The evolution of the orbit distance in the double averaged restricted 3-body problem with crossing singularities

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Abstract

We study the long term evolution of the distance between two Keplerian confocal trajectories in the framework of the averaged restricted 3-body problem. The bodies may represent the Sun, a solar system planet and an asteroid. The secular evolution of the orbital elements of the asteroid is computed by averaging the equations of motion over the mean anomalies of the asteroid and the planet. When an orbit crossing with the planet occurs the averaged equations become singular. However, it is possible to define piecewise differentiable solutions by extending the averaged vector field beyond the singularity from both sides of the orbit crossing set [8], [5]. In this paper we improve the previous results, concerning in particular the singularity extraction technique, and show that the extended vector fields are Lipschitz-continuous. Moreover, we consider the distance between the Keplerian trajectories of the small body and of the planet. Apart from exceptional cases, we can select a sign for this distance so that it becomes an analytic map of the orbital elements near to crossing configurations [11]. We prove that the evolution of the 'signed' distance along the averaged vector field is more regular than that of the elements in a neighborhood of crossing times. A comparison between averaged and non-averaged evolutions and an application of these results are shown using orbits of near-Earth asteroids.

1 Introduction

The distance between the trajectories of an asteroid (orbiting around the Sun) and our planet gives a first indication in the search for possible Earth impactors. We call it orbit distance and denote it by d_{min}.[1] A necessary condition to have a very close approach or an impact with the Earth is that d_{min} is small. Provided close approaches with the planets are avoided, the perturbations caused by the Earth make the asteroid trajectory change slowly with time. Moreover, the perturbations of the other planets produce small changes in both trajectories. The value of the semimajor axis of both is kept constant up to the first order in the small parameters (the ratio of the mass of each perturbing planet to the mass of the Sun). All these

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[1] It is often called MOID (Minimum Orbit Intersection Distance, [3]) by the astronomers.

effects are responsible of a variation of d_{min}. We can study the evolution of the asteroid in the framework of the restricted 3-body problem: Sun, planet, asteroid. Then it is easy to include more than one perturbing planet in the model, in fact the potential energy can be written as sum of terms each depending on one planet only.

If the asteroid has a close encounter with some planet, the perturbation of the latter generically produces a change in the semimajor axis of the asteroid. This can be estimated, and depends on the mass of the planet, the unperturbed planetocentric velocity of the small body and the impact parameter, see [18].

The orbits of near-Earth asteroids (NEAs, i.e. with perihelion distance ≤ 1.3 au)[2] are chaotic, with short Lyapounov times (see [19]), at most a few decades. After that period has elapsed, an orbit computed by numerical integration and the true orbit of the asteroid are practically unrelated and we can not make reliable predictions on the position of the asteroid. For this reason the averaging principle is applied to the equations of motion: it gives the average of the possible evolutions, which is useful in a statistical sense. However, the dynamical evolution often forces the trajectory of a NEA to cross that of the Earth. This produces a singularity in the averaged equations, where we take into account every possible position on the trajectories, including the collision configurations.

The problem of *averaging on planet crossing orbits* has been studied in [8] for planets on circular coplanar orbits and then generalized in [5] including nonzero eccentricities and inclinations of the planets. The work in [8] has been used to define proper elements for NEAs, that are integrals of an approximated problem, see [9]. In this paper we compute the main singular term by developing the distance between two points, one on the orbit of the Earth and the other on that of the asteroid, at its minimum points. This choice improves the results in [8], [5], where a development at the mutual nodes was used, because it avoids the artificial singularity occurring for vanishing mutual inclination of the two orbits. Moreover, we show that the averaged vector field admits two Lipschitz-continuous extensions from both sides of the orbit crossing set (see Theorem 4.2), which is useful for the numerical computation of the solutions.

The orbit distance d_{min} is a singular function of the (osculating) orbital elements when the trajectories of the Earth and the asteroid intersect. However, by suitably choosing a sign for d_{min} we obtain a map, denoted by $\tilde{d}_{min}$, which is analytic in a neighborhood of most crossing configurations (see [11]).

Here we prove that, near to crossing configurations, the averaged evolution of $\tilde{d}_{min}$ is more regular than the averaged evolution of the elements, which are piecewise differentiable functions of time.

The paper is organized as follows. Section 2 contains some preliminary results on the orbit distance. In Sections 3, 4, 5 we introduce the averaged equations, present the results on the singularity extraction method and give the definition of the generalized solutions, which go beyond crossing singularities. In Section 6 we prove the regularity of the secular evolution of the orbit distance. Section 7 is devoted to numerical experiments: we describe the algorithm for the computation of the generalized solutions and compare the averaged evolution with the solutions of the full equations of motion. We also show how this theory can be applied to estimate Earth crossing times for NEAs.

[2] 1 au (astronomical unit) $\approx$ 149,597,870 Km

2 The orbit distance

Let (E_j, v_j), $j = 1, 2$ be two sets of orbital elements of two celestial bodies on confocal Keplerian orbits. Here E_j describes the trajectory of the orbit and v_j is a parameter along the trajectory, e.g. the true anomaly. We denote by $\mathcal{E} = (E_1, E_2)$ the two-orbit configuration, moreover we set $V = (v_1, v_2)$. In this paper we consider bounded trajectories only.

Choose a reference frame, with origin in the common focus, and write $\mathcal{X}_j = \mathcal{X}_j(E_j, v_j)$, $j = 1, 2$ for the Cartesian coordinates of the two bodies.

For a given two-orbit configuration $\mathcal{E}$, we introduce the *Keplerian distance function* d, defined by

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2| \,,$$

where $\mathbb{T}^2$ is the two-dimensional torus and $|\cdot|$ is the Euclidean norm.

The local minimum points of d can be found by computing all the critical points of d^2. For this purpose in [6], [7], [12], [2] the authors have used methods of computational algebra, such as resultants and Gröbner's bases, which allow us to compute efficiently all the solutions.

Apart from the case of two concentric coplanar circles, or two overlapping ellipses, the function d^2 has finitely many stationary points. There exist configurations attaining 4 local minima of d^2: this is thought to be the maximum possible, but a proof is not known yet. A simple computation shows that, for non-overlapping trajectories, the number of crossing points is at most two, see [7].

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$. We consider the maps

$$\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) \,, \qquad \mathcal{E} \mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) \,.$$

For each choice of the two-orbit configuration $\mathcal{E}$, $d_{min}(\mathcal{E})$ gives the orbit distance.

The maps d_h and d_{min} are singular at crossing configurations, and their derivatives do not exist. We can deal with this singularity and obtain analytic maps in a neighborhood of a crossing configuration $\mathcal{E}_c$ by properly choosing a sign for these maps. We note that d_h, d_{min} can present singularities without orbit crossings. The maps d_h can have bifurcation singularities, so that the number of minimum points of d may change. Therefore the maps d_h, d_{min} are defined only locally. We say that a configuration $\mathcal{E}$ is non-degenerate if all the critical points of the Keplerian distance function are non-degenerate. If $\mathcal{E}$ is non-degenerate, there exists a neighborhood $\mathcal{W}$ of $\mathcal{E} \in \mathbb{R}^{10}$ such that the maps d_h, restricted to $\mathcal{W}$, do not have bifurcations. On the other hand, the map d_{min} can lose regularity when two local minima exchange their role as absolute minimum. There are no additional singularities apart from those mentioned above. The behavior of the maps d_h, d_{min} has been investigated in [11]. However, a detailed analysis of the occurrence of bifurcations of stationary points and exchange of minima is still lacking.

Here we summarize the procedure to deal with the crossing singularity of d_h; the procedure for d_{min} is the same. We consider the points on the two orbits corresponding to the local minimum points $V_h = (v_1^{(h)}, v_2^{(h)})$ of d^2:

$$\mathcal{X}_1^{(h)} = \mathcal{X}_1(E_1, v_1^{(h)}) \,; \qquad \mathcal{X}_2^{(h)} = \mathcal{X}_2(E_2, v_2^{(h)}) \,.$$

We introduce the vectors tangent to the trajectories E_1, E_2 at these points

$$\tau_1^{(h)} = \frac{\partial \mathcal{X}_1}{\partial v_1}(E_1, v_1^{(h)}) \,, \qquad \tau_2^{(h)} = \frac{\partial \mathcal{X}_2}{\partial v_2}(E_2, v_2^{(h)}) \,,$$

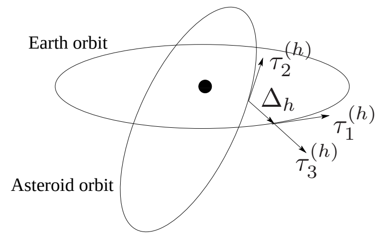

Figure 1: Geometric properties of the critical points of d^2 and regularization rule.

and their cross product

$$\tau_3^{(h)} = \tau_1^{(h)} \times \tau_2^{(h)} \ .$$

We also define

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2 \,, \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)} \ .$$

The vector Δ_h joins the points attaining a local minimum of d^2 and $|\Delta_h| = d_h$.

From the definition of critical points of d^2 both the vectors $\tau_1^{(h)}$, $\tau_2^{(h)}$ are orthogonal to Δ_h, so that $\tau_3^{(h)}$ and Δ_h are parallel, see Figure 1. Denoting by $\hat{\tau}_3^{(h)}$, $\hat{\Delta}_h$ the corresponding unit vectors and by a dot the Euclidean scalar product, the distance with sign

$$\tilde{d}_h = \big(\hat{\tau}_3^{(h)} \cdot \hat{\Delta}_h\big)\, d_h \tag{1}$$

is an analytic function in a neighborhood of most crossing configurations. Indeed, this smoothing procedure fails at crossing configurations such that $\tau_1^{(h)}$, $\tau_2^{(h)}$ are parallel. A detailed proof can be found in [11]. Note that, to obtain regularity in a neighborhood of a crossing configuration, we lose continuity at the configurations with $\tau_1^{(h)} \times \tau_2^{(h)} = 0$ and $d_h \neq 0$.

The derivatives of $\tilde{d}_h$ with respect to each component $\mathcal{E}_k$, $k = 1 \ldots 10$ of $\mathcal{E}$ are given by

$$\frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k} = \hat{\tau}_3^{(h)} \cdot \frac{\partial \Delta}{\partial \mathcal{E}_k}(\mathcal{E}, V_h) \ . \tag{2}$$

We shall call (signed) orbit distance the map $\tilde{d}_{min}$.

3 Averaged equations

Let us consider a restricted 3-body problem with the Sun, the Earth and an asteroid. The motion of the 2-body system Sun-Earth is a known function of time. We denote by $\mathcal{X}, \mathcal{X}' \in \mathbb{R}^3$ the heliocentric position of the asteroid and the planet respectively. The equations of motion for the asteroid are

$$\ddot{\mathcal{X}} = -k^2 \frac{\mathcal{X}}{|\mathcal{X}|^3} + \mu k^2 \Big[\frac{\mathcal{X}' - \mathcal{X}}{|\mathcal{X}' - \mathcal{X}|^3} - \frac{\mathcal{X}'}{|\mathcal{X}|^3} \Big] \,, \tag{3}$$

where k is Gauss' constant and μ is a small parameter representing the ratio of the Earth mass to the mass of the Sun.

We study the motion using Delaunay's elements $\mathcal{Y} = (L, G, Z, \ell, g, z)$, defined by

$$\begin{aligned} L &= k\sqrt{a}\,, & \ell &= n(t - t_0)\,,\\ G &= k\sqrt{a(1-e^2)}\,, & g &= \omega\,,\\ Z &= k\sqrt{a(1-e^2)}\cos I\,, & z &= \Omega\,, \end{aligned}$$

where $(a, e, I, \omega, \Omega, \ell)$ are Keplerian elements, n is the mean motion and t_0 is the time of passage at perihelion. Delaunay's elements of the Earth are denoted by $(L', G', Z', \ell', g', z')$. We write $\mathcal{E} = (E, E')$ for the two-orbit configuration, where E, E' are Delaunay's elements of the asteroid and the Earth respectively. Using the canonical variables $\mathcal{Y}$, equations (3) can be written in Hamiltonian form as

$$\dot{\mathcal{Y}} = \mathbb{J}_3 \nabla_{\mathcal{Y}} H\,, \tag{4}$$

where we use

$$\mathbb{J}_n = \begin{bmatrix} \mathcal{O}_n & -\mathcal{I}_n \\ \mathcal{I}_n & \mathcal{O}_n \end{bmatrix}, \qquad n \in \mathbb{N}\,,$$

for the symplectic identity matrix of order $2n$. The Hamiltonian

$$H = H_0 - R$$

is the difference of the two-body (asteroid, Sun) Hamiltonian

$$H_0 = -\frac{k^4}{2L^2}$$

and the perturbing function

$$R = \mu k^2 \left(\frac{1}{|\mathcal{X} - \mathcal{X}'|} - \frac{\mathcal{X} \cdot \mathcal{X}'}{|\mathcal{X}'|^3} \right),$$

with $\mathcal{X}, \mathcal{X}'$ considered as functions of $\mathcal{Y}, \mathcal{Y}'$.

The function R is the sum of two terms: the first is the direct part of the perturbation, due to the attraction of the Earth and singular at collisions with it. The second is called indirect perturbation, and is due to the attraction of the Sun on the Earth.

We can reduce the number of degrees of freedom of (4) by averaging over the fast angular variables ℓ, ℓ', which are the mean anomalies of the asteroid and the Earth. As a consequence, ℓ becomes a cyclic variable, so that the semimajor axis a is constant in this simplified dynamics. For a full account on averaging methods in Celestial Mechanics see [1].

The averaged equations of motion for the asteroid are given by

$$\dot{\overline{Y}} = -\mathbb{J}_2 \, \overline{\nabla_Y R}\,, \tag{5}$$

where $Y = (G, Z, g, z)^t$, $\overline{Y} = (\overline{G}, \overline{Z}, \overline{g}, \overline{z})^t$ are some of Delaunay's elements, and

$$\overline{\nabla_Y R} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \nabla_Y R \, d\ell \, d\ell'\,,$$

with $\mathbb{T}^2 = \{(\ell, \ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$, is the vector of the averaged partial derivatives of the perturbing function R. Equation (5) corresponds to the scalar equations

$$\dot{\overline{G}} = \overline{\frac{\partial R}{\partial g}}\,, \qquad \dot{\overline{Z}} = \overline{\frac{\partial R}{\partial z}}\,, \qquad \dot{\overline{g}} = -\overline{\frac{\partial R}{\partial G}}\,, \qquad \dot{\overline{z}} = -\overline{\frac{\partial R}{\partial Z}}\,.$$

We can easily include more planets in the model. In this case the perturbing function is sum of terms R_i, each depending on the coordinates of the asteroid and the planet i only, with a small parameter μ_i, representing the ratio of the mass of planet i to the mass of the Sun.

Note that, if there are mean motion resonances of low order with the planets, then the solutions of the averaged equations (5) may be not representative of the behavior of the corresponding components in the solutions of (4).

Moreover, when the planets are assumed to move on circular coplanar orbits we obtain an integrable problem. In fact the semimajor axis a, the component Z of the angular momentum orthogonal to the invariable plane[3] and the averaged Hamiltonian $\overline{H}$ are first integrals generically independent and in involution (i.e. with vanishing Poisson's brackets). Taking into account the eccentricity and the inclination of the planets the problem is not integrable any more.

In [14] the secular evolution of high eccentricity and inclination asteroids is studied in a restricted 3-body problem, with Jupiter on a circular orbit. Nevertheless, crossings with the perturbing planet are excluded in that work. In [15] there is a similar secular theory for a satellite of the Earth. The dynamical behavior described in [14], [15] is usually called *Lidov-Kozai mechanism* in the literature and an explicit solution to the related equations is given in [13].

If no orbit crossing occurs, by the theorem of differentiation under the integral sign the averaged equations of motion (5) are equal to Hamilton's equations

$$\dot{\overline{Y}} = -\mathbb{J}_2 \nabla_Y \overline{R} \tag{6}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} R \, d\ell \, d\ell' = \frac{\mu k^2}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} \, d\ell \, d\ell'$$

is the averaged perturbing function. The average of the indirect term of R is zero.

When the orbit of the asteroid crosses that of the Earth a singularity appears in (5), corresponding to the existence of a collision for particular values of the mean anomalies. We study this singularity to define generalized solutions of (5) which go beyond planet crossings. Since the semimajor axis of the asteroid is constant in the averaged dynamics, we expect that the generalized solutions can be reliable only if there are no close approaches with the planet in the dynamics of equations (4).

4 Extraction of the singularity

In the following we denote by $\mathcal{E}_c$ a non-degenerate crossing configuration with only one crossing point, and we choose the minimum point index h such that $d_h(\mathcal{E}_c) = 0$. For each $\mathcal{E}$ in a neighborhood of $\mathcal{E}_c$ we consider Taylor's development of $V \mapsto d^2(\mathcal{E}, V)$, $V = (\ell, \ell')^t$, in a neighborhood of the local minimum point $V_h = V_h(\mathcal{E})$:

$$d^2(\mathcal{E}, V) = d_h^2(\mathcal{E}) + \frac{1}{2}(V - V_h) \cdot \mathcal{H}_h(\mathcal{E})(V - V_h) + \mathcal{R}_3^{(h)}(\mathcal{E}, V)\,, \tag{7}$$

where

$$\mathcal{H}_h(\mathcal{E}) = \frac{\partial^2 d^2}{\partial V^2}(\mathcal{E}, V_h(\mathcal{E}))$$

[3] Here we mean the common plane of the planetary trajectories.

is the Hessian matrix of d^2 in $V_h = (\ell_h, \ell'_h)^t$, and

$$\mathcal{R}_3^{(h)}(\mathcal{E}, V) = \sum_{|\alpha|=3} r_\alpha^{(h)}(\mathcal{E}, V)(V - V_h)^\alpha , \tag{8}$$

$$r_\alpha^{(h)}(\mathcal{E}, V) = \frac{3}{\alpha!}\int_0^1 (1-t)^2 D^\alpha d^2(\mathcal{E}, V_h + t(V - V_h))\, dt \tag{9}$$

is Taylor's remainder in the integral form.[4] We introduce the approximated distance

$$\delta_h = \sqrt{d_h^2 + (V - V_h)\cdot \mathcal{A}_h (V - V_h)}\,, \tag{10}$$

where

$$\mathcal{A}_h = \frac{1}{2}\mathcal{H}_h = \begin{bmatrix} |\tau_h|^2 + \dfrac{\partial^2 \mathcal{X}}{\partial \ell^2}(E, \ell_h)\cdot \Delta_h & -\tau_h \cdot \tau'_h \\ -\tau_h\cdot\tau'_h & |\tau'_h|^2 - \dfrac{\partial^2 \mathcal{X}'}{\partial \ell'^2}(E', \ell'_h)\cdot\Delta_h \end{bmatrix},$$

and

$$\Delta_h = \Delta_h(\mathcal{E})\,, \qquad \tau_h = \frac{\partial \mathcal{X}}{\partial \ell}(E, \ell_h)\,, \quad \tau'_h = \frac{\partial \mathcal{X}'}{\partial \ell'}(E', \ell'_h)\ .$$

Remark 1. If the matrix $\mathcal{A}_h$ is non-degenerate, then it is positive definite because V_h is a minimum point, and this property holds in a suitably chosen neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. At a crossing configuration $\mathcal{E} = \mathcal{E}_c$ the matrix $\mathcal{A}_h$ is degenerate if and only if the tangent vectors τ_h, τ'_h are parallel (see [11]):

$$\det \mathcal{A}_h(\mathcal{E}_c) = 0 \quad \Longleftrightarrow \quad \tau_h(\mathcal{E}_c)\times\tau'_h(\mathcal{E}_c) = 0\ .$$

First we estimate the remainder function $1/d - 1/\delta_h$. To this aim we need the following:

Lemma 4.1. *There exist positive constants C_1, C_2 and a neighborhood $\mathcal{U}$ of $(\mathcal{E}_c, V_h(\mathcal{E}_c))$ such that*

$$C_1\delta_h^2 \le d^2 \le C_2\delta_h^2 \tag{11}$$

holds for $(\mathcal{E}, V)$ in $\mathcal{U}$. Moreover, there exist positive constants C_3, C_4 and a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that

$$d_h^2 + C_3|V - V_h|^2 \le \delta_h^2 \le d_h^2 + C_4|V - V_h|^2 \tag{12}$$

holds for $\mathcal{E}$ in $\mathcal{W}$ and for every $V \in \mathbb{T}^2$.

Proof. From (8), (9) we obtain the existence of a neighborhood $\mathcal{U}$ of $(\mathcal{E}_c, V_h(\mathcal{E}_c))$ and a constant $C_5 > 0$ such that

$$|\mathcal{R}_3^{(h)}(\mathcal{E}, V)| \le \sum_{|\alpha|=3} |r_\alpha^{(h)}(\mathcal{E}, V)||V - V_h|^\alpha \le C_5|V - V_h|^3\ . \tag{13}$$

[4] In (8), (9) $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N}\cup\{0\})^2$ is a multi-index, hence

$$|\alpha| = \alpha_1 + \alpha_2\,, \qquad \alpha! = \alpha_1!\alpha_2!\,, \qquad V^\alpha = v_1^{\alpha_1}v_2^{\alpha_2}\,, \qquad D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial v_1^{\alpha_1}\partial v_2^{\alpha_2}}\,,$$

for a vector $V = (v_1, v_2)$ and a smooth function $V \mapsto f(V)$.

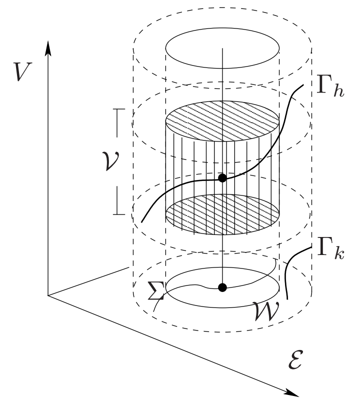

Figure 2: Sketch for the selection of the neighborhood $\mathcal{U} = \mathcal{W} \times \mathcal{V}$. Here $\Gamma_j = \{(\mathcal{E}, V_j(\mathcal{E})) : d_j(\mathcal{E}) = 0\}$ for $j = h, k$. In this case we restrict $\mathcal{W}$ to a smaller set (the inner circle), as explained in the proof of Proposition 1.

We select $\mathcal{U}$ so that no bifurcations of stationary points of d^2 occur and there exists a constant $C_6 > 0$ with $d_k(\mathcal{E}) \geq C_6, k \neq h$ for each $(\mathcal{E}, V) \in \mathcal{U}$. Relation (13) together with (7),(10) yield (11) for some $C_1, C_2 > 0$.

Moreover, we can find a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that there are no bifurcations of stationary points of d^2, and the inequalities (12) hold for some $C_3, C_4 > 0$: in fact $\mathcal{A}_h$ depends continuously on $\mathcal{E}$ and $\mathcal{A}_h(\mathcal{E}_c)$ is positive definite.

□

Proposition 1. *There exist $C > 0$ and a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that*

$$\left|\frac{1}{d} - \frac{1}{\delta_h}\right| \leq C \qquad \forall\, (\mathcal{E}, V) \in (\mathcal{W} \times \mathbb{T}^2) \setminus \mathcal{U}_\Sigma\,,$$

where $\mathcal{U}_\Sigma = \{(\mathcal{E}, V_h(\mathcal{E})) : \mathcal{E} \in \Sigma\}$ with $\Sigma = \{\mathcal{E} \in \mathcal{W} : d_h(\mathcal{E}) = 0\}$.

Proof. By Lemma 4.1 we can choose two neighborhoods $\mathcal{W}$, $\mathcal{V}$ of $\mathcal{E}_c$ and $V_h(\mathcal{E}_c)$ respectively such that both (11) and (12) hold in $\mathcal{U} = \mathcal{W} \times \mathcal{V}$. We restrict $\mathcal{W}$, if necessary, so that there exists $C_7 > 0$ with $d \geq C_7$ for each $(\mathcal{E}, V) \in \mathcal{W} \times (\mathbb{T}^2 \setminus \mathcal{V})$ (see Figure 2). In $\mathcal{U} \setminus \mathcal{U}_\Sigma$ we have

$$\left|\frac{1}{d} - \frac{1}{\delta_h}\right| = \frac{|\delta_h^2 - d^2|}{\delta_h\, d[\delta_h + d]} \leq \frac{1}{\sqrt{C_1}[1 + \sqrt{C_1}]} \frac{|\delta_h^2 - d^2|}{\delta_h^3} \leq C$$

for a constant $C > 0$. Using the boundedness of $1/d$, $1/\delta_h$ in $\mathcal{W} \times (\mathbb{T}^2 \setminus \mathcal{V})$ we conclude the proof.

□

Now we estimate the derivatives of the remainder function $1/d - 1/\delta_h$.

Proposition 2. *There exist $C > 0$ and a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that, if y_k is a component of Delaunay's elements Y, the estimate*

$$\left|\frac{\partial}{\partial y_k}\Big(\frac{1}{d} - \frac{1}{\delta_h}\Big)\right| \leq \frac{C}{d_h + |V - V_h|} \tag{14}$$

holds for each $(\mathcal{E},V)\in(\mathcal{W}\times\mathbb{T}^2)\setminus\mathcal{U}_\Sigma$, with $\mathcal{U}_\Sigma$ as in Proposition 1. Therefore the map

$$\mathcal{W}\setminus\Sigma\ni\mathcal{E}\mapsto\int_{\mathbb{T}^2}\frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell'\,, \tag{15}$$

where $\Sigma=\{\mathcal{E}\in\mathcal{W}:d_h(\mathcal{E})=0\}$, can be extended continuously to the whole set $\mathcal{W}$.

Proof. In the following we denote by $C_j, j=8\dots14$ some positive constants. We write

$$\frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)=\frac{1}{2}\Big(\frac{1}{\delta_h^3}-\frac{1}{d^3}\Big)\frac{\partial\delta_h^2}{\partial y_k}-\frac{1}{2d^3}\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_k}\,,$$

and give an estimate for the two terms at the right hand side. We choose a neighborhood $\mathcal{U}=\mathcal{W}\times\mathcal{V}$ of $(\mathcal{E}_c,V_h(\mathcal{E}_c))$ as in Proposition 1 so that, using (11), (12) and the boundedness of the remainder function, we have

$$\Big|\frac{1}{\delta_h^3}-\frac{1}{d^3}\Big| \;=\; \Big|\frac{1}{\delta_h}-\frac{1}{d}\Big|\Big(\frac{1}{\delta_h^2}+\frac{1}{\delta_h d}+\frac{1}{d^2}\Big)\le\frac{C_8}{d_h^2+|V-V_h|^2}$$

in $\mathcal{U}_0=\mathcal{U}\setminus\mathcal{U}_\Sigma$. Moreover in $\mathcal{U}_0$ we have

$$\Big|\frac{\partial\delta_h^2}{\partial y_k}\Big|\le\Big|\frac{\partial d_h^2}{\partial y_k}\Big|+C_9|V-V_h|\le C_{10}(d_h+|V-V_h|)\,,$$

since

$$\frac{\partial\delta_h^2}{\partial y_k}=\frac{\partial d_h^2}{\partial y_k}-2\frac{\partial V_h}{\partial y_k}\cdot\mathcal{A}_h(V-V_h)+(V-V_h)\cdot\frac{\partial\mathcal{A}_h}{\partial y_k}(V-V_h)\,, \tag{16}$$

and the derivatives

$$\frac{\partial V_h}{\partial y_k}(\mathcal{E})=-[\mathcal{H}_h(\mathcal{E})]^{-1}\frac{\partial}{\partial y_k}\nabla_V d^2(\mathcal{E},V_h(\mathcal{E}))$$

are uniformly bounded for $\mathcal{E}\in\mathcal{W}$ since bifurcations do not occur.

Hence the relation

$$\Big|\Big(\frac{1}{\delta_h^3}-\frac{1}{d^3}\Big)\frac{\partial\delta_h^2}{\partial y_k}\Big|\le C_{11}\frac{d_h+|V-V_h|}{d_h^2+|V-V_h|^2}\le\frac{2C_{11}}{d_h+|V-V_h|} \tag{17}$$

holds in $\mathcal{U}_0$, with $C_{11}=C_8C_{10}$. We also have

$$\Big|\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_k}\Big|\le C_{13}|V-V_h|^2\,, \tag{18}$$

for $(\mathcal{E},V)\in\mathcal{U}_0$, in fact

$$\sup_{\mathcal{U}_0}|r_\alpha^{(h)}|<+\infty\,,\qquad\sup_{\mathcal{U}_0}\Big|\frac{\partial r_\alpha^{(h)}}{\partial y_k}\Big|<+\infty\,, \tag{19}$$

for each $\alpha=(\alpha_1,\alpha_2)$ with $|\alpha|=3$. Using again (11), (12) we obtain

$$\Big|\frac{1}{d^3}\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_k}\Big|\le\frac{C_{14}}{d_h+|V-V_h|}\ . \tag{20}$$

From (17), (20) we obtain (14) and the assert of the proposition follows using the boundedness of $\frac{\partial}{\partial y_k}\big(1/d\big)$, $\frac{\partial}{\partial y_k}\big(1/\delta_h\big)$ in $\mathcal{W}\times(\mathbb{T}^2\setminus\mathcal{V})$.

□

From (14) in Proposition 2 the average over $\mathbb{T}^2$ of the derivatives of $1/d - 1/\delta_h$ in (15) is finite for each $\mathcal{E}$ in $\mathcal{W}$, and can be computed by exchanging the integral and differential operators: therefore the average of the remainder function is continuously differentiable in $\mathcal{W}$.

On the other hand, the average over $\mathbb{T}^2$ of the derivatives with respect to y_k of $1/\delta_h$ are non-convergent integrals for $\mathcal{E} \in \Sigma$: for this reason the averaged vector field in (5) is not defined at orbit crossings. Next we show that the average of these derivatives admit two analytic extensions to the whole $\mathcal{W}$ from both sides of the singular set Σ.
For this purpose, given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, we set

$$\mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h > 0\}\,, \qquad \mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h < 0\}\,,$$

with $\tilde{d}_h$ given by (1).

Proposition 3. *There exists a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that the maps*

$$\mathcal{W}^+ \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2} \frac{1}{\delta_h}\, d\ell\, d\ell'\,, \qquad \mathcal{W}^- \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2} \frac{1}{\delta_h}\, d\ell\, d\ell'\,,$$

where y_k is a component of Delaunay's elements Y, can be extended to two different analytic maps $\mathcal{G}^+_{h,k}, \mathcal{G}^-_{h,k}$, defined in $\mathcal{W}$.

Proof. We choose $\mathcal{W}$ as in Proposition 2 and, if necessary, we restrict this neighborhood by requiring that $\tau_1^{(h)} \times \tau_2^{(h)} \neq 0$ in $\mathcal{W}$, so that $\tilde{d}_h|_{\mathcal{W}}$ is analytic. To investigate the behavior close to the singularity, for each $\mathcal{E} \in \mathcal{W}$, we can restrict the integrals to the domain

$$\mathcal{D} = \mathcal{D}(V_h, r) = \{V \in \mathbb{T}^2 : (V - V_h) \cdot \mathcal{A}_h (V - V_h) \leq r^2\}\,, \tag{21}$$

for a suitable $r > 0$. By using the coordinate change $\xi = \mathcal{A}_h^{1/2}(V - V_h)$ and then polar coordinates (ρ, θ), defined by $(\rho\cos\theta, \rho\sin\theta) = \xi$, we have

$$\begin{aligned}\int_{\mathcal{D}} \frac{1}{\delta_h}\, d\ell d\ell' &= \frac{1}{\sqrt{\det \mathcal{A}_h}} \int_{\mathcal{B}} \frac{1}{\sqrt{d_h^2 + |\xi|^2}}\, d\xi \\ &= \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \int_0^r \frac{\rho}{\sqrt{d_h^2 + \rho^2}}\, d\rho = \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} (\sqrt{d_h^2 + r^2} - d_h)\,,\end{aligned}$$

with $\mathcal{B} = \{\xi \in \mathbb{R}^2 : |\xi| \leq r\}$. The term $-2\pi d_h/\sqrt{\det \mathcal{A}_h}$ is not differentiable at $\mathcal{E} = \mathcal{E}_c \in \Sigma$. We set

$$\mathcal{F}_{h,k} = \frac{\partial}{\partial y_k}\Big(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\Big)\sqrt{d_h^2 + r^2} + \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{\tilde{d}_h}{\sqrt{d_h^2 + r^2}} \frac{\partial \tilde{d}_h}{\partial y_k}$$

with $\tilde{d}_h$ as in (1), and define on $\mathcal{W}$ the two analytic maps

$$\mathcal{G}^\pm_{h,k} = \mathcal{F}_{h,k} \mp \frac{\partial}{\partial y_k}\Big(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}}\Big)\tilde{d}_h \mp \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{\partial \tilde{d}_h}{\partial y_k} + \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2 \setminus \mathcal{D}} \frac{1}{\delta_h}\, d\ell\, d\ell'\,. \tag{22}$$

We observe that $\mathcal{G}^+_{h,k}$ (resp. $\mathcal{G}^-_{h,k}$) corresponds to the derivative of $\int_{\mathbb{T}^2} 1/\delta_h\, d\ell\, d\ell'$ with respect to y_k on $\mathcal{W}^+$ (resp. $\mathcal{W}^-$).

□

Now we state the main result.

Theorem 4.2. *The averages over $\mathbb{T}^2$ of the derivatives of R with respect to Delaunay's elements y_k can be extended to two Lipschitz–continuous maps $\left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^{\pm}$ on a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. These maps, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\frac{\overline{\partial R}}{\partial y_k}$. Moreover the following relations hold:*

$$
\begin{aligned}
\mathrm{Diff}_h\left(\frac{\overline{\partial R}}{\partial y_k}\right) &\stackrel{def}{=} \left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^{-} - \left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^{+} = \\
&= \frac{\mu k^2}{\pi}\left[\frac{\partial}{\partial y_k}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial \tilde{d}_h}{\partial y_k}\right],
\end{aligned} \tag{23}
$$

with the derivatives of $\tilde{d}_h$ given by (2).

Proof. Using the results of Propositions 2, 3 we define the extended maps by

$$
\left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^{\pm} = \frac{\mu k^2}{(2\pi)^2}\left[\int_{\mathbb{T}^2}\frac{\partial}{\partial y_k}\left(\frac{1}{d} - \frac{1}{\delta_h}\right) d\ell\, d\ell' + \mathcal{G}_{h,k}^{\pm}\right],
$$

with $\mathcal{G}_{h,k}^{\pm}$ given by (22). We show that the maps $\mathcal{E} \mapsto \left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^{\pm}(\mathcal{E})$ are Lipschitz–continuous extensions to $\mathcal{W}$ of $\frac{\overline{\partial R}}{\partial y_k}$. The maps $\mathcal{G}_{h,k}^{\pm}$ are analytic in $\mathcal{W}$, thus we only have to study the integrals $\int_{\mathbb{T}^2}\frac{\partial}{\partial y_k}(1/d - 1/\delta_h)\, d\ell d\ell'$. From Proposition 2 we know that these maps are continuous.

We only need to investigate the behavior close to the singularity, therefore we restrict these integrals to the domain $\mathcal{D}$ introduced in (21). We prove that the maps

$$
\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_j}\int_{\mathcal{D}}\frac{\partial}{\partial y_k}\frac{1}{\delta_h}\, d\ell d\ell', \qquad \mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_j}\int_{\mathcal{D}}\frac{\partial}{\partial y_k}\frac{1}{d}\, d\ell d\ell',
$$

with $j = 1\ldots 4$, are bounded. First observe that the derivatives

$$
\frac{\partial}{\partial y_j}\int_{\mathcal{D}}\frac{\partial}{\partial y_k}\frac{1}{\delta_h}\, d\ell d\ell' = \int_{\mathcal{D}}\left(\frac{3}{4}\frac{1}{\delta_h^5}\frac{\partial \delta_h^2}{\partial y_j}\frac{\partial \delta_h^2}{\partial y_k} - \frac{1}{2}\frac{1}{\delta_h^3}\frac{\partial^2 \delta_h^2}{\partial y_j \partial y_k}\right) d\ell d\ell' \tag{24}
$$

are bounded in $\mathcal{W}\setminus\Sigma$, otherwise we could not find the analytic extensions $\mathcal{G}_{h,k}^{+}, \mathcal{G}_{h,k}^{-}$ introduced in Proposition 3.[5] Then we show that the maps

$$
\frac{\partial}{\partial y_j}\int_{\mathcal{D}}\frac{\partial}{\partial y_k}\frac{1}{d}\, d\ell d\ell' = \int_{\mathcal{D}}\left(\frac{3}{4}\frac{1}{d^5}\frac{\partial d^2}{\partial y_j}\frac{\partial d^2}{\partial y_k} - \frac{1}{2}\frac{1}{d^3}\frac{\partial^2 d^2}{\partial y_j \partial y_k}\right) d\ell d\ell' \tag{25}
$$

[5] Actually we can prove that

$$
\frac{3}{4}\int_{\mathcal{D}}\frac{1}{\delta_h^5}\frac{\partial \delta_h^2}{\partial y_j}\frac{\partial \delta_h^2}{\partial y_k}\, d\ell d\ell' = \mathfrak{T}_{j,k}^{(h)} + \mathfrak{U}_{j,k}^{(h)}, \qquad \frac{1}{2}\int_{\mathcal{D}}\frac{1}{\delta_h^3}\frac{\partial^2 \delta_h^2}{\partial y_j \partial y_k}\, d\ell d\ell' = \mathfrak{T}_{j,k}^{(h)} + \mathfrak{V}_{j,k}^{(h)},
$$

where $\mathfrak{U}_{j,k}^{(h)}, \mathfrak{V}_{j,k}^{(h)}$ are bounded in $\mathcal{W}\setminus\Sigma$, and

$$
\mathfrak{T}_{j,k}^{(h)} = \frac{2\pi}{d_h\sqrt{\det\mathcal{A}_h}}\left(\frac{\partial d_h}{\partial \mathcal{E}_j}\frac{\partial d_h}{\partial \mathcal{E}_k} + \frac{\partial V_h}{\partial \mathcal{E}_j}\cdot\mathcal{A}_h\frac{\partial V_h}{\partial \mathcal{E}_k}\right)
$$

is unbounded but cancels out in the difference.

are bounded in $\mathcal{W} \setminus \Sigma$. Using (7), (10) we write the integrand function in the right hand side of (25) as the sum of

$$\frac{3}{4}\frac{1}{\delta_h^5}\frac{\partial\delta_h^2}{\partial y_j}\frac{\partial\delta_h^2}{\partial y_k}\,, \qquad -\frac{1}{2}\frac{1}{\delta_h^3}\frac{\partial^2\delta_h^2}{\partial y_j\partial y_k} \tag{26}$$

and of terms of the following kind:

$$\frac{3}{4}\frac{1}{\delta_h^5}\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_j}\left[\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_k}\big(1+\mathcal{P}_5^{(h)}\big)+\mathcal{P}_5^{(h)}\frac{\partial\delta_h^2}{\partial y_k}\right]\,, \qquad -\frac{1}{2}\frac{\mathcal{P}_3^{(h)}}{\delta_h^3}\frac{\partial^2\mathcal{R}_3^{(h)}}{\partial y_j\partial y_k}\,, \tag{27}$$

$$\frac{3}{4}\frac{1}{\delta_h^5}\frac{\partial\mathcal{R}_3^{(h)}}{\partial y_j}\frac{\partial\delta_h^2}{\partial y_k}\,, \qquad -\frac{1}{2}\frac{1}{\delta_h^3}\frac{\partial^2\mathcal{R}_3^{(h)}}{\partial y_j\partial y_k}\,, \tag{28}$$

$$\frac{3}{4}\frac{\mathcal{P}_5^{(h)}}{\delta_h^5}\frac{\partial\delta_h^2}{\partial y_j}\frac{\partial\delta_h^2}{\partial y_k}\,, \qquad -\frac{1}{2}\frac{\mathcal{P}_3^{(h)}}{\delta_h^3}\frac{\partial^2\delta_h^2}{\partial y_j\partial y_k}\,. \tag{29}$$

The integrals over $\mathcal{D}$ of the terms in (26) are not bounded in $\mathcal{W} \setminus \Sigma$, but their sum is bounded and corresponds to (24). In the following we denote by $C_j, j = 15 \ldots 34$ some positive constants. Moreover we use the relation $d^2 = \delta_h^2 + \mathcal{R}_3^{(h)}$ and the developments

$$\frac{1}{d^s} = \frac{1}{(\delta_h^2+\mathcal{R}_3^{(h)})^{s/2}} = \frac{1}{\delta_h^s}\left[1+\mathcal{P}_s^{(h)}\right] \qquad\qquad (s=3,5)$$

with

$$\mathcal{P}_s^{(h)} = \mathcal{P}_s^{(h)}(\mathcal{E},V) = \sum_{|\beta|=1} p_{\beta,s}^{(h)}(\mathcal{E},V)(V-V_h)^\beta\,,$$

$$p_{\beta,s}^{(h)}(\mathcal{E},V) = \int_0^1 D^\beta\left[\left(1+\frac{\mathcal{R}_3^{(h)}}{\delta_h^2}\right)^{-s/2}\right](\mathcal{E},V_h+t(V-V_h))\,dt\,. \tag{30}$$

By developing (30) we obtain

$$\begin{aligned}
p_{\beta,s}^{(h)}(\mathcal{E},V) &= -\frac{s}{2}\int_0^1\left[\left(1+\frac{\mathcal{R}_3^{(h)}}{\delta_h^2}\right)^{-\frac{s}{2}-1}D^\beta\left(\frac{\mathcal{R}_3^{(h)}}{\delta_h^2}\right)\right](\mathcal{E},V_h+t(V-V_h))\,dt\\
&= -\frac{s}{2}\int_0^1 D^\beta\left(\frac{\mathcal{R}_3^{(h)}}{\delta_h^2}\right)(\mathcal{E},V_h+t(V-V_h))\,dt+\mathfrak{R}_s^{(h)}(\mathcal{E},V)\,,
\end{aligned}$$

with $|\mathfrak{R}_s^{(h)}(\mathcal{E},V)| \le C_{15}|V-V_h|, s=3,5$. Moreover, we have

$$D^\beta\left(\frac{\mathcal{R}_3^{(h)}}{\delta_h^2}\right) = \frac{D^\beta\mathcal{R}_3^{(h)}}{\delta_h^2} - \frac{\mathcal{R}_3^{(h)}}{\delta_h^4}D^\beta\delta_h^2\,. \tag{31}$$

We can estimate the terms in (31) as follows:

$$D^\beta\mathcal{R}_3^{(h)} = \sum_{|\alpha|=3}\left[D^\beta r_\alpha^{(h)}(V-V_h)^\alpha + r_\alpha^{(h)}D^\beta(V-V_h)^\alpha\right]$$

where

$$|D^\beta r_\alpha^{(h)}| \le C_{16}\,, \qquad |D^\beta(V-V_h)^\alpha| \le C_{17}|V-V_h|^2\,,$$

so that

$$|D^\beta \mathcal{R}_3^{(h)}| \le C_{18} |V - V_h|^2$$

Moreover

$$D^\beta \delta_h^2 = 2 D^\beta (V - V_h) \cdot \mathcal{A}_h (V - V_h)$$

so that

$$|D^\beta \delta_h| \le C_{19} |V - V_h| \ .$$

We conclude that

$$\left| D^\beta \left(\frac{\mathcal{R}_3^{(h)}}{\delta_h^2} \right) \right| \le C_{20}\,, \quad \text{so that} \ \ |p_{\beta,s}^{(h)}(\mathcal{E}, V)| \le C_{21}\,,$$

and we obtain the estimate

$$|\mathcal{P}_s^{(h)}(\mathcal{E}, V)| \le C_{22} |V - V_h| \tag{32}$$

for $(\mathcal{E}, V) \in \mathcal{U}_0$. Using (16), (18), (32) and the estimate

$$\left| \frac{\partial^2 \mathcal{R}_3^{(h)}}{\partial y_j \partial y_k} \right| \le C_{23} |V - V_h|\,,$$

that follows from the boundedness of

$$r_\alpha^{(h)}, \quad \frac{\partial r_\alpha^{(h)}}{\partial y_k}, \quad \frac{\partial^2 r_\alpha^{(h)}}{\partial y_j \partial y_k}, \quad \frac{\partial V_h}{\partial y_k}, \quad \frac{\partial^2 V_h}{\partial y_j \partial y_k}\,,$$

we can bound both terms in (27) by $C_{24}/|V - V_h|$, which has finite integral over $\mathcal{D}$.[6]

To estimate the integrals of the terms in (28) we observe that

$$\frac{\partial \mathcal{R}_3^{(h)}}{\partial y_j} = \sum_{|\alpha|=3} r_{\alpha,0}^{(h)} \frac{\partial (V - V_h)^\alpha}{\partial y_j} + \mathfrak{S}_3^{(h)}\,, \quad \frac{\partial^2 \mathcal{R}_3^{(h)}}{\partial y_j \partial y_k} = \sum_{|\alpha|=3} r_{\alpha,0}^{(h)} \frac{\partial^2 (V - V_h)^\alpha}{\partial y_j \partial y_k} + \mathfrak{S}_2^{(h)}\,,$$

with

$$r_{\alpha,0}^{(h)} = r_{\alpha,0}^{(h)}(\mathcal{E}) = r_\alpha^{(h)}(\mathcal{E}, V_h(\mathcal{E}))\,, \qquad |\mathfrak{S}_i^{(h)}| \le C_{25} |V - V_h|^i \qquad (i = 2, 3)\ .$$

Then, using (16) and writing dV for $d\ell d\ell'$, we have

$$\begin{aligned} \left| \int_{\mathcal{D}} \frac{1}{\delta_h^5} \frac{\partial \mathcal{R}_3^{(h)}}{\partial y_j} \frac{\partial \delta_h^2}{\partial y_k}\, dV \right| &\le \left| \frac{\partial d_h^2}{\partial y_k} \right| \left(\sum_{|\alpha|=3} |r_{\alpha,0}^{(h)}| \int_{\mathcal{D}} \left| \frac{1}{\delta_h^5} \frac{\partial (V - V_h)^\alpha}{\partial y_j} \right| dV + \int_{\mathcal{D}} \frac{|\mathfrak{S}_3^{(h)}|}{\delta_h^5}\, dV \right) \\ &+ 2 \sum_{|\alpha|=3} |r_{\alpha,0}^{(h)}| \left| \int_{\mathcal{D}} \frac{1}{\delta_h^5} \frac{\partial (V - V_h)^\alpha}{\partial y_j} \left[\frac{\partial V_h}{\partial y_k} \cdot \mathcal{A}_h (V - V_h) \right] dV \right| + C_{26} \int_{\mathcal{D}} \frac{|V - V_h|^4}{\delta_h^5}\, dV \end{aligned} \tag{33}$$

[6] The boundedness of $\frac{\partial^2 V_h}{\partial y_j \partial y_k}$ on $\mathcal{W}$ follows by differentiating with respect to y_j the relation

$$\mathcal{H}_h(\mathcal{E}) \frac{\partial V_h}{\partial y_k}(\mathcal{E}) = -\frac{\partial}{\partial y_k} \nabla_V d^2(\mathcal{E}, V_h(\mathcal{E}))\ .$$

and

$$\left|\int_{\mathcal{D}} \frac{1}{\delta_h^3} \frac{\partial^2 \mathcal{R}_3^{(h)}}{\partial y_j \partial y_k}\, dV\right| \le \sum_{|\alpha|=3} \left|r_{\alpha,0}^{(h)}\right| \left|\int_{\mathcal{D}} \frac{1}{\delta_h^3} \frac{\partial^2 (V-V_h)^\alpha}{\partial y_j \partial y_k}\, dV\right| + C_{27} \int_{\mathcal{D}} \frac{|V-V_h|^2}{\delta_h^3}\, dV \,. \tag{34}$$

Passing to polar coordinates (ρ,θ), defined by $(\rho\cos\theta, \rho\sin\theta) = \mathcal{A}_h^{1/2}(V-V_h)$, we find that

$$\left|\frac{\partial d_h^2}{\partial y_k}\right| \int_{\mathcal{D}} \left|\frac{1}{\delta_h^5} \frac{\partial (V-V_h)^\alpha}{\partial y_j}\right| dV \le C_{28}\,, \qquad \left|\frac{\partial d_h^2}{\partial y_k}\right| \int_{\mathcal{D}} \frac{|\mathfrak{S}_3^{(h)}|}{\delta_h^5}\, dV \le C_{29}$$

for each $\mathcal{E} \in \mathcal{W}\setminus\Sigma$ and α with $|\alpha|=3$, in fact

$$\int_0^r \frac{\rho^i}{(d_h^2+\rho^2)^{5/2}}\, d\rho \le \frac{C_{30}}{d_h} \qquad (i=3,4)\,. \tag{35}$$

Moreover, passing to polar coordinates (ρ,θ), we have

$$\begin{aligned} &\int_{\mathcal{D}} \frac{1}{\delta_h^5} \frac{\partial (V-V_h)^\alpha}{\partial y_j} \left[\frac{\partial V_h}{\partial y_k} \cdot \mathcal{A}_h (V-V_h)\right] dV = \\ = \;& \int_0^r \frac{\rho^4}{(d_h^2+\rho^2)^{5/2}}\, d\rho \sum_{|\gamma|=3} c_\gamma \int_0^{2\pi} (\cos\theta)^{\gamma_1} (\sin\theta)^{\gamma_2}\, d\theta = 0\,. \end{aligned} \tag{36}$$

for some functions $c_\gamma : \mathcal{W}\setminus\Sigma \to \mathbb{R}$, $\gamma = (\gamma_1,\gamma_2)$. Thus the integrals in (33) are uniformly bounded in $\mathcal{W}\setminus\Sigma$. In (36) we have used

$$\int_0^{2\pi} (\cos\theta)^{\gamma_1} (\sin\theta)^{\gamma_2}\, d\theta = 0 \tag{37}$$

for each γ, with odd $|\gamma| = \gamma_1+\gamma_2$. Finally, using again (37), we obtain

$$\begin{aligned} \left|\int_{\mathcal{D}} \frac{1}{\delta_h^3} \frac{\partial^2 (V-V_h)^\alpha}{\partial y_j \partial y_k}\, dV\right| \;&\le\; \left|\sum_{|\gamma|=1} b_\gamma \int_0^{2\pi} (\cos\theta)^{\gamma_1} (\sin\theta)^{\gamma_2}\, d\theta\right| \int_0^r \frac{\rho^2}{(d_h^2+\rho^2)^{3/2}}\, d\rho \\ &+\; C_{31} \int_{\mathcal{D}} \frac{1}{|V-V_h|}\, dV = C_{31} \int_{\mathcal{D}} \frac{1}{|V-V_h|}\, dV\,. \end{aligned}$$

for some functions $b_\gamma : \mathcal{W}\setminus\Sigma \to \mathbb{R}$. Hence also the integrals in (34) are uniformly bounded in $\mathcal{W}\setminus\Sigma$.

To estimate the integrals of the terms in (29) we make the following decomposition:

$$p_{\beta,s}^{(h)} = q_{\beta,s}^{(h)} + w_{\beta,s}^{(h)}\,,$$

with

$$q_{\beta,s}^{(h)} = -\frac{s}{2} \sum_{|\alpha|=3} r_{\alpha,0}^{(h)} \int_0^1 \left[\frac{1}{\delta_h^2}\left(D^\beta (V-V_h)^\alpha - \frac{D^\beta \delta_h^2}{\delta_h^2} (V-V_h)^\alpha\right)\right] (\mathcal{E}, V_h + t(V-V_h))\, dt$$

and $|w^{(h)}_{\beta,s}| \le C_{32}|V-V_h|$. For the first term in (29) we obtain

$$
\begin{aligned}
\left|\int_{\mathcal{D}} \frac{\mathcal{P}^{(h)}_5}{\delta^5_h}\frac{\partial\delta^2_h}{\partial y_j}\frac{\partial\delta^2_h}{\partial y_k}\,dV\right| &\le \left|\frac{\partial d^2_h}{\partial y_j}\frac{\partial d^2_h}{\partial y_k}\int_{\mathcal{D}}\frac{1}{\delta^5_h}\sum_{|\beta|=1} q^{(h)}_{\beta,5}(V-V_h)^\beta\,dV\right| \\
+\quad 4&\left|\int_{\mathcal{D}}\frac{1}{\delta^5_h}\sum_{|\beta|=1} q^{(h)}_{\beta,5}(V-V_h)^\beta\left[\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h(V-V_h)\right]\left[\frac{\partial V_h}{\partial y_k}\cdot\mathcal{A}_h(V-V_h)\right]dV\right| + C_{33}
\end{aligned}
\tag{38}
$$

where we have used polar coordinates and the inequalities (35). The two integrals at the right hand side of (38) vanish: in fact using Fubini-Tonelli's theorem and passing to polar coordinates (ρ,θ), by relations (37) we obtain

$$
\begin{aligned}
&\int_{\mathcal{D}}\frac{1}{\delta^5_h}\sum_{|\beta|=1} q^{(h)}_{\beta,5}(V-V_h)^\beta\,dV = \\
&=\sum_{|\beta|=1}\sum_{|\gamma|\in\{3,5\}}\int_0^1\int_0^r \phi_{\beta,\gamma}(\rho,t)\,d\rho\,dt\int_0^{2\pi}(\cos\theta)^{\gamma_1}(\sin\theta)^{\gamma_2}\,d\theta = 0
\end{aligned}
$$

for some functions $\phi_{\beta,\gamma}:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$. The computation for the other integral is analogous.
The second term in (29) is estimated in a similar way:

$$
\left|\int_{\mathcal{D}}\frac{\mathcal{P}^{(h)}_3}{\delta^3_h}\frac{\partial^2\delta^2_h}{\partial y_j\partial y_k}\,dV\right| \le \left|\left(\frac{\partial^2 d^2_h}{\partial y_j\partial y_k}+2\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h\frac{\partial V_h}{\partial y_k}\right)\int_{\mathcal{D}}\frac{1}{\delta^3_h}\sum_{|\beta|=1} q^{(h)}_{\beta,3}(V-V_h)^\beta\,dV\right| + C_{34}\,,
$$

and the integral at the right hand side vanishes as well.

We conclude the proof observing that, using (22) and the theorem of differentiation under the integral sign, the derivatives $\left(\overline{\frac{\partial R}{\partial y_k}}\right)^+_h$, $\left(\overline{\frac{\partial R}{\partial y_k}}\right)^-_h$, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\overline{\frac{\partial R}{\partial y_k}}$, and their difference in $\mathcal{W}$ is given by (23). □

Remark 2. If $\mathcal{E}_c$ is an orbit configuration with two crossings, assuming that $d_h(\mathcal{E}_c)=0$ for $h=1,2$, we can extract the singularity by considering the approximated distances δ_1,δ_2 and the remainder function $1/d-1/\delta_1-1/\delta_2$.

5 Generalized solutions

We show that generically we can uniquely extend the solutions of (5) beyond the crossing singularity $d_{min}=0$. This is obtained by patching together classical solutions defined in the domains $\mathcal{W}^+$ with solutions defined in $\mathcal{W}^-$, or *vice versa*.

Let $a>0$ be a value for the semimajor axis of the asteroid and $\overline{Y}: I\to\mathbb{R}^4$ be a continuous function defined in an open interval $I\subset\mathbb{R}$, representing a possible evolution of the asteroid orbital elements $Y=(G,Z,g,z)$. We introduce

$$
\overline{\mathcal{E}}(t) = (\overline{E}(t),\overline{E}'(t))\,, \tag{39}
$$

with

$$
\overline{E}(t) = (k\sqrt{a},\overline{Y}(t))\,, \tag{40}
$$

where k is Gauss' constant and $\overline{E}'$ is a known function of time representing the evolution of the Earth.[7]

Let $T(\overline{Y})$ be the set of times $t_c \in I$ such that $d_{min}(\overline{\mathcal{E}}(t_c)) = 0$, and assume that each t_c is isolated, so that we can represent the set

$$I \setminus T(\overline{Y}) = \sqcup_{j\in\mathcal{N}} I_j$$

as disjoint union of open intervals I_j, with $\mathcal{N}$ a countable (possibly finite) set.

Definition 5.1. We say that $\overline{Y}$ is a generalized solution of (5) if its restriction to each $I_j, j \in \mathcal{N}$ is a classical solution of (5) and, for each $t_c \in T(\overline{Y})$, there exist finite values of

$$\lim_{t\to t_c^+} \dot{\overline{Y}}(t)\,, \qquad \lim_{t\to t_c^-} \dot{\overline{Y}}(t)\ .$$

Choose $Y_0 \in \mathbb{R}^4$ and a time t_0 such that $d_{min}(\mathcal{E}_0) > 0$, with $\mathcal{E}_0 = (E_0, E_0')$, $E_0 = (k\sqrt{a}, Y_0)$, $E_0' = E'(t_0)$. We show how we can construct a generalized solution of the Cauchy problem

$$\dot{\overline{Y}} = -\mathbb{J}_2\,\overline{\nabla_Y R}\,, \qquad \overline{Y}(t_0) = Y_0\ . \tag{41}$$

Let $\overline{Y}(t)$ be the maximal classical solution of (41), defined in the maximal interval J. Assume that $t_c = \sup J < +\infty$, and $\lim_{t\to t_c^-} \overline{\mathcal{E}}(t) = \mathcal{E}_c$, with $\mathcal{E}_c$ a non-degenerate crossing configuration such that $d_{min}(\mathcal{E}_c) = d_h(\mathcal{E}_c) = 0$ for some h. Let $\mathcal{W}$, $\mathcal{W}^{\pm}$ be chosen as in Theorem 4.2. Suppose that there exists $\tau \in (t_0, t_c)$ such that $\overline{\mathcal{E}}(t) \in \mathcal{W}^+$ for $t \in (\tau, t_c)$. Let $Y_\tau = \overline{Y}(\tau)$. By Theorem 4.2 there exists $\dot{Y}_c \in \mathbb{R}^4$ such that

$$\lim_{t\to t_c^-} \dot{\overline{Y}}(t) = \dot{Y}_c\,, \tag{42}$$

in fact relation (42) is fulfilled by the solution of the Cauchy problem[8]

$$\dot{\overline{Y}} = -\mathbb{J}_2\,(\overline{\nabla_Y R})_h^+\,, \qquad \overline{Y}(\tau) = Y_\tau\,, \tag{43}$$

which corresponds to the solution of (41) in the interval (τ, t_c) and is defined also at the crossing time t_c. Let us denote by Y_c its value for $t = t_c$. Using again Theorem 4.2 we can extend $\overline{Y}(t)$ beyond the crossing singularity by considering the new problem

$$\dot{\overline{Y}} = -\mathbb{J}_2\,(\overline{\nabla_Y R})_h^-\,, \qquad \overline{Y}(t_c) = Y_c\ . \tag{44}$$

The solution of (44) fulfils

$$\lim_{t\to t_c^+} \dot{\overline{Y}}(t) = \dot{Y}_c + \mathrm{Diff}_h(\overline{\nabla_Y R})(\overline{\mathcal{E}}(t_c))\ . \tag{45}$$

The vector field in (44) corresponds to $-\mathbb{J}_2\overline{\nabla_Y R}$ on $\mathcal{W}^-$, thus we can continue the solution outside $\mathcal{W}$ and this procedure can be repeated at almost every crossing singularities. Indeed, the generalized solution is unique provided the evolution $t \mapsto \overline{\mathcal{E}}(t)$ is not tangent to the orbit crossing set Σ.

[7] In the case of one perturbing planet $\overline{E}'(t)$ is constant and represents the trajectory of a solution of the 2-body problem. If we consider more than one perturbing planet then $\overline{E}'(t)$ changes with time due to the planetary perturbations.

[8] Here $(\overline{\nabla_Y R})_h^+$ is the vector with components $(\frac{\overline{\partial R}}{\partial y_k})_h^+$, $k = 1 \ldots 4$ introduced in Theorem 4.2.

Moreover, if $\det \mathcal{A}_h = 0$ the extraction of the singularity, described in Section 4, cannot be performed.

In case $\overline{\mathcal{E}}(t) \in \mathcal{W}^-$ for $t \in (\tau, t_c)$ the previous discussion still holds if we exchange $(\overline{\nabla_Y R})_h^+$ with $(\overline{\nabla_Y R})_h^-$. In this case (45) becomes

$$\lim_{t \to t_c^+} \dot{\overline{Y}}(t) = \dot{Y}_c - \mathrm{Diff}_h(\overline{\nabla_Y R})(\overline{\mathcal{E}}(t_c)) \; .$$

6 Evolution of the orbit distance

We prove that the secular evolution of $\tilde{d}_{min}$ is more regular than that of the orbital elements in a neighborhood of a planet crossing time. We introduce the secular evolution of the distances $\tilde{d}_h$ and of the orbit distance $\tilde{d}_{min}$:

$$\overline{d}_h(t) = \tilde{d}_h(\overline{\mathcal{E}}(t)) \, , \qquad \overline{d}_{min}(t) = \tilde{d}_{min}(\overline{\mathcal{E}}(t)) \; . \tag{46}$$

Assume these maps are defined in an open interval containing a crossing time t_c, and suppose $\mathcal{E}_c = \overline{\mathcal{E}}(t_c)$ is a non-degenerate crossing configuration at time t_c, as in Section 4.

In the following we shall discuss only the case of $\tilde{d}_h$. The same result holds for $\tilde{d}_{min}$, taking care of the possible exchange of role of two local minima d_h, d_k as absolute minimum.

Proposition 4. *Let $\overline{Y}(t)$ be a generalized solution of (41) and $\overline{\mathcal{E}}(t)$ as in (39), (40). Assume $t_c \in T(\overline{Y})$ is a crossing time and $\mathcal{E}_c = \overline{\mathcal{E}}(t_c)$ is a non-degenerate crossing configuration with only one crossing point. Then there exists an interval (t_a, t_b), $t_a < t_c < t_b$ such that $\overline{d}_h \in C^1((t_a, t_b); \mathbb{R})$.*

Proof. Let the interval (t_a, t_b) be such that $\overline{\mathcal{E}}((t_a, t_b)) \subset \mathcal{W}$, where $\mathcal{W}$ is chosen as in Theorem 4.2. We can assume that $\overline{\mathcal{E}}(t) \in \mathcal{W}^+$ for $t \in (t_a, t_c)$, $\overline{\mathcal{E}}(t) \in \mathcal{W}^-$ for $t \in (t_c, t_b)$ (the proof for the opposite case is similar). For $t \in (t_a, t_b) \setminus \{t_c\}$ the time derivative of $\overline{d}_h$ is

$$\begin{aligned} \dot{\overline{d}}_h(t) &= \nabla_{\mathcal{E}} \tilde{d}_h(\overline{\mathcal{E}}(t)) \cdot \dot{\overline{\mathcal{E}}}(t) = \nabla_Y \tilde{d}_h(\overline{\mathcal{E}}(t)) \cdot \dot{\overline{Y}}(t) + \nabla_{E'} \tilde{d}_h(\overline{\mathcal{E}}(t)) \cdot \dot{\overline{E}}'(t) \\ &= -\left(\nabla_Y \tilde{d}_h \cdot \mathbb{J}_2 \overline{\nabla_Y R}\right)(\overline{\mathcal{E}}(t)) + \nabla_{E'} \tilde{d}_h(\overline{\mathcal{E}}(t)) \cdot \dot{\overline{E}}'(t) \; . \end{aligned}$$

Here $\nabla_{\mathcal{E}}$, ∇_Y, $\nabla_{E'}$ denote the vectors of partial derivatives with respect to $\mathcal{E}, Y, E'$ respectively. The derivative $\dot{\overline{E}}'(t)$ exists also for $t = t_c$. On the other hand, by Theorem 4.2, the restrictions of $\overline{\nabla_Y R}(\overline{\mathcal{E}}(t))$ to $t < t_c$ and $t > t_c$ admit two different continuous extensions to t_c. By (23), since $\tilde{d}_h(\overline{\mathcal{E}}(t_c)) = 0$, we have

$$\begin{aligned} \lim_{t \to t_c^+} \dot{\overline{d}}_h(t) - \lim_{t \to t_c^-} \dot{\overline{d}}_h(t) &= \mathrm{Diff}_h\left(\overline{\nabla_Y R}\right) \cdot \mathbb{J}_2 \nabla_Y \tilde{d}_h \Big|_{\mathcal{E} = \mathcal{E}_c} \\ &= \frac{\mu k^2}{\pi \sqrt{\det \mathcal{A}_h}} \, \{\tilde{d}_h, \tilde{d}_h\}_Y \Big|_{\mathcal{E} = \mathcal{E}_c} = 0 \, , \end{aligned}$$

where $\{,\}_Y$ is the Poisson bracket with respect to Y. Thus the time derivative of $\overline{d}_h$ exists and is continuous also in $t = t_c$. ∎

7 Numerical experiments

7.1 The secular evolution program

Using a model with 5 planets, from Venus to Saturn, we compute a planetary ephemerides database for a time span of $50,000$ yrs starting from epoch 0 MJD (November 17, 1858) with a time step of 20 yrs. The computation is performed using the FORTRAN program `orbit9`, included in the OrbFit free software[9]. From this database we can obtain, by linear interpolation, the orbital elements of the planets at any time in the specified time interval.

We describe the algorithm to compute the solutions of the averaged equations (5) beyond the singularity, where R is now the sum of the perturbing functions R_i, $i = 1 \ldots 5$, each related to a different planet. We use a Runge-Kutta-Gauss (RKG) method to perform the integration: it evaluates the averaged vector field only at intermediate points of the integration time interval. When the asteroid trajectory is close enough to an orbit crossing, then the time step is decreased to reach the crossing condition exactly.

From Theorem 4.2 we can find two Lipschitz-continuous extensions of the averaged vector field from both sides of the singular set Σ.

To compute the solution beyond the singularity we use the explicit formula (23) giving the difference between the two extended vector fields, either of which corresponds to the averaged vector field on different sides of Σ. We compute the intermediate values of the extended vector field just after crossing, then we correct these values by (23) and use them as approximations of the averaged vector field in (5) at the intermediate points of the solutions, see Figure 3. This RKG algorithm avoids the computation of the extended vector field at the singular points, which may be affected by numerical instability.

A difficulty in the application of this scheme is to estimate the size of a suitable neighborhood $\mathcal{W}$ of the crossing configuration $\mathcal{E}_c$ fulfilling the conditions given in Section 4.

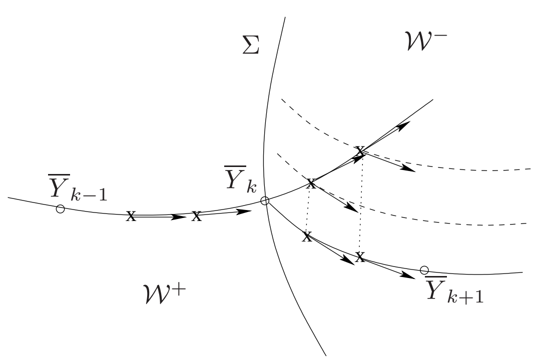

Figure 3: Runge-Kutta-Gauss method and continuation of the solutions of equations (5) beyond the singularity. The crosses correspond to the intermediate values.

[9] http://adams.dm.unipi.it/~orbmaint/orbfit/

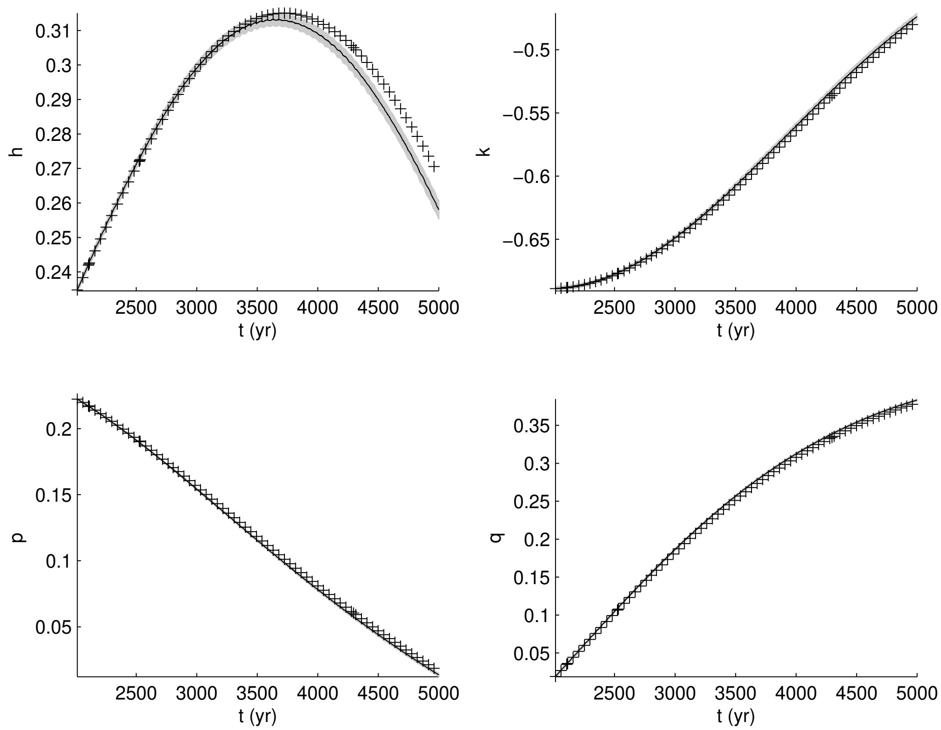

Figure 4: Averaged and non-averaged evolutions of asteroid 1979 XB.

7.2 Comparison with the solutions of the full equations

We performed some tests to compare the solutions of the averaged equations (5) with the corresponding components of the solutions of the full equations (4). Here we show two tests with the asteroids 1979 XB and 1620 (Geographos). We considered the system composed by an asteroid and 5 planets, from Venus to Saturn. We selected the 8 values $k\pi/4$, with $k = 0 \dots 7$, for the initial mean anomaly of the asteroid and the same for the planets. Using the program `orbit9`, we performed the integration of the system with these 64 different initial conditions (i.e. we selected equal initial phases for all the planets). Then we consider the arithmetic mean of the four equinoctial[10] orbital elements h, k, p, q of the asteroid over these evolutions, and compare them with the results of the secular evolution. In Figures 4, 5, we show the results: the crosses indicate the secular evolution, the continuous curve is the mean of full numerical one and the gray region represents the standard deviation from the mean. The correspondence between the solutions is good. During the evolution the distance between the asteroid and the Earth for some initial conditions attains values of the order of 10^{-4} au for 1620 (Geographos), and 10^{-3} au for 1979 XB. In Figure 5 the Earth crossing singularity is particularly evident near the epoch 3000 AD.

[10] We recall that

$$h = e\sin(\omega + \Omega)\,, \quad k = e\cos(\omega + \Omega)\,, \quad p = \tan(I/2)\sin(\Omega)\,, \quad q = \tan(I/2)\cos(\Omega)\ .$$

The equinoctial orbital elements have been introduced in [4].

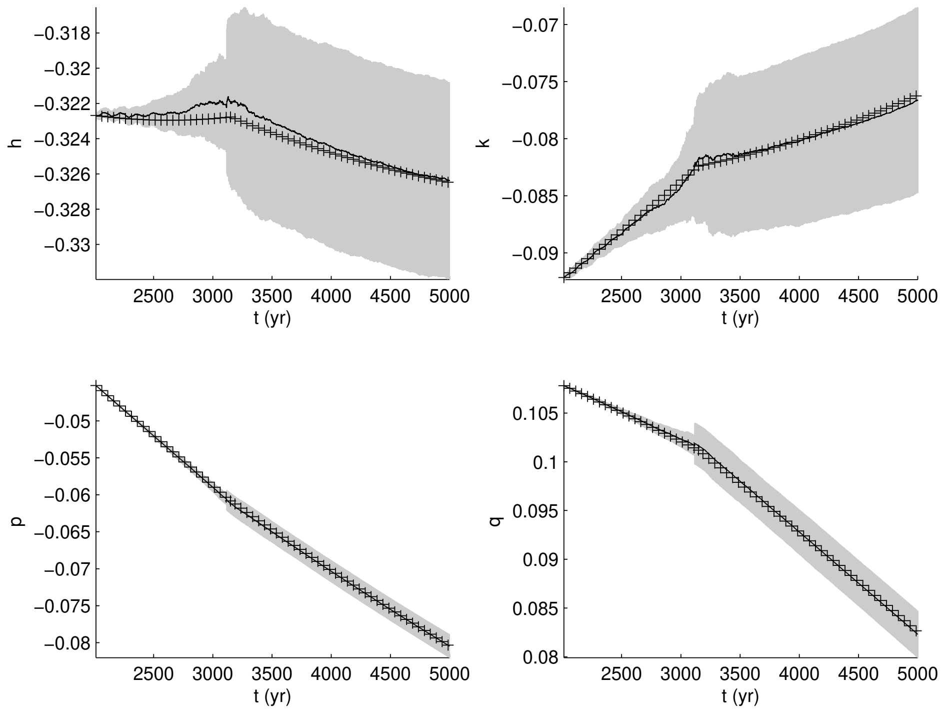

Figure 5: Averaged and non-averaged evolutions of asteroid 1620 (Geographos).

Some numerical tests of the theory introduced in [8], with the planets on circular coplanar orbits, can be found in [10].

7.3 An estimate of planet crossing times

The results of Section 6 can be used to estimate the epoch in which the orbit of a near-Earth asteroid will cross that of the Earth. We are interested in particular to study the behavior of those asteroids whose orbits will cross the Earth in the next few centuries, so that they must have a small value of d_{min} already at the present epoch. We can consider, for example, the set of potentially hazardous asteroids (PHAs), which have $d_{min} \leq 0.05$ au and absolute magnitude $H_{mag} \leq 22$, i.e. they are also large.

In Figure 6 we show 3 different evolutions of the signed orbit distance $\tilde{d}_{min}$ for the PHA 1979 XB. Here we draw the full numerical (solid line), secular (dashed) and secular linearized (dotted) evolution of $\tilde{d}_{min}$. By Proposition 4 the linearization of the secular evolution $\overline{d}_{min}(t)$ can give a good approximation also in a neighborhood of a crossing time.

We propose a method to compute an interval J of possible crossing times. We sample the *line of variation* (LOV), introduced in [16], which is a sort of 'spine' of the confidence region (see also [17]), and compute the signed orbit distance $\tilde{d}_{min}$ for each virtual asteroid (VA) of the sample. Then we compute the time derivative of $\bar{d}_{min}$ for each VA and extrapolate the crossing times by a linear approximation of the evolution. We set $J = [t_1, t_2]$, with t_1, t_2 the minimum and maximum crossing

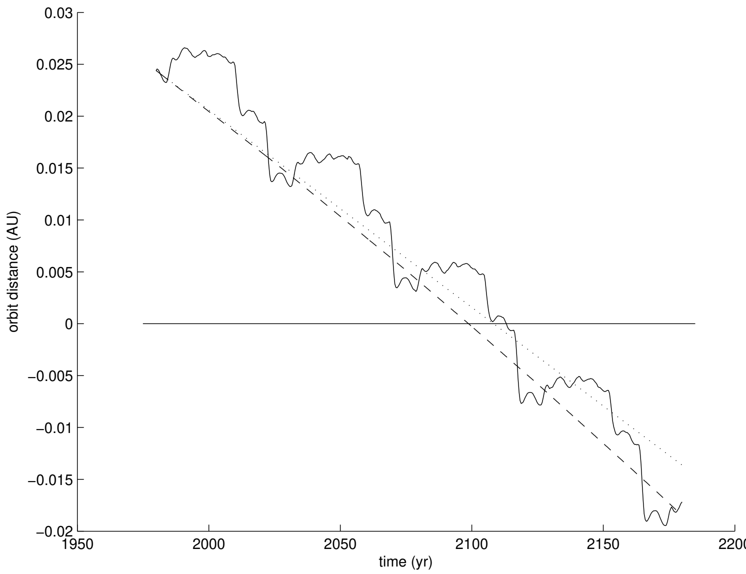

Figure 6: Different evolutions of $\tilde{d}_{min}$ for 1979 XB: full numerical (solid line), secular (dashed) and secular linearized (dotted).

times obtained (see Figure 7). In the computation of J we take into account a band centered at the Earth crossing line $d_{min} = 0$: in this test the width of the considered band is 2×10^{-3} au.

We describe a method to assign a probability of occurrence of crossings in a given time interval, which is related to the algorithm described above. For each value of the LOV parameter s we have a VA at a time t, so that we can compute $\bar{d}_{min}(t)$. Thus, using the scheme of Figure 7 we can define a map $\mathfrak{T}$ from the LOV parameter line to the time line. The map $\mathfrak{T}$ gives the crossing times (by using the linearized secular dynamics) for the VAs on the LOV that correspond to the selected values of the parameter s. Moreover, we have a probability density function $p(s)$ on the LOV. Therefore, given an interval I in the time line, we can consider the set $U_I = \mathfrak{T}^{-1}(I)$ and define the probability of having a crossing in the time interval I as

$$P(I) = \int_{U_I} p(s)\, ds \ .$$

Finally, in Figure 8 we show the corresponding interval J' obtained by computing the secular evolution (without linearization) of the orbit distance for each VA of 1979 XB. The sizes of J and J' are almost equal, but the left extremum of J' is ~ 10 years before.

8 Conclusions and future work

We have studied the double averaged restricted 3-body problem in case of orbit crossing singularities, improving and completing the results in [8], [5]. This problem is of interest to study the dynamics of near-Earth asteroids from a statistical point of view, going beyond the Lyapounov times of their orbits. We have also proved

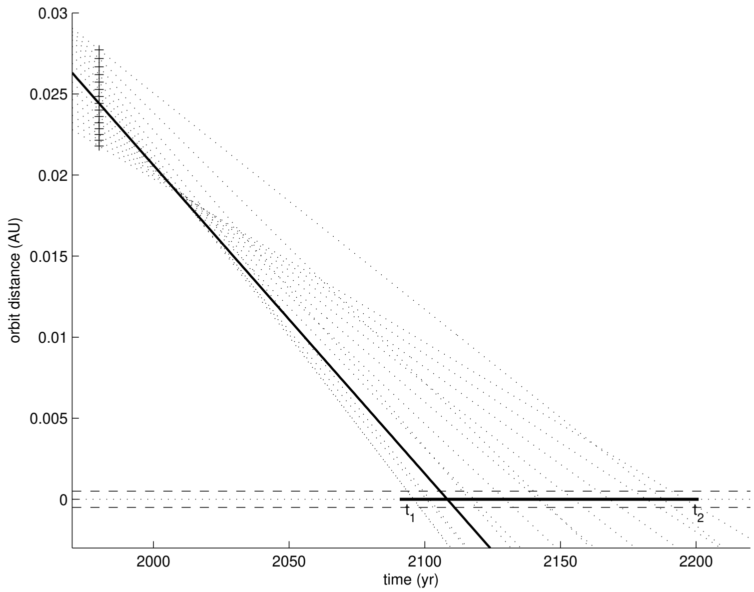

Figure 7: Computation of the interval J (horizontal solid line) for asteroid 1979 XB. The transversal solid line corresponds to the linearized secular evolution of the nominal orbit. The linearized secular evolution of the VAs are the dotted lines.

that generically, in a neighborhood of a crossing time, the secular evolution of the (signed) orbit distance is more regular than the averaged evolution of the orbital elements.

The solutions of this averaged problem have been computed by a numerical method and then compared with the solutions of the full equations in a few test cases. The results were good enough; however, we expect that the averaging technique fails in case of mean motion resonances or close encounters with a planet. We plan to perform numerical experiments with a large sample of near-Earth asteroids showing different behaviors: this will be useful to understand the applicability of the averaging technique to the whole set of NEAs.

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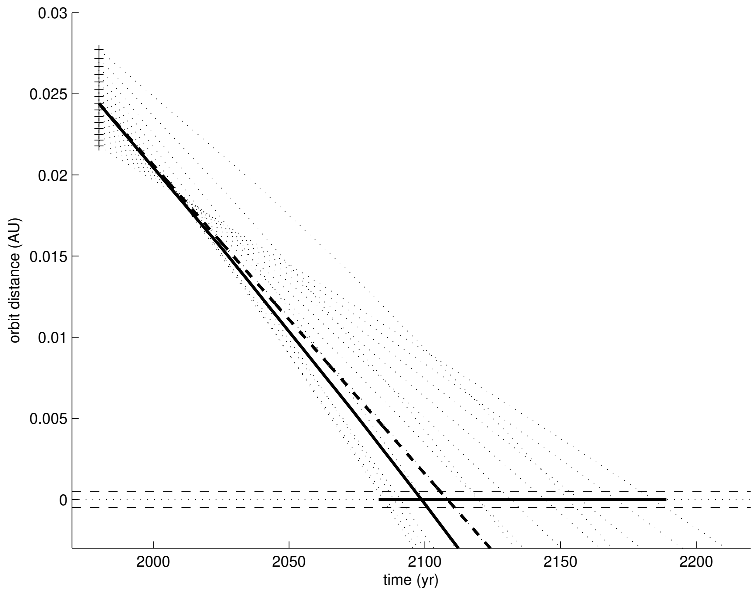

Figure 8: Computation of the interval J' (horizontal solid line) for asteroid 1979 XB. The enhanced transversal curves refer to the nominal orbit: solid line corresponds to secular evolution, linearized is dashed. The dotted curves represent the secular evolution of the VAs.

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