|\,dy)^{p(x)}$ in terms of $\fint_Q |f(x)|^{p(x)}\,dx$ (but an additional error term appears). In order to quantify this let us introduce the notation

$$\varphi(x,t) := t^{p(x)}, \quad (M_Q\varphi)(t) := \fint_Q \varphi(x,t)\,dx, \quad M_Q f := \fint_Q |f(x)|\,dx.$$

For our finite element analysis we need this estimate extended to the case of shifted Orlicz functions. For constant p this has been done in [21]. We define the shifted functions φ_a for $a \ge 0$ by

$$\varphi_a(x,t) := \int_0^t \frac{\varphi'(x,a+\tau)}{a+\tau}\tau\,d\tau,$$

where the prime denotes the partial derivative of $\varphi(x,t)$ with respect to the variable t.

Then $\varphi_a(x,\cdot)$ is the shifted N-function of $t \mapsto t^{p(x)}$, see (A.2). Note that characteristics and Δ_2-constants of $\varphi_a(x,\cdot)$ are uniformly bounded with respect to $a \ge 0$ if $1 < p^- \le p^+ < \infty$, see Section A.

We recall three fundamental results we will use in the sequel.

Theorem 2.2 (Shifted key estimate, Thm. 2.5 in [12]) *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$. Then for every $m > 0$ there exists $c_1 > 0$ only depending on m, $c_{\log}(p)$, and p^+ such that*

$$\varphi_a(x, M_Q f) \le c\, M_Q(\varphi_a(|f|)) + c\,|Q|^m,$$

for every cube (or ball) $Q \subset \mathbb{R}^n$ with $\ell(Q) \le 1$, all $x \in Q$ and all $f \in L^1(Q)$ with

$$a + \fint_Q |f|\,dy \le \max\{1, |Q|^{-m}\} = |Q|^{-m}.$$

Theorem 2.3 (Shifted Poincaré inequality, Thm. 2.4 in [12]) *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$. Then for every $m > 0$ there exists $c > 0$ only depending on m, $c_{\log}(p)$, and p^+ such that*

$$\int_Q \varphi_a\bigg(x, \frac{|u(x) - \langle u\rangle_Q|}{\ell(Q)}\bigg)\,dx \le c \int_Q \varphi_a(x, |\nabla u(x)|)\,dx + c\,|Q|^m,$$

for every cube (or ball) $Q \subset \mathbb{R}^n$ with $\ell(Q) \le 1$ and for all all $u \in W^{1,p(\cdot)}(Q)$ with

$$a + \fint_Q |\nabla u|\,dy \le \max\{1, |Q|^{-m}\} = |Q|^{-m}.$$

Theorem 2.4 (shifted Korn inequality) *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty$. Then for every $m > 0$ there exists $c > 0$ only depending on m, $c_{\log}(p)$, and p^+*

$$\int_Q \varphi_a\bigg(x, \frac{|\mathbf{u}(x) - \mathcal{R}_Q\mathbf{u}(x)|}{\ell(Q)}\bigg)\,dx \le c \int_Q \varphi_a(x, |\mathbf{Du}(x)|)\,dx + c\,|Q|^m,$$

for every cube (or ball) $Q \subset \mathbb{R}^n$ with $\ell(Q) \le 1$ and all $\mathbf{u} \in (W^{1,p(\cdot)}(Q))^n$ with

$$a + \fint_Q |\mathbf{Du}|\,dy \le \max\{1, |Q|^{-m}\} = |Q|^{-m}.$$

Here $\mathcal{R}_Q$ is a suitable rigid motion, i.e. $\mathcal{R}_Q x = \mathbf{A}x + b$ is affine linear with $\mathbf{A}^T + \mathbf{A} = 0$.

Proof Due to (2.33)-(2.39) in [31] there is a rigid motion $\mathcal{R}_Q\mathbf{u}$ such that the difference of $\mathbf{u}$ and $\mathcal{R}_Q\mathbf{u}$ can be represented as a Riesz-potential of $\mathbf{Du}$, i.e. there holds

$$|\mathbf{u}(x) - \mathcal{R}_Q\mathbf{u}(x)| \le c \int\limits_Q \frac{|\mathbf{Du}(y)|}{|x-y|^{n-1}}\,dy.$$

Due to this inequality we can prove the claim by the lines of [12], Thm. 2.4, replacing ∇u by $\mathbf{Du}$. □

3 The $p(\cdot)$-Stokes problem: notation and main results

In this section we introduce the main existence results for the $p(\cdot)$-Stokes and we describe the Finite Element formulation we will study

3.1 The $p(\cdot)$-Stokes problem

Let us briefly recall some well-known facts about the $p(\cdot)$-Stokes system (1.4). Let $\Omega \subset \mathbb{R}^n$ be a bounded, polyhedral domain. Then we define the function spaces

$$X := \big(W^{1,p(\cdot)}(\Omega)\big)^n, \qquad V := \big(W_0^{1,p(\cdot)}(\Omega)\big)^n,$$
$$Y := L^{p'(\cdot)}(\Omega)\,, \qquad Q := L_0^{p'(\cdot)}(\Omega) := \Big\{ f \in L^{p'(\cdot)}(\Omega)\, :\, \fint\limits_\Omega f\,dx = 0 \Big\}.$$

With this notation the weak formulation of problem (1.4) is the following.

Problem (Q) For $\mathbf{f} \in V^*$ find $(\mathbf{v}, q) \in V \times Q$ such that

$$\begin{aligned} \langle \mathbf{S}(\cdot, \mathbf{Dv}), \mathbf{D}\boldsymbol{\xi}\rangle - \langle \operatorname{div}\boldsymbol{\xi}, q\rangle &= \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad && \forall\, \boldsymbol{\xi} \in V, \\ \langle \operatorname{div}\mathbf{v}, \eta\rangle &= 0 && \forall\, \eta \in Y. \end{aligned}$$

Alternatively, we can reformulate the problem "hiding" the pressure:

Problem (P) For $\mathbf{f} \in L^{p'(\cdot)}(\Omega)$ find $\mathbf{v} \in V_{\mathrm{div}}$ such that

$$\langle \mathbf{S}(\cdot, \mathbf{Dv}), \mathbf{D}\boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad \forall\, \boldsymbol{\xi} \in V_{\mathrm{div}},$$

where

$$V_{\mathrm{div}} := \{\mathbf{w} \in V \,:\, \langle \operatorname{div}\mathbf{w}, \eta\rangle = 0 \quad \forall\, \eta \in Y\}.$$

The names "Problem **(Q)**" and "Problem **(P)**" are traditional, see [13,23]. By using the inf-sup condition or again the solvability of the divergence equation one easily checks that the two formulations are equivalent.

The problems **(Q)** and **(P)** have a discrete counterpart, whose analysis is the ultimate goal of this section. Let $\mathcal{T}$ be a triangulation of our domain Ω consisting of n-dimensional simplices. For a simplex $K \in \mathcal{T}$ let h_K denote its diameter and let ρ_K be the supremum of the diameters of inscribed balls. We assume that $\mathcal{T}$ is non-degenerate, i.e., $\max_{K\in\mathcal{T}} \frac{h_k}{\rho_K} \le \gamma_0$. The global mesh size h is defined by $h := \max_{K\in\mathcal{T}} h_K$. Let S_K denote the neighborhood of K, i.e., S_K is the union of all simplices of $\mathcal{T}$ touching K. One easily sees that under these assumptions we get that $|K| \sim |S_K|$ and that the number of simplices in S_K is uniformly bounded with respect to $K \in \mathcal{T}$.

We denote by $\mathfrak{P}_m(\mathcal{T})$, with $m \in \mathbb{N}_0$, the space of scalar or vector-valued continuous functions, which are polynomials of degree at most m on each simplex $K \in \mathcal{T}$. Given a triangulation of Ω with the above properties and given $k, m \in \mathbb{N}_0$ we denote by $X_h \subset (\mathfrak{P}_m(\mathcal{T}))^n$ and $Y_h \subset \mathfrak{P}_k(\mathcal{T})$ appropriate conforming finite element spaces defined on $\mathcal{T}$, i.e., X_h, Y_h satisfy

$X_h \subset X$ and $Y_h \subset Y$. Moreover, we set $V_h := X_h \cap V$ and $Q_h := Y_h \cap Q$. For the applications it is convenient to replace the exponent $p(\cdot)$ by some local approximation.

$$p_{\mathcal{T}} := \sum_{K \in \mathcal{T}} p(x_K)\chi_K = \sum_{K \in \mathcal{T}} p_K^- \chi_K,$$

where $x_K := \operatorname{argessinf}_K p(x)$, i.e. $p(x_K) = p_K^-$, and consider

$$\mathbf{S}_{\mathcal{T}}(x, \boldsymbol{\xi}) = \sum_{K \in \mathcal{T}} \chi_K(x) \mathbf{S}\big(x_K, \boldsymbol{\xi}\big),$$

instead of $\mathbf{S}$. Now the discrete counterpart of **(P)** and **(Q)** can be written as follows:

Problem $(\mathbf{Q}_h)$ For $\mathbf{f} \in L^{p'(\cdot)}(\Omega)$ find $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \langle \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h), \mathbf{D}\boldsymbol{\xi}_h \rangle - \langle \operatorname{div} \boldsymbol{\xi}_h, q_h \rangle &= \langle \mathbf{f}, \boldsymbol{\xi}_h \rangle \qquad \forall\, \boldsymbol{\xi}_h \in V_h, \\ -\langle \operatorname{div} \mathbf{v}_h, \eta_h \rangle &= 0 \qquad \forall\, \eta_h \in Q_h. \end{aligned} \tag{3.1}$$

If $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ is a solution of the "Problem (Q_h)" then $(3.1)_2$ is satisfied for all $\eta_h \in Y_h$, since $\operatorname{div} \mathbf{v}_h$ is orthogonal to constants.

Problem $(\mathbf{P}_h)$ For $\mathbf{f} \in L^{p'(\cdot)}(\Omega)$ find $\mathbf{v}_h \in V_{h,\mathrm{div}}$ such that

$$\langle \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h), \mathbf{D}\boldsymbol{\xi}_h \rangle = \langle \mathbf{f}, \boldsymbol{\xi}_h \rangle \qquad \forall\, \boldsymbol{\xi}_h \in V_{h,\mathrm{div}},$$

where

$$V_{h,\mathrm{div}} := \{ \mathbf{w}_h \in V_h \,:\, -\langle \operatorname{div} \mathbf{w}_h, \eta_h \rangle = 0 \qquad \forall\, \eta_h \in Y_h \}.$$

The coercivity of $\mathbf{S}_{\mathcal{T}}$ implies that $\mathbf{D}\mathbf{v}_h \in (L^{p_{\mathcal{T}}(\cdot)}(\Omega))^{n \times n}$ with $\|\mathbf{D}\mathbf{v}_h\|_{p_{\mathcal{T}}(\cdot)} \le c(\mathbf{f})$. The next lemma actually shows that this is equivalent to $\mathbf{D}\mathbf{v}_h \in (L^{p(\cdot)}(\Omega))^{n \times n}$.

Lemma 3.1 *On the space $\mathfrak{P}_k(\mathcal{T})$ the norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_{p_{\mathcal{T}}(\cdot)}$ are equivalent. (See also Remark 4.7 in [12].)*

Proof Let $g_h \in \mathfrak{P}_k(\mathcal{T})$ with $\|g_h\|_{p_{\mathcal{T}}(\cdot)} \le 1$ (which is equivalent to $\int_\Omega |g_h|^{p_{\mathcal{T}}(\cdot)}\, dx \le 1$ by the very definition of the Luxemburg norm). As g_h is a polynomial of order k on K we have the local estimate (recall $p(x_K) = p_K^-$)

$$\begin{aligned} \|g_h\|_{L^\infty(K)} &\le c(k) \fint_K |g_h|\, dx \le c(k) \bigg(\fint_K |g_h|^{p(x_K)}\, dx \bigg)^{\frac{1}{p(x_K)}} \\ &\le c(k) \bigg(\frac{1}{h_k^n} \int_K |g_h|^{p_{\mathcal{T}}(\cdot)}\, dx \bigg)^{\frac{1}{p(x_K)}} \le c(k)\, h_K^{-\frac{n}{p(x_K)}}. \end{aligned} \tag{3.2}$$

Thus we can apply Lemma 2.1 with $m = \frac{1}{p(x_K)}$, $\kappa = 0$ and $t = 1 + |g_h|$ to find

$$\begin{aligned} \int_\Omega |g_h|^{p(\cdot)}\, dx &= \sum_{K \in \mathcal{T}} \int_K |g_h|^{p(\cdot)}\, dx \le \sum_{K \in \mathcal{T}} \int_K (1 + |g_h|)^{p(\cdot)}\, dx \\ &\le c \sum_{K \in \mathcal{T}} \int_K (1 + |g_h|)^{p_{\mathcal{T}}}\, dx = c \int_\Omega (1 + |g_h|)^{p_{\mathcal{T}}}\, dx \le c. \end{aligned}$$

On the other hand, if $\|g_h\|_{p(\cdot)} \le 1$ there holds

$$\|g_h\|_{L^\infty(K)} \le c \bigg(\fint_K |g_h|^{p_K^-}\, dx \bigg)^{\frac{1}{p_K^-}} \le c \bigg(\fint_K 1 + |g_h|^{p(\cdot)}\, dx \bigg)^{\frac{1}{p_K^-}} \le c\, h_K^{-\frac{n}{p_K^-}},$$

and as before $\int_\Omega |g_h|^{p_{\mathcal{T}}(\cdot)}\, dx \le c$. $\square$

So we have $\mathbf{D}\mathbf{v}_h \in (L^{p(\cdot)}(\Omega))^{n\times n}$ and Korn's inequality as in [19] (Thm. 14.3.21) yields $\mathbf{v}_h \in (W^{1,p(\cdot)}(\Omega))^n$ uniformly in h.

In the following we will measure the approximation error in terms of the following adapted version of the quasi-norm

$$\|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)\|_2^2 = \sum_{K\in\mathcal{T}} \int_K |\mathbf{F}(x_K,\mathbf{D}\mathbf{v}) - \mathbf{F}(x_K,\mathbf{D}\mathbf{v}_h)|^2\,dx, \tag{3.3}$$

$$\text{where } \mathbf{F}_{\mathcal{T}}(x,\boldsymbol{\xi}) := \sum_{K\in\mathcal{T}} \chi_K(x)\mathbf{F}\big(x_K,\boldsymbol{\xi}\big). \tag{3.4}$$

Recall that $\mathbf{F}(x,\boldsymbol{\xi}) = (\kappa + |\boldsymbol{\xi}|)^{\frac{p(x)-2}{2}}\boldsymbol{\xi}$.

3.2 Main results

Throughout the paper we will make the following assumptions on our finite element spaces for approximate velocity and pressure.

Assumption 3.2 *We assume that $(\mathfrak{P}_1(\mathcal{T}))^n \subset X_h$ and there exists a linear projection operator $\Pi_h^{\mathrm{div}} : X \to X_h$ which*

(a) *preserves divergence in the Y_h^*-sense, i.e.,*

$$\langle \operatorname{div}\mathbf{w},\eta_h\rangle = \langle \operatorname{div}\Pi_h^{\mathrm{div}}\mathbf{w},\eta_h\rangle \qquad \forall\,\mathbf{w}\in X,\ \forall\,\eta_h\in Y_h\,; \tag{3.5}$$

(b) *preserves zero boundary values, i.e. $\Pi_h^{\mathrm{div}}(V)\subset V_h$;*
(c) *is locally $W^{1,1}$-stable in the sense that*

$$\fint_K |\Pi_h^{\mathrm{div}}\mathbf{w}|\,dx \le c\fint_{S_K}|\mathbf{w}|\,dx + c\fint_{S_K} h_K|\nabla\mathbf{w}|\,dx \quad \forall\,\mathbf{w}\in X,\ \forall\,K\in\mathcal{T}. \tag{3.6}$$

Assumption 3.3 *We assume that Y_h contains the constant functions, i.e. $\mathbb{R}\subset Y_h$, and that there exists a linear projection operator $\Pi_h^Y : Y\to Y_h$ which is locally L^1-stable in the sense that*

$$\fint_K |\Pi_h^Y q|\,dx \le c\fint_{S_K}|q|\,dx \qquad \forall\,q\in Y,\ \forall\,K\in\mathcal{T}. \tag{3.7}$$

Remark 3.4 Note that the Clément and the Scott–Zhang interpolation operators satisfy Assumption 3.3.

Remark 3.5 It is possible to weaken the requirements on the projection operators Π_h^{div} and Π_h^Y. In fact, we can replace the requirement $\Pi_h^{\mathrm{div}}\mathbf{w}_h = \mathbf{w}_h$ for all $\mathbf{w}_h\in X_h$ by the requirement $\Pi_h^{\mathrm{div}}\mathbf{q} = \mathbf{q}$ for all linear polynomials (not in the piecewise sense), and the requirement $\Pi_h^Y q_h = q_h$ for all $q_h\in Y_h$ by the requirement $\Pi_h^Y c = c$ for all constants c.

Certainly, the existence of Π_h^{div} depends on the choice of X_h and Y_h. Some concrete two-dimensional examples based on the Scott-Zhang interpolation and a correction of the divergence are provided in the Appendix of [5].

Let us now state our main results and shortly explain the strategy of their proofs. The first result we prove is that the error for the velocity in the natural distance is controlled by some best approximation error for the velocity (with prescribed divergence) and the pressure (cf. Lemma 4.1). We work directly with a divergence-preserving operator Π_h^{div} (cf. Assumption 3.2), which can be used (cf. [13]) to derive the inf-sup condition. From the local $W^{1,1}$-stability of Π_h^{div}, we derive its non-linear, local counterparts in terms of the natural distance (cf. Theorem 4.3). Thus we can replace the best approximation error for the velocity (with prescribed divergence) by local averages of the solution $\mathbf{v}$ in terms of the natural distance (cf. Theorem 4.5)

Once we have in hand these best approximation estimates we obtain convergence rates in terms of the mesh size. More precisely we will prove the following result (see Corollary 4.6 and Theorem 5.6):

Theorem 3.6 *Let Π_h^{div} satisfy Assumption 3.2 and Π_h^Y satisfy Assumption 3.3. Let $(\mathbf{v}, q)$ and $(\mathbf{v}_h, q_h)$ be solutions of the problems* **(Q)** *and* **($\mathbf{Q}_h$)***, respectively. Suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1]$ and $p^- > 1$. Furthermore, let $\mathbf{F}(\cdot, \mathbf{Dv}) \in (W^{1,2}(\Omega))^{n\times n}$ and also let $q \in W^{1,p'(\cdot)}(\Omega)$. Then*

$$\|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}_h)\|_2 \le c\,\big(h^{\min\{1, \frac{(p^+)'}{2}\}} + h^\alpha\big), \tag{3.8}$$

$$\|q - q_h\|_{p'(\cdot)} \le c\,\Big(h^{\frac{\min\{((p^+)')^2, 4\}}{2(p^-)'}} + h^\alpha\Big). \tag{3.9}$$

Here c depends on p^-, p^+, $[p]_\alpha$, and γ_0.

Remark 3.7 It is standard to show $\mathbf{F}(\cdot, \mathbf{Dv}) \in W^{1,2}$ in the interior of Ω. The proof follows for instance along the lines of [7, 10] where even more general constitutive relations than (1.3) were considered. Note that $\mathbf{F}(\cdot, \mathbf{Dv}) \in W^{1,2}$ implies $\mathbf{F}(\cdot, \mathbf{Dv}) \in L^{\frac{2n}{n-2}}$ (by Sobolev's Theorem) and $\mathbf{Dv} \in L^{\frac{n}{n-2}p(\cdot)}$. For the space periodic case see [16]. See also [15] for the problem with small data.

The regularity up to the boundary still seems an open challenging problem. The difficulties, even for constant exponents, are due to the combination of zero Dirichlet data with the symmetric gradients and the pressure. The closest result to $\mathbf{F}(\mathbf{Dv}) \in W^{1,2}(\Omega)$ is in [9], where it is shown for a constant p that $\mathbf{F}(\mathbf{Dv}) \in W^{1,s_1}(\Omega)$ and $q \in W^{1,s_2}$ for certain $s_1 < 2$ and $s_2 < p'$; see also the references therein for other results in this direction.

In the absence of a pressure and for constants exponents the regularity $\mathbf{F}(\mathbf{Dv}) \in W^{1,2}(\Omega)$ is shown in [38].

4 Best Approximation Error for the Velocity

In this section we prove error estimates for the velocity in terms of best approximation properties measured in the natural distance.

4.1 Equation for the error

Taking the difference between **(Q)** and **($\mathbf{Q}_h$)** we get the following equation for the numerical error

$$\langle \mathbf{S}(\cdot, \mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{Dv}_h), \mathbf{D}\boldsymbol{\xi}_h\rangle - \langle \operatorname{div} \boldsymbol{\xi}_h, q - q_h\rangle = 0 \qquad \forall\, \boldsymbol{\xi}_h \in V_h. \tag{4.1}$$

We start with a preliminary approximation result which will be improved later on in Theorem 4.5.

Lemma 4.1 *Let $(\mathbf{v}, q)$ and $(\mathbf{v}_h, q_h)$ be the solutions of the problems* **(Q)** *and* **($\mathbf{Q}_h$)***, respectively. Suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1]$ and $p^- > 1$. Then for some $s > 1$ (close to 1 for h small) we have the following estimate*

$$\begin{aligned}\|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}_h)\|_2^2 \le c \inf_{\mathbf{w}_h \in V_{h,\mathrm{div}}} &\|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dw}_h)\|_2^2 \\ &+ c \inf_{\mu_h \in Y_h} \sum_{K\in\mathcal{T}} \int_K (\varphi_{|\mathbf{Dv}|}^K)^*(\cdot, |q - \mu_h|)\,dx \\ &+ c\,h^{2\alpha}\Big(\int_\Omega (1 + |\mathbf{Dv}|^{p(x)s})\,dx\Big).\end{aligned} \tag{4.2}$$

Here c depends on p^-, p^+, $[p]_\alpha$, and γ_0, while $(\varphi_{|\mathbf{Dv}|}^K)^*$ is defined in (4.3).

Proof For $\mathbf{w}_h \in V_{h,\mathrm{div}}$ we have $\mathbf{v}_h - \mathbf{w}_h \in V_{h,\mathrm{div}}$. Consequently for all $\mu_h \in Y_h$, we obtain with Lemma A.4 and Eq. (4.1) that

$$\|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{Dv}_h)\|_2^2 \le c \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{Dv}_h)\big) : \big(\mathbf{Dv} - \mathbf{Dv}_h\big)\,dx$$

$$
\begin{aligned}
&= c \int_\Omega \big(\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\big) : \big(\mathbf{Dv} - \mathbf{Dw}_h\big)\,dx \\
&\quad + c \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{S}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dv}_h\big)\,dx \\
&\quad - c \int_\Omega \operatorname{div}\big(\mathbf{w}_h - \mathbf{v}_h\big)\,(q - \mu_h)\,dx \\
&= c \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\big) : \big(\mathbf{Dv} - \mathbf{Dw}_h\big)\,dx \\
&\quad + c \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{S}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dv}_h\big)\,dx \\
&\quad + c \int_\Omega \big(\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dw}_h\big)\,dx \\
&\quad - c \int_\Omega \operatorname{div}\big(\mathbf{w}_h - \mathbf{v}_h\big)\,(q - \mu_h)\,dx.
\end{aligned}
$$

As before this implies

$$
\begin{aligned}
\|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\|_2^2 &\le c\,\|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dw}_h)\|_2^2 \\
&\quad + c \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{S}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dv}_h\big)\,dx \\
&\quad + c \int_\Omega \big(\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dw}_h\big)\,dx \\
&\quad - c \int_\Omega \operatorname{div}\big(\mathbf{w}_h - \mathbf{v}_h\big)\,(q - \mu_h)\,dx \\
&=: (I) + (II) + (III) + (IV).
\end{aligned}
$$

We begin with the estimate for (II).

To estimate the difference between $\mathbf{S}_{\mathcal{T}}$ and $\mathbf{S}$ we need the estimate

$$
\begin{aligned}
&\big|\mathbf{S}_{\mathcal{T}}(x,\mathbf{Q}) - \mathbf{S}(x,\mathbf{Q})\big| \\
&\quad \le c\,|p_{\mathcal{T}}(x) - p(x)|\,|\ln(\kappa + |\mathbf{Q}|)|\Big((\kappa + |\mathbf{Q}|)^{p_{\mathcal{T}}(x)-2} + (\kappa + |\mathbf{Q}|)^{p(x)-2}\Big)|\mathbf{Q}| \\
&\quad \le c\,h^\alpha |\ln(\kappa + |\mathbf{Q}|)|\Big((\kappa + |\mathbf{Q}|)^{p_{\mathcal{T}}(x)-2} + (\kappa + |\mathbf{Q}|)^{p(x)-2}\Big)|\mathbf{Q}|,
\end{aligned}
$$

for all $\mathbf{Q} \in \mathbb{R}^{n\times n}_{sym}$ using also that $p \in C^{0,\alpha}$. Hence, we get

$$
\begin{aligned}
(II) &:= \int_\Omega \big(\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{S}(\cdot,\mathbf{Dv})\big) : \big(\mathbf{Dv} - \mathbf{Dv}_h\big)\,dx \\
&\le c\,h^\alpha \int_\Omega |\ln(\kappa+|\mathbf{Dv}|)|(\kappa+|\mathbf{Dv}|)^{p_{\mathcal{T}}(x)-2}|\mathbf{Dv}||\mathbf{Dv}-\mathbf{Dv}_h|\,dx \\
&\quad + c\,h^\alpha \int_\Omega |\ln(\kappa+|\mathbf{Dv}|)|(\kappa+|\mathbf{Dv}|)^{p(x)-2}|\mathbf{Dv}||\mathbf{Dv}-\mathbf{Dv}_h|\,dx \\
&=: (II)_1 + (II)_2.
\end{aligned}
$$

We begin with the estimate for $(II)_1$ on each $K \in \mathcal{T}$. Define the N-function

$$
\varphi^K(t) := \int_0^t (\kappa + s)^{p_K - 2} s\,ds. \tag{4.3}
$$

Using this definition we estimate

$$(II)_1 \le c \sum_{K\in\mathcal{T}} \int_K h^\alpha |\ln(\kappa+|\mathbf{Dv}|)| (\varphi^K)'(|\mathbf{Dv}|)|\mathbf{Dv}-\mathbf{Dv}_h|\,dx.$$

Using Young's inequality with $\varphi^K_{|\mathbf{Dv}|} := (\varphi^K)_{|\mathbf{Dv}|}$ on $|\mathbf{Dv}-\mathbf{Dv}_h|$ and its complementary function on the rest, we get

$$\begin{aligned}(II)_1 &\le \delta \sum_{K\in\mathcal{T}} \int_K (\varphi^K)_{|\mathbf{Dv}|}(|\mathbf{Dv}-\mathbf{Dv}_h|)\,dx \\ &\quad + c_\delta \sum_{K\in\mathcal{T}} \int_K \big((\varphi^K)_{|\mathbf{Dv}|}\big)^*\Big(h^\alpha |\ln(\kappa+|\mathbf{Dv}|)| (\varphi^K)'(|\mathbf{Dv}|)\Big)\,dx.\end{aligned}$$

Now we use Lemma A.4 for the first line and Lemma A.8 and Lemma A.7 (with $\lambda = h^\alpha \le 1$ using $h \le 1$) for the second line to find

$$\begin{aligned}(II)_1 &\le \delta c \|\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K\in\mathcal{T}} \int_K (1+|\ln(\kappa+|\mathbf{Dv}|)|)^{\max\{2,p_K'\}} \big((\varphi^K)_{|\mathbf{Dv}|}\big)^*\Big(h^\alpha (\varphi^K)'(|\mathbf{Dv}|)\Big)\,dx \\ &\le \delta c \|\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K\in\mathcal{T}} h^{2\alpha} \int_K (1+|\ln(\kappa+|\mathbf{Dv}|)|)^{\max\{2,p_K'\}} (\varphi^K)(|\mathbf{Dv}|)\,dx.\end{aligned}$$

The term $(II)_2$ is estimate similarly. We get

$$\begin{aligned}(II)_2 &\le \delta c \|\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K\in\mathcal{T}} h^{2\alpha} \int_K \big(1+|\ln(\kappa+|\mathbf{Dv}|)|(\kappa+|\mathbf{Dv}|)^{p(x)-p_\mathcal{T}(x)}\big)^{\max\{2,p_K'\}} (\varphi^K)(|\mathbf{Dv}|)\,dx.\end{aligned}$$

Overall, this yields

$$(II) \le \delta c\, \|\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}_h)\|_2^2 + c_\delta c_s h^{2\alpha} \int_\Omega \big(1+|\mathbf{Dv}|^{p(x)\,s}\big)\,dx,$$

Here we used $\ln(\kappa+t) \le c(\kappa) t^\kappa$ for all $t \ge 1$ and $\kappa > 0$. For h small we can choose s close to 1. For (III) the analogous estimate is

$$(III) \le c\, \|\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dw}_h)\|_2^2 + c_s h^{2\alpha} \int_\Omega \big(1+|\mathbf{Dv}|^{p(x)\,s}\big)\,dx.$$

Next, we estimate the term (IV) involving $q-\mu_h$. We add and subtract $\mathbf{Dv}$, use Young's inequality (A.1) for $\varphi^K_{|\mathbf{Dv}|}$, and apply Lemma A.4 to obtain

$$\begin{aligned}&\big|\langle \operatorname{div}(\mathbf{v}_h-\mathbf{w}_h), q-\mu_h\rangle\big| \\ &\quad\le \int_\Omega \big(|\mathbf{Dv}_h - \mathbf{Dv}| + |\mathbf{Dv}-\mathbf{Dw}_h|\big)\,|q-\mu_h|\,dx \\ &\quad= \sum_{K\in\mathcal{T}} \int_K \big(|\mathbf{Dv}_h - \mathbf{Dv}| + |\mathbf{Dv}-\mathbf{Dw}_h|\big)\,|q-\mu_h|\,dx \\ &\quad\le \varepsilon \sum_{K\in\mathcal{T}} \int_\Omega \varphi^K_{|\mathbf{Dv}|}(\cdot,|\mathbf{Dv}_h-\mathbf{Dv}|) + \varphi^K_{|\mathbf{Dv}|}(\cdot,|\mathbf{Dw}_h-\mathbf{Dv}|)\,dx \\ &\qquad + c_\varepsilon \sum_{K\in\mathcal{T}} \int_K (\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\mu_h|)\,dx\end{aligned}$$

$$\leq \varepsilon\, c \Big(\|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h)\|_2^2 + \|\mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{w}_h)\|_2^2 \Big)$$
$$+ c_\varepsilon \sum_{K \in \mathcal{T}} \int_K (\varphi^K_{|\mathbf{D}\mathbf{v}|})^*(\cdot, |q - \mu_h|)\, dx.$$

Collecting the estimates and choosing $\varepsilon > 0$ small enough we obtain the assertion by noticing that $\mathbf{w}_h \in V_{h,\mathrm{div}}$ and $\mu_h \in Y_h$ are arbitrary. □

4.2 The divergence-preserving interpolation operator

In this section we derive the non-linear estimates for Π_h^Y and the divergence preserving operator Π_h^{div}.

Theorem 4.2 (Orlicz-Continuity/Orlicz-Approximability, [12, Section 3]) *Let* $\varphi_a(x,t) := \int_0^t (\kappa + a + s)^{p(x)-2} s\, ds$. *Suppose* $p \in \mathcal{P}^{\log}(\Omega)$ *with* $p^+ < \infty$.

a) Let Π_h^Y *satisfy Assumption 3.3. Then for all* $K \in \mathcal{T}$ *and* $q \in L^{p(\cdot)}(\Omega)$ *with*

$$a + \fint_Q |q|\, dy \leq \max\{1, |Q|^{-m}\} = |Q|^{-m},$$

we have for every $m \in \mathbb{N}$ *there exists* c_m *such that*

$$\int_K \varphi_a\big(\cdot, |\Pi_h^Y q|\big)\, dx \leq c_m \int_{S_K} \varphi_a\big(|q|\big)\, dx + c_m h_K^m.$$

Moreover, for all $K \in \mathcal{T}$ *and* $q \in W^{1,p(\cdot)}(\Omega)$ *with*

$$a + \fint_Q |\nabla q|\, dy \leq \max\{1, |Q|^{-m}\} = |Q|^{-m},$$

we have

$$\int_K \varphi_a\big(\cdot, |q - \Pi_h^Y q|\big)\, dx \leq c_m \int_{S_K} \varphi_a\big(h_K |\nabla q|\big)\, dx + c_m h_K^m.$$

b) Let Π_h^{div} *satisfy Assumption 3.2. Then* Π_h^{div} *has the local continuity property*

$$\int_K \varphi_a\big(\cdot, |\nabla \Pi_h^{\mathrm{div}} \mathbf{w}|\big)\, dx \leq c \int_{S_K} \varphi_a\big(\cdot, |\nabla \mathbf{w}|\big)\, dx + c_m h_k^m,$$

for all $K \in \mathcal{T}$ *and* $\mathbf{w} \in (W^{1,p(\cdot)}(\Omega))^N$ *with*

$$a + \fint_Q |\nabla \mathbf{w}|\, dy \leq \max\{1, |Q|^{-m}\} = |Q|^{-m}.$$

The constant c_m *depends only on* n, $c_{\log}(p)$, p^+, *and the non-degeneracy constant* γ_0 *of the triangulation* $\mathcal{T}$.

Proof a) Due to Assumption 3.3 the operator Π_h^Y satisfies Assumption 1 of [12] both for $r_0 = l_0 = l = 0$ and $r_0 = l_0 = 0$, $l = 1$. The first choice and [12, Corollary 3.5] imply the first assertion, while the second one and [12, Lemma 3.4] yield the second assertion.
b) It follows from Assumption 3.2 and the usual inverse estimates that Π_h^{div} satisfies Assumption 1 of [12] with $l = l_0 = r_0 = 1$. Therefore, the local Orlicz-continuity follows from [12, Corollary 3.5] and the local Orlicz-approximability follows from [12, Lemma 3.4]. □

Next, we present the estimates concerning Π_h^{div} in terms of the natural distance.

Theorem 4.3 *Let Π_h^{div} satisfy Assumption 3.2. Suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $p^- > 1$ and let $s > 1$. Then we have uniformly with respect to $K \in \mathcal{T}$ and to $\mathbf{v} \in (W^{1,sp(\cdot)}(\Omega))^n$*

$$\int_K \big|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})\big|^2\,dx \le c\int_{S_K} \big|\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) - \langle\mathbf{F}(\cdot,\mathbf{D}\mathbf{v})\rangle_{S_K}\big|^2\,dx + c\,h_k^{2\alpha}\int_{S_K}(1+|\mathbf{D}\mathbf{v}|^{p(x)s})\,dx.$$

Here c depends on p^-, p^+, $[p]_\alpha$, s, γ_0, and $\|\mathbf{D}\mathbf{v}\|_{p(\cdot)}$.

Remark 4.4 In contrast to Lemma 4.7 in [12] we have to deal with symmetric gradients instead of full ones. So we need an appropriate version of Korn's inequality (bounding gradients by symmetric gradients). A modular version for shifted functions with variable exponents is not known in literature (but expected). Instead of this we switch to the level of functions and bound an integral depending on the function by an integral depending on the symmetric gradient (see Theorem 2.4). This is possible if we subtract a suitable rigid motion.

Proof (of Theorem 4.3) We estimate the best approximation error by the projection error using $\mathbf{w}_h = \Pi_h^{\mathrm{div}}\mathbf{v}$.

$$\begin{aligned}\int_K |\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})|^2\,dx &\le \int_K |\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})|^2\,dx \\ &\quad + \int_K |\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{D}\mathbf{v})|^2\,dx \\ &\quad + \int_K |\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})|^2\,dx \\ &=: [I] + [II] + [III].\end{aligned}$$

Note that $\mathbf{v} \in (W^{1,p(\cdot)}(\Omega))^n$ implies $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in (L^2(\Omega))^{n\times n}$. For arbitrary $\mathbf{Q} \in \mathbb{R}^{n\times n}_{sym}$ we have

$$\begin{aligned}[I] &:= \fint_K |\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})|^2\,dx \\ &\le c\fint_K |\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{Q})|^2\,dx + c\fint_K |\mathbf{F}(\cdot,\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v}) - \mathbf{F}(\cdot,\mathbf{Q})|^2\,dx \\ &=: [I]_1 + [I]_2.\end{aligned}$$

Let $\mathfrak{p} \in (\mathfrak{P}_1)^n(S_K)$ be such that $\mathbf{D}\mathfrak{p} = \mathbf{Q}$. Due to $\Pi_h^{\mathrm{div}}\mathfrak{p} = \mathfrak{p}$ there holds $\mathbf{Q} = \mathbf{D}\mathfrak{p} = \mathbf{D}\Pi_h^{\mathrm{div}}\mathfrak{p}$. We estimate by Lemma A.4 as follows

$$\begin{aligned}[I]_2 &\le c\fint_K \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot,|\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v} - \mathbf{Q}|\big)\,dx \\ &= c\fint_K \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot,|\mathbf{D}\Pi_h^{\mathrm{div}}(\mathbf{v} - \mathfrak{p})|\big)\,dx.\end{aligned}\tag{4.4}$$

Now we want to estimate $|\mathbf{D}\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p})|$. Since the function $\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p})$ belongs to a finite dimensional function space we can apply inverse estimates. So we have for every rigid motion $\mathcal{R}_K$

$$\begin{aligned}\|\mathbf{D}\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p})\|_{\infty,K} &\le c\,h_K^{-1}\|\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p}) - \mathcal{R}_K\|_{\infty,K} \\ &= c\,h_K^{-1}\,\|\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p}-\mathcal{R}_K)\|_{\infty,K} \\ &\le c\fint_K \Big|\frac{\Pi_h^{\mathrm{div}}(\mathbf{v}-\mathfrak{p}-\mathcal{R}_K)}{h_K}\Big|\,dy.\end{aligned}$$

Now applying Theorem 4.2 and Theorem 2.4 with an appropriate choice of $\mathcal{R}_K$ yields for $m_K := \max\{n(p^+_{S_K} - 2) + 2, 2\}$

$$\|\mathbf{D}\Pi_h^{\text{div}}(\mathbf{v} - \mathfrak{p})\|_{\infty,K} \le c \fint_{S_K} \Big| \frac{(\mathbf{v} - \mathfrak{p} - \mathcal{R}_K)}{h_K} \Big| \, dy + c\, h_K^{m_K}$$
$$\le c \fint_{S_K} |\mathbf{D}(\mathbf{v} - \mathfrak{p})| \, dy + c\, h_K^{m_K}.$$

Inserting this in (4.4) and using convexity of $\varphi_{|\mathbf{Q}|+\kappa}(x, \cdot)$ implies

$$[I]_2 \le c \fint_K \varphi_{|\mathbf{Q}|+\kappa}\Big(\cdot, \fint_{S_K} |\mathbf{D}(\mathbf{v} - \mathfrak{p})|\big) \, dy + c\, h_K^{m_K}\Big) \, dx$$
$$\le c \fint_K \varphi_{|\mathbf{Q}|+\kappa}\Big(\cdot, \fint_{S_K} |\mathbf{D}(\mathbf{v} - \mathfrak{p})|\big) \, dy\Big) + c \fint_K \varphi_{|\mathbf{Q}|+\kappa}(\cdot, h_K^{m_K}) \, dx.$$

As a consequence of Theorem 2.2 for $\mathbf{v} - \mathfrak{p}$, $m = 2$ and $a = |\mathbf{Q}|$ we gain

$$[I]_2 \le c \fint_{S_K} \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot, |\mathbf{D}(\mathbf{v} - \mathfrak{p})|\big) \, dx + c\, h_K^2 + c \fint_K \varphi_{|\mathbf{Q}|+\kappa}(\cdot, h_K^{m_K}) \, dx.$$

In order to proceed we need a special choice of $\mathbf{Q}$. Following the arguments from [12] (Sec. 4) one can show the existence of $\mathbf{Q} \in \mathbb{R}^{n\times n}_{sym}$ such that

$$\fint_{S_K} \mathbf{F}(\cdot, \mathbf{Q}) \, dx = \fint_{S_K} \mathbf{F}(\cdot, \mathbf{D}\mathbf{v}) \, dx, \quad |\mathbf{Q}| \le c\, h_K^{-n}, \tag{4.5}$$

$$\fint_{S_K} \big|\mathbf{F}(\cdot, \mathbf{Q}) - \langle \mathbf{F}(\cdot, \mathbf{Q})\rangle_{S_K}\big|^2 \, dx \le c\, h_K^{2\alpha} \Big(\fint_{S_K} \ln(\kappa + |\mathbf{D}\mathbf{v}|)^2 (\kappa + |\mathbf{D}\mathbf{v}|)^{p(x)} \, dx + 1\Big). \tag{4.6}$$

Due to (4.5), convexity of $\varphi_{|\mathbf{Q}|+\kappa}(x, \cdot)$, and the choice of m_K we have

$$[I]_2 \le c \fint_{S_K} \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot, |\mathbf{D}(\mathbf{v} - \mathfrak{p})|\big) \, dx + c\, h_K^2 + c \fint_{S_K} h_K^{m_K} \varphi_{|\mathbf{Q}|+\kappa}(\cdot, 1) \, dx$$
$$\le c \fint_{S_K} \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot, |\mathbf{D}(\mathbf{v} - \mathfrak{p})|\big) \, dx + c\, h_K^2 + c \fint_{S_K} h_K^{m_K} (1 + h_K^{-n(p(\cdot)-2)}) \, dx$$
$$\le c \fint_{S_K} \varphi_{|\mathbf{Q}|+\kappa}\big(\cdot, |\mathbf{D}\mathbf{v} - \mathbf{Q}|\big) \, dx + c\, h_K^2.$$

Now, with Lemma A.4

$$[I]_2 \le c \fint_{S_K} |\mathbf{F}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 \, dx + c\, h_K^2.$$

Since, $|K| \sim |S_K|$ and $K \subset S_K$ we also have

$$[I]_1 \le c \fint_{S_K} |\mathbf{F}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 \, dx.$$

Overall, we get

$$[I] \le c \fint_{S_K} \big|\mathbf{F}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})\big|^2 \, dx + c\, h_K^2,$$

which means we have to estimate the integral on the right-hand-side. Choosing $\mathbf{Q}$ via (4.5) and using (4.6) we have

$$\begin{aligned}
&\fint_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \mathbf{F}(\cdot,\mathbf{Q})\big|^2\,dx \\
&\quad\le c \fint_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \langle \mathbf{F}(\cdot,\mathbf{Dv})\rangle_{S_K}\big|^2\,dx + c \fint_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Q}) - \langle \mathbf{F}(\cdot,\mathbf{Q})\rangle_{S_K}\big|^2\,dx \\
&\quad\le c \fint_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \langle \mathbf{F}(\cdot,\mathbf{Dv})\rangle_{S_K}\big|^2\,dx + c\,h_K^{2\alpha}\bigg(\fint_{S_K} \ln(\kappa+|\mathbf{Dv}|)^2(\kappa+|\mathbf{Dv}|)^{p(x)}\,dx + 1\bigg) \\
&\quad\le c \fint_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \langle \mathbf{F}(\cdot,\mathbf{Dv})\rangle_{S_K}\big|^2\,dx + c\,h_K^{2\alpha}\bigg(\fint_{S_K} (1+|\mathbf{Dv}|)^{sp(x)}\,dx\bigg).
\end{aligned}$$

The estimate for $[II]$ and $[III]$ are similar. We have

$$\begin{aligned}
&\big|\mathbf{F}_{\mathcal{T}}(x,\mathbf{Q}) - \mathbf{F}(x,\mathbf{Q})\big| \\
&\quad\le c\,|p_{\mathcal{T}}(x) - p(x)|\,|\ln(\kappa+|\mathbf{Q}|)|\Big((\kappa+|\mathbf{Q}|)^{\frac{p_{\mathcal{T}}(x)-2}{2}} + (\kappa+|\mathbf{Q}|)^{\frac{p(x)-2}{2}}\Big)|\mathbf{Q}|.
\end{aligned}$$

This implies

$$\begin{aligned}
[II] &\le c\,h^{2\alpha}\bigg(\int_{\Omega} (1+|\mathbf{Dv}|)^{sp(x)}\,dx\bigg), \\
[III] &\le c\,h^{2\alpha}\bigg(\int_{\Omega} (1+|\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v}|)^{sp(x)}\,dx\bigg).
\end{aligned}$$

We can use the stability of Π_h^{div}, see Theorem 4.2 (for $a=0$ and the exponent $sp(\cdot)$) to get

$$[III] \le c\,h^{2\alpha}\bigg(\int_{\Omega} (1+|\mathbf{Dv}|)^{sp(x)}\,dx\bigg).$$

□

4.3 Error estimate for the velocity

Collecting the estimates and results of the previous sections we obtain the most useful error estimate.

Theorem 4.5 *Let Π_h^{div} satisfy Assumption 3.2. Let $(\mathbf{v},q)$ and $(\mathbf{v}_h,q_h)$ be solutions of the problems* $(\mathbf{Q})$ *and* $(\mathbf{Q}_h)$*, respectively. Suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $p^- > 1$ and let $s>1$. We have the following estimate*

$$\begin{aligned}
\|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\|_2^2 \le c &\sum_{K\in\mathcal{T}} \int_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \langle \mathbf{F}(\cdot,\mathbf{Dv})\rangle_{S_K}\big|^2\,dx \\
&+ c \inf_{\mu_h\in Y_h} \sum_{K\in\mathcal{T}} \int_K (\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\mu_h|)\,dx \\
&+ c\,h^{2\alpha}\int_{\Omega} (1+|\mathbf{Dv}|)^{p(x)s}\,dx.
\end{aligned}$$

Here c depends on p^-, p^+, $[p]_\alpha$, γ_0, and $\|\mathbf{Dv}\|_{p(\cdot)}$.

Proof Since Π_h^{div} is divergence-preserving (see (3.5)) $\mathbf{v}\in V_{\mathrm{div}}$ implies that $\Pi_h^{\mathrm{div}}\mathbf{v} \in V_{h,\mathrm{div}}$. The claim follows from Lemma 4.1 with $\mathbf{w}_h := \Pi_h^{\mathrm{div}}\mathbf{v}$ and Theorem 4.3. □

Corollary 4.6 *Let the assumptions of Theorem 4.5 be satisfied.*
In addition to all previous hypothesis assume that $\mathbf{F}(\cdot,\mathbf{Dv}) \in (W^{1,2}(\Omega))^{n\times n}$ *and* $q \in W^{1,p'(\cdot)}$*. Then we have*

$$\|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}) - \mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{Dv}_h)\|_2 \le c\big(h^{\frac{\min\{(p^+)',2\}}{2}} + h^{\alpha}\big).$$

Proof We estimate the three integrals which appear in Theorem 4.5 separately. By Poincaré's inequality we have

$$\begin{aligned}\sum_{K\in\mathcal{T}}\int_{S_K} \big|\mathbf{F}(\cdot,\mathbf{Dv}) - \langle\mathbf{F}(\cdot,\mathbf{Dv})\rangle_{S_K}\big|^2\,dx &\le c\sum_{K\in\mathcal{T}}\int_{S_K} h_K^2\big|\nabla\mathbf{F}(\cdot,\mathbf{Dv})\big|^2\,dx\\ &\le c\,h^2\int_{\Omega}\big|\nabla\mathbf{F}(\cdot,\mathbf{Dv})\big|^2\,dx \le c\,h^2.\end{aligned}$$

As $\mathbf{F}(\cdot,\mathbf{Dv}) \in W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ we gain

$$\int_{\Omega} |\mathbf{Dv}|^{p(\cdot)s}\,dx < \infty,$$

provided $s \le \frac{n}{n-2}$. This allows us to bound the third term by $c\,h^2$. The term involving the pressure requires more effort. We choose μ_h by $\mu_h = \Pi_h^Y q$ on K and decompose

$$\begin{aligned}\int_{\Omega}(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\mu_h|)\,dx &= \sum_{K\in\mathcal{T}}\int_K(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\Pi_h^Y q|)\,dx\\ &= \sum_{K\in\mathcal{T}^+}\int_K(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\Pi_h^Y q|)\,dx + \sum_{K\in\mathcal{T}^-}\int_K(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\Pi_h^Y q|)\,dx,\end{aligned}$$

with the abbreviations

$$\begin{aligned}\mathcal{T}^+ &:= \{K\in\mathcal{T} : p_K^- \ge 2\},\\ \mathcal{T}^- &:= \{K\in\mathcal{T} : p_K^- < 2\}.\end{aligned}$$

For $K \in \mathcal{T}^+$ we have $(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,t) \le (\varphi^K)^*(\cdot,t) \le t^{p_K'(\cdot)}$ such that

$$\int_K(\varphi^K_{|\mathbf{Dv}|})^*(\cdot,|q-\Pi_h^Y q|)\,dx \le \int_K |q-\Pi_h^Y q|^{p_K'(\cdot)}\,dx.$$

In the following we will show that

$$\fint_K |q-\Pi_h^Y q|^{p_K'(\cdot)}\,dx \le c\,h_K^{(p^+)'}\fint_K\Big(|\nabla q|^{p'(\cdot)}+1\Big)\,dx + c\,h_K^{n+2}. \tag{4.7}$$

We use the identity $q - \Pi_h^Y q = (q-\langle q\rangle_{S_K}) - \Pi_h^Y(q-\langle q\rangle_{S_K})$, the triangle inequality together with $\Delta_2(\varphi^*) < \infty$, and the local stability of Π_h^Y from Lemma 4.2 with $m = n+2$ to conclude that

$$\begin{aligned}\fint_K |q-\Pi_h^Y q|^{p_K'(\cdot)}\,dx &\le c\fint_K |q-\langle q\rangle_{S_K}|^{p_K'(\cdot)}\,dx + c\fint_K\big|\Pi_h^Y(q-\langle q\rangle_{S_K})\big|^{p_K'(\cdot)}\,dx\\ &=: \{I\} + \{II\}.\end{aligned}$$

We estimate the first term by

$$\begin{aligned}\{I\} &\le c\fint_K |q-\langle q\rangle_K|^{p_K'(\cdot)}\,dx + c\fint_K |\langle q\rangle_K - \langle q\rangle_{S_K}|^{p_K'(\cdot)}\,dx\\ &\le c\fint_K |\nabla q|^{p_K'(\cdot)}\,dx + c\,|\langle q\rangle_K - \langle q\rangle_{S_K}|^{p_K'(\cdot)}\end{aligned}$$

$$=: \{I\}_1 + \{I\}_2,$$

using Poincaré's inequality on $L^{p'_K}(K)$. If $|\langle q\rangle_K - \langle q\rangle_{S_K}| \le h_K^n$ we clearly have $\{I\}_2 \le c\, h_K^{n+2}$. Otherwise we can use Lemma 2.1 with $m = n$, Theorem 2.2 with $a = 0$ and Poincaré's inequality from Theorem 2.3 and gain

$$\begin{aligned}\{I\}_2 &\le c\,|\langle q - \langle q\rangle_{S_K}\rangle_K|^{p'(\cdot)} \le c \fint_K |q - \langle q\rangle_{S_K}|^{p'(\cdot)}\,dx + c\,h_K^{n+2} \\ &\le c \fint_{S_K} |q - \langle q\rangle_{S_K}|^{p'(\cdot)}\,dx + c\,h_K^{n+2} \le c \fint_{S_K} |h_K \nabla q|^{p'(\cdot)}\,dx + c\,h_K^{n+2}.\end{aligned}$$

Note that the application of Lemma 2.1 was possible as 4.2

$$\begin{aligned}|\langle q\rangle_K - \langle q\rangle_{S_K}| &\le \fint_K |q|\,dx + \fint_{S_K} |q|\,dx \le c \fint_{S_K} |q|\,dx \\ &\le |S_K| \|q\|_1 \le c\,h_K^{-n}.\end{aligned}$$

For $\{II\}$ again we first consider the case $|\Pi_h^Y(q - \langle q\rangle_{S_K})| \le h_K^n$ in which the estimate is obvious. Otherwise, we apply Lemma 2.1 with $m = n$ as well as Lemma 4.2 to gain

$$\begin{aligned}\{II\} &\le c \fint_K \left|\Pi_h^Y(q - \langle q\rangle_{S_K})\right|^{p'(\cdot)}\,dx \\ &\le c \fint_K |q - \langle q\rangle_{S_K}|^{p'(\cdot)}\,dx + c\,h_K^{n+2} \\ &\le c \fint_{S_K} |q - \langle q\rangle_{S_K}|^{p'(\cdot)}\,dx + c\,h_K^{n+2} \\ &\le c \fint_{S_K} |h_K \nabla q|^{p'(\cdot)}\,dx + c\,h_K^{n+2}.\end{aligned}$$

Note that the application of Lemma 2.1 is justified since

$$\begin{aligned}\|\Pi_h^Y(q - \langle q\rangle_{S_K})\|_\infty &\le \fint_K |\Pi_h^Y(q - \langle q\rangle_{S_K})|\,dx \le c \fint_{S_K} |q - \langle q\rangle_{S_K}|\,dx \\ &\le |S_K| \|q\|_1 \le c\,h_K^{-n}.\end{aligned}$$

Here, we used the inverse estimates on Y_h and Assumption 3.7. Finally $(p_K)' \le p'(x)$ on K yields the claimed inequality (4.7). This implies

$$\int_K (\varphi_{|\mathbf{D}\mathbf{v}|}^K)^*(\cdot, |q - \mu_h|)\,dx \le c\,h^{(p^+)'} \int_K (1 + |\nabla q|^{p'(\cdot)})\,dx + c\,h^{n+2},$$

for $K \in \mathcal{T}^-$. If $K \in \mathcal{T}^-$ we estimate

$$\begin{aligned}\int_K (\varphi_{|\mathbf{D}\mathbf{v}|}^K)^*(\cdot, |q - \mu_h|)\,dx &\le \int_K |q - \mu_h|^{p'_K(\cdot)}\,dx + \int_K (\kappa + |\mathbf{D}\mathbf{v}|)^{p'_K(\cdot)-2} |q - \mu_h|^2\,dx \\ &\le \int_K |q - \mu_h|^{p'_K(\cdot)}\,dx + \int_K (\kappa + |\mathbf{D}\mathbf{v}|)^{p'_K(\cdot)-2} |q - \mu_h|^2\,dx.\end{aligned}$$

The first integral can be estimated via the calculations above, whereas for the second we gain by Young's inequality and (4.7)

$$\int_K (\kappa + |\mathbf{D}\mathbf{v}|)^{p'_K(\cdot)-2} |q - \mu_h|^2\,dx = h_K^2 \int_K (\kappa + |\mathbf{D}\mathbf{v}|)^{p'_K(\cdot)-2} \left|h_K^{-1}(q - \Pi_h^Y)\right|^2\,dx$$

$$\leq h_K^2 \Bigg(\int_K (\kappa + |\mathbf{D}\mathbf{v}|)^{p_K(\cdot)}\, dx + \int_K |h_K^{-1}(q - \Pi_h^Y)|^{p'(\cdot)}\, dx \Bigg)$$
$$\leq c\, h^2 \Bigg(\int_K (1 + |\mathbf{D}\mathbf{v}|^{p(\cdot)s})\, dx + \int_K |\nabla q|^{p'(\cdot)}\, dx + h^n \Bigg).$$

Plugging all together yields

$$\int_\Omega (\varphi^K_{|\mathbf{D}\mathbf{v}|})^*(\cdot, |q - \mu_h|)\, dx$$
$$\leq c\, h^2 \int_\Omega (1 + |\mathbf{D}\mathbf{v}|^{p(\cdot)s})\, dx + c\, h^{\min\{(p^+)',2\}} \int_\Omega |\nabla q|^{p'(\cdot)}\, dx + c|\mathcal{T}|h^{2+n}$$
$$\leq c\, h^2 \int_\Omega (1 + |\mathbf{D}\mathbf{v}|^{p(\cdot)s})\, dx + c\, h^{\min\{(p^+)',2\}} \int_\Omega |\nabla q|^{p'(\cdot)}\, dx + c\, h^2$$
$$\leq c\, h^{\min\{(p^+)',2\}}.$$

Plugging all estimates together proves the claim. □

5 Best Approximation for the pressure

We are now discussing best approximation results for the pressure. As in the classical Stokes problem we need the discrete inf-sup condition to recover information on the discrete pressure. We start by extending this condition to Orlicz spaces.

5.1 Inf-sup condition on generalized Lebesgue spaces

The next lemma contains a continuous inf-sup condition. It is formulated for John domains. Note that all Lipschitz domains and in particular all polyhedral domains are John domains. We will apply the following lemmas to simplices K and their neighborhood S_K, which have uniform John constants due to the non-degeneracy of the mesh. For a precise definition of John domains we refer to [19].

Lemma 5.1 ([19], Thm. 14.3.18) *Let $G \subset \mathbb{R}^n$ be a John domain and let $p \in \mathcal{P}^{\log}(G)$ with $1 < p^- \leq p^+ < \infty$. Then, for all $q \in L_0^{p'(\cdot)}(G)$ we have*

$$\|q\|_{L_0^{p'(\cdot)}(G)} \leq c \sup_{\boldsymbol{\xi} \in W_0^{1,p(\cdot)}(G)\,:\, \|\nabla \boldsymbol{\xi}\|_{p(\cdot)} \leq 1} \langle q, \operatorname{div} \boldsymbol{\xi} \rangle,$$

where the constants depend only on p and the John constant of G.

An appropriate discrete version reads as follows.

Lemma 5.2 *Let $G \subset \mathbb{R}^n$ be a polyhedral domain, let $p \in \mathcal{P}^{\log}(G)$ with $1 < p^- \leq p^+ < \infty$ and let $\Pi_h^{\operatorname{div}}$ satisfy Assumption 3.2. Then for all $q_h \in Q_h$ holds*

$$\|q_h\|_{p'_{\mathcal{T}}(\cdot)} \leq c \sup_{\boldsymbol{\xi}_h \in V_h\,:\, \|\boldsymbol{\xi}_h\|_{1,p_{\mathcal{T}}} \leq 1} \langle q_h, \operatorname{div} \boldsymbol{\xi}_h \rangle,$$

where the constants depend [1] only on p and on G.

Remark 5.3 Note that the inf-sup condition from Lemma 5.2 only holds on the finite element space Q_h. It is not possible to extend it to the whole space $L^{p_\mathcal{T}}(G)$. This is due to the fact that the exponent $p_\mathcal{T}$ is not continuous. Log-Hölder continuity is a necessary assumption for continuity of singular integrals and the maximal function on generalized Lebesgue spaces (see [29]). The inf-sup condition is based on the negative norm theorem which follows from the continuity of (the gradient of) the Bogovskiĭ-operator.

[1] More precisely, on p and the John constant of G.

Proof (of Lemma 5.2.) We use Lemma 5.1, Assumption 3.2, and Theorem 4.2 to get

$$\begin{aligned}\|q_h\|_{Q_h} &\le c \sup_{\|\boldsymbol{\xi}\|_V \le 1} \langle q_h, \operatorname{div} \boldsymbol{\xi}\rangle = c \sup_{\|\boldsymbol{\xi}\|_V \le 1} \langle q_h, \operatorname{div} \Pi_h^{\operatorname{div}} \boldsymbol{\xi}\rangle \\ &\le c \sup_{\|\Pi_h^{\operatorname{div}}\boldsymbol{\xi}\|_{V_h} \le 1} \langle q_h, \operatorname{div} \Pi_h^{\operatorname{div}} \boldsymbol{\xi}\rangle \le c \sup_{\|\boldsymbol{\xi}_h\|_{V_h} \le 1} \langle q_h, \operatorname{div} \boldsymbol{\xi}_h\rangle.\end{aligned}$$

Due to Lemma 3.1 this is equivalent to the claim. □

5.2 Error estimate for the pressure

We now derive a best approximation result for the numerical error of the pressure.

Lemma 5.4 *Let $\Pi_h^{\operatorname{div}}$ satisfy Assumption 3.2. Let $(\mathbf{v}, q)$ and $(\mathbf{v}_h, q_h)$ be solutions of the problems* **(Q)** *and* **($\mathbf{Q}_h$)***, respectively. Suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $p^- > 1$. Then, we have the following estimate*

$$\|q - q_h\|_{p'_{\mathcal{T}}(\cdot)} \le c\|\mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h)\|_{p'_{\mathcal{T}}(\cdot)} + c \inf_{\mu_h \in Q_h} \|q - \mu_h\|_{p'_{\mathcal{T}}(\cdot)} + c\, h^{\alpha}.$$

Proof We split the error $q - q_h$ into a best approximation error $q - \mu_h$ and the remaining part $\mu_h - q_h$, which we will control by means of the equation for q_h. In particular, for all $\mu_h \in Q_h$ it holds

$$\|q - q_h\|_{p'_{\mathcal{T}}(\cdot)} \le c\,\|q - \mu_h\|_{p'_{\mathcal{T}}(\cdot)} + c\,\|\mu_h - q_h\|_{p'_{\mathcal{T}}(\cdot)},$$

by the triangle inequality. The second term is estimated with the help of Lemma 5.2 as follows

$$\|\mu_h - q_h\|_{p'_{\mathcal{T}}(\cdot)} \le \sup_{\boldsymbol{\xi}_h \in V_h \,:\, \|\boldsymbol{\xi}_h\|_{1,p_{\mathcal{T}}(\cdot)} \le 1} \langle \mu_h - q_h, \operatorname{div} \boldsymbol{\xi}_h\rangle.$$

Let us take a closer look at the term $\langle \mu_h - q_h, \operatorname{div} \boldsymbol{\xi}_h\rangle$. By using the equation (4.1) for the error, we get

$$\begin{aligned}\langle \mu_h - q_h, \operatorname{div} \boldsymbol{\xi}_h\rangle &= \langle \mu_h - q, \operatorname{div} \boldsymbol{\xi}_h\rangle + \langle q - q_h, \operatorname{div} \boldsymbol{\xi}_h\rangle \\ &= \langle \mu_h - q, \operatorname{div} \boldsymbol{\xi}_h\rangle + \langle \mathbf{S}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h), \mathbf{D}\boldsymbol{\xi}_h\rangle \\ &= \langle \mu_h - q, \operatorname{div} \boldsymbol{\xi}_h\rangle + \langle \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h), \mathbf{D}\boldsymbol{\xi}_h\rangle \\ &\quad + \langle \mathbf{S}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}), \mathbf{D}\boldsymbol{\xi}_h\rangle.\end{aligned}$$

Applying Hölder's inequality and taking the supremum with respect to $\boldsymbol{\xi}$ yields

$$\begin{aligned}\|\mu_h - q_h\|_{p'_{\mathcal{T}}(\cdot)} &\le c\,\|q - \mu_h\|_{p'_{\mathcal{T}}(\cdot)} + c\,\|\mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v}_h)\|_{p'_{\mathcal{T}}(\cdot)} \\ &\quad + c\|\mathbf{S}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\cdot, \mathbf{D}\mathbf{v})\|_{p'_{\mathcal{T}}(\cdot)}.\end{aligned}$$

The last term can be estimated as follows: we have for some $s_1 \in (1, s)$

$$\begin{aligned}&\int_{\Omega} \Big(\frac{|\mathbf{S}(\cdot, \mathbf{D}\mathbf{v}) - \mathbf{S}_{\mathcal{T}}(\mathbf{D}\mathbf{v})|}{C\, h^{\alpha}}\Big)^{p'_{\mathcal{T}}(\cdot)} dx \\ &\qquad = \sum_{K \in \mathcal{T}} \int_K \Big(\frac{|\mathbf{S}(x, \mathbf{D}\mathbf{v}) - \mathbf{S}(x_K, \mathbf{D}\mathbf{v}_h)|}{C\, h^{\alpha}}\Big)^{p'(x_K)} dx \\ &\qquad \le \sum_{K \in \mathcal{T}} \int_K \Big(\frac{\log(1 + |\mathbf{D}\mathbf{v}|)(1 + |\mathbf{D}\mathbf{v}_h|)^{p(x_K) - 1}}{C_1}\Big)^{p'(x_K)} dx \\ &\qquad \le \sum_{K \in \mathcal{T}} \int_K \Big(\frac{(1 + |\mathbf{D}\mathbf{v}|)^{s_1(p(x_K) - 1)}}{C_2}\Big)^{p'(x_K)} dx.\end{aligned}$$

Here we took into account $p \in C^{0,\alpha}(\overline{\Omega})$. An appropriate choice of C (depending on s_1 and s) implies that, for small enough $h > 0$,

$$\int_\Omega \Big(\frac{|\mathbf{S}(\cdot,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(\mathbf{Dv})|}{C\,h^\alpha}\Big)^{p'(\cdot)}\,dx \le \frac{1}{C_3}\sum_{K\in\mathcal{T}}\int_K \big(1+|\mathbf{Dv}|\big)^{p(x)s}\,dx$$
$$= \frac{1}{C_3}\int_\Omega \big(1+|\mathbf{Dv}|\big)^{p(x)s}\,dx \le 1.$$

The claim follows, since $\mu_h \in Q_h$ was arbitrary. □

Unfortunately, the estimate for the error of the pressure $q - q_h$ involves the error of the stresses $\mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}_h)$. Our error estimates for the velocity in Theorem 4.5 are however expressed in terms of $\mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{F}_\mathcal{T}(\cdot,\mathbf{Dv}_h)$. The following lemma represents the missing link between the error in terms of $\mathbf{S}_\mathcal{T}$ and the error in terms of $\mathbf{F}_\mathcal{T}$ (with an additional term with respect to the estimate for fixed p).

Lemma 5.5 *Under the assumptions of Corollary 4.6 It holds*

$$\int_\Omega |\mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}_h)|^{p'(\cdot)}\,dx \le c\,\big(h^{\min\{\frac{((p^+)')^2}{2},(p^+)'\}} + h^{\alpha\min\{2,(p^+)'\}}\big). \tag{5.1}$$

Proof By standard arguments we gain

$$\begin{aligned}\mathbf{S}_\mathcal{T}(x,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(x,\mathbf{Dv}_h) &= \int_0^1 D\mathbf{S}_\mathcal{T}(x,\mathbf{Dv}+t(\mathbf{Dv}_h-\mathbf{Dv}))\,dt : (\mathbf{Dv}-\mathbf{Dv}_h)\\ &\le c\int_0^1 (\kappa+|\mathbf{Dv}+t(\mathbf{Dv}_h-\mathbf{Dv})|)^{p_\mathcal{T}(x)-2}\,dt \quad |\mathbf{Dv}-\mathbf{Dv}_h|\\ &\le c\,(\kappa+|\mathbf{Dv}|+|\mathbf{Dv}_h-\mathbf{Dv}|)^{p_\mathcal{T}(x)-2}\,|\mathbf{Dv}-\mathbf{Dv}_h|.\end{aligned} \tag{5.2}$$

Let us decompose $\mathcal{T}$ again into $\mathcal{T}^+$ and $\mathcal{T}^-$, where

$$\mathcal{T}^+ := \{K\in\mathcal{T} : p_\mathcal{T} > 2\},$$
$$\mathcal{T}^- := \{K\in\mathcal{T} : p_\mathcal{T} \le 2\}.$$

It follows for $K \in \mathcal{T}^+$ by Young's inequality for every $\gamma > 0$

$$\begin{aligned}&\int_K |\mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}_h)|^{p'_\mathcal{T}}\,dx\\ &\le c\int_K (\kappa+|\mathbf{Dv}|+|\mathbf{Dv}_h-\mathbf{Dv}|)^{\frac{p_\mathcal{T}-2}{2}p'_\mathcal{T}}\,|\mathbf{Dv}-\mathbf{Dv}_h|^{p'_\mathcal{T}}(1+|\mathbf{Dv}|+|\mathbf{Dv}_h|)^{\frac{p_\mathcal{T}-2}{2}p'_\mathcal{T}}\,dx\\ &\le c\,\gamma^{-2}\int_K (\kappa+|\mathbf{Dv}|+|\mathbf{Dv}_h-\mathbf{Dv}|)^{p_\mathcal{T}-2}\,|\mathbf{Dv}-\mathbf{Dv}_h|^2\,dx\\ &+ c\,\gamma^{\frac{2p'_\mathcal{T}}{2-p'_\mathcal{T}}}\int_K (1+|\mathbf{Dv}|+|\mathbf{Dv}_h|)^{p_\mathcal{T}}\,dx.\end{aligned}$$

Due to Lemma 2.1 and 3.2 we gain for some $s > 1$ as a consequence of $p \in C^{0,\alpha}(\overline{\Omega})$

$$\begin{aligned}&\int_K |\mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}) - \mathbf{S}_\mathcal{T}(\cdot,\mathbf{Dv}_h)|^{p'_\mathcal{T}}\,dx\\ &\le c\,\gamma^{-2}\int_K (\kappa+|\mathbf{Dv}|+|\mathbf{Dv}_h-\mathbf{Dv}|)^{p_\mathcal{T}(x)-2}\,|\mathbf{Dv}-\mathbf{Dv}_h|^2\,dx\end{aligned}$$

$$+\ c\,\gamma^{\frac{2p'_{\mathcal{T}}}{2-p'_{\mathcal{T}}}}\int\limits_K (1+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}+|\mathbf{D}\mathbf{v}_h|^{p_{\mathcal{T}}(\cdot)})\,dx,$$

If $(p^+)' = \min_{K\in\mathcal{T}^+}\min_K p' < 2$ we obtain (in the other case the following calculations are not necessary because of $\mathcal{T}^+ = \emptyset$)

$$\begin{aligned}
\sum_{K\in\mathcal{T}^+}\int\limits_K &|\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^{p'_{\mathcal{T}}}\,dx\\
&\le c\gamma^{\frac{2(p^+)'}{2-(p^+)'}}\int\limits_\Omega(\kappa+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}+|\mathbf{D}\mathbf{v}_h|^{p_{\mathcal{T}}(\cdot)})\,dx+c\gamma^{-2}\int\limits_\Omega|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^2\,dx\\
&=: c\Big(\gamma^{\frac{2(p^+)'}{2-(p^+)'}}A+\gamma^{-2}B\Big).
\end{aligned}$$

We minimize the r.h.s. with respect to γ which leads to the optimal choice

$$\gamma=\left(\frac{2-(p^+)'}{(p^+)'}\right)^{\frac{2-(p^+)'}{4}}\left(\frac{B}{A}\right)^{\frac{2-(p^+)'}{4}}\sim\left(\frac{B}{A}\right)^{\frac{2-(p^+)'}{4}}.$$

Note that for $h \ll 1$ we can assume that $\gamma \le 1$ as a consequence of Corollary 4.6. So we end up with

$$\begin{aligned}
\sum_{K\in\mathcal{T}^+}\int\limits_K &|\mathbf{S}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{S}(\cdot,\mathbf{D}\mathbf{v}_h)|^{p'(\cdot)}\,dx\\
&\le c\bigg(\int\limits_\Omega(\kappa+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}+|\mathbf{D}\mathbf{v}_h|^{p_{\mathcal{T}}(x)})\,dx\bigg)^{\frac{2-(p^+)'}{2}}\\
&\quad\times\bigg(\int\limits_\Omega|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^2\,dx\bigg)^{\frac{(p^+)'}{2}}.
\end{aligned}$$

As a consequence of Corollary 4.6 and $\mathbf{D}\mathbf{v}_h \in L^{p_{\mathcal{T}}}(\Omega)$ uniformly we gain

$$\sum_{K\in\mathcal{T}^+}\int\limits_K|\mathbf{S}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{S}(\cdot,\mathbf{D}\mathbf{v}_h)|^{p'(\cdot)}\,dx\le c\,\big(h^{\min\{(p^+)',\frac{((p^+)')^2}{2}\}}+h^{(p^+)'\alpha}\big).$$

For $K\in\mathcal{T}^-$ we have due to $\mathbf{D}\mathbf{v}\in L^{p(\cdot)s}(\Omega)$ and $\mathbf{D}\mathbf{v}_h\in L^{p_{\mathcal{T}}(\cdot)}(\Omega)$ uniformly in h

$$\begin{aligned}
\int\limits_K &|\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^{p'_{\mathcal{T}}(\cdot)}\,dx\\
&\le c\int\limits_K(\kappa+|\mathbf{D}\mathbf{v}|+|\mathbf{D}\mathbf{v}_h-\mathbf{D}\mathbf{v}|)^{p_{\mathcal{T}}(x)-2}\,|\mathbf{D}\mathbf{v}-\mathbf{D}\mathbf{v}_h|^2\,dx\\
&\le c\int\limits_K|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^2\,dx,
\end{aligned}$$

such that

$$\begin{aligned}
\sum_{K\in\mathcal{T}^-}\int\limits_K &|\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{S}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^{p'_{\mathcal{T}}(\cdot)}\,dx\\
&\le c\int\limits_\Omega|\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v})-\mathbf{F}_{\mathcal{T}}(\cdot,\mathbf{D}\mathbf{v}_h)|^2\,dx\\
&\le c\,\big(h^{\min\{2,(p^+)'\}}+h^{2\alpha}\big).
\end{aligned}$$

Here we used again corollary 4.6. The claim follows by combining the estimates for $\mathcal{T}^+$ and $\mathcal{T}^-$. □

Combining Lemma 5.4 and Lemma 5.5 we get our desired error estimate for the pressure.

Theorem 5.6 *Let Π_h^{div} satisfy Assumption 3.2. Let $(\mathbf{v}, q)$ and $(\mathbf{v}_h, q_h)$ be solutions of the problems* $(\mathbf{Q})$ *and* $(\mathbf{Q}_h)$*, respectively. Assume further that $\mathbf{F}(\cdot, \mathbf{D}\mathbf{v}) \in (W^{1,2}(\Omega))^{n\times n}$, $q \in W^{1,p'(\cdot)}$ and suppose $p \in C^{0,\alpha}(\overline{\Omega})$ with $p^- > 1$. Then we have for*

$$\alpha_p := \frac{\alpha}{(p_-)'} \min\{2, (p^+)'\}, \quad \beta_p := \frac{1}{(p_-)'} \min\{\frac{((p^+)')^2}{2}, (p^+)'\},$$

the following estimates.

(a) $\|q - q_h\|_{L^{p'_\mathcal{T}(\cdot)}} \leq c\, h^{\min\{\alpha_p, \beta_p\}}$;

(b) $\int_\Omega |q - q_h|^{p'_\mathcal{T}(\cdot)}\, dx \leq c\, h^{(p^+)' \min\{\alpha_p, \beta_p\}}$.

Proof We choose $\mu_h := \sum_K \chi_K ⨍_K q\, dx$ and apply (4.7) such that

$$\int_K \Big|\frac{q - \Pi_h^Y q}{C h_K}\Big|^{p'_K} dx \leq c \int_{S_K} \Big(|\nabla q|^{p'(\cdot)} + 1\Big)\, dx + c\, h_K^{n+2}.$$

Since $q \in W^{1,p'(\cdot)}$ we obtain for C large enough

$$\begin{aligned}\int_\Omega \Big|\frac{q - \mu_h}{C\, h_K}\Big|^{p'_\mathcal{T}(\cdot)} dx &= \sum_K \int_K \Big|\frac{q - \mu_h}{C\, h_K}\Big|^{p'_\mathcal{T}(\cdot)} dx \leq \frac{1}{C'} \sum_K \bigg(\Big(\int_K |\nabla q|^{p'(\cdot)} + 1\Big)\, dx + h_K^{n+2}\bigg)\\ &\leq \frac{1}{C'} \bigg(\int_\Omega \Big(|\nabla q|^{p'(\cdot)} + 1\Big)\, dx + 1\bigg) = 1,\end{aligned}$$

such that a) follows from Lemma 5.4 and 5.5.
In order to show b) we define $\varkappa(h) := h^{\min\{\alpha_p, \beta_p\}}$ and estimate

$$\int_\Omega |q - q_h|^{p'_\mathcal{T}(\cdot)}\, dx \leq c\, \varkappa(h)^{(p^+)'} \int_\Omega \Big|\frac{q - q_h}{C\, \varkappa(h)}\Big|^{p'(\cdot)} dx \leq c\, \varkappa(h)^{(p^+)'},$$

using a). □

A Orlicz spaces

The following definitions and results are standard in the theory of Orlicz spaces and can for example be found in [30]. A continuous, convex function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$, and $\lim_{t\to\infty} \rho(t) = \infty$ is called a *continuous, convex φ-function.*

We say that φ satisfies the Δ_2–condition, if there exists $c > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c\, \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest such constant. Since $\varphi(t) \leq \varphi(2t)$ the Δ_2-condition is equivalent to $\varphi(2t) \sim \varphi(t)$ uniformly in t. For a family φ_λ of continuous, convex φ-functions we define $\Delta_2(\{\varphi_\lambda\}) := \sup_\lambda \Delta_2(\varphi_\lambda)$. Note that if $\Delta_2(\varphi) < \infty$ then $\varphi(t) \sim \varphi(c\, t)$ uniformly in $t \geq 0$ for any fixed $c > 0$. By L^φ and $W^{k,\varphi}$, $k \in \mathbb{N}_0$, we denote the classical Orlicz and Orlicz-Sobolev spaces, i.e. $f \in L^\varphi$ iff $\int \varphi(|f|)\, dx < \infty$ and $f \in W^{k,\varphi}$ iff $\nabla^j f \in L^\varphi$, $0 \leq j \leq k$.
A φ-function ρ is called a N-function iff it is strictly increasing and convex with

$$\lim_{t\to 0} \frac{\rho(t)}{t} = \lim_{t\to\infty} \frac{t}{\rho(t)} = 0.$$

By ρ^* we denote the conjugate N-function of ρ, which is given by $\rho^*(t) = \sup_{s\geq 0}(st - \rho(s))$. Then $\rho^{**} = \rho$.

Lemma A.1 (Young's inequality) *Let ρ be an N-function. Then for all $s, t \geq 0$ we have*

$$st \leq \rho(s) + \rho^*(t).$$

If $\Delta_2(\rho, \rho^*) < \infty$, then additionally for all $\delta > 0$

$$\begin{aligned} st &\leq \delta\, \rho(s) + c_\delta\, \rho^*(t),\\ st &\leq c_\delta\, \rho(s) + \delta\, \rho^*(t),\\ \rho'(s)t &\leq \delta\, \rho(s) + c_\delta\, \rho(t),\\ \rho'(s)t &\leq \delta\, \rho(t) + c_\delta\, \rho(s),\end{aligned}$$

where $c_\delta = c(\delta, \Delta_2(\{\rho, \rho^*\}))$.

Definition A.2 Let ρ be an N-function. We say that ρ is *elliptic*, if ρ is C^1 on $[0,\infty)$ and C^2 on $(0,\infty)$ and assume that

$$\rho'(t) \sim t\,\rho''(t), \tag{A.1}$$

uniformly in $t > 0$. The constants hidden in $\sim$ are called the *characteristics of* ρ.

Note that (A.1) is stronger than $\Delta_2(\rho,\rho^*) < \infty$. In fact, the Δ_2-constants can be estimated in terms of the characteristics of ρ.

Associated to an elliptic N-function ρ we define the tensors

$$\begin{aligned}\mathbf{A}^\rho(\boldsymbol{\xi}) &:= \frac{\rho'(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{R}^{n\times n}\\ \mathbf{F}^\rho(\boldsymbol{\xi}) &:= \sqrt{\frac{\rho'(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}\,\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{R}^{n\times n}.\end{aligned}$$

We define the *shifted* N-function ρ_a for $a \geq 0$ by

$$\rho_a(t) := \int_0^t \frac{\rho'(a+\tau)}{a+\tau}\tau\, d\tau. \tag{A.2}$$

The following auxiliary result can be found in [17,21].

Lemma A.3 *For all $a, b, t \geq 0$ we have*

$$\begin{aligned}\rho_a(t) &\sim \begin{cases}\rho''(a)t^2 & \text{if } t \lesssim a\\ \rho(t) & \text{if } t \gtrsim a,\end{cases}\\ (\rho_a)_b(t) &\sim \rho_{a+b}(t).\end{aligned}$$

Lemma A.4 ([17, Lemma 2.3]) *We have*

$$\begin{aligned}\big(\mathbf{A}^\rho(\mathbf{P}) - \mathbf{A}^\rho(\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) &\sim \big|\mathbf{F}^\rho(\mathbf{P}) - \mathbf{F}^\rho(\mathbf{Q})\big|^2\\ &\sim \rho_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\\ &\sim \rho''\big(|\mathbf{P}|+|\mathbf{Q}|\big)|\mathbf{P}-\mathbf{Q}|^2,\end{aligned}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$. Moreover, uniformly in $\mathbf{Q} \in \mathbb{R}^{n\times n}$,

$$\begin{aligned}\mathbf{A}^\rho(\mathbf{Q})\cdot\mathbf{Q} &\sim |\mathbf{F}^\rho(\mathbf{Q})|^2 \sim \rho(|\mathbf{Q}|)\\ |\mathbf{A}^\rho(\mathbf{P}) - \mathbf{A}^\rho(\mathbf{Q})| &\sim \big(\rho_{|\mathbf{P}|}\big)'(|\mathbf{P}-\mathbf{Q}|).\end{aligned}$$

The constants depend only on the characteristics of ρ.

Lemma A.5 (Change of Shift) *Let ρ be an elliptic N-function. Then for each $\delta > 0$ there exists $C_\delta \geq 1$ (only depending on δ and the characteristics of ρ) such that*

$$\begin{aligned}\rho_{|\mathbf{a}|}(t) &\leq C_\delta\,\rho_{|\mathbf{b}|}(t) + \delta\,\rho_{|\mathbf{a}|}(|\mathbf{a}-\mathbf{b}|),\\ (\rho_{|\mathbf{a}|})^*(t) &\leq C_\delta\,(\rho_{|\mathbf{b}|})^*(t) + \delta\,\rho_{|\mathbf{a}|}(|\mathbf{a}-\mathbf{b}|),\end{aligned}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $t \geq 0$.

The case $\mathbf{a} = 0$ or $\mathbf{b} = 0$ implies the following corollary.

Corollary A.6 (Removal of Shift) *Let ρ be an elliptic N-function. Then for each $\delta > 0$ there exists $C_\delta \geq 1$ (only depending on δ and the characteristics of ρ) such that*

$$\begin{aligned}\rho_{|\mathbf{a}|}(t) &\leq C_\delta\,\rho(t) + \delta\,\rho(|\mathbf{a}|),\\ \rho(t) &\leq C_\delta\,\rho_{|\mathbf{a}|}(t) + \delta\,\rho(|\mathbf{a}|),\end{aligned}$$

for all $\mathbf{a} \in \mathbb{R}^n$ and $t \geq 0$.

Lemma A.7 *Let ρ be an elliptic N-function. Then $(\rho_a)^*(t) \sim (\rho^*)_{\rho'(a)}(t)$ uniformly in $a, t \geq 0$. Moreover, for all $\lambda \in [0,1]$ we have*

$$\rho_a(\lambda a) \sim \lambda^2\rho(a) \sim (\rho_a)^*(\lambda\rho'(a)).$$

Lemma A.8 *Let $\rho(t) := \int_0^t(\kappa+s)^{q-2}s\,ds$ with $q \in (1,\infty)$ and $t \geq 0$. Then*

$$\begin{aligned}\rho_a(\lambda t) &\leq c\,\max\{\lambda^q,\lambda^2\}\rho(t),\\ (\rho_a)^*(\lambda t) &\leq c\,\max\{\lambda^{q'},\lambda^2\}\rho(t),\end{aligned}$$

uniformly in $a, \lambda \geq 0$.

Remark A.9 Let $p \in \mathcal{P}(\Omega)$ with $p^- > 1$ and $p^+ < \infty$. The results above extend to the function $\varphi(x,t) = \int_0^t(\kappa+s)^{p(x)-2}s\,ds$ uniformly in $x \in \Omega$, where the constants only depend on p^- and p^+.

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