

A GROMOV-HAUSDORFF DISTANCE BETWEEN VON NEUMANN ALGEBRAS AND AN APPLICATION TO FREE QUANTUM FIELDS

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Dedicated to Sergio Doplicher, on the occasion of his 75th birthday.

ABSTRACT. A distance between von Neumann algebras is introduced, depending on a further norm inducing the w^* -topology on bounded sets. Such notion is related both with the Gromov-Hausdorff distance for quantum metric spaces of Rieffel [24] and with the Effros-Maréchal topology [10, 19] on the von Neumann algebras acting on a Hilbert space. This construction is tested on the local algebras of free quantum fields endowed with norms related with the Buchholz-Wichmann nuclearity condition [3], showing the continuity of such algebras w.r.t. the mass parameter.

0. INTRODUCTION

In this note we introduce a suitable notion of distance between von Neumann algebras endowed with a further norm inducing the w^* -topology on bounded sets, and apply this construction to the local algebras of the free massive quantum field, showing their convergence to the local algebras of the massless free quantum field.

On the one hand, the mentioned notion of distance between von Neumann algebras is a sort of dual version of the quantum Gromov-Hausdorff distance of Rieffel [24]. On the other hand, it is clearly related to the Effros-Marechal topology ([10, 19, 12, 13]) on the set of von Neumann algebras acting on a given Hilbert space \mathcal{H} .

Let us recall that, according to Rieffel [22], a quantum metric space is a C^* -algebra, or more generally an order unit space, endowed with a *Lip-seminorm*¹, namely a densely defined seminorm vanishing only on multiples of the identity which induces a distance on the state space compatible with the w^* -topology or, equivalently, whose closed balls modulo scalars are totally bounded w.r.t. the C^* -norm [21, 22, 23]. Alternatively [11], one may assign a densely defined norm L on the C^* -algebra \mathcal{A} whose closed balls are norm compact, so that the dual

¹The original terminology of [22] is actually *Lip-norm*, but we will reserve it for a slightly different object, see Def. 1.1 below.

space \mathcal{A}^* is endowed with a dual norm L' inducing the w^* -topology on bounded sets.

In our setting, we call Lip-von Neumann algebra (LvNA) a von Neumann algebra M endowed with a dual-Lip-norm L' , namely a norm inducing the w^* -topology on bounded sets. In this way the predual M_* is endowed with a Lip-norm L , which is a densely defined norm whose closed balls are norm compact. We note that L' gives rise to a Hausdorff distance between w^* -compact subsets of M . Then, following the ideas of Rieffel [24], one may proceed to define a Gromov-Hausdorff pseudo-distance between pairs (M_1, L'_1) , (M_2, L'_2) as a distance between the corresponding unit balls, embedded in the direct-sum von Neumann algebra, but, as in his case, distance zero does not imply isomorphic von Neumann algebras. Therefore, following [14, 15], we replace unit balls with the positive part of the unit balls of $\mathcal{M}_2(M_i)$, thus getting a distance between isomorphism classes of Lip-von Neumann algebras (cf. Definition 2.4).

Let us mention here that, besides the method proposed by Kerr and Li, other proposals have been made in order to get a notion of Gromov-Hausdorff distance more tailored for C^* -algebras, cf. e.g. [18, 16, 17].

Recall [10, 19, 12, 13] also that, for a separable Hilbert space \mathcal{H} , the Effros-Marechal topology on the set of von Neumann algebras acting on a given Hilbert space \mathcal{H} may be metrized as follows: one chooses a distance on $\mathcal{B}(\mathcal{H})$ which metrizes the w^* -topology on bounded subsets and considers the corresponding Hausdorff distance on w^* -compact sets. Then the distance between two von Neumann algebras on \mathcal{H} may be defined as the Hausdorff distance between their unit balls. In this sense our construction is a local version of the Effros-Marechal topology, since it metrizes the w^* -compact sets of a given von Neumann algebra, instead of all the w^* -compact sets in $\mathcal{B}(\mathcal{H})$, and our distance between isomorphism classes of LvNA is at the same time a Gromov-Hausdorff version of the Effros-Marechal topology.

In algebraic quantum field theory, namely the description of relativistic quantum physics by means of operator algebras, various notions of compactness or, more strongly, nuclearity properties have been considered. We focus here on the Buchholz-Wichmann nuclearity condition [3]. It provides natural, physically meaningful dual-Lip-norms for the algebras of local observables, which can be used to study various notions of distance or convergence. In this paper we use them to prove that the local algebras of the free quantum field are continuous in the mass parameter and, in particular, that, for any bounded region \mathcal{O} , the algebras $\mathcal{A}_m(\mathcal{O})$ converge to $\mathcal{A}_0(\mathcal{O})$ when m goes to zero.

We conclude this introduction mentioning that the idea of some kind of topological convergence for the algebras of local observables in quantum field theory originally came from early discussions of the first named author with R. Verch and concerned the scaling limit [4], however with the expectation that Rieffel’s quantum Gromov-Hausdorff distance between C^* -algebras could be used. Only later, during discussions of the first named author with H. Bostelmann, the idea rose that the notion had to be dualized and adapted to von Neumann algebras. The related technical work is contained in the PhD thesis of the last named author, under the supervision of the first named author. Finally, in the master thesis of the second named author, under the supervision of the first and third author, the convergence of the free local algebras w.r.t. the mass parameter was studied.

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1. LIP-VON NEUMANN ALGEBRAS

In this section, we will introduce the notion of Lip-von Neumann Algebra². The reason why we use the term “Lip” (which stands for “Lipschitz”) will be clear in a while. The leading idea is in fact to “dualize” the notion of quantum metric spaces given by Rieffel in [24]. One may reformulate Rieffel’s approach for C^* -algebras as follows: assign a further norm L on the C^* -algebra \mathcal{A} in such a way that the dual norm L' metrizes the w^* -topology on bounded subsets of \mathcal{A}^* . If instead one starts with a von Neumann algebra, which is canonically a dual space, it becomes quite natural to assign a norm L on its pre-dual M_* in such a way that the corresponding dual norm L' on M metrizes the w^* -topology on bounded subsets of M .

Before translating these ideas into a definition, we briefly recall some basic facts about the notion of Lip-space introduced in [11].

1.1. Definition. *We call Lip-space (LS) a triple $(X, \|\cdot\|, L)$, where:*

- (i) $(X, \|\cdot\|)$ is a Banach space,
- (ii) $L : X \rightarrow [0, +\infty]$ is finite on a dense vector subspace \mathcal{L} , where it is a norm,
- (iii) the unit ball w.r.t. L , namely $\mathcal{L}_1 := \{x \in X : L(x) \leq 1\}$, is compact w.r.t. the Banach norm of X .

We call Lip-norm a norm L satisfying properties (ii) and (iii) above.

²In what follows, we will consider concrete von Neumann algebras, but the same definitions may refer also to abstract W^* -algebras, as long as we do not make any reference to the representing Hilbert space.

We then call *radius* of the Lip-space $(X, \|\cdot\|, L)$, and denote it by R , the maximum value of the (Banach) norm on \mathcal{L}_1 , namely

$$(1.1) \quad R := \max\{\|x\| : x \in \mathcal{L}_1\}.$$

It then follows that

$$(1.2) \quad \|x\| \leq RL(x), \quad \forall x \in X.$$

For the reader convenience, we also recall Prop. 2.3 of [11].

1.2. Proposition. *Let $(X, \|\cdot\|, L)$ be a Lip-space. Then, the norm L' on the Banach dual X^* given by*

$$(1.3) \quad L'(\xi) := \sup\{|\langle \xi, x \rangle| : x \in X, L(x) \leq 1\}$$

induces the w^ -topology on the bounded subsets of X^* , and the radius R is also equal to the radius of the unit ball of $(X^*, \|\cdot\|)$ w.r.t. the L' -norm.*

Our aim is to give an intrinsic characterization of the norm L' on X^* . Given a seminorm L on a subspace \mathcal{L} of a Banach space X , we can of course extend it to X by setting it equal to ∞ on $X \setminus \mathcal{L}$. For simplicity, we will still refer to such an extension as a seminorm on X . Moreover, if L' is a seminorm (in this sense) on X^* , we can associate to it a seminorm L on X by a formula which is “dual” to (1.3):

$$(1.4) \quad L(x) := \sup\{|\langle \xi, x \rangle| : \xi \in X^*, L'(\xi) \leq 1\}.$$

1.3. Lemma. *Let X be a Banach space, X^* its Banach dual, L a seminorm on X and L' a seminorm on X^* . Then $C := \{x \in X : L(x) \leq 1\}$ is norm closed and (1.3) is satisfied if and only if $Q := \{\xi \in X^* : L'(\xi) \leq 1\}$ is norm closed and (1.4) is satisfied.*

Proof. The set C is convex and balanced, and if (1.3) holds its polar set C° satisfies

$$\begin{aligned} C^\circ &= \{\xi \in X^* : |\langle \xi, x \rangle| \leq 1, L(x) \leq 1\} \\ &= \{\xi \in X^* : L'(\xi) \leq 1\} = Q. \end{aligned}$$

Therefore Q , as a polar set, is norm closed, and if C is norm closed too, by the Bipolar Theorem [8] $Q^\circ = C^{\circ\circ} = C$, which entails (1.4). The proof of the converse statement is completely analogous. \blacksquare

We can now give the notion of dual Lip-space.

1.4. Definition. *A dual Lip-space is a Banach space Y endowed with a norm L' which induces a compact topology on the closed unit ball. We will call such a norm a dual-Lip-norm.*

1.5. Proposition. *Let X be a Banach space, X^* its Banach dual.*

- (i) Given a Lip-norm L on X , formula (1.3) defines a dual-Lip-norm on X^* .
- (ii) Given a dual-Lip-norm L' on X^* , formula (1.4) defines a Lip-norm on X .
- (iii) The constructions in (i) and (ii) are inverse each other, namely L on X produces L' on X^* iff L' on X^* produces L on X .
- (iv) A dual-Lip-norm L' on X^* induces the w^* -topology on the bounded subsets.

Proof. We will use the notations $C := \{x \in X : L(x) \leq 1\}$, $Q := \{\xi \in X^* : L'(\xi) \leq 1\}$ as in Lemma 1.3.

(i) It follows by Proposition 1.2.

(ii) Since L' is a norm (everywhere finite on X^*), it is clear that $L(x) = 0$ only if $x = 0$. The normic unit ball $X_1^* := \{\xi \in X^* : \|\xi\| \leq 1\}$ is L' -compact by hypothesis, hence there exists $R > 0$ such that $L'(\xi) \leq R\|\xi\|$, which implies that Q is norm closed. Therefore by Lemma 1.3, C is norm closed. We now prove that it is norm compact. Identifying X with its isometric image in the bidual X^{**} of X , we may consider the set C as a family of functions on the L' -compact set $X_1^* \subseteq R \cdot Q$. Since $|\langle \xi, x \rangle| \leq L(x)L'(\xi)$, we see that C is uniformly bounded by R on X_1^* . Moreover, as $|\langle \xi_1, x \rangle - \langle \xi_2, x \rangle| \leq L'(\xi_1 - \xi_2)$ on C , then C is also L' -equicontinuous. Therefore, by the Ascoli-Arzelà Theorem (see, e.g., [25]), C is compact in the sup-norm $\|\cdot\|_\infty$, which coincides on it with the original (Banach) norm of X . It remains to check that $\mathcal{L} = \{x \in X : L(x) < \infty\}$ is dense. If not, by the Hahn-Banach Theorem one could find $\xi \in X^* \setminus \{0\}$ vanishing on \mathcal{L} , but since (1.3) holds by Lemma 1.3, this is incompatible with the norm property of L' .

(iii) The result follows by Lemma 1.3: starting from (X, L) , by hypothesis C is norm compact, hence norm closed, therefore (1.4) holds; starting from (X^*, L') , Q is norm closed as observed in the proof of (ii), and then one gets (1.3).

(iv) By points (ii) and (iii), any L' is generated by an L on X , therefore the thesis follows by Proposition 1.2. ■

We now specialize to the case in which X^* is a von Neumann algebra.

1.6. Definition. A Lip-von Neumann algebra (LvNA) is a von Neumann algebra M endowed with a further norm L' which metrizes the w^* -topology on bounded subsets.

The LvNA's (M, L'_M) and (N, L'_N) are said to be isomorphic (as Lip-von Neumann algebras) if there is an isometric $*$ -isomorphism $\varphi :$

$M \rightarrow N$, such that

$$(1.5) \quad L'_N(\varphi(x)) = L'_M(x), \quad \text{for any } x \in M.$$

By the previous discussion it turns out that a Lip-von Neumann algebra is a von Neumann algebra M which is also a dual Lip-space, or equivalently, it is a vNA whose predual M_* is a Lip-space. Also, to assign a dual-Lip-norm on M is equivalent to assign a Lip-norm on its predual M_* . Finally, the request that L' metrizes the w^* -topology on bounded subsets can be replaced by the request that $M_1 = \{x \in M : \|x\| \leq 1\}$ is L' -compact.

2. THE DUAL QUANTUM GROMOV-HAUSDORFF DISTANCE

dist_{qGH^*} .

2.1. The Effros-Maréchal Topology. Let \mathcal{H} be a (fixed) Hilbert space, and let $\text{vN}(\mathcal{H})$ be the set of all von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. We can endow the space $\text{vN}(\mathcal{H})$ with a certain natural topological structure. The definition of this topology goes back to the works of Effros [10] and Maréchal [19], and has been further studied by Haagerup and Winslow in the papers [12, 13]. The original definition of Maréchal is the following:

2.1. Definition. *The Effros-Maréchal topology is the weakest topology on $\text{vN}(\mathcal{H})$ in which the maps*

$$\text{vN}(\mathcal{H}) \ni M \mapsto \|\varphi|_M\|$$

are continuous on $\text{vN}(\mathcal{H})$ for every $\varphi \in \mathcal{B}(\mathcal{H})_$.*

We now specialize to the separable case. If \mathcal{H} is separable, then $\text{vN}(\mathcal{H})$ is a Polish space in this topology [19], *i.e.*, the Effros-Maréchal topology is metrizable, second countable and complete. In order to construct a metric on $\text{vN}(\mathcal{H})$ which induces the Effros-Maréchal topology, one takes any distance ρ on $B(\mathcal{H})$ inducing the wo -topology on bounded subsets of $B(\mathcal{H})$ (which coincides with the σ -weak or w^* -topology on bounded sets). The corresponding Hausdorff distance between unit balls of von Neumann algebras in $\text{vN}(\mathcal{H})$ will be then a metric for the Effros-Maréchal topology (this is a consequence of [9]), that is, for $M_1, M_2 \in \text{vN}(\mathcal{H})$,

$$\text{dist}_{EM}(M_1, M_2) := \text{dist}_H^\rho((M_1)_1, (M_2)_1),$$

where we recall that, for C_1, C_2 closed subsets of a compact metric space (X, ρ) , their Hausdorff distance (see e.g. [5]) is given by

$$(2.1) \quad \text{dist}_H^\rho(C_1, C_2) = \max \left(\sup_{x \in C_1} \inf_{y \in C_2} \rho(x, y), \sup_{y \in C_2} \inf_{x \in C_1} \rho(x, y) \right).$$

2.2. The Distance dist_{qGH^*} . As seen in the previous paragraph, it is possible to define a Hausdorff-like distance between two von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space \mathcal{H} , provided one chooses a norm on $\mathcal{B}(\mathcal{H})$ inducing the w^* -topology on bounded sets. As in the case of ordinary compact metric spaces, one may proceed from the Hausdorff distance between closed subsets of a given compact metric space to the Gromov-Hausdorff distance, which is a pseudo-distance between compact metric spaces considered *per se*. This pseudo-distance then becomes a true distance on the space of isometry equivalence classes of compact metric spaces.

Let M, N be two Lip-von Neumann algebras with dual-Lip-norms L'_M, L'_N . We want to introduce a Gromov-Hausdorff-type notion of distance between them. In order to have isomorphic Lip-von Neumann algebras (according to Def. 1.6) whenever they are at distance zero we follow Kerr [14] and consider not only the original algebras M and N , but also the 2×2 -matrix algebras $\mathcal{M}_2(M)$ and $\mathcal{M}_2(N)$ with entries in M and N , respectively. We introduce some notation.

2.2. Notation. From now on, we will write L instead of L' when dealing with dual-Lip-norms on von Neumann algebras. Moreover, given a LvNA (M, L_M) , we still denote by L_M the dual-Lip-norm on $\mathcal{M}_2(M)$ induced by that on M as follows:

$$(2.2) \quad L_M((a_{ij})) := \max_{i,j=1,2} (L_M(a_{ij})), \quad (a_{ij}) \in \mathcal{M}_2(M).$$

Notice that the extended L_M gives back the original Lip-norm, when restricted to the copy of M diagonally embedded in $\mathcal{M}_2(M)$.

2.3. Lemma. *If (M, L_M) is a Lip-von Neumann algebra, then $(\mathcal{M}_2(M), L_M)$ is a LvNA, too. Moreover, they have the same radius.*

Proof. Indeed, $\mathcal{M}_2(M_1) \cong M_1 \otimes F_2$, where F_2 is the type I_2 factor, and so, if L_M induces the w^* -topology on the unit ball M_1 of M , then its “lift” to $\mathcal{M}_2(M)$ will induce the same topology on $\mathcal{M}_2(M_1)$. Since $\mathcal{M}_2(M)_1$ is a w^* -closed subset of $\mathcal{M}_2(M_1)$, the dual Lip-norm property follows.

As for the radii, as $R_M = \max_{x \in M_1} L_M(x)$, and by w^* -compactness the maximum is realized by some element $x_0 \in M_1$, then

$$R_{\mathcal{M}_2(M)} = \max_{(x_{ij}) \in \mathcal{M}_2(M)_1} L_M((x_{ij})) \geq L_M \left(\begin{pmatrix} x_0 & 0 \\ 0 & 0 \end{pmatrix} \right) = R_M;$$

on the other hand, as $(x_{ij}) \in \mathcal{M}_2(M)_1$ implies $x_{ij} \in M_1$, $i, j = 1, 2$, one has

$$L_M((x_{ij})) = \max_{ij} L_M(x_{ij}) \leq R_M,$$

hence, $R_{\mathcal{M}_2(M)} \leq R_M$, and the claim follows. \blacksquare

We now introduce our notion of distance between Lip-von Neumann algebras.

2.4. Definition. Let (M, L_M) and (N, L_N) be Lip-von Neumann algebras, and denote by $\mathcal{L}((M, L_M), (N, L_N))$ the set of all the seminorms $L = L_{M \oplus N}$ on the direct sum $M \oplus N$, s.t. $L|_{M \oplus \{0\}} = L_M$ and $L|_{\{0\} \oplus N} = L_N$. We define the dual quantum Gromov-Hausdorff distance between (M, L_M) and (N, L_N) , by setting

$$(2.3) \quad \text{dist}_{qGH^*}((M, L_M), (N, L_N)) := \inf_{L \in \mathcal{L}((M, L_M), (N, L_N))} \text{dist}_H^L(X_M, X_N),$$

where X_M (resp., X_N) denotes the positive part of the unit ball of 2×2 matrices on M (resp., N).

To simplify the notation, we will drop the indication of the Lip-norms L_M, L_N from the set $\mathcal{L}((M, L_M), (N, L_N))$ and from the distance $\text{dist}_{qGH^*}((M, L_M), (N, L_N))$ whenever no ambiguity can arise. In analogy with Lm. 13.6 in [24], we have the following.

2.5. Lemma. Let $(M, L_M), (N, L_N)$ be Lip-von Neumann algebras, and let R_M, R_N be the respective radii. Then,

$$(2.4) \quad |R_M - R_N| \leq \text{dist}_{qGH^*}(M, N) \leq R_M + R_N.$$

Proof. Let us prove the second inequality, from which the finiteness of the distance follows. For $L \in \mathcal{L}(M, N)$ we have

$$L(x, -y) \leq L_M(x) + L_N(y) \leq R_M + R_N, \quad x \in X_M, y \in X_N,$$

where we used the equality of the radii in Lemma 2.3. The second inequality directly follows from the definition of Hausdorff distance (2.1).

Now, let $d = \text{dist}_{qGH^*}(M, N)$. Given $\varepsilon > 0$, we can find an $L \in \mathcal{L}(M, N)$ s.t. $\text{dist}_H^L(X_M, X_N) < d + \varepsilon$. Then, for any $x \in X_M$, there is an $y \in X_N$ such that

$$L_M(x) \leq L(x, -y) + L_N(y) < d + \varepsilon + R_N.$$

Since ε is arbitrary, it follows that

$$R_M \leq d + R_N.$$

Reversing the roles of M and N , we obtain also

$$R_N \leq d + R_M,$$

and the first inequality is proven. \blacksquare

We need to show that dist_{qGH^*} is a metric. It is clearly symmetric in M and N .

2.6. Theorem (Triangle Inequality). *Let (M_1, L_1) , (M_2, L_2) , and (M_3, L_3) be Lip-von Neumann algebras. Then*

$$(2.5) \quad \text{dist}_{qGH^*}(M_1, M_3) \leq \text{dist}_{qGH^*}(M_1, M_2) + \text{dist}_{qGH^*}(M_2, M_3).$$

Proof. Let $1 \geq \varepsilon > 0$ be given. Then, we can find an $L_{12} \in \mathcal{L}(M_1, M_2)$ s.t.

$$\text{dist}_H^{L_{12}}(X_{M_1}, X_{M_2}) \leq \text{dist}_{qGH^*}(M_1, M_2) + \varepsilon/2.$$

Similarly, we can find $L_{23} \in \mathcal{L}(M_2, M_3)$ s.t.

$$\text{dist}_H^{L_{23}}(X_{M_2}, X_{M_3}) \leq \text{dist}_{qGH^*}(M_2, M_3) + \varepsilon/2.$$

We define

$$L_{13}(x_1, x_3) := \inf_{x_2 \in M_2} (L_{12}(x_1, -x_2) + L_{23}(x_2, x_3)).$$

We shall prove that it is a seminorm, whose restrictions to M_1 and M_3 are L_1 and L_3 , respectively. Indeed, the positive homogeneity is clear, and we have

$$\begin{aligned} & L_{13}(x_1 + y_1, x_3 + y_3) \\ &= \inf_{x_2 \in M_2} (L_{12}(x_1 + y_1, -x_2) + L_{23}(x_2, x_3 + y_3)) \\ &= \inf_{x_2, y_2 \in M_2} (L_{12}(x_1 + y_1, -(x_2 + y_2)) + L_{23}(x_2 + y_2, x_3 + y_3)) \\ &\leq \inf_{x_2, y_2 \in M_2} (L_{12}(x_1, -x_2) + L_{12}(y_1, -y_2) + L_{23}(x_2, x_3) + L_{23}(y_2, y_3)) \\ &= \inf_{x_2 \in M_2} (L_{12}(x_1, -x_2) + L_{23}(x_2, x_3)) + \inf_{y_2 \in M_2} (L_{12}(y_1, -y_2) + L_{23}(y_2, y_3)) \\ &= L_{13}(x_1, x_3) + L_{13}(y_1, y_3). \end{aligned}$$

Then, let us check the restriction requirement: since $L_{23}(x_2, 0) = L_2(x_2) = L_{12}(0, x_2)$, we have

$$\begin{aligned} L_{13}(x_1, 0) &= \inf_{x_2 \in M_2} (L_{12}(x_1, -x_2) + L_{23}(x_2, 0)) \\ &= \inf_{x_2 \in M_2} (L_{12}(x_1, -x_2) + L_2(x_2)) \leq L_{12}(x_1, 0) = L_1(x_1), \end{aligned}$$

and, since $L_{12}(x_1, -x_2) = L_{12}((x_1, 0) + (0, -x_2)) \geq |L_1(x_1) - L_2(x_2)|$,

$$L_{13}(x_1, 0) \geq \inf_{x_2 \in M_2} (|L_1(x_1) - L_2(x_2)| + L_2(x_2)) = L_1(x_1),$$

and so $L_{13}(x_1, 0) = L_1(x_1)$. Similarly, one shows that $L_{13}(0, x_3) = L_3(x_3)$.

Now, suppose that $\text{dist}_H^{L_{12}}(X_{M_1}, X_{M_2}) = d_{12}$ and $\text{dist}_H^{L_{23}}(X_{M_2}, X_{M_3}) = d_{23}$. By definition of Hausdorff distance and by w^* -compactness of the

positive part of the unit ball, for any given $x_1 \in X_{M_1}$ we can find an $x_2 \in X_{M_2}$ - call it $f(x_1)$ - s.t. $L_{12}(x_1, -x_2) = L_{12}(x_1, -f(x_1)) \leq d_{12}$, and, analogously, for any $x_2 \in X_{M_2}$, we can find a corresponding $x_3 \in X_{M_3}$ - call it $g(x_2)$ - s.t. $L_{23}(x_2, -x_3) = L_{23}(x_2, -g(x_2)) \leq d_{23}$. In other words, for any given $x_1 \in X_{M_1}$, we can find an $x_3 = g(f(x_1)) \in X_{M_3}$, s.t.

$$\begin{aligned} L_{13}(x_1, -g(f(x_1))) &\leq L_{12}(x_1, -f(x_1)) + L_{23}(f(x_1), -g(f(x_1))) \\ &\leq d_{12} + d_{23}. \end{aligned}$$

Similarly, for any given $x_3 \in X_{M_3}$, we can find an $x_1 = h(k(x_3)) \in X_{M_1}$, s.t.

$$\begin{aligned} L_{13}(h(k(x_3)), -x_3) &\leq L_{12}(h(k(x_3)), -k(x_3)) + L_{23}(k(x_3), -x_3) \\ &\leq d_{12} + d_{23}. \end{aligned}$$

Since this holds for any x_1 in X_{M_1} and for any x_3 in X_{M_3} , we obtain

$$\begin{aligned} \text{dist}_H^{L_{13}}(X_{M_1}, X_{M_3}) &\leq \text{dist}_H^{L_{12}}(X_{M_1}, X_{M_2}) + \text{dist}_H^{L_{23}}(X_{M_2}, X_{M_3}) \\ &\leq \text{dist}_{qGH^*}(M_1, M_2) + \text{dist}_{qGH^*}(M_2, M_3) + \varepsilon. \end{aligned}$$

Therefore, taking the infimum on the l.h.s., we obtain

$$\text{dist}_{qGH^*}(M_1, M_3) \leq \text{dist}_{qGH^*}(M_1, M_2) + \text{dist}_{qGH^*}(M_2, M_3) + \varepsilon,$$

and so, by the arbitrariness of ε , the thesis follows. \blacksquare

Finally, we want to show that, if two Lip-von Neumann algebras have distance dist_{qGH^*} equal to zero, then they are isomorphic as LvNA (see Definition 1.6, and also Definition 2.4 and Thm. 4.1 in [14]). The proof is inspired by Rieffel's proof of the analogous property for the quantum Gromov-Hausdorff distance between compact quantum metric spaces, completed by the 2×2 matrix trick of Kerr [14].

2.7. Lemma. *The family $\mathcal{L}(M, N)$ is uniformly w^* -equicontinuous on the unit ball of $M \oplus N$.*

Proof. For any $\varepsilon > 0$, and for any given $x_0 \in M_1$, $y_0 \in N_1$, let

$$\begin{aligned} \mathcal{N}(x_0, \varepsilon/2) &= \{x \in M : L_M(x - x_0) < \varepsilon/2\}, \\ \mathcal{N}(y_0, \varepsilon/2) &= \{y \in N : L_N(y - y_0) < \varepsilon/2\}, \end{aligned}$$

so that the subset $\mathcal{N}(x_0, y_0; \varepsilon/2) := \{(x, -y) \in M \oplus N : x \in \mathcal{N}(x_0, \varepsilon/2), y \in \mathcal{N}(y_0, \varepsilon/2)\}$ is a w^* -neighborhood of $(x_0, -y_0)$. If $(x, -y) \in \mathcal{N}(x_0, y_0; \varepsilon/2)$, then, for any $L \in \mathcal{L}(M, N)$, we have

$$\begin{aligned} |L(x, -y) - L(x_0, -y_0)| &\leq |L(x, -y) - L(x, -y_0)| + \\ &\quad |L(x, -y_0) - L(x_0, -y_0)| \\ &\leq L_M(x - x_0) + L_N(y - y_0) < \varepsilon, \end{aligned}$$

hence, $\mathcal{L}(M, N)$ is uniformly w^* -equicontinuous. \blacksquare

2.8. Lemma. *Let $L \in \mathcal{L}(M, N)$. For each $x \in \mathfrak{M}_2(M)$ there is at most one $y \in \mathfrak{M}_2(N)$ such that $L(x, -y) = 0$, and similarly for each $y \in \mathfrak{M}_2(N)$.*

Proof. If $L(x, -y) = 0 = L(x, -y')$, then

$$\begin{aligned} L_N(y - y') &= L((0, -y) - (0, -y')) = L((x, -y) - (x, -y')) \\ &\leq L(x, -y) + L(x, -y') = 0, \end{aligned}$$

and thus $y' = y$. \blacksquare

Now, we can prove the following

2.9. Theorem. *Let (M, L_M) and (N, L_N) be Lip-von Neumann algebras. If*

$$\text{dist}_{qGH^*}(M, N) = 0,$$

then (M, L_M) and (N, L_N) are isomorphic Lip-von Neumann algebras.

Proof. We shall proceed by steps.

Claim 1. *There exists a seminorm $L_0 \in \mathcal{L}(M, N)$ and a unital positive bijective map $\varphi : \mathfrak{M}_2(M) \rightarrow \mathfrak{M}_2(N)$ with unital positive inverse φ^{-1} s.t. $L_0(x, -\varphi(x)) = 0$ for any $x \in \mathfrak{M}_2(M)$.*

In fact, if $\text{dist}_{qGH^*}(M, N) = 0$, then we can find a sequence $\{L_n\}_{n \in \mathbb{N}}$ of seminorms in $\mathcal{L}(M, N)$ s.t.

$$\text{dist}_H^{L_n}(X_M, X_N) < \frac{1}{n}.$$

Clearly, the sequence $\{L_n\}$ is uniformly bounded on the unit ball of $M \oplus N$ by $R_M + R_N$ (see the proof of Lm. 2.5), where R_M (resp., R_N) is the radius of (M, L_M) (resp., (N, L_N)). Since it is also w^* -equicontinuous by Lm. 2.7, we can apply the Ascoli-Arzelà Theorem (see, *e.g.*, [25]) to conclude that it admits a uniformly convergent subsequence, which we will still denote by $\{L_n\}$ for simplicity. It is then clear that, by homogeneity, this sequence converges uniformly on bounded subsets, and therefore pointwise everywhere on $M \oplus N$, and we denote by L_0 its limit. Then obviously $L_0 \in \mathcal{L}(M, N)$, and for all $\varepsilon > 0$ we can find $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$

$$|L_n(x) - L_0(x)| < \varepsilon \|x\|, \quad \forall x \in M \oplus N.$$

We now observe that the liftings of L_n to $\mathfrak{M}_2(M) \oplus \mathfrak{M}_2(N)$ are again converging to the lifting of L_0 to $\mathfrak{M}_2(M) \oplus \mathfrak{M}_2(N)$, uniformly on

bounded sets. In fact, given $x := (x_{ij}) \in \mathcal{M}_2(M) \oplus \mathcal{M}_2(N) = \mathcal{M}_2(M \oplus N)$, we have

$$\begin{aligned} L_n(x) - L_0(x) &= \max_{ij} L_n(x_{ij}) - \max_{ij} L_0(x_{ij}) \\ &= L_n(x_{i_1 j_1}) - L_0(x_{i_2 j_2}) \\ &= (L_n(x_{i_1 j_1}) - L_0(x_{i_1 j_1})) + (L_0(x_{i_1 j_1}) - L_0(x_{i_2 j_2})) \\ &\leq \varepsilon \|x_{i_1 j_1}\| \leq \varepsilon \|x\|, \end{aligned}$$

where i_1, j_1 realize the maximum for $L_n(x_{ij})$ and i_2, j_2 realize the maximum for $L_0(x_{ij})$. The inequality $L_n(x) - L_0(x) \geq -\varepsilon \|x\|$ is obtained analogously. We now show that $\text{dist}_H^{L_0}(X_M, X_N) = 0$. In fact, given $\varepsilon > 0$, we can find an n_ε such that for all $n \geq n_\varepsilon$, we have $|L_0(x, -y) - L_n(x, -y)| < \varepsilon/2$ for all $x \in X_M, y \in X_N$, and thus

$$\begin{aligned} L_0(x, -y) &\leq |L_0(x, -y) - L_n(x, -y)| + L_n(x, -y) \\ &\leq \varepsilon/2 + L_n(x, -y). \end{aligned}$$

Hence, keeping $x \in X_M$ fixed, if we take the infimum over all $y \in X_N$, we obtain, for n sufficiently large (with $1/n < \varepsilon/2$),

$$\inf_{y \in X_N} L_0(x, -y) \leq \varepsilon/2 + \inf_{y \in X_N} L_n(x, -y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By arbitrariness of ε we get $\inf_{y \in X_N} L_0(x, -y) = 0$ and, by compactness, the infimum is actually a minimum, namely for any $x \in X_M$ we find a unique $y := \varphi(x) \in X_N$ such that $L_0(x, -y) = 0$, where uniqueness follows by Lm. 2.8. Reversing the role of M and N one sees that such map has an inverse defined on X_N , namely φ is surjective. This proves $\text{dist}_H^{L_0}(X_M, X_N) = 0$.

Notice that φ is by construction an isometry w.r.t. the Lip-norms L_M, L_N , since $|L_M(x) - L_N(\varphi(x))| \leq L_0(x, -\varphi(x)) = 0$. We want to show that φ is an affine map. To this aim, let $x_1, x_2 \in X_M$ and let y_1, y_2 be the corresponding elements in X_N for which $L_0(x_i, -y_i) = 0$, $i = 1, 2$. Then, for any $t \in [0, 1]$, we have

$$\begin{aligned} &L_0(tx_1 + (1-t)x_2, -(ty_1 + (1-t)y_2)) \\ &= L_0(t(x_1, -y_1) + (1-t)(x_2, -y_2)) \\ &\leq tL_0(x_1, -y_1) + (1-t)L_0(x_2, -y_2) = 0, \end{aligned}$$

and thus

$$\varphi(tx_1 + (1-t)x_2) = ty_1 + (1-t)y_2 = t\varphi(x_1) + (1-t)\varphi(x_2),$$

showing that φ is affine.

Now, since φ is an affine bijective map from X_M onto X_N , and $\varphi(0) = 0$ by Lm. 2.8, it extends to a bijective³ linear map from $\mathcal{M}_2(M)$ onto $\mathcal{M}_2(N)$ (we denote it still by φ), which is automatically positive and unital. In fact, we have

$$\begin{aligned} 0 \leq x_1 \leq x_2 \leq I_{\mathcal{M}_2(M)} &\Rightarrow 0 \leq x_2 - x_1 \leq I_{\mathcal{M}_2(M)} \Rightarrow \\ 0 \leq \varphi(x_2 - x_1) \leq I_{\mathcal{M}_2(N)} &\Rightarrow \varphi(x_1) \leq \varphi(x_2), \end{aligned}$$

and, analogously, $0 \leq y_1 \leq y_2 \leq I_{\mathcal{M}_2(N)}$ implies $\varphi^{-1}(y_1) \leq \varphi^{-1}(y_2)$. Thus,

$$\begin{aligned} \varphi^{-1}(y) \leq I_{\mathcal{M}_2(M)} \quad \forall y \in X_N &\Rightarrow \varphi^{-1}(I_{\mathcal{M}_2(N)}) \leq I_{\mathcal{M}_2(M)} \\ \varphi(x) \leq I_{\mathcal{M}_2(N)} \quad \forall x \in X_M &\Rightarrow \varphi(I_{\mathcal{M}_2(M)}) \leq I_{\mathcal{M}_2(N)}, \end{aligned}$$

i. e., $\varphi(I_{\mathcal{M}_2(M)}) = I_{\mathcal{M}_2(N)}$ and $\varphi^{-1}(I_{\mathcal{M}_2(N)}) = I_{\mathcal{M}_2(M)}$.

Finally, since any $x \in \mathcal{M}_2(M)_1$ admits a canonical decomposition $x = x_{1,+} - x_{1,-} + i(x_{2,+} - x_{2,-})$ with $x_{\pm,1}, x_{\pm,2} \in \mathcal{M}_2(M)_{1,+}$, we get

$$L_0(x, -\varphi(x)) \leq \sum_{i=1,2} (L_0(x_{i,+}, -\varphi(x_{i,+})) + L_0(x_{i,-}, -\varphi(x_{i,-}))) = 0,$$

and the claim follows, as $x \in \mathcal{M}_2(M)$ implies $\frac{x}{\|x\|} \in \mathcal{M}_2(M)_1$ and clearly $\|x\| \varphi(\frac{x}{\|x\|}) = \varphi(x)$.

Claim 2. Let $x \in \mathcal{M}_2(M)$, $x = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{ij} \in M$, and set $\varphi(x) = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$ with $a'_{ij} = \varphi_{ij} \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) \in N$. Then,

$$L_0 \left(a_{ij}, -\varphi_{ij} \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) \right) = 0, \quad i, j = 1, 2.$$

It is evident, as $\max_{ij} (L_0(a_{ij}, -a'_{ij})) = L_0(x, -\varphi(x)) = 0$.

Claim 3. The map φ is of the form $\text{id}_2 \otimes \phi$, with ϕ a bijective positive linear map between M and N .

Notice that $L_0(0, -x) = 0$, $x \in N$, implies $x = 0$ by the analog of Lm. 2.8 with M, N replacing $\mathcal{M}_2(M), \mathcal{M}_2(N)$. Therefore introducing the matrix units e_{ij} , $i, j = 1, 2$, one sees, thanks to *Claim 2*, that

³The surjectivity simply follows by construction. As for the injectivity, let $x_1, x_2 \in \mathcal{M}_2(M)_+$, and assume that $\varphi(x_1 - x_2) = 0$; then, $L_0(x_1 - x_2, -\varphi(x_1 - x_2)) \leq L_0(x_1, -\varphi(x_1)) + L_0(x_2, -\varphi(x_2)) = 0$, and thus, as $L_0(x_1 - x_2, 0) = 0 = L_0(0, 0)$, by Lm. 2.8 we get $x_1 - x_2 = 0$. Finally, let $x, y \in \mathcal{M}_2(M)_{sa}$; then, since $\varphi(x + iy) = 0$ implies $\varphi(x) = \varphi(y) = 0$, by the first part the claim follows.

$\varphi_{hk}(e_{ij} \otimes a_{ij}) = 0$ if $(h, k) \neq (i, j)$. Moreover, again by *Claim 2*, $\varphi_{ij}(e_{ij} \otimes a)$, $a \in M$, is uniquely determined by

$$L_0(a, -\varphi_{ij}(e_{ij} \otimes a)) = 0,$$

and therefore $\phi(a) := \varphi_{ij}(e_{ij} \otimes a)$ is independent of (i, j) and linear. Then one has, by linearity of φ ,

$$\varphi(x) = \sum_{i,j=1}^2 \varphi(e_{ij} \otimes a_{ij}) = \sum_{i,j=1}^2 e_{ij} \otimes \phi(a_{ij}) = (\text{id}_2 \otimes \phi)(x),$$

which easily implies bijectivity and positivity of ϕ .

As ϕ is injective, unital and positive, it is an order-isomorphism onto its image; furthermore, being surjective, it is a C^* -homomorphism by Thm. 6.4 in [26]. Reversing the roles of M and N , we see that also $\phi^{-1} : N \rightarrow M$ is a unital, 2-positive bijective C^* -homomorphism. Therefore, by a result of Choi (see Corollary 3.2 in [6]), ϕ is indeed a $*$ -isomorphism and there holds

$$L_N(\phi(x)) = L_N(\varphi(e_{11} \otimes x)) = L_M(e_{11} \otimes x) = L_M(x), \quad x \in M.$$

The proof is now complete. ■

Let us observe that we might define the distance dist_{qGH^*} in the following equivalent way. Given two Lip-von Neumann algebras (M, L_M) , (N, L_N) , we consider all the pairs (R, L_R) with R a von Neumann algebra and L_R a seminorm on R such that there exist 2-positive isometric embeddings

$$\begin{aligned} \phi_M : M &\rightarrow R, & L_R(\phi_M(\cdot)) &= L_M(\cdot), \\ \phi_N : N &\rightarrow R, & L_R(\phi_N(\cdot)) &= L_N(\cdot). \end{aligned}$$

We set $\varphi_M = \text{id}_2 \otimes \phi_M$ and $\varphi_N = \text{id}_2 \otimes \phi_N$, and denote by $\mathcal{L}_{ie} \equiv \mathcal{L}_{ie}(M, N)$ the set of all such triples (R, ϕ_M, ϕ_N) . We then define

$$\text{dist}_{qGH^*}^{ie}(M, N) := \inf \{ \text{dist}_H^R(\varphi_M(X_M), \varphi_N(X_N)) : (R, \phi_M, \phi_N) \in \mathcal{L}_{ie} \},$$

where “ ie ” stands for “isometric embedding”.

2.10. Proposition. *For any pair of Lip-von Neumann algebras (M, L_M) , (N, L_N) , we have:*

$$(2.6) \quad \text{dist}_{qGH^*}(M, N) = \text{dist}_{qGH^*}^{ie}(M, N).$$

Proof. Clearly, $\text{dist}_{qGH^*}(M, N) \geq \text{dist}_{qGH^*}^{ie}(M, N)$, since $R = M \oplus N$, with $\phi_M = \iota_M$, $\phi_N = \iota_N$ and ι_M, ι_N the canonical embeddings, is just a particular choice, and, on the r.h.s., we take the infimum over all such choices. For the reverse inequality, let $(R, \phi_M, \phi_N) \in \mathcal{L}_{ie}$ be given. We

then get a seminorm $L \in \mathcal{L}(R, R)$ simply by setting $L(x, y) := L_R(x + y)$. L is clearly a seminorm that restricts to L_R on each summand. We define $L_{M \oplus N}$ as the restriction of L to $\phi_M(M) \oplus \phi_N(N)$. Then, since $L_{M \oplus N} \in \mathcal{L}(\varphi_M(M), \varphi_N(N))$ implies $L_{M \oplus N} \circ (\varphi_M \oplus \varphi_N) \in \mathcal{L}(M, N)$, we have

$$\begin{aligned} \text{dist}_{qGH^*}(M, N) &\leq \text{dist}_H^{L_{M \oplus N}}(\varphi_M(X_M), \varphi_N(X_N)) \\ &= \text{dist}_H^{(R \oplus R, L)}(\varphi_M(X_M) \oplus \{0\}, \{0\} \oplus \varphi_N(X_N)) \\ &\leq \text{dist}_H^{(R \oplus R, L)}(\varphi_M(X_M) \oplus \{0\}, \{0\} \oplus \varphi_M(X_M)) \\ &\quad + \text{dist}_H^{(R \oplus R, L)}(\{0\} \oplus \varphi_M(X_M), \{0\} \oplus \varphi_N(X_N)) \\ &= \text{dist}_H^R(\varphi_M(X_M), \varphi_N(X_N)), \end{aligned}$$

as $\text{dist}_H^{(R \oplus R, L)}(\varphi_M(X_M) \oplus \{0\}, \{0\} \oplus \varphi_M(X_M)) = 0$. By taking the infimum over all the triples (R, ϕ_M, ϕ_N) , we have $\text{dist}_{qGH^*}(M, N) \leq \text{dist}_{qGH^*}^{ie}(M, N)$, and the thesis follows. \blacksquare

2.11. Theorem. dist_{qGH^*} is a metric on the space of isomorphism classes of Lip-von Neumann algebras.

Proof. By Thm. 2.9, we already know that, if $\text{dist}_{qGH^*}(M, N) = 0$, then M and N are isomorphic as LvNA's.

We show now the reverse implication. Let $\psi : M \rightarrow N$ be an isomorphism of LvNA's from (M, L_M) onto (N, L_N) . We set $R := N \oplus N$, $\phi_M := \psi \oplus 0$, $\phi_N := 0 \oplus \iota_N$, where ι_N is the identity map on N , and we define the following seminorm on R :

$$L_R(y_1, y_2) := L_N(y_1 + y_2).$$

Notice that $L_R(\phi_M(x)) = L_M(x)$ for any $x \in M$, and $L_R(\phi_N(y)) = L_N(y)$ for any $y \in N$. Then, by the previous Proposition, we have

$$\text{dist}_{qGH^*}(M, N) \leq \text{dist}_H^R(\varphi_M(X_M), \varphi_N(X_N)).$$

As $\text{dist}_H^R(\varphi_M(X_M), \varphi_N(X_N)) = 0$, we get the claim. \blacksquare

2.12. Remark. Let us notice that the distance dist_{qGH^*} does not appear to be complete, essentially because we do not have an estimate for the Lip-norm of products of elements, much like in the Rieffel's setting (see [11]).

As mentioned briefly at the beginning of the present subsection, if one restricts to the von Neumann algebras acting on a separable Hilbert space \mathcal{H} , the relation between the distance just introduced and the one inducing the Effros-Maréchal topology is analogous to the one

between the Gromov-Hausdorff and the Hausdorff distance between compact subsets of a metric space.

More in detail, the distance $\rho(x, y) = \sum_{m,n} \frac{1}{2^{m+n}} |(\xi_m, (x-y)\xi_n)|$, $x, y \in \mathcal{B}(\mathcal{H})_1$, with $\{\xi_n\}_{n \in \mathbb{N}}$ a dense subset in \mathcal{H}_1 , metrizes the w_0 -topology on bounded subsets of $\mathcal{B}(\mathcal{H})$ (see, e.g., [27], Prop. II.2.7), and therefore it can be used to define dist_{EM} as in Sec. 2.1. On the other hand, by translation-invariance (and positive homogeneity), $L_{\mathcal{H}}(x) := \rho(x, 0)$, $x \in \mathcal{B}(\mathcal{H})$, is a dual-Lip-norm on $\mathcal{B}(\mathcal{H})$, and the restriction $L_M := L_{\mathcal{H}}|_M$, $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra, is a dual-Lip-norm on M .

It is then clear that one can not expect that convergence in dist_{qGH^*} (with respect to these dual-Lip-norms) implies convergence in dist_{EM} as there can be distinct von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ which are isomorphic as LvNA's. Conversely, if one defines

$$\text{dist}_{EM}^+(M_1, M_2) := \text{dist}_H^{\rho}((M_1)_{1,+}, (M_2)_{1,+})$$

it is not difficult to see that if a sequence $\{M_n\}$ of von Neumann algebras on \mathcal{H} converges to M w.r.t. dist_{EM}^+ it converges also w.r.t. both dist_{EM} and dist_{qGH^*} .

2.3. A class of dual-Lip-norms. In view of the subsequent application to the study of limits of vNA's associated to (bosonic) free fields, we now specialize to the case when we are given the same von Neumann algebra (as a subalgebra of some $\mathcal{B}(\mathcal{H})$ with \mathcal{H} separable) endowed with different dual-Lip-norms of a specific type.

2.13. Lemma. *Let L_1 and L_2 be two dual-Lip-norms on the same von Neumann algebra M , and let $J \in \mathcal{L}((M, L_1), (M, L_2))$. Then,*

$$\text{dist}_{qGH^*}((M, L_1), (M, L_2)) \leq \sup_{x \in M, \|x\|=1} J(x, -x).$$

Proof. By definition, we have

$$\text{dist}_{qGH^*}((M, L_1), (M, L_2)) \leq \text{dist}_H^J((M, L_1), (M, L_2)),$$

and, by definition of Hausdorff distance,

$$\begin{aligned} \text{dist}_H^J((M, L_1), (M, L_2)) &= \sup_{x \in X_M} \inf_{y \in X_M} J(x, -y) \leq \sup_{x \in X_M} J(x, -x) \\ &= \sup_{(x_{ij}) \in X_M} \max_{ij} J(x_{ij}, -x_{ij}) \\ &\leq \sup_{x \in M, \|x\|=1} J(x, -x), \end{aligned}$$

where the last inequality follows from that fact that $X_M \subseteq \mathcal{M}_2(M)_1 \subseteq \mathcal{M}_2(M_1)$. ■

2.14. Proposition. *Let M be a von Neumann algebra acting on a separable Hilbert space \mathcal{H} and $\Omega \in \mathcal{H}$ be a separating vector for M . Consider furthermore $T \in \mathcal{B}(\mathcal{H})$ s.t. $\text{Ker } T = \{0\}$ and the mapping $M \ni x \mapsto Tx\Omega \in \mathcal{H}$ is compact. Then, setting $L(x) := \|Tx\Omega\|$, L is a dual Lip-norm on M .*

Proof. L is clearly a seminorm; moreover, $L(x) = 0$ iff $\|Tx\Omega\| = 0$ iff $Tx\Omega = 0$ iff $x\Omega = 0$ iff $x = 0$, as Ω is separating for M and T is injective, showing that L is indeed a norm.

Finally, by definition, the space (M_1, L) is omeomorphic to the space $(\{Tx\Omega, x \in M_1\}, \|\cdot\|)$, which is compact by hypothesis. The result follows by Proposition 1.5(iv). \blacksquare

2.15. Lemma. *Let M be a von Neumann algebra acting on a separable Hilbert space \mathcal{H} , and let L_1, L_2 be dual-Lip-norms on M with $L_1(\cdot) := \|T \cdot \Omega\|$ and $L_2(\cdot) := \|S \cdot \Omega\|$, and assume that the operators T and S satisfy the same conditions as in the previous Proposition. Then,*

$$(2.7) \quad J(x, y) := \|Tx\Omega + Sy\Omega\|$$

belongs to $\mathcal{L}((M, L_1), (M, L_2))$, and we have

$$\text{dist}_{qGH^*}((M, L_1), (M, L_2)) \leq \sup_{x \in M, \|x\|=1} \|(T - S)x\Omega\|.$$

Proof. J is clearly a seminorm, and the restrictions of J to the subalgebras $M \oplus \{0\}$ and $\{0\} \oplus M$ coincide with L_1 and L_2 , respectively, implying $J \in \mathcal{L}((M, L_1), (M, L_2))$. Finally, as $J(x, -x) = \|(T - S)x\Omega\|$, Lemma 2.13 yields the last statement. \blacksquare

3. AN APPLICATION TO FREE QUANTUM FIELDS

As an application of the theory developed in the previous sections, we are going to show that the local von Neumann algebras of the free quantum scalar field endowed with suitable dual-Lip-norms depend continuously on the field mass m with respect to the dual quantum Gromov-Hausdorff distance.

For the convenience of the reader, and to fix notations, we recall here briefly the main definitions, see, *e.g.*, [20, Sec. X.7] or [1, Sec. 3.5] for more details. For an open bounded set $O \subset \mathbb{R}^4$, we denote by $\mathcal{A}_m(O)$ the local von Neumann algebra of the free quantum scalar field ϕ_m of mass $m \geq 0$, *i.e.*, the von Neumann algebra generated by the Weyl operators $W_m(f) = e^{i\phi_m(f)}$, $f \in \mathcal{D}_{\mathbb{R}}(O)$ (the real valued, $C^\infty(\mathbb{R}^4)$ functions with support in O), acting on the (separable) symmetric Fock

space \mathcal{H} over $L^2(\mathbb{R}^3)$,

$$\mathcal{H} := \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} L^2(\mathbb{R}^3)^{\otimes n}.$$

Moreover, we denote by H_m the corresponding hamiltonian operator, *i.e.*, the self-adjoint generator of the strongly continuous one-parameter unitary group obtained through second quantization from the unitary group $t \in \mathbb{R} \mapsto u_m(t)$ on $L^2(\mathbb{R}^3)$ defined by

$$(u_m(t)\psi)(p) := e^{it\omega_m(p)}\psi(p), \quad \psi \in L^2(\mathbb{R}^3),$$

where $\omega_m(p) := \sqrt{m^2 + p^2}$, $p \in \mathbb{R}^3$. The vector $\Omega = (1, 0, 0, \dots) \in \mathcal{H}$, called the vacuum vector, is the (up to a phase) unique unit vector invariant for e^{itH_m} for all $t \in \mathbb{R}$, and it is separating for the local algebras $\mathcal{A}_m(O)$, $m \geq 0$, by the Reeh-Schlieder theorem [1, Thm. 4.14]. Moreover, it is well known that given $O \subset \mathbb{R}^4$ and $\beta > 0$, the map

$$\Theta_m : \mathcal{A}_m(O) \rightarrow \mathcal{H}, \quad \Theta_m(A) := e^{-\beta H_m} A \Omega,$$

is compact (and actually nuclear) for all $m \geq 0$ [2].

It is also well known [7] that for each O and $m > 0$ there exists an isomorphism of von Neumann algebras $\tau_m : \mathcal{A}_m(O) \rightarrow \mathcal{A}_0(O)$ such that $\tau_m(W_m(f)) = W_0(f)$ for all $f \in \mathcal{D}_{\mathbb{R}}(O)$. We also denote by $\tau_0 : \mathcal{A}_0(O) \rightarrow \mathcal{A}_0(O)$ the identity isomorphism.

The above facts suggest the definition of a natural family of Lip-norms L_m on $\mathcal{A}_m(O)$, $m \geq 0$.

3.1. Proposition. *The map L_m , $m \geq 0$, defined by*

$$L_m(A) := \|e^{-\beta H_m} \tau_m(A) \Omega\|, \quad A \in \mathcal{A}_m(O),$$

is a dual-Lip-norm on $\mathcal{A}_m(O)$.

Proof. Consider first the case $m = 0$. Then L_0 is a dual-Lip-norm thanks to the separating property of Ω and to the compactness of Θ_0 , as follows from Prop. 2.14.

The analogous statement for L_m , $m > 0$, is obtained by observing that, for all $\Psi \in \mathcal{H}$,

$$(e^{-\beta H_m} e^{\beta H_0} \Psi)_n(p_1, \dots, p_n) = e^{-\beta \sum_{j=1}^n [\omega_m(p_j) - \omega_0(p_j)]} \Psi_n(p_1, \dots, p_n),$$

which, together with $\omega_m(p) \geq \omega_0(p)$, $p \in \mathbb{R}^3$, implies $\|e^{-\beta H_m} e^{\beta H_0}\| \leq 1$. Therefore

$$L_m \circ \tau_m^{-1}(A) = \|e^{-\beta H_m} A \Omega\| \leq \|\Theta_0(A)\|, \quad A \in \mathcal{A}_0(O),$$

which implies that $A \in \mathcal{A}_0(O) \mapsto e^{-\beta H_m} A \Omega$ is also compact, and then $L_m \circ \tau_m^{-1}$ is a dual-Lip-norm on $\mathcal{A}_0(O)$, again by Prop. 2.14. Being

τ_m isometric and bi- w^* -continuous, we conclude that L_m is a dual-Lip-norm on $\mathcal{A}_m(O)$. \blacksquare

In order to prove convergence of $\mathcal{A}_m(O)$ to $\mathcal{A}_0(O)$ with respect to the above defined dual-Lip-norms, we note the following general fact.

3.2. Lemma. *Let X, Y be Banach spaces, $\Theta : X \rightarrow Y$ a compact linear map, and $T(s)$, $s > 0$, a uniformly bounded family of operators on Y such that $T(s) \rightarrow 0$ strongly as $s \rightarrow 0$. Then*

$$\limsup_{s \rightarrow 0} \sup_{x \in X_1} \|T(s)\Theta(x)\| = 0.$$

Proof. It is not restrictive to assume that $\|T(s)\| \leq 1$ for all $s > 0$. Since Θ is compact, it maps the unit ball X_1 to a totally bounded subset of Y , namely for each given $\varepsilon > 0$ there exist finitely many elements $y_1, \dots, y_n \in \Theta(X_1)$ such that

$$\Theta(X_1) \subset \bigcup_{j=1}^n B_{\varepsilon/2}(y_j),$$

and for each $j = 1, \dots, n$ we can find $\delta_j > 0$ such that $\|T(s)y_j\| < \varepsilon/2$ if $s < \delta_j$. Therefore if $s < \delta := \min_j \delta_j$ and $x \in X_1$, given y_j such that $\|\Theta(x) - y_j\| < \varepsilon/2$, one has

$$\|T(s)\Theta(x)\| \leq \|\Theta(x) - y_j\| + \|T(s)y_j\| < \varepsilon,$$

and hence the statement. \blacksquare

The main result of this Section is therefore the following.

3.3. Theorem. *The family of von Neumann algebras $\mathcal{A}_m(O)$, $m \geq 0$, is continuous with respect to the dual quantum Gromov-Hausdorff distance defined by the dual-Lip-norms L_m defined in Prop. 3.1, namely*

$$\lim_{m' \rightarrow m} \text{dist}_{qGH^*}(\mathcal{A}_{m'}(O), \mathcal{A}_m(O)) = 0, \quad m \geq 0.$$

Proof. Since $L_m \circ \tau_m^{-1} = \|e^{-\beta H_m}(\cdot)\Omega\|$ is a dual-Lip-norm on $\mathcal{A}_0(O)$, as observed in the proof of Prop. 3.1, the Lip-von Neumann algebras $(\mathcal{A}_m(O), L_m)$ and $(\mathcal{A}_0(O), L_m \circ \tau_m^{-1})$ are isomorphic, and then, by Thm. 2.11

$$\begin{aligned} & \text{dist}_{qGH^*}(\mathcal{A}_{m'}(O), \mathcal{A}_m(O)) \\ &= \text{dist}_{qGH^*}((\mathcal{A}_0(O), L_{m'} \circ \tau_{m'}^{-1}), (\mathcal{A}_0(O), L_m \circ \tau_m^{-1})). \end{aligned}$$

Moreover, setting

$$J(A, B) := \|e^{-\beta H_{m'}} A \Omega + e^{-\beta H_m} B \Omega\|, \quad (A, B) \in \mathcal{A}_0(O) \oplus \mathcal{A}_0(O),$$

by Lm. 2.15, J is a seminorm on $\mathcal{A}_0(O) \oplus \mathcal{A}_0(O)$ which restricts to $L_{m'} \circ \tau_{m'}^{-1}$ and $L_m \circ \tau_m^{-1}$, and therefore, again by Lemma 2.15,

$$\begin{aligned} \text{dist}_{qGH^*}(\mathcal{A}_{m'}(O), \mathcal{A}_m(O)) &\leq \sup_{A \in \mathcal{A}_0(O)_1} \|(e^{-\beta H_{m'}} - e^{-\beta H_m})A\Omega\| \\ &= \sup_{A \in \mathcal{A}_0(O)_1} \|(e^{-\beta H_{m'}} e^{\beta H_0} - e^{-\beta H_m} e^{\beta H_0})\Theta_0(A)\|. \end{aligned}$$

Thanks to Lm. 3.2, the statement is then achieved if we can show that $e^{-\beta H_{m'}} e^{\beta H_0} - e^{-\beta H_m} e^{\beta H_0} \rightarrow 0$ strongly as $m' \rightarrow m$. In order to do that, take $\Psi_n \in \mathcal{H}$, a vector whose only non-vanishing component lies in $L^2(\mathbb{R}^3)^{\otimes s^n}$. Then

$$\begin{aligned} &\|(e^{-\beta H_{m'}} e^{\beta H_0} - e^{-\beta H_m} e^{\beta H_0})\Psi_n\|^2 \\ &= \int_{\mathbb{R}^{3n}} dp_1 \dots dp_n \left| (e^{-\beta \sum_{j=1}^n [\omega_{m'}(p_j) - \omega_0(p_j)]} \right. \\ &\quad \left. - e^{-\beta \sum_{j=1}^n [\omega_m(p_j) - \omega_0(p_j)]}) \Psi_n(p_1, \dots, p_n) \right|^2, \end{aligned}$$

and an application of the dominated convergence theorem shows that $\|(e^{-\beta H_{m'}} e^{\beta H_0} - e^{-\beta H_m} e^{\beta H_0})\Psi_n\| \rightarrow 0$. The required strong convergence is then obtained by observing that vectors of the form Ψ_n span a dense subspace of \mathcal{H} , and thanks to the uniform boundedness, in $m' \geq 0$, of $\|e^{-\beta H_{m'}} e^{\beta H_0} - e^{-\beta H_m} e^{\beta H_0}\|$. \blacksquare

It is an interesting open question whether an analogous result holds with respect to other natural Lip-norms, such as, e.g., $A \in \mathcal{A}_m(O) \mapsto \|e^{-\beta H_m} A\Omega\|$, $m \geq 0$.

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REFERENCES

- [1] H. Araki, *Mathematical theory of quantum fields*. Oxford University Press, 1999.
- [2] D. Buchholz, M. Poppmann. *How small is the phase space in Quantum Field Theory?* Ann. Inst. H. Poincaré 52 (1990), 237–257.

- [3] D. Buchholz, E. Wichmann *Causal independence and the energy-level density of states in local quantum field theory* Comm. Math. Phys. 106 (1986), 321-344.
- [4] D. Buchholz, R. Verch. *Scaling algebras and renormalization group in algebraic quantum field theory.* Rev. Math. Phys. 7 (1995), 1195–1239.
- [5] D. Burago, Yu. Burago, S. Ivanov. *A Course in Metric Geometry.* American Mathematical Society, 2001.
- [6] M.-D. Choi. *A Schwarz inequality for positive linear maps on C^* -algebras.* Illinois J. Math. 18 (1974), 565–574.
- [7] J.P. Eckmann, J. Fröhlich. *Unitary equivalence of local algebras in the quasi-free representation.* Ann. Inst. H. Poincaré A 20 (1974), 201–209.
- [8] R.E. Edwards. *Functional analysis. Theory and applications. Corrected reprint of the 1965 original.* Dover Publications, New York, 1995.
- [9] E. Effros. *Convergence of closed subsets in a topological space.* Proc. Amer. Math. Soc. 16 (1965), 929–931.
- [10] E. Effros. *Global structure in von Neumann algebras.* Trans. Amer. Math. Soc. 121 (1966), 434–454.
- [11] D. Guido, T. Isola. *The problem of completeness for Gromov-Hausdorff metrics on C^* -algebras.* J. Funct. Anal. 233 (2006), 173-205.
- [12] U. Haagerup, C. Winslow. *The Effros-Maréchal topology in the space of von Neumann algebras.* Amer. J. Math. 120 (1998), 567–617.
- [13] U. Haagerup, C. Winslow. *The Effros-Maréchal topology in the space of von Neumann algebras, II.* J. Funct. Anal. 171 (2000), 401–431.
- [14] D. Kerr. *Matricial quantum Gromov-Hausdorff distance,* J. of Funct. Anal. **205** (2003), 132-167.
- [15] D. Kerr, H. Li, *On Gromov-Hausdorff convergence of operator metric spaces,* J. Oper. Theory **1** (2009), 83-109.
- [16] F. Latrémolière, *The dual Gromov-Hausdorff Propinquity,* Journal de Mathématiques Pures et Appliquées **103** (2015), no. 2, 303-351.
- [17] F. Latrémolière, *The quantum Gromov-Hausdorff propinquity,* Trans. Amer. Math. Soc. **368** (2016), no. 1, 365-411.
- [18] H. Li, *C^* -algebraic quantum Gromov-Hausdorff distance,* (2003), arXiv: math.OA/0312003
- [19] O. Maréchal. *Topologie et structure borélienne sur l'ensemble des algèbres de von Neumann.* Compt. Rend. Acad. Sc. Paris 276 (1973), 847–850.
- [20] M. Reed, B. Simon. *Methods of Mathematical Physics. Vol. 2. Fourier Analysis, Self-adjointness.* Academic Press, 1975.
- [21] M.A. Rieffel. *Metrics on states from actions of compact groups.* Doc. Math. 3 (1998), 215–229.
- [22] M.A. Rieffel. *Metrics on state spaces.* Doc. Math. 4 (1999), 559–600.
- [23] M.A. Rieffel. *Group C^* -algebras as compact quantum metric spaces.* Doc. Math. 7 (2002), 605–651.
- [24] M.A. Rieffel. *Gromov-Hausdorff Distance for Quantum Metric Spaces.* Mem. Amer. Math. Soc. 168, no. 796 (2004), 1–65.
- [25] W. Rudin. *Functional Analysis.* 2nd ed., McGraw-Hill, 1961.
- [26] E. Stoermer. *Positive linear maps of operator algebras.* Acta Mathematica 110 (1963), 233-278.

- [27] M. Takesaki. *Theory of operator algebras. I*. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002.

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