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Roundness: a closed form upper bound for the centroid to minimum zone center distance by worst-case analysis

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Abstract

The minimum zone tolerance (MZT) meets the ISO 1101 definition of roundness error: it determines two concentric circles that contain the roundness profile and such that the difference in radii is the least possible value.

This article provides theoretical evidence that the minimum size of the neighborhood of the centroid containing the minimum zone center is $\pi^1 E_C$, where E_C is the roundness error related to the centroid, which can be evaluated in closed form.

The implications of such linear estimating are twofold: (i) locating the part center with a given tolerance, e.g. for manufacturing tasks, such as handling (peg-hole) or machining (centering) and (ii) providing a search area for minimum zone center-based algorithms, such as metaheuristics (GA, PSO etc.).

Keywords: minimum zone tolerance (MZT), centroid neighborhood, upper bound, MZC evaluation, circularity.

1. Introduction

The evaluation of form errors of machined parts is fundamental in quality inspection to verify their conformance to the expected tolerances. The form tolerance is evaluated with reference to a Euclidean geometric feature, i.e. a circle in the case of roundness. Roundness is a typical geometric form to be inspected.

The most used criteria to establish the reference circle are: the least-squares method (LSQ), the maximum inscribed circle (MIC), the minimum circumscribed circle (MCC) and the minimum zone tolerance (MZT). The performance of minimum zone fitting algorithms has been reviewed in [1]. The use of a particular data fitting method depends on the required application; e.g., MIC and MCC can be used when mating is involved. The MZT meets the standard definition of roundness error, as reported in ISO 1101 [2] and therefore it prevents good parts to be rejected by data fitting and economic loss.

The MZT determines two concentric circles that contain the roundness profile and such that the difference in radii is the least possible value. This difference is the minimum zone roundness error E_{MZ} . Figure 1 shows two pairs of concentric circles that include the sample points centered respectively at C_1 and C_2 and where $(R_1 - r_1)$ and $(R_2 - r_2)$ are their difference in radii. If the minimum zone center is found, the minimum zone error can be determined as the corresponding roundness error.

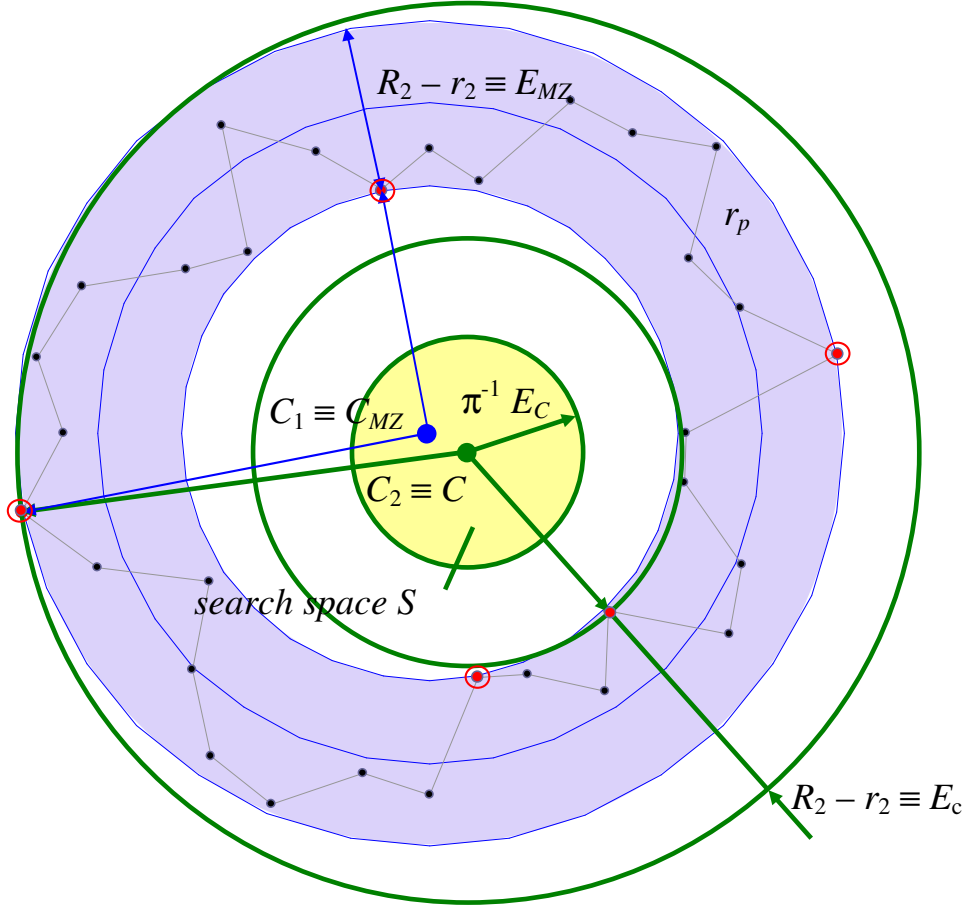


Figure 1: Minimum zone error E_{MZ} . C_1 and C_2 are possible locations of the centers of the two concentric circles. $(R_1 - r_1)$ and $(R_2 - r_2)$ are the differences in radii. C_1 becomes the minimum zone center C_{MZ} if the minimal difference in radii $(R_1 - r_1)$ coincides with E_{MZ} . Center-based algorithms (like metaheuristics) search the (unknown) neighborhood S for C_{MZ} . The main purpose of this work is to provide center and radius of S , as indicated, where C is centroid of the roundness profile r_p .

Errors are scaled for clarity.

About here

The MZT is the solution of the following optimization problem [3]:

$$\min \left[\max_{\theta_i = i \times \frac{2\pi}{n}, i=1, \dots, n} r_p(x, y, \theta_i) - \min_{\theta_i = i \times \frac{2\pi}{n}, i=1, \dots, n} r_p(x, y, \theta_i) \right] \quad (1)$$

subject to $(x, y) \in S$

where S is the search space, $\theta_i = i \times \frac{2\pi}{n}$, $i=1, \dots, n$ are the angular locations of n equiangular data of the roundness profile $r_p(x, y, \theta_i)$ of the reference circle of center (x, y) .

The solution of problem (1) is the minimum zone error defined as:

$$\begin{aligned}
E_{MZ} &= R_{\max}(C_{MZ}) - R_{\min}(C_{MZ}) = \\
&= \max_{\theta_i = i \times \frac{2\pi}{n}, i=1, \dots, n} r_p(C_{MZ}, \theta_i) - \min_{\theta_i = i \times \frac{2\pi}{n}, i=1, \dots, n} r_p(C_{MZ}, \theta_i)
\end{aligned} \tag{2}$$

where $C_{MZ} = (x_{MZ}, y_{MZ})$ is the minimum zone center.

The main purpose of this work is to provide a closed form upper bound of the distance between centroid C_n and minimum zone center C_{MZ} . This evaluation can be used *tout court* as a first estimation of the minimum zone center position or as the starting point for a local search, e.g. as a search neighborhood of metaheuristics, such as genetic algorithms, particle swarm optimization etc. By reducing the search area the algorithm complexity and the computation time can be reduced.

It seems that a theoretical analysis that provides a closed form expression for the roundness problem with the MZT criterion is only available by the authors [4], where the center of the search space S is the centroid and the radius is the upper bound of the distance between centroid and minimum zone center; current lowest upper bound of about $0.43E_{C_n}$ has been extrapolated asymptotically.

The search space definition represents the focus of current work. It will be shown that in the proposed approach it has been decreased by 27%.

2. Literature

Two approaches to the MZT problem have been proposed in the literature: computational geometry techniques and solutions of a non linear optimization problem. The first approach is, in general, very computationally intensive, especially, when the number of data points is large. One of these methods is based on the Voronoi diagram [5] [6]. The second approach is based on the minimization of the E_{MZ} as a function of C_{MZ} (Figure 1). The inconvenience is that this function has several local minima consequently the exploration is computation intensive; for this reason it takes advantage of the search area reduction, which is the main purpose of current work. Some examples of center-based approaches are: the Chebyshev approximation [7], the simplex search / linear approximation [8] [9], the steepest descent algorithm [10], and metaheuristics like the particle swarm optimization (PSO) [11] [12], the simulated annealing (SA) [13], and genetic algorithms (GAs) [3] [14] [15].

Wang et al. [16] and Jywe et al. [17] presented a generalized non-linear optimization procedure based on the developed necessary and sufficient conditions to evaluate the roundness error. In order to meet the standards, the minimum zone reference circles should pass through at least four points of the roundness profile. This can occur in two cases: a) when three points lie on a circle and one point lies on the other circle (the 1-3 and the 3-1 criteria); b) when two points lie on each of the

concentric circles (the 2-2 criterion, represented by the circled red points in Figure 1). The computation time required to meet these conditions increases exponentially with the dataset size. Gadelmawla [18] uses a heuristic approach to drastically reduce the number of sample points used by the min-max 1-3, 3-1 and 2-2 criteria.

Samuel and Shunmugam [19] established a minimum zone limaçon based on computational geometry to evaluate roundness error; with geometric methods, global optima are found by exhaustively checking every local minimum candidate. A mesh based method with starting center on the least square center, where the convergence depends on the number of mesh cross points, representing a compromise between accuracy and speed, was proposed by Xianqing et al. [20]. This and similar approaches require the approximate size of the search space.

The strategy to equiangular data on the roundness profile is generally adopted in the literature. Conversely, in a previous paper the authors developed a cross-validation method for small samples to assess the kind of manufacturing signature on the roundness profile in order to detect critical points such as peaks and valleys [21].

Only few contributions are currently available in the literature regarding the setting of the search space of the non linear optimization problem. The centroid is usually considered as the center of the search space. In [14] the search space is a square of fixed 0.2 mm side, in [3] it is 5% of the circle diameter [22], the side is determined by the distance of the farthest point and the nearest point from the mean center which is approximated to $2E_{C_n}$ where E_{C_n} is the roundness error related to the centroid of n equiangular data. In [23] it is the rectangle circumscribed to the sample points.

3. An upper bound for the centroid to minimum zone center distance

In this section, we define a geometrical feature $F(\alpha)$ that represents a worst-case for the examined roundness problem. A formal theory is developed and presented below to find a closed form expression for a theoretical (*local*) maximum (upper bound) of the distance between the centroid, $C = (x_0, y_0)$, and the minimum zone center, $C_{MZ} = (x_{MZ}, y_{MZ})$.

The proposed theory is general, it does not take into account form deviations, e.g. it is not signature specific.

The proposed geometrical feature $F(\alpha)$ inspired from [4] is formed by two concentric-opposite arcs of circle shown in Figure 2 and described by:

$$F(\alpha) = \begin{cases} (x,0), & \theta = 0 \\ (R \cdot \cos \theta, R \cdot \sin \theta), & \theta \in \left(0, \frac{\pi}{2} + \alpha\right] \\ (r \cdot \cos \theta, r \cdot \sin \theta), & \theta \in \left(\frac{\pi}{2} + \alpha, \frac{3\pi}{2} - \alpha\right) \\ (R \cdot \cos \theta, R \cdot \sin \theta), & \theta \in \left[\frac{3\pi}{2} - \alpha, 0\right) \end{cases} \quad (3)$$

where $0 < r < R$, $-\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}$ and $(x,0)$ is a *control point*, which makes the feature open in $\theta = 0$.

By construction $\alpha = \theta + \frac{\pi}{2}$.

In Appendix, the effect of the variation of $x \in [0, R]$ on the minimum zone error is discussed.

It will be proven that for $x=r$ the control point forces the minimum zone center in $(x_{MZ}, y_{MZ}) = (0,0)$. This property is exploited by using the feature $F(\alpha)$ with the control point at $(r,0)$ as displayed in Figure 2.

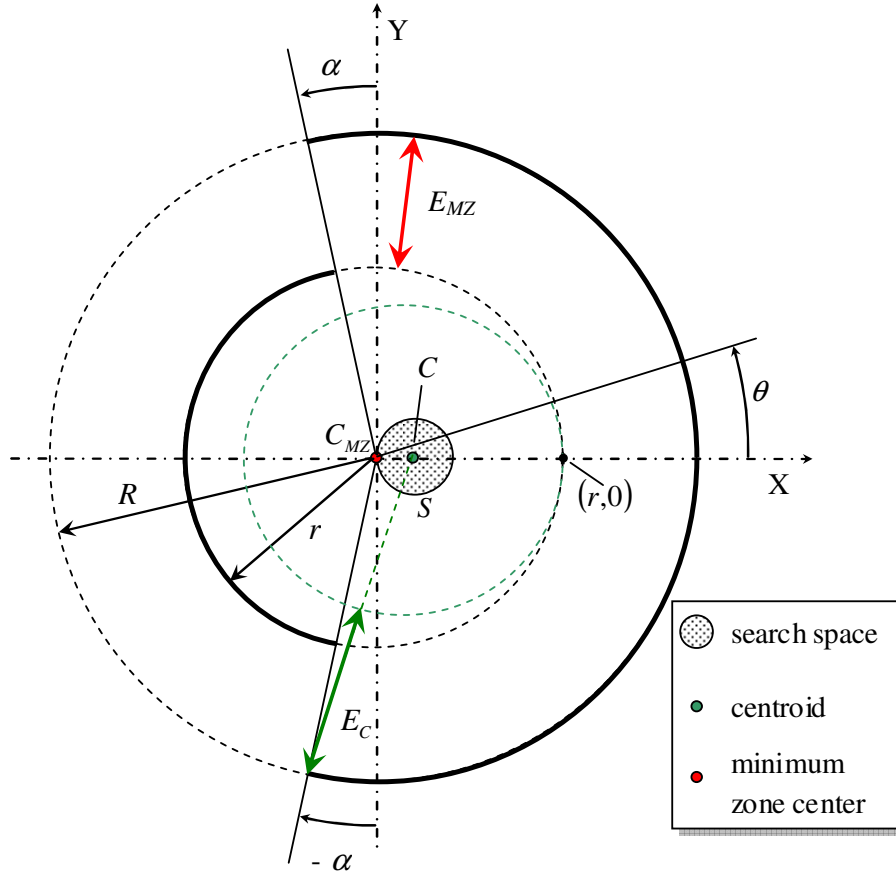


Figure 2: Worst-case for the distance (enhanced for clarity) between centroid C and minimum zone center C_{MZ} , with the control point in $(r,0)$

About here

It results that $F(0)$ is formed by *two concentric-opposite semicircular arcs*.

The feature $F\left(\frac{\pi}{2}\right)$ is a circumference of radius R ; similarly, $F\left(-\frac{\pi}{2}\right)$ is a circumference of radius r .

As the considered feature is symmetrical with respect to the X axis by a rotation of axes, without loss of generality in the remainder the Y component is neglected.

A number of properties of $F(\alpha)$ is given. They can be applied for each feature defined by expression (3) and rotated by $\frac{k}{2\pi}$, $0 \leq k < 2\pi$.

Lemma 1. Let C be the centroid of $F(0)$, then $C = \left(\frac{R-r}{\pi}, 0\right)$ \square

■ The proof follows the fact that $F(0)$ is formed by two semicircular arcs of radii R and r and the centroids of these semicircular arcs are located on the X axis respectively at $C_R = \frac{2 \cdot R}{\pi}$ and $C_r = -\left(\frac{2 \cdot r}{\pi}\right)$. Obviously, the control point $(r,0)$ has a negligible weight in the evaluation of C □

Let $F_R(\alpha)$ and $F_r(\alpha)$ be the component features of radii R and r , respectively, at the right and at the left side of $F(\alpha)$ as shown in Figure 3. Figure 3 (top) shows that $F(\alpha)$ is obtained by $F(0)$ removing (adding) a feature symmetric on the X^- axis (X^+ axis), if $\alpha > 0$ ($\alpha < 0$). The maximum asymmetry of $F(\alpha)$ about the Y axis is achieved by setting $\alpha = 0$. In fact if $\alpha > 0$, $F_r(\alpha)$ evaluated for $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \alpha$ is mirrored to $F_R(\alpha)$ evaluated for $\frac{\pi}{2} - \alpha \leq \theta \leq \frac{\pi}{2}$ (Figure 3 bottom).

Analogously if $\alpha < 0$ $F_R(\alpha)$ evaluated for $\alpha \leq \theta \leq \frac{\pi}{2}$ is mirrored to $F_r(\alpha)$ evaluated for

$$\frac{\pi}{2} + \alpha \leq \theta \leq \pi.$$

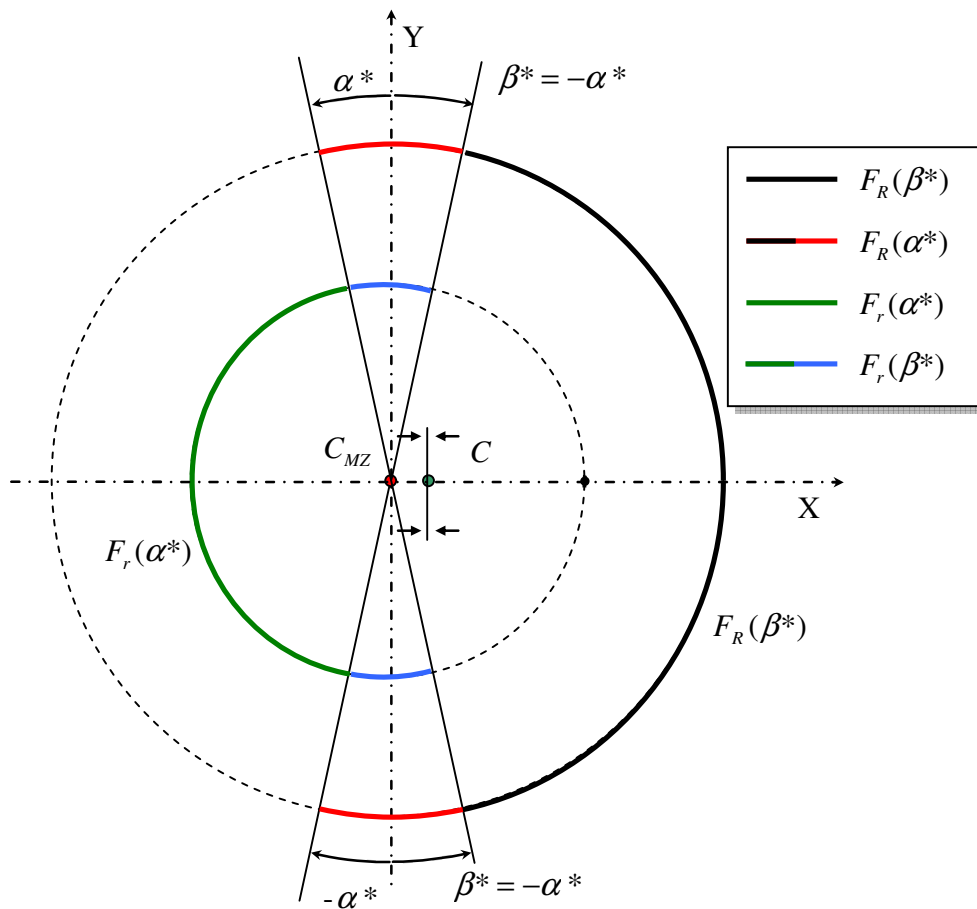
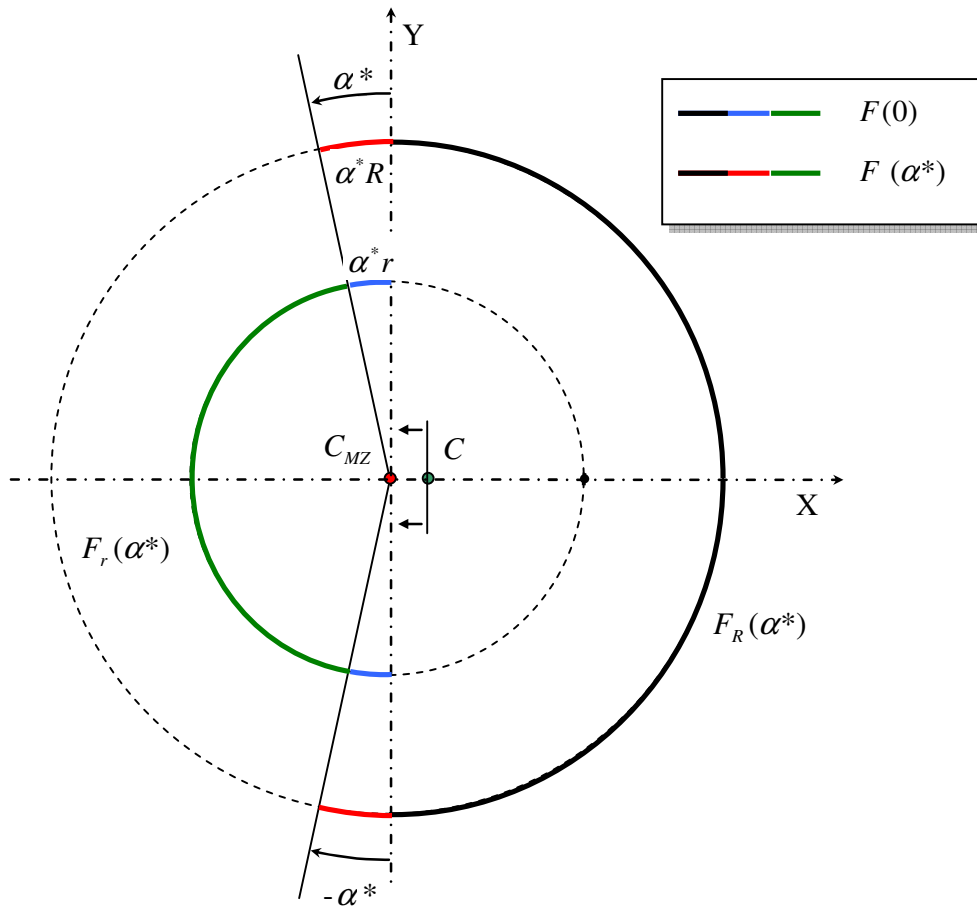


Figure 3: Graphical proof of the theorem for $-\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}$. Top: composition of feature $F(\alpha)$

starting from $F(0)$. Bottom: symmetry condition about the Y axis between $F(\alpha^*)$ and

$$F(\beta^*), \beta^* = -\alpha^*.$$

About here

Reducing the asymmetry of $F(\alpha)$ about the Y axis, the centroid C moves towards the minimum zone center C_{MZ} (i.e. to the left according to Figure 3).

Theorem 1. The two concentric-opposite semicircular arcs of $F(0)$ maximize the distance $\frac{R-r}{\pi}$

given by *Lemma 1* between the centroid and the origin (0,0) \square

■ Let us suppose that a better upper bound exists, by reducing the asymmetry of the feature with

$\alpha^* \neq 0$ and let us show that the new centroid abscissa C^* is less than $\frac{R-r}{\pi}$ from (0,0) thus

contradicting the fact that reducing the asymmetry of the feature, the distance between the centroid and the minimum zone center increases.

In fact, as $\alpha^* \neq 0$ and $-\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}$ it results:

$$C^* = \frac{\int_{-\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} R \cdot \cos \theta \cdot d\theta + \int_{\frac{\pi}{2}+\alpha}^{\frac{3\pi}{2}-\alpha} r \cdot \cos \theta \cdot d\theta}{\int_0^{2\pi} d\theta} = \frac{R \cdot \sin \theta \Big|_{-\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} + r \cdot \sin \theta \Big|_{\frac{\pi}{2}+\alpha}^{\frac{3\pi}{2}-\alpha}}{2\pi} = \frac{2R \cdot \cos \alpha - 2r \cdot \cos \alpha}{2\pi} =$$

$$= \frac{(R-r) \cos \alpha}{\pi} < C$$

(4)

\square

In the case of $\alpha^* = 0$, it results $C^* = C$.

From *Theorem 1* it follows that, in the worst-case, the distance from the centroid to the origin (0,0) is achieved when r is low. This distance D can be formulated as a function of the roundness error E_c related to the centroid C . As a consequence, in the remainder only $F(0)$ is considered.

It can be noticed that if r is high enough with respect to R , C_{MZ} is located at $p_I=(0,0)$ because any given point belonging to $F(\alpha)$ is on the inner circumference (of radius r) or on the outer

circumference (of radius R); E_{MZ} is equal to $(R-r)$. Vice versa, if r is low with respect to R , the smaller of the two concentric circles with minimal radial separation crosses the points $(r,0)$ and $(R,0)$: C_{MZ} is located on $p_2 = (\frac{R+r}{2}, 0)$.

Lemma 2. The minimum zone center C_{MZ} of $F(0)$ is at $(0,0)$ if and only if the ratio $\frac{r}{R}$ is a solution

of the equation $2\left(\frac{r}{R}\right)^2 - 5\frac{r}{R} + 1 \leq 0$ \square

■ By the definition (2) of minimum zone center, $p_1 = C_{MZ}$ if the following condition is verified:

$$R_{\max}(p_1) - R_{\min}(p_1) \leq R_{\max}(p_2) - R_{\min}(p_2) \quad (5)$$

Besides, if $p_2 = C_{MZ}$ the greater of the two concentric circles with minimal radial separation crosses the point $(0,R)$. Hence in this case:

$$E_{MZ} = \sqrt{\left(\frac{R+r}{2}\right)^2 + R^2} - \left(\frac{R-r}{2}\right) \quad (6)$$

and the expression (5) becomes:

$$\begin{aligned} R-r &\leq \sqrt{\left(\frac{R+r}{2}\right)^2 + R^2} - \left(\frac{R-r}{2}\right) \\ 3\left(1-\frac{r}{R}\right) &\leq \sqrt{5+2\frac{r}{R}+\left(\frac{r}{R}\right)^2} \\ 2\left(\frac{r}{R}\right)^2 - 5\frac{r}{R} + 1 &\leq 0 \end{aligned} \quad (7)$$

\square

Corollary. The interval of feasible solutions of expression (7) is $\frac{5-\sqrt{17}}{4} \leq \frac{r}{R} < 1$ because the

contiguous interval $1 \leq \frac{r}{R} \leq \frac{5+\sqrt{17}}{4}$ includes unfeasible solutions for $F(\alpha)$.

Theorem 2. $D = \max_{F(0)} |C - C_{MZ}| \leq \pi^{-1} E_C$ \square

■ From *Lemma 1* the distance D between centroid and minimum zone center is maximized when r is minimized.

From *Theorem 2* and its *Corollary*, if $C_{MZ} = (0,0)$ it results:

$$D = \max_{F(0)} |C - C_{MZ}| = \max_{r \geq \left(\frac{5-\sqrt{17}}{4}\right)R} \frac{R-r}{\pi} \leq \frac{\sqrt{17}-1}{4\pi} R \quad (8)$$

Equation (6) can be used to evaluate E_C . Let RC_C and RI_C be the radii of, respectively, the minimum circumscribed and the maximum inscribed concentric circles centered at the centroid. By expression (6) it results:

$$RC_C = \sqrt{\left(\frac{R+r}{2}\right)^2 + R^2} \quad (9)$$

$$RI_C = \frac{R-r}{2} \quad (10)$$

Therefore:

$$\begin{aligned} E_C &= RC_C - RI_C = \\ &= R \left(\sqrt{\left(\frac{1+\frac{r}{R}}{2}\right)^2 + 1} - \left(\frac{1-\frac{r}{R}}{2}\right) \right) = \\ &= \frac{R}{2} \left(\sqrt{5+2\frac{r}{R} + \left(\frac{r}{R}\right)^2} - 1 + \frac{r}{R} \right) \geq \\ &\geq \frac{R}{2} \left(\sqrt{5 + \frac{5-\sqrt{17}}{2} + \left(\frac{5-\sqrt{17}}{4}\right)^2} - \frac{\sqrt{17}-1}{4} \right) = \\ &= \frac{R}{2} \left(\sqrt{\frac{81-9\sqrt{17}}{8}} - \frac{\sqrt{17}-1}{4} \right) = \\ &= \frac{\sqrt{17}-1}{4} R \end{aligned} \quad (11)$$

From equations (8) and (11):

$$\frac{D}{E_C} \leq \frac{k}{\pi k}, \text{ where } k = \frac{\sqrt{17}-1}{4} \quad (12)$$

Finally, if $C_{MZ} \neq (0,0)$ it is located on $p_2 = \left(\frac{R+r}{2}, 0\right)$, which is closer to C than $p_1 = (0,0)$. In fact:

$$|p_2 - C| = \frac{R+r}{2} - \frac{R-r}{\pi} \leq \frac{R-r}{\pi} = |p_1 - C| \Leftrightarrow \frac{r}{R} \leq \frac{4-\pi}{4+\pi} = 0.12 \quad (13)$$

Expression (13) is verified because for the *Corollary* in this case $\frac{r}{R} < \frac{5-\sqrt{17}}{4} = 0.219$

□

4. Application

To show a possible application of the method and to provide some orders of magnitude of the estimation of the minimum zone center C_{MZ} found by different authors, its distance $|C-C_{MZ}|$ from the centroid C on datasets from the literature has been compared in Table 1 with the upper bound $\pi^1 E_{C_n}$ predicted by the theory proven in the previous section. A similar comparison is reported in Table 2 with datasets obtained by NPL Chebyshev best fit circle certified software [24], where the *exact* minimum zone center is known by construction. By this analysis on concrete data, the position of the minimum zone center within the (theoretical) search area S is estimated.

Table 1: The proposed upper bound ($\pi^1 E_{C_n}$) compared with the distance between centroid C and minimum zone center C_{MZ} (estimated) on roundness profiles taken from the literature.

case #	reference	dataset		algorithm	centroid C			minimum zone center C_{MZ}		$ C-C_{MZ} $	$\pi^1 E_{C_n}$	difference (14)	
		author	n		notes	x_0	y_0	E_{C_n}	x_{MZ}				y_{MZ}
1.	[3]	[3]	1800	Drilling/aluminum alloy/CMM data	GA	-0.0002	0.0n022	0.0301	-0.0026	0.0044	0.003256	0.009581	+0.006325
2.	[3]	[3]	1800	Milling/steel/CMM data	GA	0.0042	0.0008	0.0242	0.0071	-0.0018	0.003895	0.007703	+0.003808
3.	[3]	[3]	1800	Turning/steel/CMM data	GA	0.0007	-0.0048	0.0270	0.0010	-0.0049	0.000308	0.008594	+0.008287
4.	[3]	[3]	1800	Drilling/marble CMM data	GA	0.0040	-0.0199	0.0688	0.0031	-0.0181	0.002012	0.021263	+0.019251
5.	[3]	[3]	1800	Turning/steel CMM data	GA	0.0037	-0.0199	0.0709	-0.0060	-0.0137	0.009801	0.022568	+0.012768
6.	[3]	[3]	1800	Turning/aluminum alloy CMM data	GA	-0.0115	-0.0112	0.0935	-0.0178	-0.0068	0.007684	0.029762	+0.022078
7.	[3]	[3]	1800	Turning/steel CMM data	GA	-0.0021	0.0010	0.0423	-0.0071	0.0014	0.005016	0.013465	+0.008449
8.	[14]	[17]	39	Artificial data	GA	0.0356	-0.0536	0.0092	0.0356	-0.0529	0.0007	0.002928451	+0.002228451
9.	[14]	[25]	100	Artificial data	GA	0.02908	-0.00634	0.9868	0.00536	0.00788	0.027655864	0.314108196	+0.286452332
10.	[14]	[10]	24	CMM data	GA	82.989744	97.009116	0.039058	82.990941	97.008387	0.001401517	0.012432548	+0.01103103

11.	[26]	[26]	8	Artificial data	Computational Geometric	29.997	40.0606	0.30177	29.96886	40.10828	0.055342	0.096056	+0.040715
12.	[10]	[19]	24	CMM data	Steepest Descend	82.989744	97.009116	0.039058	82.99094	97.00839	0.001404	0.012433	+0.011029
13.	[10]	[10]	25	CMM data	Steepest Descend	40.0007	50.0015	0.0293	40.00074	50.00153	0.0000492	0.009326	+0.009277
14.	[20]	[20]	80	CMM/Bearing ring	Polar Coordinate Transform	0	0	0.029	0.001599	-0.00165	0.0023	0.009231	+0.006931
15.	[20]	[20]	80	CMM/Bearing ring	PTC 8×80 mesh	0	0	0.029	0.001422	-0.0023	0.002704	0.009231	+0.006527
16.	[20]	[20]	80	CMM/Bearing ring	PTC 10×120 mesh	0	0	0.029	0.001592	-0.00166	0.0023	0.009231	+0.006931
17.	[20]	[20]	80	CMM/Bearing ring	PTC 8×100 mesh	0	0	0.029	0.001372	-0.00233	0.002704	0.009231	+0.006527
18.	[20]	[20]	80	CMM/Bearing ring	PTC 20×180 mesh	0	0	0.029	0.001573	-0.00168	0.002301	0.009231	+0.006930
19.	[27]	[27]	100	Artificial data	Approximation Algorithm	0	0	0.04	-	-	0.00000002	0.012732	+0.012732
20.	[27]	[27]	1000	Artificial data	Approximation Algorithm	5	-4	0.02	5.00000016	-	0.00000006	0.006366	+0.006366

Table 2: The proposed upper bound ($\pi^1 E_{C_n}$) compared with the distance between centroid C and minimum zone center C_{MZ} (known) on roundness profiles obtained by certified software [24] with increasing dataset size n , predefined minimum zone error E_{MZ} and center $C_{MZ}=(x_{MZ}, y_{MZ})\equiv(0,0)$.

case n.	n	E_{MZ}	C			$ C-C_{MZ} $	$\pi^1 E_{C_n}$	difference (14)
			x_0	y_0	E_{C_n}			
1.	8	0.01	-0.0009	-0.0002	0.0113	0.0007	0.0036	+0.0029
2.	16	0.03	-0.0001	-0.0025	0.0333	0.0024	0.0106	+0.0082
3.	32	0.06	-0.0007	0.0030	0.0632	0.0031	0.0201	+0.0170
4.	64	0.09	-0.0030	0.0008	0.0940	0.0031	0.0299	+0.0268
5.	128	0.01	-0.0002	-0.0001	0.0102	0.0002	0.0032	+0.0030
6.	256	0.03	0.0004	0.0002	0.0308	0.0004	0.0098	+0.0094
7.	512	0.06	-0.004	-0.0007	0.0612	0.0008	0.0195	+0.0187
8.	1024	0.09	0.0010	0.0001	0.0915	0.0010	0.0291	+0.0281
9.	2048	0.01	0.0000	0.0000	0.0101	0.0000	0.0032	+0.0032
10.	4096	0.03	0.0001	0.0000	0.0301	0.0001	0.0096	+0.0095
11.	8192	0.06	0.0001	-0.0001	0.0602	0.0001	0.0192	+0.0190
12.	16384	0.09	-0.0002	0.0000	0.0903	0.0002	0.0287	+0.0285

As anticipated in the introduction, not all the minimum zone methods are based on the minimum zone center C_{MZ} to determine the minimum zone roundness error and are able to provide its position. From our extensive search, all center-based methods found that provide the C_{MZ} have been included in Table 1. The listed C_{MZ} values are not necessarily optima; they have been obtained by the respective authors by their proposed method, as specified. Table 2 includes datasets where the C_{MZ} is known. The estimate is quantified by the following parameter

$$difference = \pi^{-1} E_{C_n} - |C - C_{MZ}| \quad (14)$$

and is always positive as an experimental evidence of the proposed theory.

It can be noticed that the proposed upper bound $\pi^{-1} E_C$ ranges from the double to one order of magnitude above the estimated C to C_{MZ} distance in most cases, regardless of the magnitude of E_C .

The same can be observed for Table 2.

It can also be observed for both tables that the difference increases both with the dataset size and with the minimum zone error.

The first seven roundness profiles in Table 1 have been taken by the authors from real samples on a variety of materials and manufacturing processes, are visually shown in [3] and include different shapes, like flattened circle, three lobed part, elliptic form, circle with random deviations etc.

Form deviations are not considered by the proposed theory, which provides a general result, e.g. it is not signature specific, however it is speculated that the upper bound provides a larger overestimate (*difference* by expression (14)) with high form errors (high E_{C_n}) and low profile deviations (C_{MZ} close to C).

5. Profile sampling

In Figure 4, the same comparison of previous section is repeated on roundness profiles at increasing sampling size n .

36 datasets have been generated with certified software [24] and minimum zone error 0.01, 0.03 and 0.06 mm for 12 values of the sampling size n in the range 8 to 16,384. The deviation type with generated profiles is random, i.e. the error is uniformly distributed around the 360°.

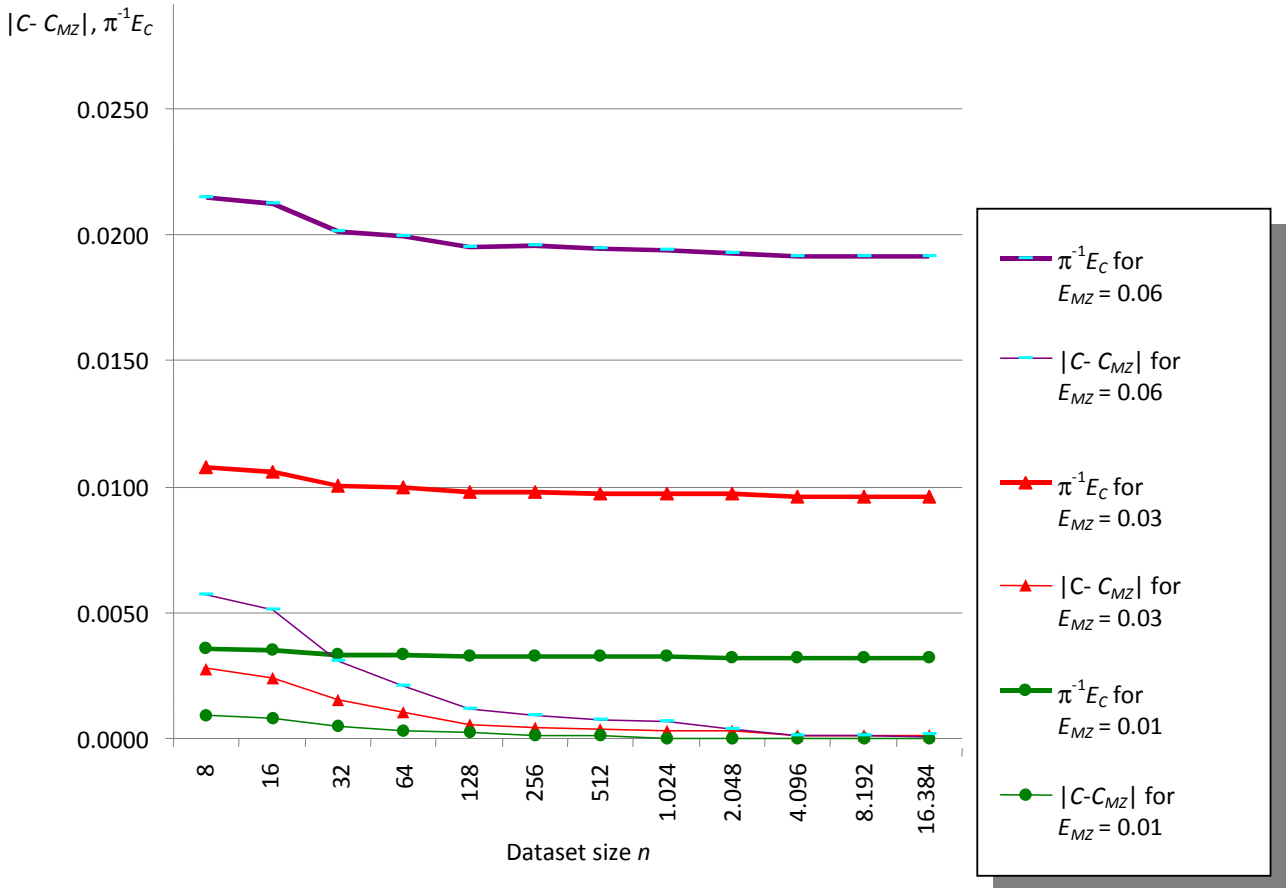


Figure 4: The proposed upper bound $\pi^{-1} E_C$ compared to the distance between the centroid and the minimum zone center fixed in (0,0) of roundness profiles generated by certified software [24].

About here

As in Table 2, the minimum zone center in Figure 4 is known; it has been fixed at the origin of coordinates (0,0) in the profile generation; its distance from the centroid and the respective upper bound $\pi^{-1} E_C$ proposed in this paper have been compared.

The developed theory is valid for continuous profiles and consequently at increasing sampling size, when dealing with discrete data; however, it is satisfied already for n as low as 8. It should also be considered that in applications requiring better accuracy on the MZE estimation, n should be higher, up to thousands datapoints, which are available with optical scanning techniques. The trend on exact data is clearly monotonously decreasing as expected.

6. Discussion and conclusions

The worst-case described in Figure 2 has been built to achieve a closed form evaluation of the centroid to minimum zone center distance. By knowing the centroid of the roundness profile, the minimum zone center position can be estimated with a tolerance whose upper bound has been given in closed form by $\pi^{-1} E_C$.

The worst-case also expresses concrete manufacturing cases, such as casting or forming where two dies or molds have a different radius.

The roundness error E_{C_n} related to the centroid of a dataset with n equiangular data can be evaluated in linear function of the sampling data; increasing the number of sampled points, the estimate accuracy increases.

This theory has extensive applications.

- The part center can be located with a given tolerance in manufacturing tasks, such as handling (peg-hole) or machining (centering).
- Based on the worst-case approach, these conclusions can be extended to any sampling profile also defined as blind or not manufacturing signature-specific.
- By a transformation from polar to Cartesian coordinates, the proposed upper bound can be directly applied to the estimation of straightness.
- Minimum zone based algorithms and algorithms that approach the minimum zone method by iterative center evaluations (e.g. metaheuristics like genetic algorithms or particle swarm optimization and others cited above) can benefit of the lower search space size to a neighbor of the centroid C .

For these latter center-based minimum zone methods, current result represents a new opportunity for experimental research.

The theoretical upper bound proposed is not signature specific, however it is speculated that a larger overestimate is obtained with high errors (high E_{C_n}) and low profile deviations (C_{MZ} close to C).

The correlation between the proposed upper bound and form deviations can be investigated in future research.

Future work includes the extension of the proposed theory to other (symmetrical) form errors, such as sphericity and planarity.

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Appendix

The two concentric-opposite arcs feature can be parameterized by:

$$F(x, \alpha) = \begin{cases} (x \cdot \cos \theta, 0), & \theta = 0 \\ (R \cdot \cos \theta, R \cdot \sin \theta), & \theta \in \left(0, \frac{\pi}{2} + \alpha\right] \\ (r \cdot \cos \theta, r \cdot \sin \theta), & \theta \in \left(\frac{\pi}{2} + \alpha, \frac{3\pi}{2} + \alpha\right) \\ (R \cdot \cos \theta, R \cdot \sin \theta), & \theta \in \left[\frac{3\pi}{2} + \alpha, 0\right) \end{cases} \quad (9)$$

where $-\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}$ and $0 \leq x \leq R$.

Proposition. E_{MZ} is an upper bound of $(R - r)$ when a control point $(x, 0)$ moves from r to 0 on the X axis \square

■ The (1-3) minimum zone condition is considered: the one-point taken on the inner circumference is exactly $(x, 0)$; the three-points taken on the outer circumference have center C_{MZ} . Therefore $E_{MZ} = R - x \geq R - r$ \square

Besides, it can be observed that E_{MZ} is included in the region delimited by $(R - x)$ and $(R - r)$ when the point x moves from r to G on the X axis, where G is the circumcenter of vertexes $\{(r \cdot \sin \alpha, r \cdot \cos \alpha), (r \cdot \sin \alpha, -r \cdot \cos \alpha), (R, 0)\}$ (not proven here). Finally, E_{MZ} is equal to $(R - r)$ for $G \leq x \leq R$.

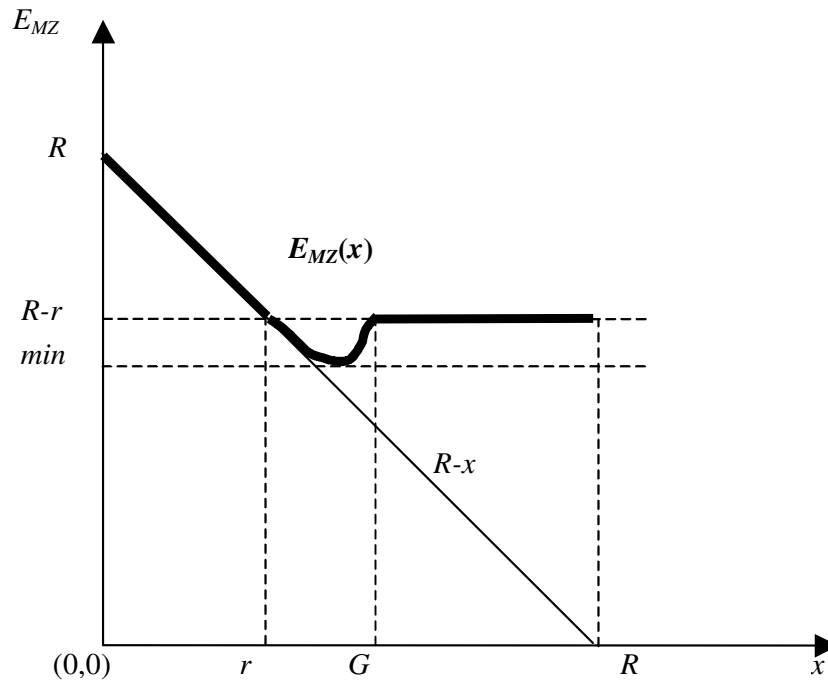


Figure A: Qualitative effect of the variation of the control point $x \in [0, R]$ on the minimum zone error

$$E_{MZ} \cdot$$

About here

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