SPECTRAL OPTIMIZATION PROBLEMS FOR POTENTIALS AND MEASURES[*]

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Abstract. In the present paper we consider spectral optimization problems involving the Schrödinger operator $-\Delta + \mu$ on $\mathbb{R}^d$, the prototype being the minimization of the k-th eigenvalue $\lambda_k(\mu)$. Here μ may be a capacitary measure with prescribed torsional rigidity (like in the Kohler–Jobin problem) or a classical nonnegative potential V which satisfies the integral constraint $\int V^{-p}dx \leq m$ with $0 < p < 1$. We prove the existence of global solutions in $\mathbb{R}^d$ and that the optimal potentials or measures are equal to $+\infty$ outside a compact set.

Key words. shape optimization, eigenvalues, Kohler–Jobin, Schrödinger operator, torsional rigidity

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1. Introduction. In shape optimization problems, as in all general optimization problems, proving the existence of a solution is a crucial step, which may turn out to be, in some cases, particularly difficult, due to the lack of compactness of minimizing sequences. In the case of shape optimization problems of spectral type, the existence issue was studied by Buttazzo and Dal Maso in [13] (see also [2, 8, 11, 22, 23] for a survey on the field), who proved that when the competing domains Ω are constrained to stay in a given *bounding box* $D \subset \mathbb{R}^d$, the optimization problem for a shape cost functional F

$$(1.1) \qquad \min\big\{F(\Omega) : \Omega \subset D, \ |\Omega| \leq m\big\}$$

admits a solution, provided the assumptions below are satisfied:

(i) F is lower semicontinuous with respect to a suitable variational convergence of sets (called γ-convergence, see section 2.5), that is,

$$F(\Omega) \leq \liminf_n F(\Omega_n) \qquad \text{whenever } \Omega_n \to \Omega;$$

(ii) F is monotone decreasing with respect to the set inclusion, that is,

$$F(\Omega_2) \leq F(\Omega_1) \qquad \text{whenever } \Omega_1 \subset \Omega_2.$$

Removing the bounded box constraint $\Omega \subset D$ in (1.1) creates additional difficulties, and a general existence result, similar to the Buttazzo and Dal Maso one, is not available. The particular case of spectral optimization problems, in which

$$F(\Omega) = \Phi\big(\lambda(\Omega)\big),$$

$\lambda(\Omega) = \big(\lambda_1(\Omega), \lambda_2(\Omega), \dots\big)$ being the spectrum of the Dirichlet Laplacian in Ω, made by the eigenvalues $\lambda_k(\Omega)$ of the operator $-\Delta$ on the space $H_0^1(\Omega)$, was considered in [7] and in [27]. In these papers, by using two different approaches, an optimal solution for the problem

$$\min\left\{\Phi\big(\lambda_1(\Omega), \dots, \lambda_k(\Omega)\big) : \Omega \subset \mathbb{R}^d,\ |\Omega| \leq m\right\}$$

is shown to exist, provided Φ is increasing in each variable and Lipschitz continuous. In addition, these solutions are proved to be bounded domains of $\mathbb{R}^d$ of finite perimeter. In particular, this applies to the case

(1.2) $$\Phi(\lambda) = \lambda_k,$$

which provides the optimal shape for the k-th eigenvalue $\lambda_k(\Omega)$ under the sole volume constraint.

The purpose of this paper is to consider spectral optimization problems similar to (1.2), but involving the Schrödinger operator

$$-\Delta + \mu \quad \text{on } \mathbb{R}^d,$$

where μ is either a general nonnegative Borel measure absolutely continuous with respect to capacity (these measures are the so-called capacitary measures and they may possibly be infinite; see section 2.1) or a classical potential V. The potential V, which is a nonnegative Borel function in our framework, can be interpreted as a capacitary measure by setting $\mu := V(x)dx$.

In our first main result (Theorem 5.4), we prove that the problem

$$\min\left\{\lambda_k(V) : V \geq 0 \text{ on } \mathbb{R}^d,\ \int_{\mathbb{R}^d} V^{-p}dx \leq m\right\}$$

when $0 < p < 1$ has a solution and, moreover, that the optimal potential V equals $+\infty$ outside a compact set. We remark that the condition $\int_{\mathbb{R}^d} V^{-p}dx < +\infty$ with $0 < p < 1$ implies the compactness of the resolvent of the Schrödinger operator with potential V on $\mathbb{R}^d$ (see Example 3.10 of [9]), so that the corresponding spectrum is discrete. The techniques we use rely on new tools, such as concentration-compactness results for capacitary measures (section 3), on the concept of subsolutions for measure functionals (section 4), and on a De Giorgi type argument (Lemma 4.4).

The second result of the paper is concerned with the minimization of the k-th eigenvalue under a torsion constraint, in the spirit of the Kohler–Jobin result for the first eigenvalue (see [25]), precisely,

$$\min\left\{\lambda_k(\mu) : \mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d),\ P(\mu) \leq m\right\},$$

where $\mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ is the class of capacitary measures and $P(\mu)$ is the torsion of the measure μ (see section 2.2). In Theorem 6.2 we prove that optimal capacitary measures exist and that they are $+\infty$ outside a compact set. Nevertheless, we are not able to prove that the optimal measure is a domain, as in the particular cases $k = 1, 2$. An interesting property we use in the proof is concerned with the behavior of the heat equation solutions in unbounded sets: as soon as the heat source is positive outside a compact set, the corresponding temperature has the same property.

As final remarks in the introduction, we point out that the results of Theorems 5.4 and 6.2 are valid for a general function of eigenvalues

$$\Phi\big(\lambda_1, \dots, \lambda_k\big),$$

where $\Phi : (\mathbb{R}_+)^k \to \mathbb{R}$ is nondecreasing and locally bi-Lipschitz in each variable. We chose to present only the case $\Phi(\lambda_1, \ldots, \lambda_k) = \lambda_k$ for the simplicity of the exposition. The extension to a general function Φ is based on an induction argument which is not specific to potentials. (See, for instance, [19, Theorem 1.2] for the case of shapes.)

As well, we notice that if one restricts the class of potentials in Theorem 5.4, e.g., to competing potentials that take the value $+\infty$ outside *a given bounded* set $D \subset \mathbb{R}^d$ (so that the corresponding solutions vanish on $\mathbb{R}^d \setminus D$), a recent result (see [14, Theorem 4.1]) provides the existence of an optimal potential when minimizing a quite general functional $F(V)$, which is not necessarily of spectral type. Removing the boundedness of D introduces the difficulties that we are able here to overcome only for spectral functionals.

2. Preliminaries.

2.1. Sobolev spaces and capacitary measures. We define the capacity of a generic set $E \subset \mathbb{R}^d$ as

$$\mathrm{cap}(E) = \inf \left\{ \|u\|_{H^1}^2 : u \in H^1(\mathbb{R}^d), \ u = 1 \text{ a.e. in a neighbourhood of } E \right\},$$

where $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$. We say that a property $\mathcal{P}(x)$ holds *quasi-everywhere* if the set where $\mathcal{P}(x)$ does not hold has zero capacity, i.e.,

$$\mathrm{cap}\left(\left\{ x \in \mathbb{R}^d : \mathcal{P}(x) \text{ does not hold} \right\} \right) = 0.$$

We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *quasi-continuous* if there is a sequence of open set $\omega_n \subset \mathbb{R}^d$ such that

$$\mathrm{cap}(\omega_n) \to 0 \qquad \text{and} \qquad f : (\mathbb{R}^d \setminus \omega_n) \to \mathbb{R} \quad \text{is continuous.}$$

It is well known (see, for example, [24, 21, 29]) that every Sobolev function $u \in H^1(\mathbb{R}^d)$ has a quasi-continuous representative $\widetilde{u} : \mathbb{R}^d \to \mathbb{R}$. Moreover, if $\widetilde{u}_1$ and $\widetilde{u}_2$ are two quasi-continuous representatives of the same class of equivalence $u \in H^1(\mathbb{R}^d)$, then $\widetilde{u}_1 = \widetilde{u}_2$ quasi-everywhere. From now on we identify the Sobolev space $H^1(\mathbb{R}^d)$ with the space of quasi-continuous representatives

$$H^1(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R} : u \text{ quasi-continuous}, \ \int_{\mathbb{R}^d} \left(u^2 + |\nabla u|^2 \right) dx < +\infty \right\},$$

and we note that each element of $H^1(\mathbb{R}^d)$ is a function defined up to a set of zero capacity. Moreover, we recall that if the sequence $u_n \in H^1(\mathbb{R}^d)$ converges in norm to $u \in H^1(\mathbb{R}^d)$, then u_n converges quasi-everywhere (up to a subsequence) to u.

We say that a regular Borel measure (possibly $+\infty$ valued) μ in $\mathbb{R}^d$ is a *capacitary measure* if

$$\left(\mathrm{cap}(E) = 0 \right) \Rightarrow \left(\mu(E) = 0 \right) \qquad \text{for every } E \subset \mathbb{R}^d.$$

Remark 2.1. Any measure μ which is absolutely continuous with respect to the Lebesgue measure is a capacitary measure. Indeed, if $\mathrm{cap}(E) = 0$, then $|E| = 0$ and so $\mu(E) = 0$.

Since the Sobolev functions are defined up to a set of zero capacity, the integral $\int_{\mathbb{R}^d} u^2 \, d\mu$ is well defined, when μ is a capacitary measure, for every $u \in H^1(R^d)$. We say that the capacitary measures μ and ν are equivalent if

$$\int_{\mathbb{R}^d} u^2 \, d\mu = \int_{\mathbb{R}^d} u^2 \, d\nu \quad \text{for every } u \in H^1(\mathbb{R}^d).$$

We denote by $\mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ the space obtained as a quotient of the family of capacitary measures on $\mathbb{R}^d$ with respect to this equivalence relation. From now on we will identify a capacitary measure with its class of equivalence (see also [3]). On the space of capacitary measures $\mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$, there is a partial order, induced by the testing with Sobolev functions, i.e., we say that $\mu \prec \nu$ if

$$\int_{\mathbb{R}^d} u^2 \, d\mu \le \int_{\mathbb{R}^d} u^2 \, d\nu \quad \text{for every } u \in H^1(\mathbb{R}^d).$$

We define the Sobolev space H^1_μ as

$$H^1_\mu(\mathbb{R}^d) = \left\{ u \in H^1(\mathbb{R}^d) : \int u^2 \, d\mu < +\infty \right\}.$$

As was proved in [13], the space H^1_μ, equipped with the norm

$$\|u\|^2_{H^1_\mu} = \|u\|^2_{H^1} + \|u\|^2_{L^2(\mu)},$$

is a Hilbert space. Notice that if $\mu \prec \nu$, then $H^1_\mu \supset H^1_\nu$.

A typical example of a capacitary measure is the measure I_Ω associated to a Borel set Ω

$$I_\Omega(E) = \begin{cases} 0 & \text{if } \mathrm{cap}(E \cap \Omega^c) = 0, \\ +\infty & \text{if } \mathrm{cap}(E \cap \Omega^c) > 0 \end{cases}$$

for every $E \subset \mathbb{R}^d$. For a Borel set $\Omega \subset \mathbb{R}^d$, we use the notation $H^1_0(\Omega) := H^1_{I_\Omega}(\mathbb{R}^d)$. We note that (see [8, 24]) if Ω is an open set, then $H^1_0(\Omega)$ is the usual Sobolev space, obtained as the closure, with respect to the norm $\| \cdot \|_{H^1}$, of the smooth functions with compact support in Ω, which we denote by $C^\infty_c(\Omega)$.

In what follows we will often be interested to the action of a capacitary measure $\mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ inside a quasi-open set $\Omega \subset \mathbb{R}^d$. For this purpose we define the capacitary measure $\mu \vee I_\Omega$ as

$$\mu \vee I_\Omega(E) = \begin{cases} \mu(E) & \text{if } \mathrm{cap}(E \cap \Omega^c) = 0, \\ +\infty & \text{if } \mathrm{cap}(E \cap \Omega^c) > 0. \end{cases}$$

It is easy to see that $\mu \vee I_\Omega \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ and that the corresponding Sobolev space satisfies

$$H^1_{\mu \vee I_\Omega} = H^1_\mu \cap H^1_0(\Omega).$$

2.2. Torsional rigidity and torsion function. Let $\mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ and fix $f \in L^p(\mathbb{R}^d)$ with $p \in [1, +\infty]$. For $u \in H^1_\mu \cap L^{p'}(\mathbb{R}^d)$, where $p' = p/(p-1)$, we define the functional

$$J_{\mu,f}(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} u^2 \, d\mu - \int_{\mathbb{R}^d} fu \, dx$$

and the *torsional energy* of μ as

$$E(\mu) = \inf \left\{ J_{\mu,1}(u) : u \in H^1_\mu \cap L^1(\mathbb{R}^d) \right\}.$$

Since $J_{\mu,1}(0) = 0$, we have that $E(\mu) \leq 0$. We call torsion of μ the nonnegative quantity $P(\mu) := -E(\mu)$, extending in this way the classical notion of torsional rigidity of a two-dimensional simply connected domain up to multiplicative constant. We note that $P(\mu)$ can be $+\infty$, for example, in the case $\mu \equiv 0$. On the other hand, if $\mu = I_\Omega$ for some set Ω of finite Lebesgue measure, then $P(\mu) < +\infty$ and the functional $J_{\mu,1}$ has a unique minimizer in $H_0^1(\Omega)$.

We define the *torsion function* w_μ for a generic $\mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ as

$$w_\mu = \sup_{R>0} w_R,$$

where w_R is the unique minimizer of $J_{\mu \vee I_{B_R},1}$, i.e., the solution of

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d}u^2\,d\mu - \int_{\mathbb{R}^d}u\,dx : u \in H_\mu^1 \cap H_0^1(B_R)\right\}.$$

For every capacitary measure μ, we denote by Ω_μ *the set of finiteness of* μ (also known in the literature as *the regular set of* μ), that is,

$$\Omega_\mu = \{w_\mu > 0\}.$$

In the following we denote by $\mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ the subclass of capacitary measures μ whose torsion $P(\mu)$ is finite.

The following result was proved in [9] and [28] and relates the integrability of w_μ to the finiteness of the torsion $P(\mu)$ and to the compact embedding of H_μ^1 into $L^1(\mathbb{R}^d)$.

THEOREM 2.2. *Let $\mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ and let w_μ be its torsion function. Then the following conditions are equivalent:*

(1) *The inclusion $H_\mu^1 \subset L^1(\mathbb{R}^d)$ is continuous, and there is a constant $C > 0$ such that*

$$(2.1) \qquad \|u\|_{L^1} \leq C\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu)}^2\right)^{1/2} \qquad \text{for every } u \in H_\mu^1.$$

(2) *The inclusion $H_\mu^1 \subset L^1$ is compact and (2.1) holds.*
(3) *The torsion function w_μ is in $L^1(\mathbb{R}^d)$.*
(4) *The torsion $P(\mu)$ is finite.*

Moreover, if the above conditions hold, then $w_\mu \in H_\mu^1 \cap L^1(\mathbb{R}^d)$ is the unique minimizer of $J_{\mu,1}$ in H_μ^1 and

$$C^2 \leq \int_{\mathbb{R}^d} w_\mu\,dx = 2P(\mu).$$

Proof. We first prove that (3) and (4) are equivalent.

(3) $\Rightarrow$ (4). Since the functions in $H_\mu^1 \cap L^1$ with compact support are dense in $H_\mu^1 \cap L^1$, we have

$$\inf\left\{J_{\mu,1}(u) : u \in H_\mu^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\right\} = \inf_{R>0}\inf\left\{J_{\mu,1}(u) : u \in H_{\mu \vee I_{B_R}}^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\right\}$$

$$(2.2) \hspace{4cm} = \inf_{R>0} J_{\mu,1}(w_R) = \inf_{R>0}\left\{-\frac{1}{2}\int_{\mathbb{R}^d} w_R\,dx\right\}$$

$$= -\frac{1}{2}\int_{\mathbb{R}^d} w_\mu\,dx > -\infty,$$

where the last equality is due to the fact that w_R is increasing in R and converges to w_μ. Moreover, we have that $w_\mu \in H^1_\mu \cap L^1(\mathbb{R}^d)$ and w_μ minimizes $J_{\mu,1}$. Indeed, since w_R converges to w_μ in $L^1(\mathbb{R}^d)$ and w_R is uniformly bounded in H^1_μ by the inequality

$$\int_{\mathbb{R}^d} |\nabla w_R|^2 \, dx + \int_{\mathbb{R}^d} w_R^2 \, dx \leq 2 \int_{\mathbb{R}^d} w_R \, dx \leq 2 \int_{\mathbb{R}^d} w_\mu \, dx,$$

we have that $w_\mu \in H^1_\mu$ and $J_{\mu,1}(w_\mu) \leq \liminf_{R \to \infty} J_{\mu,1}(w_R)$.

(4) $\Rightarrow$ (3). By (2.2), we have that for every $R > 0$,

$$\int_{\mathbb{R}^d} w_R \, dx \leq -2 \inf \left\{ J_{\mu,1}(u) : u \in H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \right\} < +\infty.$$

Taking the limit as $R \to \infty$, and taking in consideration again (2.2), we obtain

$$\int_{\mathbb{R}^d} w_\mu \, dx = -2 \inf \left\{ J_{\mu,1}(u) : u \in H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \right\} < +\infty.$$

Since the implication (2) $\Rightarrow$ (1) is clear, it is sufficient to prove that (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (4). Let $u_n \in H^1_\mu$ be a minimizing sequence for $J_{\mu,1}$ such that $u_n \geq 0$ and $J_{\mu,1}(u_n) \leq 0$ for every $n \in \mathbb{N}$. Then we have

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} u_n^2 \, d\mu \leq \int_{\mathbb{R}^d} u_n \, dx \leq C \left(\|\nabla u_n\|_{L^2}^2 + \|u_n\|_{L^2(\mu)}^2 \right)^{1/2},$$

and so u_n is bounded in $H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Suppose that u is the weak limit of u_n in H^1_μ. Then

$$\|u\|_{H^1_\mu} \leq \liminf_{n \to \infty} \|u_n\|_{H^1_\mu}, \qquad \int_{\mathbb{R}^d} u \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} u_n \, dx,$$

where the last equality is due to the fact that the functional $\{u \mapsto \int u \, dx\}$ is continuous in H^1_μ. Thus, $u \in H^1_\mu \cap L^1(\mathbb{R}^d)$ is the (unique, due to the strict convexity of $J_{\mu,1}$) minimizer of $J_{\mu,1}$, and so $E(\mu) = \inf J_{\mu,1} > -\infty$.

We now prove (3) $\Rightarrow$ (1). Since $w_\mu \in H^1_\mu \cap L^1(\mathbb{R}^d)$ is the minimizer of $J_{\mu,1}$ in $H^1_\mu \cap L^1(\mathbb{R}^d)$, we have that the following Euler–Lagrange equation holds:

$$(2.3) \qquad \int_{\mathbb{R}^d} \nabla w_\mu \cdot \nabla u \, dx + \int_{\mathbb{R}^d} w_\mu u \, d\mu = \int_{\mathbb{R}^d} u \, dx \qquad \forall u \in H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).$$

Thus, for every $u \in H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we obtain

$$\|u\|_{L^1} \leq \left(\|\nabla w_\mu\|_{L^2}^2 + \|w_\mu\|_{L^2(\mu)}^2 \right)^{1/2} \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu)}^2 \right)^{1/2}$$
$$= \|w_\mu\|_{L^1}^{1/2} \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu)}^2 \right)^{1/2}.$$

Since $H^1_\mu(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is dense in $H^1_\mu(\mathbb{R}^d)$, we obtain (1).

(3) $\Rightarrow$ (2). Following [9, Theorem 3.2], consider a sequence $u_n \in H^1_\mu$ weakly converging to zero in H^1_μ and suppose that $u_n \geq 0$ for every $n \in \mathbb{N}$. Since the injection $H^1(\mathbb{R}^d) \hookrightarrow L^1_{loc}(\mathbb{R}^d)$ is locally compact, we only have to prove that for every $\varepsilon > 0$

there is some $R > 0$ such that $\int_{B_R^c} u_n \, dx \leq \varepsilon$. Consider the function $\eta_R(x) := \eta(x/R)$, where

$$\eta \in C_c^\infty(\mathbb{R}^d), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } B_1, \quad \eta = 0 \text{ on } \mathbb{R}^d \setminus B_2.$$

Testing (2.3) with $(1 - \eta_R)u_n$, we have

$$\int_{\mathbb{R}^d} \left[u_n \nabla w_\mu \cdot \nabla(1 - \eta_R) + (1 - \eta_R)\nabla w_\mu \cdot \nabla u_n \right] dx + \int_{\mathbb{R}^d} w_\mu (1 - \eta_R) u_n \, d\mu$$

$$= \int_{\mathbb{R}^d} (1 - \eta_R) u_n \, dx,$$

and using the identity $\|\nabla \eta_R\|_{L^\infty} = R^{-1} \|\nabla \eta\|_{L^\infty}$ and the Cauchy–Schwartz inequality, we have

$$\int_{B_{2R}^c} u_n \, dx \leq R^{-1} \|u_n\|_{L^2} \|\nabla w_\mu\|_{L^2} + \|\nabla u_n\|_{L^2} \|\nabla w_\mu\|_{L^2(B_R^c)}$$

$$+ \|u_n\|_{L^2(\mu)} \left(\int_{B_R^c} w_\mu^2 \, d\mu \right)^{1/2},$$

which for R large enough gives the desired ε. $\qquad\square$

Remark 2.3. We note that if the conditions of Theorem 2.2 hold, then $H_\mu^1 \subset L^1(\mathbb{R}^d) \cap L^{2^*}(\mathbb{R}^d)$, so that by interpolation $H_\mu^1 \subset L^p(\mathbb{R}^d)$ for every $1 \leq p < 2d/(d-2)$, and

$$\|u\|_{L^p} \leq C \big(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu)}^2 \big)^{1/2} \quad \text{for every } u \in H_\mu^1.$$

Moreover, the embedding is compact.

2.3. Infinity estimates.

LEMMA 2.4. *Let $\mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ and consider a nonnegative function $f \in L^p(\mathbb{R}^d)$, where $p \in (d/2, +\infty]$. Suppose that $u \in H_\mu^1 \cap L^{p'}(\mathbb{R}^d)$ minimizes $J_{\mu,f}$ in $H_\mu^1 \cap L^{p'}(\mathbb{R}^d)$. Then we have for some constant C, depending on d and p,*

$$\|(u - t)^+\|_{L^\infty} \leq C \|f\|_{L^p} |\{u > t\}|^{2/d - 1/p} \qquad \forall t \geq 0.$$

Proof. We first notice that, being $u \in L^{p'}(\mathbb{R}^d)$, we have

$$|\{u > t\}| \leq \frac{1}{t^{p'}} \int_{\mathbb{R}^d} u^{p'} \, dx < +\infty \qquad \forall t > 0.$$

For every $t \in (0, \|u\|_{L^\infty})$ and $\varepsilon > 0$, we consider the test function

$$u_{t,\varepsilon} = u \wedge t + (u - t - \varepsilon)^+.$$

Since $u_{t,\varepsilon} \leq u$ and $J_{\mu,f}(u) \leq J_{\mu,f}(u_{t,\varepsilon})$, we get

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} f u \, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_{t,\varepsilon}|^2 \, dx - \int_{\mathbb{R}^d} f u_{t,\varepsilon} \, dx,$$

and after some calculations

$$\frac{1}{2} \int_{\{t < u \leq t + \varepsilon\}} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^d} f \, (u - u_{t,\varepsilon}) \, dx \leq \varepsilon \int_{\{u > t\}} f \, dx.$$

By the co-area formula, for a.e. $t > 0$ we have

$$\int_{\{u=t\}} |\nabla u| \, d\mathcal{H}^{d-1} \leq 2 \int_{\{u>t\}} f \, dx \leq 2\|f\|_{L^p} |\{u > t\}|^{1/p'}.$$

Setting $\varphi(t) = |\{u > t\}|$, for a.e. $t > 0$ we have that

$$
\begin{aligned}
\varphi'(t) &= - \int_{\{u=t\}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{d-1} \\
&\leq - \left(\int_{\{u=t\}} |\nabla u| \, d\mathcal{H}^{d-1} \right)^{-1} P(\{u > t\})^2 \\
&\leq -\|f\|_{L^p}^{-1} \varphi(t)^{-1+1/p} C_d \varphi(t)^{2(d-1)/d} = -\|f\|_{L^p}^{-1} C_d \varphi(t)^{(d-2)/d+1/p},
\end{aligned}
$$

where C_d is the constant from the isoperimetric inequality in $\mathbb{R}^d$. Setting $\alpha = \frac{d-2}{d} + \frac{1}{p}$ and $A = C_d/\|f\|_{L^p}$, we have that $\alpha < 1$ and the solution of the ODE

$$y' = -Ay^\alpha, \qquad y(t_0) = y_0,$$

where $t_0 > 0$, is given by

$$y(t) = \left(y_0^{1-\alpha} - (1-\alpha)A(t-t_0) \right)^{1/(1-\alpha)}.$$

Note that $\phi(t) \geq 0$ for every $t \geq 0$, and $y(t) \geq \phi(t)$ if $\phi(t) > 0$. Thus, we have that there is some $t_{\max}$ such that $\phi(t) = 0$ for every $t \geq t_{\max}$. Taking $y_0 = \phi(t_0) = |\{u > t_0\}|$, we have the estimate

$$\|(u - t_0)^+\|_{L^\infty} \leq t_{\max} - t_0 \leq \frac{pd}{(2p-d)C_d} \|f\|_{L^p} |\{u > t_0\}|^{2/d-1/p},$$

as required. $\quad \square$

PROPOSITION 2.5. *Let $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$, $d \geq 2$, $p \in (d/2, +\infty]$, and $f \in L^p(\mathbb{R}^d)$. Then there is a unique minimizer $u \in H_\mu^1$ of the functional $J_{\mu,f} : H_\mu^1 \to \mathbb{R}$. Moreover, u satisfies the inequality*

$$(2.4) \qquad \|u\|_{L^\infty} \leq C P(\mu)^\alpha \|f\|_{L^p}$$

for some constants C and α, depending on the dimension d and the exponent p.

Proof. We first note that for any $v \in H_\mu^1$ such that $J_{\mu,f}(v) \leq 0$, we have

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \int_{\mathbb{R}^d} v^2 \, d\mu \leq 2 \int_{\mathbb{R}^d} fv \, dx \leq 2\|f\|_{L^p} \|v\|_{L^{p'}}.$$

On the other hand, $p > d/2$ implies $p' < \frac{d}{d-2}$, and so $p' \in [1, \frac{2d}{d-2})$. Thus, using (2.1) with $C = P(\mu)^{1/2}$ and an interpolation as in Remark 2.3, we obtain

$$(2.5) \qquad \int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \int_{\mathbb{R}^d} v^2 \, d\mu \leq C_d P(\mu)^\alpha \|f\|_{L^p}^2,$$

which in turn implies the existence of a minimizer u of $J_{\mu,f}$, satisfying the same estimate.

In order to prove (2.4), it is sufficient to consider the case $f \geq 0$. In this case the solution is nonnegative $u \geq 0$ (since the minimizer is unique and $J_{\mu,f}(|u|) \leq J_{\mu,f}(u)$)

and, by Lemma 2.4, we have that $u \in L^\infty$. We set $M := \|u\|_{L^\infty} < +\infty$ and again apply Lemma 2.4 to obtain

$$\frac{M^2}{2} = \int_0^M (M - t)\, dt \le C\|f\|_{L^p} \int_0^M |\{u > t\}|^\beta\, dt \le C\|f\|_{L^p} M^{1-\beta}\|u\|_{L^1}^\beta,$$

where we set $\beta = 2/d - 1/p \le 1$. Thus we obtain

$$M^{1+\beta} \le C\|f\|_{L^p}\|u\|_{L^1}^\beta,$$

and using (2.5) with $v = u$, we get (2.4).   □

PROPOSITION 2.6. *Suppose that μ is a capacitary measure such that $w_\mu \in L^1(\mathbb{R}^d)$. Then $w_\mu \in L^\infty(\mathbb{R}^d)$ and vanishes at infinity:*

$$\|w_\mu\|_{L^\infty} \le C_d\|w_\mu\|_{L^1}^{\frac{2}{d+2}} \qquad and \qquad \lim_{R \to \infty} \|w_\mu \mathbb{1}_{B_R^c}\|_{L^\infty} = 0,$$

where C_d is a dimensional constant.

Proof. We set $w := w_\mu$. Taking $f \equiv 1$ and $p = +\infty$ in Lemma 2.4, we obtain

$$\|(w - t)_+\|_{L^\infty} \le C_d|\{w > t\}|^{2/d}.$$

Thus, $w \in L^\infty(\mathbb{R}^d)$ and setting $M = \|w\|_{L^\infty}$, we have

$$(M - t)^{d/2} \le C_d^{d/2}|\{w > t\}|,$$

and integrating for $t \in (0, M]$, we get

$$\frac{2}{2+d} M^{\frac{d+2}{2}} = \int_0^M (M - t)^{d/2}\, dt \le C_d^{d/2} \int_0^M |\{w > t\}|\, dt = C_d^{d/2}\|w_\mu\|_{L^1},$$

which gives the first part of the claim. The second part was proved in [9].   □

2.4. Schrödinger operators for capacitary measures. Suppose that $f \in L^p(\mathbb{R}^d)$ and $p \in [2, +\infty]$. Arguing as in the proof of the first part of Proposition 2.5, we can show that for every capacitary measure $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ there is a unique minimizer $w_{\mu,f} \in H_\mu^1$ of the functional $J_{\mu,f}$. Moreover, $w_{\mu,f}$ satisfies

$$(2.6) \qquad \int_{\mathbb{R}^d} \nabla w_{\mu,f} \cdot \nabla v\, dx + \int_{\mathbb{R}^d} w_{\mu,f} v\, d\mu = \int_{\mathbb{R}^d} fv\, dx \qquad \text{for every } v \in H_\mu^1,$$

where the last integral is well defined thanks to Remark 2.3. By definition, we say that $w_{\mu,f}$ solves the equation

$$-\Delta w_{\mu,f} + \mu w_{\mu,f} = f, \qquad w_{\mu,f} \in H_\mu^1.$$

Using $v = w_{\mu,f}$ as a test function in (2.6), we get

$$\int_{\mathbb{R}^d} |\nabla w_{\mu,f}|^2\, dx + \int_{\mathbb{R}^d} w_{\mu,f}^2\, d\mu = \int_{\mathbb{R}^d} w_{\mu,f} f\, dx \le \|f\|_{L^p}\|w_{\mu,f}\|_{L^{p'}},$$

which in turn gives that there is a constant C (depending on the dimension d, on the exponent p, and on the torsion $P(\mu)$) such that

$$\|w_{\mu,f}\|_{L^{p'}} \le C\|f\|_{L^p}.$$

We call resolvent of μ the linear continuous operator

$$R_\mu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad R_\mu(f) = w_{\mu,f}.$$

Since $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$, by Theorem 2.2 the operator R_μ is compact and so it has a discrete spectrum $0 \le \cdots \le \Lambda_k(\mu) \le \cdots \le \Lambda_2(\mu) \le \Lambda_1(\mu)$. Thus the spectrum of the unbounded Schrödinger operator $(-\Delta + \mu)$, associated to the bilinear form $Q_\mu(u) := \|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu)}^2$, is given by

$$0 < \lambda_1(\mu) \le \lambda_2(\mu) \le \cdots \le \lambda_k(\mu) \le \cdots,$$

where $\lambda_k(\mu) = \Lambda_k(\mu)^{-1}$. Moreover, we have the variational characterization

$$\lambda_k(\mu) = \min_{S_k \subset H_\mu^1} \max_{u \in S_k} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 \, d\mu}{\int_{\mathbb{R}^d} u^2 \, dx},$$

where the minimum is over all k-dimensional subspaces S_k of H_μ^1.

There is a sequence of eigenfunctions $u_k \in H_\mu^1$, orthonormal in $L^2(\mathbb{R}^d)$ and satisfying

$$-\Delta u_k + u_k \mu = \lambda_k(\mu) u_k, \qquad u_k \in H_\mu^1.$$

Moreover, $u_k \in L^\infty(\mathbb{R}^d)$, and we have the estimate (see [18] and [28])

$$(2.7) \qquad \|u_k\|_{L^\infty} \le e^{1/(8\pi)} \lambda_k(\mu)^{d/4}.$$

Remark 2.7 (scaling). Let $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ be a capacitary measure of finite torsion and let $u_k \in H_\mu^1$ be the kth eigenfunction of $(-\Delta + \mu)$. Then we have

$$-\Delta u_k + \mu u_k = \lambda_k(\mu) u_k,$$

and rescaling the eigenfunction u_k with $t > 0$, we have

$$-\Delta\big(u_k(x/t)\big) + \mu_t u_k(x/t) = t^{-2} \lambda_k(\mu) u_k(x/t), \qquad u_k(\cdot/t) \in H_{\mu_t}^1,$$

where the measure μ_t is defined as $\mu_t := t^{d-2}\mu(\cdot/t)$, i.e., for every $\phi \in L^1(\mu)$, we have

$$\int_{\mathbb{R}^d} \phi(x/t) \, d\mu_t(x) := t^{d-2} \int_{\mathbb{R}^d} \phi \, d\mu.$$

Repeating the same argument for every eigenfunction, we have that

$$\lambda_k(\mu_t) = t^{-2} \lambda_k(\mu).$$

Analogously, for the energy function w_μ, we obtain

$$-\Delta\big(w_\mu(x/t)\big) + t^{d-2}\mu(x/t)w_\mu(x/t) = t^{-2}, \qquad w_\mu(\cdot/t) \in H_{\mu_t}^1,$$

and, in particular, we have

$$w_{\mu_t}(x) = t^2 w_\mu(x/t) \qquad \text{and} \qquad E(\mu_t) = t^{d+2} E(\mu).$$

Remark 2.8 (scaling of potentials). We note that if $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ is of the form $\mu = V(x)dx$, then $\mu_t = V_t(x)dx$, where $V_t(x) := t^{-2}V(x/t)$.

2.5. The γ-distance on the space of capacitary measures. We define the γ-distance between $\mu, \nu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ as

$$d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^1},$$

where w_μ and w_ν are the torsion functions of μ and ν, which are integrable by Theorem 2.2. In particular, we say that the sequence of capacitary measures $\mu_n \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ γ-converges to $\mu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ if the sequence of energy functions w_{μ_n} converges in $L^1(\mathbb{R}^d)$ to the energy function w_μ. It was first proved in [16] and [17] (see also [15] for a different approach) that if Ω is a bounded open set, then the space of capacitary measures in Ω

$$\left\{\mu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d) : I_\Omega \prec \mu\right\}$$

is compact and, in particular, complete with respect to the γ-distance. Using this result, it was proved in [6] that the space $\mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ endowed with the distance d_γ is complete. (We also refer to [28] for a more direct approach.)

Remark 2.9. The γ-convergence implies the norm convergence of the resolvents R_μ and the Γ-convergence in $L^2(\mathbb{R}^d)$ of the norms $\|\cdot\|_{H_\mu^1}$. More precisely, we have the following:

(i) If the sequence $\mu_n \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ γ-converges to $\mu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$, then the sequence of resolvents R_{μ_n} converges in norm to R_μ, i.e.,

$$\lim_{n \to \infty} \|R_{\mu_n} - R_\mu\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = 0.$$

(ii) If the sequence $\mu_n \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ γ-converges to $\mu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$, then the sequence of functionals $\|\cdot\|_{H_{\mu_n}^1}^2 : L^2(\mathbb{R}^d) \to [0, +\infty]$ defined by

$$\|u\|_{H_{\mu_n}^1}^2 = \begin{cases} \|\nabla u\|_{L^2}^2 + \|u\|_{L^2(\mu_n)}^2 + \|u\|_{L^2}^2 & \text{if } u \in H_{\mu_n}^1, \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converges to $\|\cdot\|_{H_\mu^1}^2 : L^2(\mathbb{R}^d) \to [0, +\infty]$, i.e., the following conditions are satisfied:

(Γ1) for every sequence $u_n \in L^2(\mathbb{R}^d)$, converging in $L^2(\mathbb{R}^d)$ to $u \in L^2(\mathbb{R}^d)$, we have

$$\|u\|_{H_\mu^1} \le \liminf_{n \to \infty} \|u_n\|_{H_{\mu_n}^1};$$

(Γ2) for every $u \in H_\mu^1$, there is a sequence $u_n \in H_{\mu_n}^1$, converging to u strongly in $L^2(\mathbb{R}^d)$ and such that

$$\|u\|_{H_\mu^1} = \lim_{n \to \infty} \|u_n\|_{H_{\mu_n}^1}.$$

For a proof of these two facts we refer to [28].

Remark 2.10. We note that the γ-convergence is not equivalent to the norm convergence of the resolvent operators $R_\mu \in \mathcal{L}(L^2(\mathbb{R}^d))$. In fact, one can construct a sequence of capacitary measures $\mu_n \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ such that

(2.8) $$\|w_{\mu_n}\|_{L^1} = 1 \quad \text{and} \quad \|R_{\mu_n}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \to 0.$$

For example, let $\mu_n = I_{\Omega_n}$, where Ω_n is a disjoint union of n balls $B_{n,k} := B_{r_n}(x_k^n)$, $k = 1, \ldots, n$, of the same radius equal to $r_n > 0$. Since $w_{\mu_n} = \sum_{k=1}^n w_{I_{B_{n,k}}}$, we can choose r_n such that $\|w_{\mu_n}\|_{L^1} = 1$. Since $r_n \to 0$, as $n \to \infty$, we have that

$$\|R_{\mu_n}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \lambda_1^{-1}(\mu_n) = C_d r_n^2 \to 0,$$

which completes the construction of the sequence satisfying (2.8).

2.6. $(\Delta - \mu)$-harmonic functions. In order to prove the boundedness of the local subsolutions for functionals of the form $E_f - E_1$, we will need the notion of $(\Delta - \mu)$-harmonic function.

DEFINITION 2.11. *Let $\mu \in \mathcal{M}^P_{\mathrm{cap}}(\mathbb{R}^d)$ be a capacitary measure with finite torsion and let $B_R \subset \mathbb{R}^d$ be a given ball. For every $u \in H^1_\mu$, we will denote with h_u the solution of the problem*

$$(2.9) \qquad \min\left\{\int_{B_r} |\nabla v|^2\, dx + \int_{B_R} v^2\, d\mu : v \in H^1_\mu,\ u - v \in H^1_0(B_R)\right\}.$$

We will refer to h_u as the $(\Delta - \mu)$-harmonic function on B_R with boundary data u on ∂B_R.

The following remark summarizes the main properties of the harmonic functions, which we will use in what follows.

Remark 2.12. Properties of the $(\Delta - \mu)$-harmonic functions:

- *(Uniqueness)* By the strict convexity of the functional in (2.9), we have that the problem (2.9) has a unique minimizer, i.e., h_u is uniquely determined.
- *(First variation)* Calculating the first variation of the functional from (2.9), we have

$$(2.10) \qquad \int_{\mathbb{R}^d} \nabla h_u \cdot \nabla \psi\, dx + \int_{\mathbb{R}^d} h_u \psi\, d\mu = 0 \qquad \forall \psi \in H^1_\mu \cap H^1_0(B_R),$$

  and conversely, if the function $h_u \in H^1_\mu$ satisfies (2.10), then it minimizes (2.9).
- *(Comparison principle)* If $u, w \in H^1_\mu$ are two functions such that $w \geq u$ on ∂B_R, then $h_u \leq h_w$. Indeed, using $h_u \vee h_w \in H^1_\mu$ and $h_w \wedge h_u \in H^1_\mu$ to test the minimality of h_w and h_u, respectively, we get

$$\int_{\{h_u > h_w\}} |\nabla h_u|^2\, dx + \int_{\{h_u > h_w\}} h_u^2\, d\mu = \int_{\{h_u > h_w\}} |\nabla h_w|^2\, dx + \int_{\{h_u > h_w\}} h_w^2\, d\mu,$$

  which implies that $h_w \wedge h_u$ is also a minimizer of (2.9) and so $h_w \wedge h_u = h_u$.

3. Concentration-compactness principle for capacitary measures. In this section we introduce our main tools for studying the behavior of minimizing sequences of functionals involving capacitary measures. Our main result is a concentration-compactness principle for capacitary measures, analogous to the concentration-compactness theorem proved by Bucur in [6], which was the key argument in the proof of existence of optimal domains for λ_k under measure constraint. Before we state the main theorem, we need some preliminary results. We start by recalling a classical result due to Lions (see [26]).

THEOREM 3.1. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative functions, uniformly bounded in $L^1(\mathbb{R}^d)$. Then, up to a subsequence, one of the following properties holds:*

(i) *Concentration. There exists a sequence $(x_n)_{n \geq 1} \subset \mathbb{R}^d$ with the property that for all $\epsilon > 0$ there is some $R > 0$ such that*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d \setminus B_R(x_n)} f_n\, dx \leq \epsilon.$$

(ii) *Vanishing. For each $R > 0$*

$$\lim_{n \to \infty} \left(\sup_{x \in \mathbb{R}^d} \int_{B_R(x)} f_n\, dx\right) = 0.$$

(iii) *Dichotomy.* For every $\alpha > 1$, there is a sequence $x_n \in \mathbb{R}^d$ and an increasing sequence $R_n \to +\infty$ such that

$$\lim_{n \to \infty} \int_{B_{\alpha R_n}(x_n) \setminus B_{R_n}(x_n)} f_n \, dx = 0,$$

$$\liminf_{n \to \infty} \int_{B_{R_n}(x_n)} f_n \, dx > 0 \qquad and \qquad \liminf_{n \to \infty} \int_{\mathbb{R}^d \setminus B_{\alpha R_n}(x_n)} f_n \, dx > 0.$$

Remark 3.2. Since the inclusion $H^1(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$ is compact, we have that if a sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ and has the concentration property, then there is a subsequence converging strongly in L^1.

LEMMA 3.3. *Let $\mu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$. Then, for every $1 < R_1 < R_2 \le +\infty$, we have*

$$(3.1) \qquad d_\gamma\big(\mu, \mu \vee I_{B_{R_1} \cup B^c_{R_2}}\big) \le \int_{B_{2R_2} \setminus B_{R_1/2}} w_\mu \, dx + C(R_1^{-2} + R_2^{-2}),$$

where the constant C depends only on the torsion $P(\mu)$ and the dimension d.

Proof. We first consider the case $R_2 = +\infty$. We set for simplicity $R = R_1$, $w_R = w_{\mu \vee I_{B_R}}$, and $\eta_R(x) = \eta(x/R)$, where

$$\eta \in C^\infty_c(\mathbb{R}^d), \quad 0 \le \eta \le 1, \quad \eta = 1 \text{ on } B_1, \quad \eta = 0 \text{ on } \mathbb{R}^d \setminus B_2.$$

Then we have

$$\begin{aligned}
d_\gamma(\mu, \mu \vee I_{B_{2R}}) &= \int_{\mathbb{R}^d} (w_\mu - w_{2R}) \, dx \\
&= \int_{\mathbb{R}^d} w_\mu \, dx + 2J_\mu(w_{2R}) \le \int_{\mathbb{R}^d} w_\mu \, dx + 2J_\mu(\eta_R w_\mu) \\
&= \int_{\mathbb{R}^d} |\nabla(\eta_R w_\mu)|^2 \, dx + \int_{\mathbb{R}^d} \eta_R^2 w_\mu^2 \, d\mu - 2\int_{\mathbb{R}^d} \eta_R w_\mu \, dx + \int_{\mathbb{R}^d} w_\mu \, dx \\
&= \int_{\mathbb{R}^d} \Big(w_\mu^2 |\nabla \eta_R|^2 + \nabla w_\mu \cdot \nabla(\eta_R^2 w_\mu)\Big) \, dx + \int_{\mathbb{R}^d} \eta_R^2 w_\mu^2 \, d\mu \\
&\quad - 2\int_{\mathbb{R}^d} \eta_R w_\mu \, dx + \int_{\mathbb{R}^d} w_\mu \, dx \\
&= \int_{\mathbb{R}^d} w_\mu^2 |\nabla \eta_R|^2 \, dx + \int_{\mathbb{R}^d} \eta_R^2 w_\mu \, dx - 2\int_{\mathbb{R}^d} \eta_R w_\mu \, dx + \int_{\mathbb{R}^d} w_\mu \, dx \\
&= \int_{\mathbb{R}^d} w_\mu^2 |\nabla \eta_R|^2 \, dx + \int_{\mathbb{R}^d} (1 - \eta_R)^2 w_\mu \, dx \\
&\le \frac{\|\nabla \eta\|^2_{L^\infty}}{R^2} \|w_\mu\|_{L^2} + \int_{\mathbb{R}^d \setminus B_R} w_\mu \, dx,
\end{aligned}$$

which concludes the proof. The case $R_2 < +\infty$ is analogous. $\square$

LEMMA 3.4. *Let $\mu, \mu' \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$ be such that $\mu \prec \mu'$. Then, we have*

$$\|R_\mu - R_{\mu'}\|_{\mathcal{L}(L^2)} \le C \left[d_\gamma(\mu, \mu')\right]^{(d-1)/d^2},$$

where $C_{d,\mu}$ is a constant depending only on the dimension d and the torsion $P(\mu)$.

Proof. The proof follows the same argument as in [6, Lemma 3.6] and we report it here for the sake of completeness. Let $p \geq d \geq 2$ and $f \in L^p$, $f \geq 0$. Then by the choice of p, we have that $R_\mu(f) \in L^\infty(\mathbb{R}^d)$ and

$$(3.2) \quad \int_{\mathbb{R}^d} |R_\mu(f) - R_{\mu'}(f)|^p \, dx \leq \|R_\mu(f) - R_{\mu'}(f)\|_{L^\infty}^{p-1} \int_{\mathbb{R}^d} R_\mu(f) - R_{\mu'}(f) \, dx$$

$$\leq C^{p-1} \|f\|_{L^p}^{p-1} \int_{\mathbb{R}^d} f(w_\mu - w_{\mu'}) \, dx$$

$$\leq C^{p-1} \|f\|_{L^p}^p \|w_\mu - w_{\mu'}\|_{L^{p'}},$$

and so, $R_\mu - R_{\mu'}$ is a linear operator from L^p to L^p such that

$$\|R_\mu - R_{\mu'}\|_{\mathcal{L}(L^p; L^p)} \leq C^{1-1/p} \|w_\mu - w_{\mu'}\|_{L^{p'}}^{1/p},$$

where, by Proposition 2.5, the constant C depends on the dimension d and the torsion $T(\mu) = \|w_\mu\|_{L^1}$. Since $R_\mu - R_{\mu'}$ is a self-adjoint operator in L^2, we can extend it to an operator on $L^{p'}$. Indeed, let $f \in L^2 \cap L^{p'}$, where $p' = p/(p-1)$. Since $L^{p'}$ is the dual of L^p and $L^2 \cap L^p$ and and $L^2 \cap L^{p'}$ are dense, respectively, in L^p and $L^{p'}$, we have

$$\|R_\mu(f) - R_{\mu'}(f)\|_{L^{p'}} = \sup \left\{ \int_{\mathbb{R}^d} \left(R_\mu(f) - R_{\mu'}(f) \right) g \, dx : g \in L^2 \cap L^p, \ \|g\|_{L^p} = 1 \right\}.$$

On the other hand, by the self-adjointness of $R_\mu - R_{\mu'}$ in L^2, for f and g as above, we have

$$\int_{\mathbb{R}^d} \left(R_\mu(f) - R_{\mu'}(f) \right) g \, dx = \int_{\mathbb{R}^d} \left(R_\mu(g) - R_{\mu'}(g) \right) f \, dx$$

$$\leq \|R_\mu(g) - R_{\mu'}(g)\|_{L^p} \|f\|_{L^{p'}}$$

$$\leq C^{1-1/p} \|w_\mu - w_{\mu'}\|_{L^{p'}}^{1/p} \|g\|_{L^p} \|f\|_{L^{p'}},$$

which gives that for every $f \in L^2 \cap L^{p'}$

$$\|R_\mu(f) - R_{\mu'}(f)\|_{L^{p'}} \leq C^{1-1/p} \|w_\mu - w_{\mu'}\|_{L^{p'}}^{1/p} \|f\|_{L^{p'}},$$

and so $R_\mu - R_{\mu'}$ can be extended to a linear operator on $L^{p'}$ such that

$$\|R_\mu - R_{\mu'}\|_{\mathcal{L}(L^{p'}; L^{p'})} \leq C^{1-1/p} \|w_\mu - w_{\mu'}\|_{L^{p'}}^{1/p}.$$

By the classical Riesz–Thorin interpolation theorem we get

$$|R_\mu - R_{\mu'}\|_{\mathcal{L}(L^2)} \leq C^{1-1/p} \|w_\mu - w_{\mu'}\|_{L^{p'}}^{1/p}$$

$$\leq C^{1-1/p} \|w_\mu\|_{L^\infty}^{1/p^2} \|w_\mu - w_{\mu'}\|_{L^1}^{(p-1)/p^2}.$$

Now using the L^∞ estimate on w_μ, and taking $p = d$, we have the claim. $\square$

We are now in position to state the concentration-compactness principle for capacitary measures. In the theorem below we will use the notion of infimum of two capacitary measures μ and ν with disjoint sets of finiteness, i.e., $\operatorname{cap}(\Omega_\mu \cap \Omega_\nu) = 0$, namely,

$$\mu \wedge \nu(E) = \begin{cases} \mu(\Omega_\mu \cap E) + \nu(\Omega_\nu \cap E) & \text{if } \operatorname{cap}\left(E \setminus (\Omega_\mu \cup \Omega_\nu) \right) = 0, \\ +\infty & \text{if } \operatorname{cap}\left(E \setminus (\Omega_\mu \cup \Omega_\nu) \right) > 0. \end{cases}$$

THEOREM 3.5. *Let $\mu_n \in \mathcal{M}_{cap}^P(\mathbb{R}^d)$ be a sequence of capacitary measures of uniformly bounded torsion $P(\mu_n)$. Then, up to a subsequence, one of the following situations occurs:*

(i) *(Compactness) There is a sequence $x_n \in \mathbb{R}^d$ such that $\mu_n(x_n + \cdot)$ γ-converges.*
(ii) *(Vanishing) The sequence of resolvents R_{μ_n} converges to zero in the operator norm of $\mathcal{L}(L^2(\mathbb{R}^d))$. Moreover, we have $\|w_{\mu_n}\|_{L^\infty} \to 0$ and $\lambda_1(\mu_n) \to +\infty$, as $n \to \infty$.*
(iii) *(Dichotomy) There are capacitary measures μ_n^1 and μ_n^2 such that*
   - *$dist(\Omega_{\mu_n^1}, \Omega_{\mu_n^2}) \to +\infty$, as $n \to \infty$, where $\Omega_{\mu_n^1}$ and $\Omega_{\mu_n^2}$ are the sets of finiteness defined in section 2.2[1];*
   - *$\mu_n \le \mu_n^1 \wedge \mu_n^2$ for every $n \in \mathbb{N}$;*
   - *$d_\gamma(\mu_n, \mu_n^1 \wedge \mu_n^2) \to 0$, as $n \to \infty$;*
   - *$\|R_{\mu_n} - R_{\mu_n^1 \wedge \mu_n^2}\|_{\mathcal{L}(L^2)} \to 0$, as $n \to \infty$;*
   - *$\liminf\limits_{n \to +\infty} P(\mu_n^1) > 0$ and $\liminf\limits_{n \to +\infty} P(\mu_n^2) > 0$.*

Proof. Consider the sequence of corresponding energy functions $w_n := w_{\mu_n}$. Since

$$\|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2(\mu_n)}^2 = \|w_n\|_{L^1} = 2P(\mu_n),$$

we have that w_n is bounded in $H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. We now apply the concentration compactness principle (Theorem 3.1) to the sequence w_n.

If the concentration (Theorem 3.1(i)) occurs, then by the compactness of the embedding $H^1(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$, up to a subsequence $w_n(\cdot + x_n)$ is compact in $L^1(\mathbb{R}^d)$ for some sequence $x_n \in \mathbb{R}^d$, and so we have (i).

Suppose now that the vanishing (Theorem 3.1(ii)) holds. We prove that (ii) holds. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ and let $\varepsilon > 0$. We choose $R > \varepsilon^{-d^2/2(d-1)}$ large enough and $N \in \mathbb{N}$ such that for every $n \ge N$, we have

$$\int_{B_R} w_n \, dx \le \varepsilon^{d^2/(d-1)}.$$

By Lemmas 3.3 and 3.4, we have

$$\|R_{\mu_n}(\varphi) - R_{\mu_n \vee I_{B_R}}(\varphi)\|_{L^2} \le C\varepsilon\|\varphi\|_{L^2}$$

for some constant C, and by the vanishing property,

$$\|R_{\mu_n \vee I_{B_R}}(\varphi)\|_{L^2} \le C\varepsilon\|\varphi\|_{L^2}.$$

Thus,

$$\|R_{\mu_n}(\varphi)\|_{L^2} \le \|R_{\mu_n}(\varphi) - R_{\mu_n \vee I_{B_R}}(\varphi)\|_{L^2} + \|R_{\mu_n \vee I_{B_R}}(\varphi)\|_{L^2} \le C\varepsilon\|\varphi\|_{L^2},$$

and we obtain the strong convergence in (ii).

We now prove that $\|w_n\|_{L^\infty} \to 0$. Suppose by contradiction that there is $\delta > 0$ and a sequence $x_n \in \mathbb{R}^d$ such that $w_n(x_n) > \delta$. Since $\Delta w_n + 1 \ge 0$ on $\mathbb{R}^d$ (see Proposition 3.4 of [12]), we have that the function

$$x \mapsto w_n(x) - \frac{r^2 - |x - x_n|^2}{2d}$$

---
[1] We use the notation $dist(A, B)$ to denote the quantity $\inf\{|x - y| : x \in A, \ y \in B\}$.

is subharmonic. Thus, choosing $r = \sqrt{d\delta}$, we have

$$\int_{B_r(x_n)} w_n \, dx \geq w_n(x_n) - \frac{r^2}{2d} \geq \delta/2,$$

which contradicts Theorem 3.1(ii).

Let $u_n \in H^1_{\mu_n}$ be the first eigenfunction for the operator $-\Delta + \mu_n$, normalized in $L^2(\mathbb{R}^d)$. By (2.7), we have

$$-\Delta u_n + \mu_n u_n = \lambda_1(\mu_n) u_n \leq \lambda_1(\mu_n) \|u_n\|_{L^\infty} \leq e^{1/(8\pi)} \lambda_1(\mu_n)^{(d+4)/4}.$$

Suppose that the sequence $\lambda_1(\mu_n)$ is bounded. Then by the weak maximum principle (see Proposition 3.4 of [12]), we have $u_n \leq C w_n$ for some constant C. Thus, we have

$$1 = \int_{\mathbb{R}^d} u_n^2 \, dx \leq C^2 \int_{\mathbb{R}^d} w_n^2 \, dx \leq C^2 \|w_n\|_{L^\infty} \|w_n\|_{L^1} \to 0,$$

which is a contradiction.

Suppose that the *dichotomy* (Theorem 3.1(iii)) occurs. Choose $\alpha = 8$ and let $x_n \in \mathbb{R}^d$ and $R_n \to \infty$ be as in Theorem 3.1(iii). Setting

$$\mu_n^1 = \mu_n \vee I_{B_{2R_n}(x_n)} \qquad \text{and} \qquad \mu_n^2 = \mu_n \vee I_{B_{4R_n}(x_n)^c},$$

we have that μ_n^1 and μ_n^2 have disjoint sets of finiteness and it is immediate to check that

$$\mu_n^1 \wedge \mu_n^2 = \mu_n \vee I_{B_{2R_n}(x_n) \cup B^c_{4R_n}(x_n)}.$$

Since $R_n \to +\infty$, by the estimate (3.1) from Lemma 3.3, we obtain

$$\lim_{n \to \infty} d_\gamma(\mu_n, \mu_n^1 \wedge \mu_n^2) = 0.$$

By Lemma 3.4, we have

$$\liminf_{n \to \infty} \|R_{\mu_n} - R_{\mu_n^1 \wedge \mu_n^2}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \lim_{n \to \infty} d_\gamma(\mu_n, \mu_n^1 \wedge \mu_n^2)^{(d-1)/d^2} = 0,$$

where C is a constant depending on the dimension and on $\sup_n P(\mu_n)$.

For last claim of (iii), we note that by Theorem 3.1(iii)

$$\liminf_{n \to \infty} \int_{B_{2R_n}(x_n)} w_n \, dx > 0 \qquad \text{and} \qquad \liminf_{n \to \infty} \int_{B^c_{4R_n}(x_n)} w_n \, dx,$$

and, on the other hand, by Lemma 3.3 we have

$$0 \leq \int_{B_{2R_n}(x_n)} w_n \, dx - \int_{\mathbb{R}^d} w_{\mu_n^1} \, dx \leq \int_{B_{8R_n} \setminus B_{R_n}} w_n \, dx + C R_n^{-2},$$

$$0 \leq \int_{B^c_{4R_n}(x_n)} w_n \, dx - \int_{\mathbb{R}^d} w_{\mu_n^2} \, dx \leq \int_{B_{8R_n} \setminus B_{R_n}} w_n \, dx + C R_n^{-2}$$

for some constant $C > 0$, which gives the claim since the right-hand side of both inequalities converges to zero as $n \to +\infty$. $\quad\square$

4. Subsolutions of measure functionals. Consider a functional

$$\mathcal{F} : \mathcal{M}^P_{\text{cap}}(\mathbb{R}^d) \to \mathbb{R}.$$

We say that the capacitary measure μ is a subsolution for $\mathcal{F}$ if we have

$$\mathcal{F}(\mu) \leq \mathcal{F}(\nu) \qquad \text{for every } \nu \in \mathcal{M}^P_{\text{cap}}(\mathbb{R}^d) \text{ such that } \mu \prec \nu.$$

We say that the capacitary measure μ is a local subsolution for $\mathcal{F}$ if

$$\exists \varepsilon > 0 : \mathcal{F}(\mu) \leq \mathcal{F}(\nu) \qquad \text{for every } \nu \in \mathcal{M}^P_{\text{cap}}(\mathbb{R}^d) \text{ such that } \mu \prec \nu \text{ and } d_\gamma(\mu, \nu) < \varepsilon.$$

In the rest of this section we study the torsion function w_μ of a subsolution μ for a special class of functionals $\mathcal{F}$. Precisely, we consider spectral functionals with energy and mass penalization

$$\mathcal{F}(\mu) = \lambda_k(\mu) + P(\mu) \qquad \text{and} \qquad \mathcal{F}(\mu) = \lambda_k(\mu) + \int_{\mathbb{R}^d} \mu_{ac}^{-p}(x) \, dx,$$

where $p \in (0, 1)$ and μ_{ac} denotes the absolute continuous part of μ with respect to the Lebesgue measure. Our main qualitative results, Theorems 4.5 and 4.9, state that if μ is a subsolution, then it is constantly equal to infinity, outside a compact set. In other words, the sets of finiteness $\Omega_\mu := \{w_\mu > 0\}$ is bounded.

4.1. From spectral to energy functionals. In general, the spectral functionals are difficult to treat when the aim is to study the qualitative properties of the optimal measures. This difficulty is due to the fact that the eigenvalues $\lambda_k(\mu)$ are defined through a min-max principle, which makes any perturbation argument quite involved. In [7] a technique was introduced that allows us to concentrate only on energy functionals, at least when we aim to study the boundedness of the set of finiteness.

The following result is just a slight improvement of [7, Lemma 3], but it is one of the crucial steps in the proof of existence of optimal measures for spectral-torsion functionals of the form $\mathcal{F}(\mu) = \lambda_k(\mu) + P(\mu)$.

LEMMA 4.1. *Let μ be a capacitary measure such that $w_\mu \in L^1(\mathbb{R}^d)$. Then for every capacitary measure $\mu \prec \nu$ and every $k \in \mathbb{N}$, denoting by $\Lambda_j(\mu) := \frac{1}{\lambda_j(\mu)}$ the eigenvalues of the resolvent operator R_μ, we have for every $1 \leq j \leq k$*

$$\Lambda_j(\mu) - \Lambda_j(\nu) \leq k^2 e^{1/(4\pi)} \lambda_k(\mu)^{(d+4)/2} \int_{\mathbb{R}^d} \left(R_\mu(w_\mu) w_\mu - R_\nu(w_\mu) w_\mu \right) dx.$$

Proof. Consider the orthonormal in $L^2(\mathbb{R}^d)$ family of eigenfunctions $u_1, \ldots, u_k \in H^1_\mu$ corresponding to the compact self-adjoint operator $R_\mu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Let P_k be the projection

$$P_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad P_k(u) = \sum_{j=1}^{k} \left(\int_{\mathbb{R}^d} u u_j \, dx \right) u_j.$$

Consider the linear space $V = Im(P_k)$, generated by $u_1, \ldots, u_k$, and the operators T_μ and T_ν on V, defined by

$$T_\mu = P_k \circ R_\mu \circ P_k \qquad \text{and} \qquad T_\nu = P_k \circ R_\nu \circ P_k.$$

It is immediate to check that $u_1, \ldots, u_k$ and $\Lambda_1(\mu), \ldots, \Lambda_1(\mu)$ are the eigenvectors and the corresponding eigenvalues of T_μ. On the other hand, for the eigenvalues $\Lambda_1(T_\nu), \ldots, \Lambda_k(T_\nu)$ of T_ν, we have the inequality

$$\Lambda_j(T_\nu) \leq \Lambda_j(\nu) \quad \forall j = 1, \ldots, k.$$

Indeed, by the min-max theorem, we have

$$\Lambda_j(T_\nu) = \min_{V_j \subset V} \max_{u \in V, u \perp V_j} \frac{\langle P_k \circ R_\nu \circ P_k(u), u \rangle_{L^2}}{\|u\|_{L^2}^2}$$

$$= \min_{V_j \subset L^2} \max_{u \in V, u \perp V_j} \frac{\langle R_\nu(u), u \rangle_{L^2}}{\|u\|_{L^2}^2}$$

$$\leq \min_{V_j \subset L^2} \max_{u \in L^2, u \perp V_j} \frac{\langle R_\nu(u), u \rangle_{L^2}}{\|u\|_{L^2}^2} = \Lambda_j(\nu),$$

where with V_j we denote a generic $(j-1)$-dimensional subspace of $L^2(\mathbb{R}^d)$.

In conclusion, we obtain the estimate

$$0 \leq \Lambda_j(\mu) - \Lambda_j(\nu) \leq \Lambda_j(T_\mu) - \Lambda_j(T_\nu) \leq \|T_\mu - T_\nu\|_{\mathcal{L}(V)}.$$

On the other hand, by definition of the norm $\| \cdot \|_{\mathcal{L}(V)}$, we have

$$(4.1) \qquad \|T_\mu - T_\nu\|_{\mathcal{L}(V)} = \sup_{u \in V} \frac{\langle (T_\mu - T_\nu)u, u \rangle_{L^2}}{\|u\|_{L^2}^2} = \sup_{u \in V} \frac{\langle (R_\mu - R_\nu)u, u \rangle_{L^2}}{\|u\|_{L^2}^2}$$

$$= \sup_{u \in V} \frac{1}{\|u\|_{L^2}^2} \int_{\mathbb{R}^d} \left(R_\mu(u) - R_\nu(u) \right) u \, dx.$$

Let $u \in V$ be the function for which the supremum in the right-hand side of (4.1) is achieved. We can suppose that $\|u\|_{L^2} = 1$, i.e., that there are real numbers $\alpha_1, \ldots, \alpha_k$, such that

$$u = \alpha_1 u_1 + \ldots + \alpha_k u_k \qquad \text{and} \qquad \alpha_1^2 + \ldots + \alpha_k^2 = 1.$$

Thus, we have

$$(4.2) \qquad \|T_\mu - T_\nu\|_{\mathcal{L}(V)} \leq \int_{\mathbb{R}^d} |R_\mu(u) - R_\nu(u)| \cdot |u| \, dx$$

$$\leq \int_{\mathbb{R}^d} \left| \sum_{j=1}^k \alpha_j \left(R_\mu(u_j) - R_\nu(u_j) \right) \right| \cdot \left(\sum_{j=1}^k |u_j| \right) dx$$

$$\leq \int_{\mathbb{R}^d} \left(\sum_{j=1}^k |(R_\mu(u_j) - R_\nu(u_j))| \right) \cdot \left(\sum_{j=1}^k |u_j| \right) dx$$

$$\leq \int_{\mathbb{R}^d} \left(\sum_{j=1}^k \left((R_\mu(|u_j|) - R_\nu(|u_j|)) \right) \right) \cdot \left(\sum_{j=1}^k |u_j| \right) dx,$$

where the last inequality is due to the linearity and the positivity of $R_\mu - R_\nu$, which easily follows by the maximum principle. We now recall that by (2.7), we have $\|u_j\|_{L^\infty} \leq e^{\frac{1}{8\pi}} \lambda_k(\mu)^{d/4}$ for each $j = 1, \ldots, k$. By the weak maximum principle applied for u_j and w_μ, we have

$$(4.3) \qquad |u_j| \leq e^{1/(8\pi)} \lambda_k(\mu)^{(d+4)/4} w_\mu \qquad \text{for every } 1 \leq j \leq k.$$

Using again the positivity of $R_\mu - R_\nu$ and substituting (4.3) in (4.2), we obtain the claim. $\square$

For a capacitary measure $\mu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$, we denote by $E_f(\mu)$ the Dirichlet energy with respect to the function $f \in L^2$, i.e.,

$$E_f(\mu) = \min_{u \in H^1_\mu} J_{\mu,f} = J_{\mu,f}(w_{\mu,f}) = -\frac{1}{2} \int_{\mathbb{R}^d} f w_{\mu,f},$$

which can be written in terms of the resolvent R_μ as

(4.4)
$$E_f(\mu) = -\frac{1}{2} \int_{\mathbb{R}^d} f R_\mu(f) \, dx.$$

PROPOSITION 4.2. *Suppose that $\mathcal{G} : \mathcal{M}^P_{cap}(\mathbb{R}^d) \to \mathbb{R}$ is a given functional and that the capacitary measure $\mu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$ is a subsolution for the functional $\mathcal{F} = \lambda_k + \mathcal{G}$. Then there is a nonnegative function $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ vanishing at infinity, such that μ is a local subsolution for the functional $E_f(\mu) + \mathcal{G}(\mu)$.*

Proof. We first note that by Lemma 3.4, we can choose $\varepsilon > 0$ such that, for every $\nu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$ with $\mu \prec \nu$ and $d_\gamma(\mu, \nu) < \varepsilon$, we have $\lambda_k(\mu) \le \lambda_k(\nu) \le 2\lambda_k(\mu)$. We now consider $\nu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$ with this property. Since μ is a subsolution for $\mathcal{F}$, we have

$$\begin{aligned}
\mathcal{G}(\mu) - \mathcal{G}(\nu) &\le \lambda_k(\nu) - \lambda_k(\mu) \\
&\le 2\lambda_k(\mu)^2 \big(\Lambda_k(\mu) - \Lambda_k(\nu)\big) \\
&\le 4k^2 e^{\frac{1}{4\pi}} \lambda_k(\mu)^{\frac{d+8}{2}} \big(E_{w_\mu}(\mu) - E_{w_\mu}(\nu)\big),
\end{aligned}$$

where the last inequality is due to Lemma 4.1 and the representation (4.4). Note that for a generic $\rho \in L^2$ and $t > 0$, we have $E_{t\rho}(\mu) = t^2 E_\rho(\mu)$. Thus, setting $f = 2k e^{\frac{1}{8\pi}} \lambda_k(\mu)^{\frac{d+8}{4}} w_\mu$, we have the claim. $\square$

4.2. Subsolutions of spectral functionals with mass penalization. In this subsection we prove that the subsolutions for the functionals of the form

(4.5)
$$\mathcal{F}(\mu) = \lambda_k(\mu) + \int_{\mathbb{R}^d} \mu_{ac}(x)^{-p} \, dx$$

have bounded sets of finiteness whenever $p \in (0, 1)$. Our argument is based on Proposition 4.2 and the following Lemma, which is implicitly contained in [19, Lemma 3.1].

LEMMA 4.3. *Suppose that $\mu \in \mathcal{M}^P_{cap}(\mathbb{R}^d)$ is a capacitary measure of finite torsion. For the half-space $H = \{x \in \mathbb{R}^d : c + x \cdot \xi > 0\}$, where the constant $c \in \mathbb{R}$ and the vector $\xi \in \mathbb{R}^d$ are given, we have*

$$\begin{aligned}
d_\gamma(\mu, \mu \vee I_H) &\le \sqrt{8\|w_\mu\|_{L^\infty}} \int_{\partial H} w_\mu \, d\mathcal{H}^{d-1} - \int_{\mathbb{R}^d \backslash H} |\nabla w_\mu|^2 \, dx \\
&\quad - \int_{\mathbb{R}^d \backslash H} w_\mu^2 \, d\mu + 2 \int_{\mathbb{R}^d \backslash H} w_\mu \, dx.
\end{aligned}$$

Proof. For the sake of simplicity, set $w := w_\mu$, $M = \|w\|_{L^\infty}$, $c = 0$, and $\xi = (0, \ldots, 0, -1)$. Consider the function

$$v(x_1, \ldots, x_d) = \begin{cases} M & \text{if } x_1 \le -\sqrt{M}, \\ \frac{1}{2}\big(2M - (x_1 + \sqrt{2M})^2\big) & \text{if } -\sqrt{2M} \le x_1 \le 0, \\ 0 & \text{if } 0 \le x_1, \end{cases}$$

and let $w_H = w \wedge v \in H_0^1(H) \cap H_\mu^1$. We have

$$
\begin{aligned}
d_\gamma(\mu, \mu \vee I_H) &= \int_{\mathbb{R}^d} (w - w_{\mu \vee I_H})\, dx = 2(J_\mu(w_{\mu \vee I_H}) - J_\mu(w)) \\
&\leq 2(J_\mu(w_H) - J_\mu(w)) \\
&\leq \int_{\mathbb{R}^d} |\nabla(w_H)|^2 - |\nabla w|^2\, dx - \int_{\mathbb{R}^d \setminus H} w^2\, d\mu + 2\int_{\mathbb{R}^d} (w - w_H)\, dx \\
&\leq \int_{\{-\sqrt{2M} < x_1 \leq 0\}} |\nabla(w_H)|^2 - |\nabla w|^2\, dx - \int_{\mathbb{R}^d \setminus H} |\nabla w|^2\, dx \\
&\quad - \int_{\mathbb{R}^d \setminus H} w^2\, d\mu + 2\int_{\mathbb{R}^d} (w - w_H)\, dx \\
&\leq 2\int_{\{-\sqrt{2M} < x_1 \leq 0\}} \nabla w_H \cdot \nabla(w_H - w)\, dx + 2\int_{\{-\sqrt{2M} < x_1 \leq 0\}} (w - w_H)\, dx \\
&\quad - \int_{\mathbb{R}^d \setminus H} |\nabla w|^2\, dx - \int_{\mathbb{R}^d \setminus H} w^2\, d\mu + 2\int_{\mathbb{R}^d \setminus H} w\, dx \\
&= 2\int_{\{-\sqrt{2M} < x_1 \leq 0\}} \nabla v \cdot \nabla(w_H - w)\, dx + 2\int_{\{-\sqrt{2M} < x_1 \leq 0\}} (w - w_H)\, dx \\
&\quad - \int_{\mathbb{R}^d \setminus H} |\nabla w|^2\, dx - \int_{\mathbb{R}^d \setminus H} w^2\, d\mu + 2\int_{\mathbb{R}^d \setminus H} w\, dx \\
&= \sqrt{8M} \int_{\partial H} w\, d\mathcal{H}^{d-1} - \int_{\mathbb{R}^d \setminus H} |\nabla w|^2\, dx - \int_{\mathbb{R}^d \setminus H} w^2\, d\mu + 2\int_{\mathbb{R}^d \setminus H} w\, dx
\end{aligned}
$$

as required.   ☐

The next result has a double implication: on the one side it plays a fundamental role in the proof of the existence of optimal potentials, and on the other side it gives a first qualitative result on them. The spirit of the proof follows a classical argument introduced by De Giorgi, associated to an Alt–Caffarelli truncation argument [1].

LEMMA 4.4. *Consider a nonnegative function $f \in L^\infty(\mathbb{R}^d)$ and a real number $p \in (0, 1)$. Suppose that $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ is a local subsolution for the functional*

$$
\mathcal{F}(\mu) = E_f(\mu) + \int_{\mathbb{R}^d} \mu_{ac}(x)^{-p}\, dx.
$$

Then the set $\Omega_\mu = \{w_\mu > 0\}$ is bounded.

Proof. We first recall that if $\nu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ is such that $\mu \prec \nu$, then the functional $R_\mu - R_\nu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is positive. Thus, we have

$$
E_f(\nu) - E_f(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} \big(fR_\mu(f) - fR_\nu(f)\big)\, dx \leq \frac{1}{2}\|f\|_{L^\infty}^2 \big(E(\nu) - E(\mu)\big),
$$

and so we can restrict our attention to the case $f \equiv 1$.

For each $t \in \mathbb{R}$, we set

$$
H_t = \{x \in \mathbb{R}^d : x_1 = t\}, \qquad H_t^+ = \{x \in \mathbb{R}^d : x_1 > t\}, \qquad H_t^- = \{x \in \mathbb{R}^d : x_1 < t\}.
$$

We prove that there is some $t \in \mathbb{R}$ such that $|H_t^+ \cap \Omega| = 0$. For the sake of simplicity, set

$$
w := w_\mu, \qquad M = \|w\|_{L^\infty}, \qquad \text{and} \qquad V(x)dx := \mu_{ac}.
$$

By Lemma 4.3 and the subminimality of Ω, we have

$$\frac{1}{2} \int_{H_t^+} |\nabla w|^2 \, dx + \frac{1}{2} \int_{H_t^+} w^2 V \, dx + \int_{H_t^+} V^{-p} \, dx \leq \sqrt{2M} \int_{H_t} w \, d\mathcal{H}^{d-1} + \int_{H_t^+} w \, dx$$

for every $t \in \mathbb{R}$. We aim to prove that the left-hand side is greater than a power of $\int_{H_t^+} w \, dx$. Indeed, by the Hölder and Young inequalities, we have

$$\int_{H_t^+} w^{\frac{2p}{p+1}} \, dx \leq \left(\int_{H_t^+} w^2 V \, dx \right)^{\frac{p}{1+p}} \left(\int_{H_t^+} V^{-p} \, dx \right)^{\frac{1}{1+p}}$$

$$\leq \frac{p}{1+p} \int_{H_t^+} w^2 V \, dx + \frac{1}{1+p} \int_{H_t^+} V^{-p} \, dx.$$

If $d \geq 3$, using the Hölder, Sobolev, and Young inequalities, we get

$$\left(\int_{H_t^+} w \, dx \right)^{\frac{d+2p}{d+1+p}} \leq \left(\int_{H_t^+} w^{\frac{2p}{p+1}} \, dx \right)^{\frac{(1+p)(d+2)}{2(d+1+p)}} \left(\int_{H_t^+} w^{\frac{2d}{d-2}} \, dx \right)^{\frac{(1-p)(d-2)}{2(d+1+p)}}$$

$$\leq \left(\int_{H_t^+} w^{\frac{2p}{p+1}} \, dx \right)^{\frac{(1+p)(d+2)}{2(d+1+p)}} \left(\int_{H_t^+} |\nabla w|^2 \, dx \right)^{\frac{(1-p)d}{2(d+1+p)}}$$

$$\leq \frac{(1+p)(d+2)}{2(d+1+p)} \int_{H_t^+} w^{\frac{2p}{p+1}} \, dx + \frac{(1-p)d}{2(d+1+p)} \int_{H_t^+} |\nabla w|^2 \, dx,$$

which finally gives

$$\left(\int_{H_t^+} w \, dx \right)^{\alpha} \leq \frac{p(d+2)}{2(d+1+p)} \int_{H_t^+} w^2 V \, dx + \frac{d+2}{2(d+1+p)} \int_{H_t^+} V^{-p} \, dx$$

$$+ \frac{(1-p)d}{2(d+1+p)} \int_{H_t^+} |\nabla w|^2 \, dx$$

$$\leq C\sqrt{2M} \int_{H_t} w \, d\mathcal{H}^{d-1} + C \int_{H_t^+} w \, dx,$$

where $\alpha = \frac{d+2p}{d+1+p} < 1$ and C is a constant depending on the dimension d and the exponent p. Setting

$$\phi(t) := \int_{H_t^+} w \, dx,$$

we have that

$$\phi'(t) = - \int_{H_t} w \, d\mathcal{H}^{d-1},$$

and finally

$$\phi(t)^{\beta} \leq -C\sqrt{2M}\phi'(t) + C\phi(t),$$

which gives that ϕ vanishes in a finite time. Repeating this argument in any direction, we obtain that the support of w is bounded.

If $d = 2$, the same reasoning can be repeated replacing the Sobolev inequality by

$$\|u\|_{L^3(\mathbb{R}^2)} \leq \frac{3}{2}\|u\|_{L^1(\mathbb{R}^2)}^{\frac{1}{3}}\|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{2}{3}}. \qquad \square$$

THEOREM 4.5. *Suppose that $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ is a subsolution for the functional $\mathcal{F}$ defined in* (4.5). *Then the set of finiteness $\Omega_\mu = \{w_\mu > 0\}$ is bounded.*

Proof. By Proposition 4.2, we have that μ is a local subsolution for a functional of the form $E_f(\mu) + \int_{\mathbb{R}^d}\mu_{ac}^{-\alpha}\,dx$. The conclusion follows by Lemma 4.4. $\qquad \square$

4.3. Subsolutions for spectral-torsion functionals. In this subsection we consider spectral functionals with torsion penalization of the form

$$(4.6) \qquad\qquad \mathcal{F}(\mu) = \lambda_k(\mu) + P(\mu).$$

We prove that any subsolution μ for $\mathcal{F}$ has a bounded set of finiteness $\Omega_\mu = \{w_\mu > 0\}$. As in the case of functionals with mass penalization (4.5), we will reduce our study to subsolutions of energy functionals. Our main instrument in proving the boundedness of Ω_μ will be the following comparison principle "at infinity."

LEMMA 4.6. *Consider a capacitary measure of finite torsion $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$. Suppose that $u \in H_\mu^1$ is a solution of*

$$-\Delta u + \mu u = f, \qquad u \in H_\mu^1,$$

where $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\lim_{x\to\infty} f(x) = 0$. Then, there is some $R > 0$ large enough such that $u \leq w_\mu$ on $\mathbb{R}^d \setminus B_R$.

Proof. Set $v = u - w_\mu$. We will prove that the set $\{v > 0\}$ is bounded. Taking v^+ instead of v and $\mu \vee I_{\{v>0\}}$ instead of μ, we note that it is sufficient to restrict our attention to the case $v \geq 0$ on $\mathbb{R}^d$. We will prove the Lemma in four steps.

Step 1. There are constants $R_0 > 0$, $C_d > 0$, and $\delta > 0$ such that

$$(4.7) \qquad \left(\int_{\mathbb{R}^d} v^2\varphi^{2(1+\delta)}\right)^{\frac{1}{1+\delta}} \leq C_d \int_{\mathbb{R}^d} |\nabla\varphi|^2 v^2\,dx \qquad \forall \varphi \in W_0^{1,\infty}(B_{R_0}^c).$$

For any $\varphi \in W^{1,\infty}(\mathbb{R}^d)$, we have that $v\varphi^2 \in H_\mu^1$, and so we may use it as a test function in

$$-\Delta v + \mu v = f - 1, \qquad v \in H_\mu^1,$$

obtaining the identity

$$(4.8)$$
$$\int_{\mathbb{R}^d} |\nabla(\varphi v)|^2\,dx + \int_{\mathbb{R}^d} \varphi^2 v^2\,d\mu = \int_{\mathbb{R}^d} |\nabla\varphi|^2 v^2\,dx + \int_{\mathbb{R}^d} v\varphi^2(f-1)\,dx \quad \forall \varphi \in W^{1,\infty}(\mathbb{R}^d).$$

Let $R_0 > 0$ be large enough such that $1 - f > \frac{4}{d+4}$. Then for any $\varphi \in W_0^{1,\infty}(\mathbb{R}^d \setminus B_{R_0})$, we use the Hölder, Young, and Sobolev's inequalities together with (4.8) to obtain

$$\left(\int_{\mathbb{R}^d} v^2 \varphi^{\frac{2d+8}{d+2}}\, dx\right)^{\frac{d+2}{d+4}} \le \left(\int_{\mathbb{R}^d} (\varphi v)^{\frac{2d}{d-2}}\, dx\right)^{\frac{d-2}{d+4}} \left(\int_{\mathbb{R}^d} v\varphi^2\, dx\right)^{\frac{4}{d+4}}$$

$$\le \frac{d}{d+4}\left(\int_{\mathbb{R}^d}(\varphi v)^{\frac{2d}{d-2}}\, dx\right)^{\frac{d-2}{d}} + \frac{4}{d+4}\int_{\mathbb{R}^d} v\varphi^2\, dx$$

$$\le C_d\left(\int_{\mathbb{R}^d}|\nabla(\varphi v)|^2\, dx + \int_{\mathbb{R}^d} v\varphi^2(1-f)\, dx\right)$$

$$\le C_d\int_{\mathbb{R}^d}|\nabla\varphi|^2 v^2\, dx,$$

where C_d is a dimensional constant.

Step 2. There is some $R_1 > 0$ such that the function $M(r) := \frac{1}{d\omega_d r^{d-1}}\int_{\partial B_r} v^2\, d\mathcal{H}^{d-1}$ is decreasing and convex on the interval $(R_1, +\infty)$.

We first note that for $R > 0$ large enough, $\Delta v \ge (1-f)\chi_{\{v>0\}} \ge 0$ as an element of $H^{-1}(B_R^c)$. Since $\Delta(v^2) = 2v\Delta v + 2|\nabla v|^2$, we get that the function $U := v^2$ is subharmonic on $\mathbb{R}^d \setminus B_R$. Now, the formal derivation of the mean M gives

$$M'(r) = \frac{1}{d\omega_d r^{d-1}}\int_{\partial B_r} \nu \cdot \nabla U\, d\mathcal{H}^{d-1},$$

where ν_r is the external normal to ∂B_r. Let $R_1 > 0$ be such that $1 \ge f$ on $\mathbb{R}^d \setminus B_{R_1}$. Then for any $R_1 < r < R < +\infty$, we have

$$d\omega_d\big(R^{d-1}M'(R) - r^{d-1}M'(r)\big) = \int_{\partial B_R} \nu_R \cdot \nabla U\, d\mathcal{H}^{d-1} - \int_{\partial B_r} \nu_r \cdot \nabla U\, d\mathcal{H}^{d-1}$$

$$= \int_{B_{R_2}\setminus B_{R_1}} \Delta U\, dx \ge 0.$$

If we have that $M'(r) > 0$ for some $r > R_1$, then $M'(R) > 0$ for each $R > r$ and so M is increasing on $[r, +\infty)$, which is a contradiction of the fact that v (and so, M) vanishes at infinity. Thus, $M'(r) \le 0$ for all $r \in (R_1, +\infty)$ and so for every $R_1 < r < R < +\infty$, we have

$$R^{d-1}\big(M'(R) - M'(r)\big) \ge R^{d-1}M'(R) - r^{d-1}M'(r) \ge 0,$$

which proves that $M'(r)$ is also increasing.

Step 3. There are constants $R_2 > 0$, $C > 0$, and $0 < \delta < 1/(d-1)$ such that the mean value function $M(r)$ satisfies the differential inequality

$$(4.9)\qquad M(r) \le C\big(r|M'(r)| + M(r)\big)^{\frac{d-1}{2}\delta}|M'(r)|^{1-\frac{d-2}{2}\delta}\qquad \forall r \in (R_2, +\infty).$$

We first test the inequality (4.7) with radial functions of the form $\varphi(x) = \phi(|x|)$, where

$$\phi(r) = 0 \text{ for } r \le R,\quad \phi(r) = \frac{r-R}{\varepsilon(R)} \text{ for } R \le r \le R+\varepsilon(R),\quad \phi(r) = 1 \text{ for } r \ge R+\varepsilon(R),$$

where $R > 0$ is large enough and $\varepsilon(R) > 0$ is a given constant. As a consequence, we obtain

$$(4.10)\qquad \left(\int_{R+\varepsilon(R)}^{+\infty} r^{d-1}M(r)\, dr\right)^{\frac{1}{1+\delta}} \le C_d\varepsilon(R)^{-2}\int_R^{R+\varepsilon(R)} r^{d-1}M(r)\, dr.$$
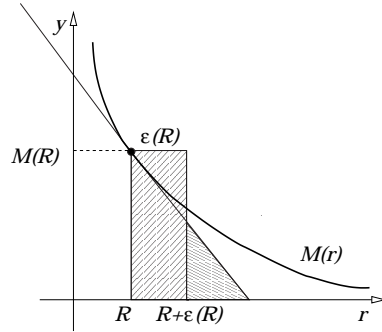
FIG. 1. *We estimate the integral $\int_R^{R+\varepsilon(R)} M(r)\,dr$ by the area of the rectangle on the right, while the integral $\int_{R+\varepsilon(R)}^{+\infty} M(r)\,dr$ is bounded from below by the area of the triangle on the right.*

By Step 2, we have that for R large enough
- M is monotone, i.e., $M(r) \le M(R)$ for $r \ge R$;
- M is convex, i.e., $M(r) \ge M'(R)(r - R) + M(R)$ for $r \ge R$.

We now take $\varepsilon(R) = \frac{1}{2}\frac{M(R)}{|M'(R)|}$, i.e., $2\varepsilon(R)$ is exactly the distance between $(R, 0)$ and the intersection point of the x-axis with the line tangent to the graph of M in $(R, M(R))$ (see Figure 1). With this choice of $\varepsilon(R)$, we estimate both sides of (4.10), obtaining

$$\left(R + \varepsilon(R)\right)^{\frac{d-1}{1+\delta}} \left(\frac{1}{4}M(R)\varepsilon(R)\right)^{\frac{1}{1+\delta}} \le C_d\left(R + \varepsilon(R)\right)^{d-1}\varepsilon(R)^{-2}M(R),$$

which, after substituting $\varepsilon(R)$ with $\frac{1}{2}\frac{M(R)}{|M'(R)|}$, gives (4.9).

Step 4. Each nonnegative (differentiable a.e.) function $M(r)$, which vanishes at infinity and satisfies the inequality (4.9) for some $\delta > 0$ small enough, has compact support.

Let $r \in (R_2, +\infty)$, where R_2 is as in Step 3. We have two cases:

(a) If $r|M'(r)| \ge M(r)$, then $M(r) \le C_1 r^{\frac{(d-1)\delta}{2}}|M'(r)|^{1+\frac{\delta}{2}}$.

(b) If $r|M'(r)| \le M(r)$, then $M(r) \le C_2 |M'(r)|^{1+\frac{\delta}{2}\left(1 - \frac{(d-1)\delta}{2}\right)}$.

Choosing δ small enough, we get that in both cases M satisfies the differential inequality

$$M(r)^{1-\delta_1} \le -Cr^{\delta_2}M'(r)$$

for appropriate constants $C > 0$ and $0 < \delta_1, \delta_2 < 1$. After integration, we have

$$C' - C''r^{1-\delta_2} \ge M(r)^{\delta_1}$$

for some constants $C', C'' > 0$, which concludes the proof.   □

Remark 4.7. An alternative shorter proof of Lemma 4.6 could be made by using viscosity solutions. For the sake of completeness, we report this alternative proof in the appendix.

LEMMA 4.8. *Consider a capacitary measure of finite torsion $\mu \in \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$. Let f be a bounded measurable function converging to zero at infinity, i.e.,*

$\lim_{R \to +\infty} \|f\|_{L^\infty(B_R^c)} = 0$. If μ is a local subsolution for the functional $E_f(\mu) + P(\mu)$, then the finiteness set $\Omega_\mu = \{w_\mu > 0\}$ is bounded.

Proof. Let $\nu \in \mathcal{M}_{\text{cap}}^P(\mathbb{R}^d)$ be such that $\mu \prec \nu$ and $d_\gamma(\mu, \nu) < \varepsilon$. The subminimality of μ gives

$$E_f(\mu) - E(\mu) \le E_f(\nu) - E(\nu),$$

which can be stated in terms of R_μ and R_ν as

$$(4.11) \qquad \int_{\mathbb{R}^d} \big(R_\mu(1) - f R_\mu(f)\big)\, dx \le \int_{\mathbb{R}^d} \big(R_\nu(1) - f R_\nu(f)\big)\, dx.$$

Moreover, by considering $f/2$ instead of f, we can suppose that the above inequality is strict whenever $w_\mu \ne w_\nu$.

We now show that choosing $\nu = \mu \vee I_{B_R}$ for some R large enough, we can obtain equality in (4.11). Indeed, we have

$$\begin{aligned}
0 &\ge \int_{\mathbb{R}^d} \big(R_\mu(1) - R_\nu(1)\big) - f\big(R_\mu(f) - R_\nu(f)\big)\, dx \\
&\ge \int_{\mathbb{R}^d} \big(R_\mu(1) - R_\nu(1)\big) - \big(R_\mu(\|f\|_{L^\infty} f) - R_\nu(\|f\|_{L^\infty} f)\big)\, dx \\
&= \int_{B_R} \big(R_\mu(1) - R_\nu(1)\big) - \big(R_\mu(\|f\|_{L^\infty} f) - R_\nu(\|f\|_{L^\infty} f)\big)\, dx \\
&\quad + \int_{B_R^c} \big(R_\mu(1) - R_\mu(\|f\|_{L^\infty} f)\big)\, dx \\
&\ge \int_{B_R} \big(R_\mu(1) - R_\nu(1)\big) - \big(R_\mu(\|f\|_{L^\infty} f) - R_\nu(\|f\|_{L^\infty} f)\big)\, dx,
\end{aligned}$$

where the last inequality holds for $R > 0$ large enough and is due to Lemma 4.6. We now set for simplicity $w, u \in H_\mu^1$ to be, respectively, the solutions of

$$-\Delta w + \mu w = 1 \qquad \text{and} \qquad -\Delta u + \mu u = \|f\|_{L^\infty} f.$$

Thus, the functions

$$h_w = R_\Omega(1) - R_\omega(1) \in H_\mu^1 \qquad \text{and} \qquad h_u = R_\Omega(\|f\|_{L^\infty} f) - R_\omega(\|f\|_{L^\infty} f)$$

are $(\Delta - \mu)$-harmonic on the ball B_R. By the comparison principle, since $w \ge u$ on ∂B_R, we have that $h_w \ge h_u$ in B_R. Thus, for R large enough and $\nu = \mu \vee I_{B_R}$, we have an equality in (4.11), which gives that $w_\mu = w_\nu$ and so Ω_μ is bounded. $\quad \square$

THEOREM 4.9. *Suppose that $\mu \in \mathcal{M}_{\text{cap}}(\mathbb{R}^d)$ is a subsolution for the functional $\mathcal{F}$ from* (4.6). *Then the set of finiteness $\Omega_\mu = \{w_\mu > 0\}$ is bounded.*

Proof. By Proposition 4.2, we have that μ is a subsolution for a functional of the form $E_f(\mu) + P(\mu)$. By Lemma 4.8, we conclude that Ω_μ is bounded. $\quad \square$

5. Optimal potentials for Schrödinger operators. In this subsection we consider optimization problems for spectral functionals in $\mathbb{R}^d$. In particular, we consider the problem

$$(5.1) \qquad \min\left\{\lambda_k(V) : V : \mathbb{R}^d \to [0, +\infty] \text{ measurable}, \int_{\mathbb{R}^d} V^{-p}\, dx = 1\right\},$$

where $p \in (0,1)$. In the following proposition we prove that, under the integrability constraint in (5.1), the spectrum of $-\Delta + V$ is discrete and thus $\lambda_k(V)$ is well defined.

PROPOSITION 5.1 (compactness of the embedding $H^1_V \hookrightarrow L^1$). *Let $V : \mathbb{R}^d \to [0, +\infty]$ be a measurable function such that $\int_{\mathbb{R}^d} V^{-p} \, dx < +\infty$, where $p \in (0,1]$. Then the torsion function w_V, related to the measure $V dx$, is integrable. In particular, the embedding $H^1_V \hookrightarrow L^1(\mathbb{R}^d)$ is compact and the spectrum of the operator $-\Delta + V$ is discrete.*

Proof. See Example 3.10 in [9]. ☐

By Remark 2.8, the cost functional $\lambda_k(V)$ and the constraint $\int_{\mathbb{R}^d} V^{-p} \, dx$ have the following rescaling properties:

$$\lambda_k(V_t) = t^{-2}\lambda_k(V) \qquad \text{and} \qquad \int_{\mathbb{R}^d} V_t^{-p} \, dx = t^{2p+d} \int_{\mathbb{R}^d} V^{-p} \, dx,$$

where

$$(5.2) \qquad\qquad V_t(x) := t^{-2}V(x/t).$$

This rescaling property allows us to make the following remark.

Remark 5.2 (measure penalization). The potential $\widetilde{V} : \mathbb{R}^d \to [0, +\infty]$ is a solution of

$$(5.3) \qquad \min\left\{ \lambda_k(V) + m \int_{\mathbb{R}^d} V^{-p} \, dx : V : \mathbb{R}^d \to [0, +\infty] \text{ measurable} \right\}$$

if and only if, for every $t > 0$, we have that $\widetilde{V}_t$, defined as in (5.2), is a solution of

$$\min\left\{ \lambda_k(V) : V : \mathbb{R}^d \to [0, +\infty] \text{ measurable}, \int_{\mathbb{R}^d} V^{-p} \, dx = \int_{\mathbb{R}^d} \widetilde{V}_t^{-p} \, dx \right\},$$

and the function

$$f(t) := t^{-2}\lambda_k(\widetilde{V}) + mt^{2p+d} \int_{\mathbb{R}^d} \widetilde{V}^{-p} \, dx$$

achieves its minimum, on the interval $(0, +\infty)$, in the point $t = 1$.

In the case $k = 1$, the existence holds for every $p > 0$. The following result was proved in [14].

PROPOSITION 5.3 (Faber–Krahn inequality for potentials). *For every $p > 0$ there is a solution V_p of the problem (5.1) with $k = 1$. Moreover, there is an optimal potential V_p given by*

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p+1)} \, dx \right)^{1/p} |u_p|^{-2/(1+p)},$$

where u_p is a radially decreasing minimizer of

$$\min\left\{ \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)} \, dx \right)^{(p+1)/p} : u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\}.$$

Moreover, u_p has a compact support, and hence the set $\{V_p < +\infty\}$ is a ball of finite radius in $\mathbb{R}^d$.

We now prove the existence of an optimal potential in the general case $k \geq 2$.

THEOREM 5.4. *Suppose that $p \in (0,1)$. Then, for every $k \in \mathbb{N}$, there is a solution of the problem* (5.1). *Moreover, any solution V of* (5.1) *is constantly equal to $+\infty$ outside a ball of finite radius.*

Proof. By Remark 5.2, every solution of (5.1) is a solution also of the penalized problem (5.3) for some appropriately chosen Lagrange multiplier $m > 0$. Thus, by Theorem 4.5 and Lemma 4.4, we have that if V is optimal for (5.3), then it is constantly $+\infty$ outside a ball of finite radius.

The proof of the existence part follows by induction on k. The first step $k = 1$ was proved in Proposition (5.3). We prove the claim for $k > 1$, provided that the existence holds for all $1, \dots, k - 1$.

Let V_n be a minimizing sequence for (5.1). By Proposition 5.1, we have that the sequence w_{V_n} is uniformly bounded in $L^1(\mathbb{R}^d)$ and so, by Theorem 3.5, we have two possibilities for the sequence of capacitary measures $V_n dx$: *compactness* and *dichotomy*.

If the compactness occurs, then there is a capacitary measure μ such that the sequence $V_n dx$ γ-converges to μ. The sequence $v_n := V_n^{-p/2}$ is a bounded sequence in $L^2(\mathbb{R}^d)$ and so, up to a subsequence, we have that v_n converges weakly in L^2 to some $v \in L^2(\mathbb{R}^d)$. We will prove that the function $V := v^{-2/p}$ is a solution of (5.1). The function V satisfies the constraint from (5.1), and so it is sufficient to prove the inequality

$$(5.4) \qquad \lambda_k(V) \leq \lambda_k(\mu) = \lim_{n \to \infty} \lambda_k(V_n),$$

where the equality is just the continuity of λ_k with respect to the γ-convergence. Since $V_n dx$ γ-converges to μ, we have that the sequence of functionals $\|\cdot\|_{H^1_{V_n}}$ Γ-converges in $L^2(\mathbb{R}^d)$ to the functional $\|\cdot\|_{H^1_\mu}$ (see Remark 2.9). In particular, for every $u \in H^1_\mu$, there is a sequence $u_n \in H^1_{V_n}$ which converges to u in $L^2(\mathbb{R}^d)$ and is such that

$$(5.5) \qquad \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^d} u_n^2 V_n \, dx$$
$$= \lim_{n \to \infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^d} u_n^2 v_n^{-2/p} \, dx$$
$$\geq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 v^{-2/p} \, dx$$
$$= \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 V \, dx,$$

where the inequality in (5.5) is due to strong-weak lower semicontinuity of integral functionals (see, for instance, [10]). Thus, for any $u \in H^1_\mu$, we have that

$$\int_{\mathbb{R}^d} u^2 \, d\mu \geq \int_{\mathbb{R}^d} u^2 V \, dx,$$

and so $V \prec \mu$. Since λ_k is an increasing functional, we obtain the first inequality in (5.4), and so V is a solution of (5.1).

If the dichotomy occurs, then we can suppose that $V_n = V_n^+ \vee V_n^-$, where

$$1/V_n = 1/V_n^+ + 1/V_n^-, \qquad dist\big(\{V_n^+ < \infty\}, \{V_n^- < \infty\}\big) \to +\infty.$$

Since V_n is minimizing, there is $1 \leq l \leq k-1$ such that

$$\lambda_k(V_n) = \lambda_l(V_n^+) \geq \lambda_{k-l}(V_n^-).$$

Taking the solutions V^+ and V^-, respectively, of

$$\min \left\{ \lambda_l(V) : V : \mathbb{R}^d \to [0, +\infty] \text{ measurable}, \int_{\mathbb{R}^d} V^{-p} \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} V_n^+ \, dx \right\},$$

$$\min \left\{ \lambda_{k-l}(V) : V : \mathbb{R}^d \to [0, +\infty] \text{ measurable}, \int_{\mathbb{R}^d} V^{-p} \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} V_n^- \, dx \right\}$$

in such a way that $dist(\{V^+ < \infty\}, \{V^- < \infty\}) > 0$, we have that $V = V^+ \wedge V^-$ is a solution of (5.1). □

6. Optimal measures for spectral-torsion functionals. In this section we consider consider the problem

$$(6.1) \qquad \min \left\{ \lambda_k(\mu) : \mu \in \mathcal{M}_{cap}^P(\mathbb{R}^d), \ P(\mu) = c \right\},$$

where $c > 0$ is a given constant. As in the case of potentials, we can substitute the constraint by a penalization.

Remark 6.1 (measure penalization). The capacitary measure $\widetilde{\mu} \in \mathcal{M}_{cap}(\mathbb{R}^d)$ is a solution of

$$(6.2) \qquad \min \left\{ \lambda_k(\mu) + mP(\mu) : \mu \in \mathcal{M}_{cap}^P(\mathbb{R}^d) \right\}$$

if and only if, for every $t > 0$, the capacitary measure $\widetilde{\mu}_t$, defined as in Remark 2.7, is a solution of

$$\min \left\{ \lambda_k(\mu) : \mu \in \mathcal{M}_{cap}^P(\mathbb{R}^d), \ P(\mu) = P(\widetilde{\mu}_t) \right\},$$

and the function

$$f(t) := t^{-2} \lambda_k(\widetilde{\mu}) + mt^{2+d} P(\widetilde{\mu})$$

achieves its minimum, on the interval $(0, +\infty)$, for $t = 1$.

THEOREM 6.2. *For every $k \in \mathbb{N}$ and $c > 0$, there is a solution of the problem* (6.1). *Moreover, for any solution μ of* (6.1), *there is a ball B_R such that $I_{B_R} \prec \mu$.*

Proof. Suppose first that μ is a solution of (6.1). By Remark 6.1, μ is also a solution of the problem (6.2) for some constant $m > 0$. In particular, μ is a subsolution for the functional

$$\mathcal{F}(\mu) = \lambda_k(\mu) + mP(\mu).$$

By Theorem 4.9, we have that the set of finiteness $\Omega_\mu = \{w_\mu > 0\}$ is bounded, and so there is a ball B_R such that $I_{B_R} \prec \mu$.

The proof of the existence part follows by induction on k. Suppose that $k = 1$ and let μ_n be a minimizing sequence for the problem

$$(6.3) \qquad \min \left\{ \lambda_1(\mu) + mP(\mu) : \mu \in \mathcal{M}_{cap}^P(\mathbb{R}^d) \right\}.$$

By the concentration-compactness principle (Theorem 3.5), we have two possibilities: compactness and dichotomy. If the compactness occurs, we have that, up to a subsequence, μ_n γ-converges to some $\mu \in \mathcal{M}_{cap}^P(\mathbb{R}^d)$. Thus, by the continuity of λ_1 and

T, we have that μ is a solution of (6.3). We now show that the dichotomy cannot occur. Indeed, if we suppose that $\mu_n = \mu_n^+ \wedge \mu_n^-$, where μ_n^+ and μ_n^- have distant sets of finiteness $\Omega_{\mu_n^+}$ and $\Omega_{\mu_n^-}$, then

$$\lambda_1(\mu_n) = \min\{\lambda_1(\mu_n^+), \lambda_1(\mu_n^-)\} \qquad \text{and} \qquad E(\mu_n) = E(\mu_n^+) + E(\mu_n^-).$$

Since, by Theorem 3.5

$$\liminf_{n\to\infty} P(\mu_n^+) > 0 \qquad \text{and} \qquad \liminf_{n\to\infty} P(\mu_n^-) > 0,$$

we obtain that one of the sequences μ_n^+ and μ_n^-, say, μ_n^+, is such that

$$\liminf_{n\to\infty} \left\{\lambda_1(\mu_n^+) + mP(\mu_n^+)\right\} < \liminf_{n\to\infty} \left\{\lambda_1(\mu_n) + mP(\mu_n)\right\},$$

which is a contradiction, and so the compactness is the only possible case for μ_n.

We now prove the claim for $k > 1$, provided that the existence holds for all $1, \ldots, k-1$.

Let μ_n be a minimizing sequence for (5.1). The sequence w_{μ_n} is uniformly bounded in $L^1(\mathbb{R}^d)$, and so by Theorem 3.5, we have two possibilities for the sequence of capacitary measures μ_n: compactness and dichotomy.

If the compactness occurs, then there is a capacitary measure μ such that the sequence μ_n γ-converges to μ, which by the continuity of λ_k and the torsion T is a solution of (6.1).

If the dichotomy occurs, then we can suppose that $\mu_n = \mu_n^+ \wedge \mu_n^-$, where the sets of finiteness $\Omega_{\mu_n^+}$ and $\Omega_{\mu_n^-}$ are such that

$$dist\big(\Omega_{\mu_n^+}, \Omega_{\mu_n^-}\big) \to +\infty, \qquad P(\mu_n) = P(\mu_n^+) + P(\mu_n^-),$$
$$\lim_{n\to\infty} P(\mu_n^+) > 0 \qquad \text{and} \qquad \lim_{n\to\infty} P(\mu_n^-) > 0.$$

Since μ_n is a minimizing sequence, there is a constant $1 \le l \le k-1$ such that

$$\lambda_k(\mu_n) = \lambda_l(\mu_n^+) \ge \lambda_{k-l}(\mu_n^-).$$

Taking the solutions μ^+ and μ^-, respectively, of

$$\min\left\{\lambda_l(\mu) : \mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d),\ P(\mu) = \lim_{n\to\infty} P(\mu_n^+)\right\},$$
$$\min\left\{\lambda_{k-l}(\mu) : \mu \in \mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d),\ P(\mu) = \lim_{n\to\infty} P(\mu_n^-)\right\}$$

in such a way that $dist\big(\Omega_{\mu^+}, \Omega_{\mu^-}\big) > 0$, we have that $\mu = \mu^+ \wedge \mu^-$ is a solution of (6.1). $\quad\square$

Remark 6.3. The Kohler–Jobin inequality (we refer to [5] and the references therein for more details on this isoperimetric inequality) states that the ball B, such that $E(B) = c$, minimizes the first eigenvalue $\lambda_1(\Omega)$ under the constraint $E(\Omega) = c$, among all open sets $\Omega \subset \mathbb{R}^d$. Since the set $\{I_\Omega : \Omega \subset \mathbb{R}^d \text{ open}\} \subset \mathcal{M}_{\mathrm{cap}}^P(\mathbb{R}^d)$ is dense in $\mathcal{M}_{\mathrm{cap}}(\mathbb{R}^d)$ (see [13]), we have that the measure I_B solves (6.1) for $k = 1$.

Open problem. It would be interesting to establish whether the optimal measure μ given by Theorem 6.2 is actually a domain. Some numerical computations made by Beniamin Bogosel [4] and Ioana Durus [20] seem to indicate that this is true and that, at least in dimension two and for low k, the optimal set is made by k disjoint equal disks.

Appendix A. An alternative proof of Lemma 4.6.

Proof of Lemma 4.6. Set $v = u - w_\mu$. We will prove that the set $\{v > 0\}$ is bounded. Taking v^+ instead of v and $\mu \vee I_{\{v>0\}}$ instead of μ, we note that it is sufficient to restrict our attention to the case $v \geq 0$ on $\mathbb{R}^d$. We now prove that if $v \in H^1(\mathbb{R}^d)$ is a nonnegative function such that

$$(A.1) \qquad -\Delta v + \mu v = f - 1, \qquad v \in H^1_\mu,$$

where $\mu \in \mathcal{M}^P_{\mathrm{cap}}(\mathbb{R}^d)$, $f \in L^\infty(\mathbb{R}^d)$, and $\lim_{|x| \to \infty} f(x) = 0$, then $\{v > 0\}$ is bounded.

We first prove that there is some $R_0 > 0$ large enough such that the function v satisfies the inequality $\Delta v \geq 1/2$ on $\mathbb{R}^d \setminus B_{R_0}$ in the viscosity sense, i.e., for each $x \in \mathbb{R}^d \setminus B_{R_0}$ and each $\varphi \in C^\infty(\mathbb{R}^d)$, satisfying $v \leq \phi$ and $\varphi(x) = v(x)$, we have that $\Delta\varphi(x) \geq 1/2$.

Suppose that $\varphi \in C^\infty(\mathbb{R}^d)$ is such that $v \leq \phi$, $\varphi(x) = v(x)$, and $\Delta\varphi(x) < 1/2 - \varepsilon$. By modifying φ and considering $\varepsilon/2$ instead of ε, we may suppose that, for $\delta > 0$ small enough, $\{v + \delta > \varphi\} \subset B^c_{R_0}$ and $\Delta\varphi < 1/2 - \varepsilon$ on the set $\{v + \delta > \varphi\}$. Now taking $(v - \varphi + \delta)^+ \in H^1_\mu$ as a test function in (A.1), we get that

$$\int_{\mathbb{R}^d} (f-1)(v - \varphi + \delta)^+ \, dx = \int_{\mathbb{R}^d} \nabla v \cdot \nabla(v - \varphi + \delta)^+ \, dx + \int_{\mathbb{R}^d} v(v - \varphi + \delta)^+ \, d\mu$$

$$\geq \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla(v - \varphi + \delta)^+ \, dx$$

$$= -\int_{\mathbb{R}^d} (v - \varphi + \delta)^+ \Delta\varphi \, dx$$

$$> \left(-\frac{1}{2} + \varepsilon \right) \int_{\mathbb{R}^d} (v - \varphi + \delta)^+ \, dx,$$

which gives a contradiction, once we choose $R_0 > 0$ large enough such that $f < 1/4$ on $\mathbb{R}^d \setminus B_{R_0}$.

For $r \in (R_0, +\infty)$, we consider the function $M(r) = \sup_{\partial B_r} v$. Then $M : (R_0, +\infty) \to \mathbb{R}$ satisfies the inequality

$$(A.2) \qquad M''(r) + \frac{d-1}{r} M'(r) \geq \frac{1}{2} \qquad \text{in the viscosity sense.}$$

Indeed, let $r \in (R_0, +\infty)$ and $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(r) = M(r)$ and $\phi \geq M$. Then, taking a point $x_0 \in \partial B_r$ such that $v(x) = M(r)$ (which exists due to the upper semicontinuity of v) and the function $\varphi(x) := \phi(|x|)$, we have that $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x_0) = v(r)$, and $\varphi \geq v$, which implies $\Delta\varphi \geq 1/2$ and so (A.2) holds.

There is a constant $\varepsilon_0 > 0$, depending on R_0, the dimension d, and $\|v\|_{L^\infty}$, such that the function $\phi \in C^\infty(\mathbb{R})$, which solves

$$\phi''(r) + \frac{d-1}{r} \phi'(r) = \frac{1}{3}, \qquad \phi(R_0) = \phi(R_0 + \varepsilon_0) = 2\|v\|_{L^\infty},$$

changes sign on the interval $(R_0, R_0 + \varepsilon_0)$. We set

$$t_0 = \sup \{ t : \{M \geq \phi + t\} \neq \emptyset \} > 0.$$

Since M is upper semicontinuous, there is some $r \in (R_0, R_0 + \varepsilon_0)$ such that $M(r) = \phi(r) + t_0$ and $M \leq \phi + t_0$, which is a contradiction with (A.2). $\square$

REFERENCES

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., 325 (1981), pp. 105–144.

[2] M. S. Ashbaugh, *Open problems on eigenvalues of the Laplacian*, in Analytic and Geometric Inequalities and Applications, Math. Appl. 478, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1999, pp. 13–28.

[3] J. Baxter, G. Dal Maso, and U. Mosco, *Stopping times and Γ-convergence*, Trans. Amer. Math. Soc., 303 (1987), pp. 1–38.

[4] B. Bogosel, Private communication (2013).

[5] L. Brasco, *On torsional rigidity and principal frequencies: an invitation to the Kohler–Jobin rearrangement technique*, ESAIM Control Optim. Calc. Var., 20 (2014), pp. 315–338.

[6] D. Bucur, *Uniform concentration-compactness for Sobolev spaces on variable domains*, J. Differential Equations, 162 (2000), pp. 427–450.

[7] D. Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Ration. Mech. Anal., 206 (2012), pp. 1073–1083.

[8] D. Bucur and G. Buttazzo, *Variational Methods in Shape Optimization Problems*, Progr. Nonlinear Differential Equations Appl. 65, Birkhäuser-Verlag, Basel, Switzerland, 2005.

[9] D. Bucur and G. Buttazzo, *On the characterization of the compact embedding of Sobolev spaces*, Calc. Var. Partial Differential Equations, 44 (2012), pp. 455–475.

[10] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Pitman Res. Notes Math. 207, Longman Scientific and Technical, Harlow, UK, 1989.

[11] G. Buttazzo, *Spectral optimization problems*, Rev. Mat. Complut., 24 (2011), pp. 277–322.

[12] G. Buttazzo and G. Dal Maso, *Shape optimization for Dirichlet problems: Relaxed formulation and optimality conditions*, Appl. Math. Optim., 23 (1991), pp. 17–49.

[13] G. Buttazzo and G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Ration. Mech. Anal., 122 (1993), pp. 183–195.

[14] G. Buttazzo, A. Gerolin, B. Ruffini, and B. Velichkov, *Optimal Potentials for Schrödinger Operators*, preprint, Dipartimento di Matematica, Università di Pisa, Pisa, Italy, 2013; also available online from http://cvgmt.sns.it.

[15] G. Dal Maso and A. Garroni, *New results on the asymptotic behaviour of Dirichlet problems in perforated domains*, Math. Models Methods Appl. Sci., 3 (1994), pp. 373–407.

[16] G. Dal Maso and U. Mosco, *Wiener criteria and energy decay for relaxed Dirichlet problems*, Arch. Ration. Mech. Anal., 95 (1986), pp. 345–387.

[17] G. Dal Maso and U. Mosco, *Wiener's criterion and Γ-convergence*, Appl. Math. Optim., 15 (1987), pp. 15–63.

[18] E. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[19] G. De Philippis and B. Velichkov, *Existence and regularity of minimizers for some spectral optimization problems with perimeter constraint*, Appl. Math. Optim., 69 (2014), pp. 199–231.

[20] I. Durus, Private communication (2008).

[21] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.

[22] A. Henrot, *Minimization problems for eigenvalues of the Laplacian*, J. Evol. Equ., 3 (2003), pp. 443–461.

[23] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Front. Math., Birkhäuser-Verlag, Basel, Switzerland, 2006.

[24] A. Henrot and M. Pierre, *Variation et Optimisation de Formes. Une Analyse Géométrique*, Math. Appl. 48, Springer-Verlag, Berlin, 2005.

[25] M.-T. Kohler-Jobin, *Une mthode de comparaison isoprimtrique de fonctionnelles de domaines de la physique mathmatique. I. Une dmonstration de la conjecture isoprimtrique $P\lambda^2 \geq \pi j_0^4/2$ de Plya et Szegö*, Z. Angew. Math. Phys., 29 (1978), pp. 757–766.

[26] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–145.

[27] D. Mazzoleni and A. Pratelli, *Existence of minimizers for spectral problems*, J. Math. Pures Appl., 100 (2013), pp. 433–453.

[28] B. Velichkov, *Existence and Regularity Results for Some Shape Optimization Problems*, Ph.D. thesis, http://cvgmt.sns.it.

[29] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, Berlin, 1989.