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Théorie des groupes/*Group Theory*

An invariant for subfactors in the von Neumann algebra of a free group

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Abstract — In this Note we are considering a new invariant for subfactors in the von Neumann algebra $\mathscr{L}(F_k)$ of a free group. This invariant is obtained by computing the Connes's χ invariant for the enveloping von Neumann algebra in the iteration of the Jone's basic construction for the given inclusion. In the case of the subfactors considered in [22], [24] this invariant is easily computed as a relative χ invariant, in the form considered in [14].

One considers the inclusion $\mathscr{L}(F_n) \subseteqq \mathscr{A} = \mathscr{L}(F_n) \rtimes_\theta \mathbb{Z}_{k^2}$, where θ is an injective homomorphism from $\mathbb{Z}_k$ into $\mathrm{Out}(\mathscr{L}(F_n))$ (*i.e.* a $\mathbb{Z}_n$-kernel) with minimal period k^2 [in $\mathrm{Aut}(\mathscr{L}(F_n))$]. Then there exists a canonical copy $\hat\theta$ of $\mathbb{Z}_k$ in $\chi(\mathscr{A})$ which can be lifted to $\mathrm{Aut}(\mathscr{A})$ [4]. The decomposition of the generator of the dual action of $\mathbb{Z}_k$ on $\mathscr{A} \rtimes_{\hat\theta} \mathbb{Z}_k$ as the product of a centrally trivial automorphism and an almost inner automorphism, gives an action of $\mathbb{Z}_{k^2} \oplus \mathbb{Z}_{k^2}$ on the algebra $\mathscr{A} \rtimes_{\hat\theta} \mathbb{Z}_k$. The algebraic invariants [9] for this last action give a more subtle invariant for θ.

As an application we show that, contrary to the case of finite group actions (or more general G-kernels) on the hyperfine II_1 factor (settled in [2], [9], [18]), there exists non outer conjugate, injective homomorphisms (*i.e.* two $\mathbb{Z}_2$-kernels) from $\mathbb{Z}_2$ into $\mathrm{Out}(\mathscr{L}(F_k))$, with non-trivial obstruction to lifting to an action on $\mathscr{L}(F_k)$. Moreover the algebraic invariants [3] do not distinguish between these two $\mathbb{Z}_2$-kernels. Also, there exists two non-outer conjugate, outer actions of $\mathbb{Z}_2$ on $\mathscr{L}(F_k) \otimes R$ that are neither almost inner or centrally trivial.

Un invariant pour les sous-facteurs de l'algèbre de von Neumann associée à un groupe libre

Résumé — *Nous introduisons un nouvel invariant pour les sous-facteurs de l'algèbre de von Neumann* $\mathscr{L}(F_k)$ *d'un groupe libre* F_k, *en calculant l'invariant* χ *de Connes pour l'algèbre enveloppante de la construction de base de Jones associée à l'inclusion donnée. Dans le cas des sous-facteurs considérés dans* [21], [24] *cet invariant peut être facilement calculé en utilisant une version relative pour* χ, *considérée dans* [14].

On considère une inclusion de type $\mathscr{L}(F_n) \subseteqq \mathscr{A} = \mathscr{L}(F_n) \rtimes_\theta \mathbb{K}_{k^2}$, *où* θ *est un homomorphisme injectif de* $\mathbb{Z}_k$ *dans* $\mathrm{Out}(\mathscr{L}(F_n))$, *de période minimale* k^2 [*dans* $\mathrm{Aut}(\mathscr{L}(F_n))$]. *Alors il existe un sous-groupe canonique* $\mathbb{Z}_k \subseteqq \mathrm{Aut}(\mathscr{A})$ *dans* $\chi(\mathscr{A})$ [4]. *La décomposition du générateur de l'action duale de* $\mathbb{Z}_k$ *sur* $\underline{\mathscr{A} \rtimes \mathbb{Z}_k}$, *comme le produit d'un automorphisme centralement trivial et d'un automorphisme dans* $\overline{\mathrm{Int}(\mathscr{A})}$, *donne une action de* $\mathbb{Z}_{k^2} \oplus \mathbb{Z}_{k^2}$ *sur* $\mathscr{A} \rtimes \mathbb{Z}_k$. *On obtient ainsi un invariant plus subtil pour* θ, *en considérant les invariants algébrique* [9] *de l'action de* $\mathbb{Z}_{k^2} \oplus \mathbb{Z}_{k^2}$.

Comme application nous montrons que, contrairement au cas du facteur hyperfini de type II_1 (*analysé dans* [2], [3], [9], [18]), *il existe deux homomorphismes injectifs de* $\mathbb{Z}_2$ *dans* $\mathrm{Out}(\mathscr{L}(F_k)) = \mathrm{Aut}(\mathscr{L}(F_k))/\mathrm{Int}(\mathscr{L}(F_k))$, *qui ont les mêmes invariants algébrique* [3], *mais qui ne sont pas extérieurement conjuguées.*

Version française abrégée — Le but de cette Note est d'introduire un nouvel invariant pour les sous-facteurs des facteurs de type II_1 associés à un groupe libre. L'invariant que nous proposons est obtenu en calculant l'invariant χ de Connes [4] pour l'algèbre de von Neumann enveloppante de la construction de base de Jones, pour l'inclusion donnée. L'invariant χ a été introduit par A. Connes ([4], [5]) pour résoudre un problème très difficile posé par Sakai [27] concernant l'existence d'un facteur de type II_1 non anti-isomorphe à lui-même.

On rappelle que si M est un facteur de type II_1, alors l'invariant $\chi(M)$ est le groupe (commutatif, *voir* [4]) $\chi(M) = (\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)})/\mathrm{Int}(M) \subseteqq \mathrm{Out}(M)$ où $\mathrm{Aut}(M)$ est le groupe d'automorphismes de M, $\mathrm{Int}(M)$ est le groupe des automorphismes intérieurs et $\mathrm{Ct}(M)$ est défini par

$$\mathrm{Ct}(M) = \{ \alpha \in \mathrm{Aut}(M),\ \| \alpha(x_n) - x_n \|_2 \to 0, \text{ pour toute suite centralisante } x_n \}.$$

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L'INVARIANT. — *Soit* $A \subseteq B$ *une inclusion d'algèbres de von Neumann finies, soit* $E_A^B : B \to A$ *une espérance conditionnelle de* B *dans* A *par rapport à une trace fidèle, normale sur* B (*invariante par* E) *et soit* $\mathscr{A}_\infty$ *l'algèbre enveloppante de l'inclusion* $A \subseteq B$ i.e. *le terme à l'infini dans l'itération de la construction de base de Jones* [7] *pour l'inclusion donnée*). *Dans ces conditions,*

$$\chi(\mathscr{A}_\infty) \subseteq \mathrm{Out}(\mathscr{A}_\infty),$$

est un invariant pour l'inclusion $A \subseteq B$. (Comme dans [4], la force de l'invariant vient du fait que l'on considère $\chi(\mathscr{A}_\infty)$, non seulement comme un groupe abstrait, mais comme un sous-groupe de $\mathrm{Out}(\mathscr{A}_\infty)$).

Dans les exemples de sous-facteurs des algèbres de von Neumann des groupes libres considérés dans [24], on peut facilement montrer que cet invariant coïncide avec l'invariant relatif χ [14], pour une inclusion des facteurs hyperfinis associés à l'inclusion donnée.

THÉORÈME. — *Soit* $N \subseteq M$ *une inclusion d'algèbres finies, à prédual séparable et telle qu'il existe une base orthonormale de* M *sur* N, *de type Pimsner et Popa* [20], *pour une espérance conditionnelle de* M *sur* N. *Soit* $\mathscr{L}(F_k)$ *le facteur de type* II_1 *associé à un groupe libre* F_k, $k \geqq 2$. *Si* $\mathscr{A} = (\mathscr{L}(F_k) *_N M$ *est le produit libre, amalgamé réduit des algèbres de von Neumann introduit dans* [22], *alors*

$$\chi(\mathscr{A}) \cong \chi(N, M).$$

Ici $\chi(N, M)$ est l'invariant relatif de Connes pour l'inclusion $N \subseteq M$, considéré dans [14].

Comme application des résultats précédents, nous allons construire deux homomorphismes injectifs de $\mathbb{Z}_2$ dans $\mathrm{Out}(\mathscr{L}(F_k))$, non extérieurement conjugués, tels que tous les invariants algébriques (*voir* [9], [3], [18]) coïncident.

THÉORÈME. — *Pour tout* $k \in \mathbb{Z}$, $k \geqq 2$, *il existe deux automorphismes* $\alpha_i \in \mathrm{Aut}(\mathscr{L}(F_k))$, *tels que les images de* α_i *dans* $\mathrm{Out}(\mathscr{L}(F_k))$ *sont non conjugués et d'ordre deux. De plus, tous les invariants algébriques* [3] *pour les deux homomorphismes injectifs de* $\mathbb{Z}_2$ *dans* $\mathrm{Out}(\mathscr{L}(F_k))$ *coïncident.*

COROLLARY. — *Il existe des sous-facteurs de* $\mathscr{L}(F_k)$, *de profondeur finie* ([7], [17]), *non conjugués et tels que tous les invariants des deux inclusions provenant des commutants relatifs dans la construction de base* ([7], [18], [21]) *sont identiques.*

THE INVARIANT. — *Given* $A \subseteq B$ *an inclusion of type* II_1 *factors, of finite index, we let* B_∞ *be the enveloping algebra for the tower of algebras in the* (*iterated*) *Jones's basic construction* [7] *for* $A \subseteq B$. *Then*

$$\chi(B_\infty) \subseteq \mathrm{Out}(B_\infty)$$

is a conjugacy invariant for $A \subseteq B$.

Here, for a type II_1 factor M, $\chi(M)$ denotes the Connes's invariant for M [4]. The $\chi(M)$ invariant was introduced by Connes [4] in connection to a breakthrough construction that showed the existence of a type II_1 factor not antiisomorphic to itself.

Recall that $\chi(M) = (\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)})/\mathrm{Int}(M)$. Here $\mathrm{Ct}(M)$ is the set of all automorphisms α of M that have the property that $\|\alpha(x_n) - x_n\|_2 \to 0$, whenever x_n, $n \in \mathbb{Z}$ is a central sequence in M (*i.e.* a bounded sequence in M with $\|[x_n, y]\|_2 \to 0$ for any $y \in M$). $\overline{\mathrm{Int}(M)}$ is the closure of $\mathrm{Int}(M)$ in the pointwise convergence topology on M, while $\mathrm{Int}(M)$ is the set of inner automorphisms of M.

We will show that for the subfactors in free group algebras, considered in [24] (or for those considered in [22]), this invariant may be computed and it coincides with the relative form of the invariant χ [14] for an associated pair of hyperfinite factors.

Using the full strength of the invariant (which comes as in [4] by considering $\chi(B_\infty)$ as a subgroup in $\mathrm{Out}(B_\infty)=\mathrm{Aut}(B_\infty)/\mathrm{Int}(B_\infty)$, rather than as an abstract abelian group (*see* also [8], [12], [27])) we obtain as an application that the classification theory for finite group actions (and for G-kernels) on type II_1 factors such as $\mathscr{L}(F_k)$ (the von Neumann algebra of a free group F_k) is different from the corresponding classification theory on the hyperfinite II_1 factor ([2], [3], [9], [18]).

THEOREM. — *There exists two injective homomorphisms from* $\mathbb{Z}_2$ *into* $\mathrm{Out}(\mathscr{L}(F_k))$, *for any* $k\in\mathbb{Z}$, $k\geqq 2$, *non-liftable to an action of* $\mathbb{Z}_2$ *on* $\mathscr{L}(F_k)$ [i.e. *two* $\mathbb{Z}_2$-*kernels on* $\mathscr{L}(F_k)$ *with non-trivial obstruction*] *that are not outer conjugate* (*in* Out) *and for which all the algebraic invariants* ([2], [9]) *coincide.*

Explicitly this means that there exists two (non-inner) automorphisms α_i, $i=1, 2$ on $\mathscr{L}(F_k)$, having non conjugate images (of order two) in $\mathrm{Out}(\mathscr{L}(F_k))$. Moreover, both automorphisms are not liftable to actions of $\mathbb{Z}_2$ on $\mathscr{L}(F_k)$ (*i.e* $\alpha_i^2=\mathrm{Ad}\, g_i$, $\alpha_i(g_i)=-g_i$, $g_i=g_i^+-g_i^-$, with $g_i^\pm$ projections of trace 1/2 in $\mathscr{L}(F_k)$ (*see* [3], [9])).

By [19], the actions on $\mathscr{L}(F_\infty)$, obtained by multiplying the generators by complex numbers of modulus 1 are distinguished (modulo outer conjugacy) by the topology induced on $\mathbb{Z}$ from $\mathrm{Aut}\,\mathscr{L}(F_\infty)$. This is easily seen not to work in the case of finite cyclic groups. For example all $\mathbb{Z}_2$ actions on $\mathscr{L}(F_2)$ that are obtained by multiplying the generators with ± 1 or by switching the generators, are conjugate via elements in $\mathrm{Aut}\,\mathscr{L}(F_2)$, which belong to the image of the O(2) action on $\mathscr{L}(F_2)$, considered in [29].

COROLLARY. — *There exists two non-conjugate, finite depth subfactors in* $\mathscr{L}(F_k)$ *having the same higher relative commutant invariants* ([17], [7], [21]).

It is plausible that an invariant of the type considered in this Note could be used to settle Kadison's problem in [13] (*see* also Sakai's book [26], Problem 4.4.44) on the isomorphism class of the algebras associated to free groups.

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For the proof of the next lemma see the proof of lemma 4.3.3 in [26].

LEMMA 1. — *Let* $N\subseteq M$ *be a pair of separable, finite von Neumann algebras so that there exists an orthonormal Pimsner-Popa* [20] *basis for the inclusion* $N\subseteq M$, *with respect to some faithful, normal conditional expectation from* M *onto* N. *Assume that* N, M *have centers with trivial intersection.*

Then the von Neumann algebra reduced free product $\mathscr{A}=(\mathscr{L}(F_k)\otimes N)*_N M$ (*which by* [22] *is a type* II_1 *factor*) *has the property that* $\|E_N^{\mathscr{A}}(y)-y\|_2\leqq 14 \max_{i=1,2}\|[g_i, y]\|_2$, *with respect to the induced normal faithful conditional expectation* $E_N^{\mathscr{A}}$ *from* $\mathscr{A}$ *into* N. *Here* g_i, $i=1, 2$ *are two of the k generators in* F_k, *while* $\|x\|^2=\tau(x^*x)$, $x\in\mathscr{A}$ *and* τ *is the trace on* $\mathscr{A}$. *In particular all central sequences in* $\mathscr{A}$ *are asymptotically contained in* N.

This lemma together with lemma 1.4 in [22] and lemma 3 in [11] gives:

THEOREM 2. — *Let* $N\subseteq M$ *be a pair of separable, finite von Neumann algebras so that, with respect some faithful normal conditional expectation from* M *into* N, *there exists an orthonormal Pimsner Popa* [20] *basis for the given inclusion. Assume that* N, M *have* *centers with trivial intersection. Let* $\mathscr{A}=(\mathscr{L}(F_k)\otimes N)*_N M$. *Then*

$$\chi((\mathscr{L}(F_k)\otimes N)*_N M)\cong\chi(N, M).$$

Moreover in this identification, corresponding elements in $\chi(\mathscr{A})$ *and* $\chi(N, M)$ *have the same obstructions to lifting from the quotient.*

Here $\chi(N, M)$ is the relative χ invariant [14] for the inclusion $N\subseteq M$, where N, M of separable finite von Neumann algebras:

$$\chi(N, M)=(Ct(N, M)\cap\overline{Int(N, M)})/Int(N, M),$$

where $Ct(N, M)$ is the set of all automorphisms of M, leaving N invariant and acting asymptotically trivial on central sequences for M that are contained in N and $Int(N, M)$ is the set of all inner automorphisms of M that are implemented by unitaries in N.

Corollary 3. — *Let* $\mathscr{C}=(A\supseteq B\supseteq C;\ A\supseteq C\supseteq D)$ *be a commuting square* [21] *of finite dimensional algebras, which is irreducible* (i.e. *the centers of* A, B *and respectively* C, D *have trivial intersection*) *and* λ-*Markov* ([7], [6]) i.e. *there exists a* λ-*Markov trace, in the sense of Jones* [7] *for* $C\subseteq A$, *which restricts to a* λ-*Markov trace for* $D\subseteq B$).

By [24] $(\mathscr{L}(F_k)\otimes B)*_B A\supseteq(\mathscr{L}(F_k))\otimes D)*_D C$ *is a finite index inclusion of type* II_1 *factors* [*that are isomorphic* $\mathscr{L}(F_N)$ *for some* $N>1$]. *Let* $\mathscr{A}_\infty$ *be the envoloping von Neumann algebra for the tower of algebras in the iterated Jones's basic construction of the given inclusion. Note that* $\mathscr{A}_\infty$ *is isomorphic to* $(\mathscr{L}(F_k)\otimes B_\infty)*_{B_\infty}A_\infty$. *Here* B_∞, A_∞ *are obtained by iterating the basic construction for the inclusions* $D\subseteq B$ *and respectively* $C\subseteq A$. *Then*

$$\chi(\mathscr{A}_\infty)=\chi(\mathscr{L}(F_k)\otimes B_\infty*_{B_\infty}A_\infty)\cong\chi(B_\infty, A_\infty).$$

Lemma 4 ([17], [14], [15], [1], [23]). — *Let* θ *be a* $\mathbb{Z}_4$ *action on a copy* R_{-1} *of the hyperfinite* II_1 *factor so that* θ *induces a* $\mathbb{Z}_2$-*kernel on* R_{-1} *with obstruction* -1 [3]. *Let* $R_0=R_{-1}\rtimes_\theta\mathbb{Z}_4$. *Then* $\chi(R_{-1}, R_0)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$.

Moreover, for exactly one of the two copies of $\mathbb{Z}_2$ *in* $\chi(R_{-1}, R_0)$, *there exists no obstruction to lifting from the quotient.*

We may assume (*see* [10]) that

$$R_{-1}=\{gU_0^2, U_1, U_2, \ldots\}''\subseteq\{g, U_0, U_1, \ldots,\}''=R_0.$$

The k-th step in the iterated basic construction for this construction is:

$$R_k=\{gU_{-1}^2\ldots U_{-(k-1)}^2, U_{-k}, U_{-(k+1)}, \ldots\}'',$$

where $(U_k)_{k\in\mathbb{Z}}$ is a family of unitaries with four point spectrum so that each spectral projection for U_k has trace 1/4 and $g=g_+-g_-$ is a selfadjoint unitary whose spectral projections $g_\pm$ have trace 1/2. We have the following relations $U_k^4=1$, $k\in\mathbb{Z}$, $U_k g U_k^*=-g$, if $k=0, 1$, $U_k g U_k^*=g$, if $k\in\mathbb{Z}/\{0, 1\}$ and $U_k U_{k+1} U_k^*=\sqrt{-1}\,U_{k+1}$ (the trace on this construction is specified by requiring that each non-trivial monomial in the U_k's and g have zero trace). With this notations, a commuting square for the inclusion $R_{-1}\subseteq R_0$ is given by

$$D=\{g\}''\subseteq C=\{gU_{-1}^2, U_{-1}\}'';$$
$$B=\{gU_0^2, U_0\}''\subseteq A=\{gU_0^2U_{-1}^2, U_{-1}, U_0\}''.$$

Since $U_0\in B$ normalizes A and $\mathrm{Ad}\,U_0^2|_C=\mathrm{Ad}\,g|_C$, while $g\in D$, we get:

Lemma 5. — *With* A, B, C, D *as before,* $(\mathscr{L}(F_k)\otimes B)*_B A$ *is isomorphic to the cross product of* $(\mathscr{L}(F_k)\otimes D)*_D C$ *by the* $\mathbb{Z}_4$ *action on* $(\mathscr{L}(F_k)\otimes D)*_D C$ *induced by* $\mathrm{Ad}\,U_0$. [*Note that this action is in fact an injective* $\mathbb{Z}_2$-*kernel on* $(\mathscr{L}(F_k)\otimes D)*_D C$ *with obstruction*-1 *to lifting* (*to* Aut).]

Moreover, by Corollary 3, *the enveloping algebra for the inclusion before is* $\mathscr{A}=(\mathscr{L}(F_k)\otimes R_{-1})*_{R_{-1}} R_0$ *and hence, by the preceding lemma,* $\chi(\mathscr{A})\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$ *is generated by* $\mathrm{Ad}\,U_{-1}^2|_{\mathscr{A}}$ *and by* $\mathrm{Ad}\,U_0 U_{-1}|_{\mathscr{A}}$. *There exists no obstruction to lifting to* Aut $(\mathscr{A})$ *for the first copy of* $\mathbb{Z}_2$, *while for the second copy of* $\mathbb{Z}_2$, *the obstruction is* -1.

Lemma 6. — *With the notations before, let* $\mathscr{B}=\mathscr{A}\rtimes\mathbb{Z}_2$, *where the* $\mathbb{Z}_2$ *action on* $\mathscr{A}$ *is induced by* $\mathrm{Ad}\,U_{-1}^2|_{\mathscr{A}}\in\chi(\mathscr{A})$ *(this action is an invariant for* $\mathscr{A}$*). Let s be the dual action on the cross product. Then* $\mathscr{B}$ *isomorphic to* $(\mathscr{L}(F_k)\otimes Q_{-1})*_{Q_{-1}} Q_0$, *where* $Q_-=\{R_{-1}, U_{-1}^2\}''$, $Q_0=\{R_0, U_{-1}^2\}''$. *Moreover we have the following decomposition for s:*

$$s=(s\,\mathrm{Ad}\,U_0 U_{-1})(\mathrm{Ad}\,U_0^* U_{-1}^*)=s_1 s_2,$$

where $s_1=s\,\mathrm{Ad}\,U_0 U_{-1}\in\overline{\mathrm{Int}(\mathscr{B})}$, $s_2=\mathrm{Ad}\,U_0^* U_{-1}^*\in\mathrm{Ct}(\mathscr{B})$. *In addition* $s_i^2=\mathrm{Ad}\,h$, *with* $h=U_0^2 g\in\mathscr{A}$ *and* $s_i(h)=-h$.

The decomposition above prevents $\mathscr{B}$ from being isomorphic to Jones's example ([8], in analogy with the construction in [4]). We recall that in Jones's example one considers $N=(\mathscr{L}(F_N)\otimes R_0)\rtimes_\gamma\mathbb{Z}_2$ for a suitable action γ. Then $\chi(N)=\mathbb{Z}_2\oplus\mathbb{Z}_2$, where, for exactly one of the two copies of $\mathbb{Z}_2$ in $\chi(N)\subseteq\mathrm{Out}(N)$, there exists no obstruction for lifting to an action on N. By [4] (*see* also [8]) this copy (denoted by $\hat{\gamma}$) corresponds to the dual $\mathbb{Z}_2$ action for $N=(\mathscr{L}(F_N)\otimes R_0)\rtimes_\gamma\mathbb{Z}_2$. By Takesaki duality we get:

Proposition 7. — *With the notations before, the dual action* $\tilde{\gamma}$ *on the cross product* $N\rtimes_{\hat{\gamma}}\mathbb{Z}_2$, *has decomposition of the form:* $\tilde{\gamma}=t_1 t_2\,\mathrm{Ad}\,W$, *where* $t_1\in\mathrm{Ct}(N\rtimes_{\hat{\gamma}}\mathbb{Z}_2)$, $t_2\in\overline{\mathrm{Int}(N\rtimes_{\hat{\gamma}}\mathbb{Z}_2)}$. *Moreover there exists selfadjoint unitaries* $e^i=e_+^i-e_-^i$, *with* $e_\pm^i$ *projections of trace* 1/2, *so that* $t_i^2=\mathrm{Ad}\,e^i$, $t_i(e^i)=-e^i$, $t_i(e_j)=e_j$, $i\neq j$.

Corollary 8. — *With the notations above,* $N\cong\mathscr{A}=(\mathscr{L}(F_k)\otimes R_{-1})*_{R_{-1}} R_0$.

Proof. — Assume the contrary. Then $\mathscr{B}=\mathscr{A}\rtimes\mathbb{Z}_2$, $N=(\mathscr{L}(F_N)\otimes R_0)\rtimes_\gamma\mathbb{Z}_2$ would be isomorphic (since in each of the two cases the $\mathbb{Z}_2$ action is canonical). Moreover it would follow that the corresponding dual actions have the same image in Out, and thus by [3], we may assume that they are equal.

Since in $N=(\mathscr{L}(F_N)\otimes R_0)\rtimes_\gamma\mathbb{Z}_2$, the decomposition of an automorphism as the product of a centrally trivial automorphism and an almost inner automorphism, is unique (modulo inner automorphism), it would follow that the decomposition $s=s_1 s_2$ (regarded as an action of $\mathbb{Z}_4\oplus\mathbb{Z}_4$) is outer conjugate to the similar action described by t_1, t_2 in the preceding lemma. But this is impossible because the two actions have different characteristic invariants (*see* [9]). [Since $t_1(e_2)=e_2$, $(t_i)^2=\mathrm{Ad}\,e^i$, while, with the notations in Proposition 5, $s_i(h)=-h$, $s_i^2=\mathrm{Ad}\,h$.]

Proposition 9. — *Let* N *be the factor constructed in* [8]. *Then there exists subalgebras* $A\subseteq B$ *in* N *so that* A *is isomorphic to* $(\mathscr{L}(F_k)$, B *is the cross product* $A\rtimes_\theta\mathbb{Z}_4$, *where* θ *is a* $\mathbb{Z}_2$*-kernel on* A *with non-trivial obstruction (to lifting). Moreover* N *is the enveloping algebra for the inclusion* $A\subseteq B$ *and* A *is isomorphic to* $\mathscr{L}(F_k)$.

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References

[1] M. Choda and H. Kosadi, Strongly outer actions for an inclusion of factors, *Preprint*, 1992.
[2] A. Connes, Periodic automorphisms of the hyperfinite II_1 factor, *Acta Sci. Math.*, 39, 1977, pp. 39-66.
[3] A. Connes, Outer conjugacy classes of automorphisms of factors, *Ann. Sci. École Norm. Sup.*, 8, fasc. 3, 1975.

[4] A. CONNES, Sur la classification des facteurs de type II, *C. R. Acad. Sci., Paris*, 281, Series A, 1975, pp. 13-15.

[5] A. CONNES, A factor not anti-isomorphic to itself, *Ann. of Math.*, 104, 1976, pp. 536-554.

[6] F. GOODMAN, P. DE LA HARPE and V. F. R. JONES, *Coxeter graphs and towers of algebras, Coxeter graphs and towers of algebras*, MSRI Publications 14, Springer Verlag, Berlin, Heidelberg, New York, 1989.

[7] V. F. R. JONES, Index for subfactors, *Invent. Math.*, 72, 1983, pp. 1-25.

[8] V. F. R. JONES, A II_1 factor anti-isomorphic to itself but without involutory antiautomorphisms, *Math. Scand.*, 46, 1980, pp. 103-117.

[9] V. F. R. JONES, Actions of finite groups, in *Springer Lect. Notes in Math.*, 725, pp. 237-253.

[10] V. F. R. JONES, Baxterization, *Intnl. J. Modern Phys. A*, 6, No. 12, 1991, pp. 2035-2043.

[11] V. F. R. JONES, *Notes on Connes invariant* χ(M), unpublished.

[12] V. F. R. JONES and C. E. SUTHERLAND, unpublished manuscript.

[13] R. V. KADISON, List of open problems at the Baton Rouge conference, 1967.

[14] Y. KAWAHIGASHI, *Centrally trivial automorphisms and an analogue of Connes's* χ(M) *for subfactors*, Preprint, Berkeley, 1992.

[15] Ph. H. LOI, *On the derived towers of certain inclusions of type* $\text{III}=\lambda$ *factors of index* 4, Preprint.

[16] D. MCDUFF, Central sequences and the hyperfinite factor, *Proc. London Math. Soc.*, 21, 1976, pp. 443-461.

[17] A. OCNEANU, in *Operator Algebras and Appl.*, 2, London Math. Soc. Lect. Notes, 136, 1988, pp. 202-236.

[18] A. OCNEANU, *Actions of discrete amenable groups on von Neumann algebras*, Springer Lect. Notes in Math., 1138, 1985.

[19] J. PHILLIPS, Automorphisms of full II_1 factors, *Duke J. Math.*, 43, 1976, pp. 375-385.

[20] M. PIMSNER and S. POPA, Entropy and index for subfactors, *Ann. Sci. Ec. Norm. Sup.*, 19, 1986, pp. 57-106.

[21] S. POPA, Classification of subfactors: the finite depth case, *Invent. Math.*, 101, 1990, pp. 19-43.

[22] S. POPA, Markov traces on universal Jones algebras and subfactors of finite index, *Invent. Math.*

[23] S. POPA, *Classification of amenable subfactors of type II*, IHES Preprint, No. 43, June 1992.

[24] F. RĂDULESCU, *Random matrices, amalgameted free products and subfactors of the von Neumann algebra of a free group of non-integer index*, Preprint IHES, 1991.

[25] E. STORMER, Conjugacy of Involutive Antiautomorphisms of von Neumann algebras, *J. Funct. Analysis*, 66, 1986, pp. 54-66.

[26] D. VOICULESCU, Circular and semicircular systems and free product factors, *in Operator Alg., Unitary Repr., Enveloping Algebras and Inv. Theory*, Progress in Mathematics, 92, Birkhäuser, Boston, 1990.

[27] D. VOICULESCU, in *Operator Algebras and connections with Ergodic Theory*, Springer Lecture Notes in Math., 1138, 1989.

[28] C. E. SUTHERLAND, Cohomology and extensions of von Neumann algebras, *Publ. RIMS*, 16, 1980, pp. 105-133.

[29] M. TAKESAKI, *Duality and von Neumann Algebras*, Springer Lecture Notes, No. 247.

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