New results on old spectral triples for fractals

Daniele Guido and Tommaso Isola

Abstract. It is shown that many important features of nested fractals, such as the Hausdorff dimension and measure, the geodesic distance induced by the immersion in $\mathbb{R}^n$ (when it exists), and the self-similar energy can be recovered by the description of the fractal in terms of spectral triples. We describe in particular the case of the Vicsek square, showing that all self-similar energies can be described through a deformation of the square to a rhombus.

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0. Introduction

In this note, we review some notions relative to a noncommutative description of some fractals which goes back to [5] for the case of the Cantor set, and to [8] for a wide class of fractals, specialising the analysis to the class of nested fractals [10, 15, 12, 13], which allows a more precise study of the various properties that can be recovered via the (noncommutative) geometric treatment.

Nested fractals are described here via a discrete spectral triple, consisting of the algebra of (continuous) functions on the fractal acting on the ℓ^2 space H on the oriented edges of the fractal, where the Dirac operator D on H maps an oriented edge to its opposite, multiplied by the inverse length of the edge itself.

The tools of noncommutative geometry [5] may associate with these data a notion of dimension, an integral on the elements of the algebra, a distance on the state space of the algebra, and also an energy form.

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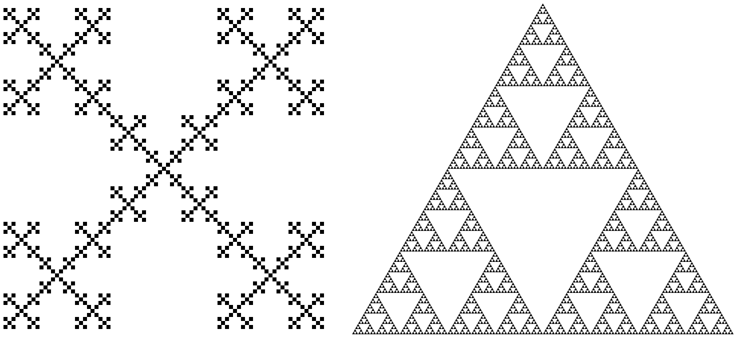

Figure 1. Nested fractals

We prove in Sections 1 and 2 that, for nested fractals, the noncommutative dimension coincides with the Hausdorff dimension and the noncommutative integral coincides (up to a multiplicative constant) with the integral w.r.t. the Hausdorff measure. Less stringent results in this direction were proved in [8] for a wider class of fractals. We recall here that the noncommutative dimension d is given by the abscissa of convergence of the zeta function $z(s) = \mathrm{Tr}(|D|^{-s})$, while the integral of f is given by the residue in d of the zeta function $z_f(s) = \mathrm{Tr}(f|D|^{-s})$.

As for the distance, in the case of the Sierpinski gasket, we stated (without proof) in [8] that the geodesic distance induced by the natural immersion of the gasket in the plane is also recovered via the discrete spectral triple. The same result was proved in [3] with a completely different spectral triple.

A noncommutative energy form for fractals was first considered in [4] for the case of the gasket, with a spectral triple given by a deformation of those considered in [3], and was proved to coincide with the unique self-similar energy on the gasket up to a constant. In that paper a residue formula for a suitable zeta function was used, and the abscissa of convergence, which we called energy dimension, was shown to be different from the Hausdorff dimension.

Recently, we could prove that this is a general result, when the discrete spectral triple is considered on nested fractals, see [9] for the proof. More precisely the energy dimension δ, or the abscissa of convergence of the zeta function $Z_f(s) = \mathrm{Tr}(|[D,f]|^2|D|^{-s})$, is always given by $2 - \frac{\log\rho}{\log\lambda}$ for finite energy funcions, where ρ is the scaling factor for the energy, and λ is the (unique) scaling factor for the contractions. Moreover, the residue in δ of $Z_f(s)$ always coincides with a self similar energy of f on the fractal.

We discuss these results in Section 3, describing in particular the case of the Vicsek square, for which the self-similar energy is not unique. Since our triples are completely determined by the geometry, or better by the embedding of the fractal in $\mathbb{R}^n$, we expected to be able to recover only one of such energies (up to multiplicative constants), in particular the one which is invariant under all symmetries of the square. We prove here that all self-similar energies for the Vicsek are recovered by our methods, if we are allowed to replace the squared Vicsek with a rhombic Vicsek, where some symmetries are violated.

In Section 4 we report on another recent result of ours, namely the fact that the geodesic distance induced by the natural immersion of the nested fractal in $\mathbb{R}^n$ coincides with a suitable noncommutative distance induced by the discrete spectral triple. More precisely, Connes noncommutative distance is induced by the semi-norm given by the norm of the commutator of the Dirac operator with an element of the algebra, a function on the fractal in our case. Our variant of the noncommutative distance consists of replacing the norm with a norm up to compacts, namely with a norm in the Calkin algebra. For these reasons such distance does not always coincide with the noncommutative distance considered by Connes, however it does for the gasket. The proof of this result will be given in [9].

1. Nested fractals

Let $\Omega := \{w_i : i = 1, \dots, k\}$ be a family of contracting similarities of $\mathbb{R}^N$, *i.e.* there are $\lambda_i \in (0,1)$ such that $\|w_i(x) - w_i(y)\| = \lambda_i \|x - y\|$, $x, y \in \mathbb{R}^N$. The unique non-empty compact subset K of $\mathbb{R}^N$ such that $K = \bigcup_{i=1}^k w_i(K)$ is called the *self-similar fractal* defined by $\{w_i\}_{i=1,\dots,k}$. For any $i \in \{1, \dots, k\}$, let $p_i \in \mathbb{R}^N$ be the unique fixed-point of w_i, and say that p_i is an essential fixed-point of Ω if there are $i', j, j' \in \{1, \dots, k\}$, $i' \neq i$, such that $w_j(p_i) = w_{j'}(p_{i'})$. Denote by V_0 the set of essential fixed-points of Ω, and we assume that it has at least two elements, and let $E_0 := \{(p, q) : p, q \in V_0, p \neq q\}$. For any $n \in \mathbb{N}$, set $\Sigma_n := \{\sigma : \{1, \dots, n\} \to \{1, \dots, k\}\}$, $w_\sigma := w_{\sigma(1)} \circ \cdots \circ w_{\sigma(n)}$, $\forall \sigma \in \Sigma_n$, $V_n := \cup_{\sigma \in \Sigma_n} w_\sigma(V_0)$, and $w_\emptyset := id$, $\Sigma_0 := \{\emptyset\}$, $\Sigma := \cup_{n=0}^\infty \Sigma_n$. Then, $V_{n-1} \subset V_n$, $\forall n \in \mathbb{N}$. Sets of the form $w_\sigma(V_0)$, for $\sigma \in \Sigma_n$, are called *combinatorial n-cells*. For any $n \in \mathbb{N}$, define $E_n := \{(w_\sigma(p), w_\sigma(q)) : \sigma \in \Sigma_n, p, q \in V_0, p \neq q\}$, and, for any $\sigma \in \Sigma_n$, $i \in \{1, \dots, k\}$, denote by $\sigma \cdot i \in \Sigma_{n+1}$ the map defined by $\sigma \cdot i(j) = \sigma(j)$, $j \in \{1, \dots, n\}$, $\sigma \cdot i(n+1) = i$. The couple (K, Ω) is said to be a *nested fractal* in the sense of Lindstrøm [10] if

(1) $\lambda_i = \lambda$, for all $i \in \{1, \dots, k\}$,
(2) there is an open bounded set $U \subset \mathbb{R}^N$, such that $\cup_{i=1}^k w_i(U) \subset U$, and $w_i(U) \cap w_j(U) = \emptyset$, for all $i, j \in \{1, \dots, k\}$, $i \neq j$ (open set condition),
(3) the graph (V_1, E_1) is connected, that is, for any $p, q \in V_1$, there are $p_0, \dots, p_s \in V_1$, such that $p_0 = p$, $p_s = q$, and $(p_{i-1}, p_i) \in E_1$, for all $i = 1, \dots, s$,
(4) if $\sigma, \sigma' \in \Sigma_n$, $\sigma \neq \sigma'$, then $w_\sigma(V_0) \neq w_{\sigma'}(V_0)$, and $w_\sigma(K) \cap w_{\sigma'}(K) = w_\sigma(V_0) \cap w_{\sigma'}(V_0)$ (nesting property),
(5) if $p, q \in V_0$, $p \neq q$, then the symmetry with respect to $\Pi_{pq} := \{z \in \mathbb{R}^N : \|z - p\| = \|z - q\|\}$ maps combinatorial n-cells to combinatorial n-cells,

for any $n \in \mathbb{N} \cup \{0\}$, and maps an n-cell lying on both sides of Π_{pq} to itself (symmetry property).

The following definitions are taken from [12, 13], with slight modifications.

If V is a finite set, let us denote by $C(V)$ the set of functions from V to $\mathbb{R}$, and by $\mathscr{D}$ the set of functionals $\mathcal{E} : C(V_0) \to \mathbb{R}$ such that

(1) there exists $c_{pq} = c_{qp} > 0$, $\forall (p,q) \in E_0$, for which, $\forall f \in C(V_0)$, one has $\mathcal{E}[f] := \sum_{(p,q)\in E_0} c_{pq}(f(p) - f(q))^2$,

(2) $\mathcal{E}[f] = 0 \iff f$ is constant.

Define, for $\mathcal{E} \in \mathscr{D}$, $n \in \mathbb{N}$, the functionals

$$S_n(\mathcal{E})[f] := \sum_{\sigma\in\Sigma_n} \mathcal{E}[f \circ w_\sigma], \qquad \forall f \in C(V_n),$$

$$M_n(\mathcal{E})[f] := \inf\{S_n(\mathcal{E})[g] : g \in C(V_n), g|_{V_0} = f\}, \qquad \forall f \in C(V_0).$$

Definition 1.1. A functional $\mathcal{E} \in \mathscr{D}$ is said an eigenform, with eigenvalue $\rho > 0$, if $M_1(\mathcal{E}) = \rho\mathcal{E}$.

Lindstrøm proved that there is an eigenform $\widehat{\mathcal{E}} \in \mathscr{D}$. Note that all eigenforms have the same eigenvalue ρ, which satisfies $\rho \in (0,1)$, see [15], Proposition 3.8. It is known that $\widehat{\mathcal{E}}_\infty[f] := \lim_{n\to\infty} \rho^{-n} S_n(\widehat{\mathcal{E}})[f]$ defines a Dirichlet form on the fractal K. Define $\mathscr{F} := \{f \in C(K) : \widehat{\mathcal{E}}_\infty[f] < \infty\}$.

Theorem 1.2. *Let $\mathcal{E} \in \mathscr{D}$. Then there exists*

$$\mathcal{E}_\infty[f] := \lim_{n\to\infty} \rho^{-n} S_n(\mathcal{E})[f], \qquad f \in \mathscr{F}.$$

Moreover, there is an eigenform $\mathcal{E}' \in \mathscr{D}$ such that

$$\mathcal{E}_\infty = \lim_{n\to\infty} \rho^{-n} S_n(\mathcal{E}')[f].$$

Proof. See [13], Theorem 4.11, and Remark 4.1. □

2. Singular traces and spectral triples on self-similar fractals

Let us recall that $(\mathcal{A}, \mathcal{H}, D)$ is called a *spectral triple* when $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$, D is a self adjoint operator on the same Hilbert space such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and D has compact resolvent. In the following we shall assume that 0 is not an eigenvalue of D, the general case being recovered by replacing D with $D|_{\ker(D)^\perp}$. Such a triple is called d^+-summable, $d \in (0,\infty)$, when $|D|^{-d}$ belongs to the Macaev ideal $\mathcal{L}^{1,\infty} = \{a : \frac{S_n(a)}{\log n} < \infty\}$, where $S_n(a) := \sum_{k=1}^n \mu_k(a)$ is the sum of the first n largest eigenvalues (counted with multiplicity) of $|a|$.

The noncommutative version of the integral on functions is given by the formula $\mathrm{Tr}_\omega(a|D|^{-d})$, where $\mathrm{Tr}_\omega(a) := \lim_\omega \frac{S_n(a)}{\log n}$ is the Dixmier trace, i.e. a singular trace summing logarithmic divergences. By the arguments below, such integral can be non-trivial only if d is the Hausdorff dimension of the spectral triple, but even this choice does not guarantee non-triviality.

Definition 2.1. [7] Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, Tr_ω the Dixmier trace.
(i) We call α-dimensional Hausdorff functional the map $a \mapsto Tr_\omega(a|D|^{-\alpha})$;
(ii) we call (Hausdorff) dimension of the spectral triple the number

$$d(\mathcal{A}, \mathcal{H}, D) = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\},$$

where $\mathcal{L}_0^{1,\infty} = \{a : \frac{S_n(a)}{\log n} \to 0\}$.

Theorem 2.2. [7] *$d := d(\mathcal{A}, \mathcal{H}, D)$ is the unique exponent, if any, such that the d-dimensional Hausdorff functional is non-trivial.*

In general, a self-similar fractal (in $\mathbb{R}^n$) is described by a finite set of similitudes $w_1, \dots w_k$, with scaling parameters $\lambda_1, \dots \lambda_k$, $\lambda_i < 1$, as the unique compact set K such that

$$\bigcup_{i=1}^{k} w_i(K) = K.$$

A standard way to construct spectral triples on such fractals is the following:

- Select a subset $S \subset K$ together with a triple $\mathcal{T}_o = (\pi_o, \mathcal{H}_o, D_o)$ on $\mathcal{C}(S)$.
- Set $\mathcal{T}_\emptyset = (\pi_\emptyset, \mathcal{H}_\emptyset, D_\emptyset)$ on $\mathcal{C}(K)$, where $\pi_\emptyset(f) = \pi_o(f|_S)$, $\mathcal{H}_\emptyset = \mathcal{H}_o$, $D_\emptyset = D_o$.
- Set $\mathcal{T}_\sigma := (\pi_\sigma, \mathcal{H}_\emptyset, D_\sigma)$ on $\mathcal{C}(K)$, with $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$, $D_\sigma = \lambda_\sigma^{-1} D_\emptyset$, $\lambda_\sigma = \prod_{i=1}^{|\sigma|} \lambda_{\sigma(i)}$.
- Set $\mathcal{T} = \bigoplus_\sigma \mathcal{T}_\sigma$ on $\mathcal{C}(K)$ and consider the *-algebra $\mathcal{A} = \{f \in \mathcal{C}(K) : [D, f]$ is bdd$\}$.

Definition 2.3. [Discrete triple on nested fractals] Assume K to be a nested fractal in $\mathbb{R}^n$, and construct a triple $\mathcal{T}_o = (\pi_o, \mathcal{H}_o, D_o)$ on $\mathcal{C}(V_0)$ as follows:

$$\mathcal{H}_o = \bigoplus_{e \in E_0} \ell^2(\partial e), \quad \pi(f) = \bigoplus_{e \in E_0} \begin{pmatrix} f(e^+) & 0 \\ 0 & f(e^-) \end{pmatrix} \quad D_o = \frac{1}{\ell(e)} \bigoplus_{e \in E_0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then construct the triples $\mathcal{T}_\sigma$ and $\mathcal{T} = \bigoplus_\sigma \mathcal{T}_\sigma$ as above.

Theorem 2.4. *The zeta function $\mathcal{Z}_D$ of $(\mathcal{A}, \mathcal{H}, D)$, i.e. the meromorphic extension of the function $s \in \mathbb{C} \mapsto \mathrm{Tr}(|D|^{-s})$, is given by*

$$\mathcal{Z}_D(s) = \frac{2 \sum_{e \in E_0} \ell(e)^s}{1 - k\lambda^s}.$$

Therefore, the dimensional spectrum of the spectral triple is

$$\mathcal{S}_{dim} = \{d\left(1 + \frac{2\pi i}{\log k} n\right) : n \in \mathbb{Z}\} \subset \mathbb{C}.$$

As a consequence, the metric dimension d_D of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, namely the abscissa of convergence of its zeta function, is $d_D = d = \frac{\log k}{\log 1/\lambda}$.

$\mathcal{Z}_D$ has a simple pole in d_D, and the measure associated via Riesz theorem with the functional $f \to \oint f$ coincides with a multiple of the Hausdorff measure H_d (normalized on K):

$$vol(f) \equiv \int_K f \, d\,\mathrm{vol} := tr_\omega(f|D|^{-d}) = \frac{2}{\log 1/\lambda} \sum_{e \in E_0} \ell(e)^d \int_K f \, dH_d \qquad f \in C(K).$$

Proof. The non vanishing eigenvalues of $|D_\sigma|$ are exactly $\{\frac{1}{\ell(e)\lambda^{|\sigma|}}\}_{e\in E_0}$, each one with multiplicity 2.

Hence $\mathrm{Tr}(|D_\sigma|^{-s}) = 2\lambda^{s|\sigma|}\sum_{e\in E_0}\ell(e)^s$ and for $\mathrm{Re}\, s > d$ we have

$$\begin{aligned}\mathrm{Tr}(|D|^{-s}) &= \sum_\sigma \mathrm{Tr}(|D_\sigma|^{-s}) = 2\sum_{e\in E_0}\ell(e)^s\sum_\sigma \lambda^{s|\sigma|}\\ &= 2\sum_{e\in E_0}\ell(e)^s\sum_{n\geq 0}\sum_{|\sigma|=n}\lambda^{sn}\\ &= 2\sum_{e\in E_0}\ell(e)^s\sum_{n\geq 0}k^n\lambda^{sn} = 2\sum_{e\in E_0}\ell(e)^s(1-k\lambda^s)^{-1}.\end{aligned}$$

Therefore, we have $\mathcal{S}_{dim} = \{d\left(1+\frac{2\pi i}{\log k}n\right) : n\in\mathbb{Z}\}\subset\mathbb{C}$. Now we prove that the volume measure is a multiple of the Hausdorff measure H_d.

Clearly, the functional $\mathrm{vol}(f) = tr_\omega(f|D|^{-d})$ makes sense also for bounded Borel functions on K, and we recall that the logarithmic Dixmier trace may be calculated as a residue (cf. [5]): $tr_\omega(f|D|^{-d}) = \mathrm{Res}_{s=d}\ \mathrm{Tr}(f|D|^{-s})$, when the latter exists. Then, for any multi-index τ,

$$\begin{aligned}tr_\omega(\chi_{C_\tau}|D|^{-d}) &= \mathrm{Res}_{s=d}\ \mathrm{Tr}(\chi_{C_\tau}|D|^{-s})\\ &= \lim_{s\to d^+}(s-d)\ \mathrm{Tr}(\chi_{C_\tau}|D|^{-s})\\ &= \lim_{s\to d^+}(s-d)\sum_\sigma \mathrm{Tr}(\chi_{C_\tau}\circ w_\sigma|D_\sigma|^{-s}),\end{aligned}$$

and we note that $\chi_{C_\tau}\circ w_\sigma$ is not zero either when $\sigma<\tau$ or when $\sigma\geq\tau$. In the latter case, $\chi_{C_\tau}\circ w_\sigma = 1$. Observe that $\mathrm{Tr}(\chi_{C_\tau}|D_\sigma|^{-s}) \leq \mathrm{Tr}(|D_\sigma|^{-s}) = 2\lambda^{s|\sigma|}\sum_{e\in E_0}\ell(e)^s \to 2\lambda^{d|\sigma|}\sum_{e\in E_0}\ell(e)^d$ when $s\to d^+$, hence $\lim_{s\to d^+}(s-d)\,\mathrm{Tr}(\chi_{C_\tau}|D_\sigma|^{-s}) = 0$. Therefore we may forget about the finitely many $\sigma<\tau$, and get

$$\begin{aligned}tr_\omega(\chi_{C_\tau}|D|^{-d}) &= \lim_{s\to d^+}(s-d)\sum_{\sigma\geq\tau}\mathrm{Tr}(|D_\sigma|^{-s})\\ &= \lim_{s\to d^+}(s-d)\sum_{n=0}^\infty 2k^n\lambda^{s(|\tau|+n)}\sum_{e\in E_0}\ell(e)^s\\ &= 2\lambda^{d|\tau|}\sum_{e\in E_0}\ell(e)^d\lim_{s\to d^+}\frac{s-d}{1-k\lambda^s}\\ &= \frac{2}{k^{|\tau|}\log 1/\lambda}\sum_{e\in E_0}\ell(e)^d = \frac{2}{\log 1/\lambda}\sum_{e\in E_0}\ell(e)^d H_d(C_\tau).\end{aligned}$$

This implies that, for any $f\in\mathcal{C}(K)$ for which $f\leq\chi_{C_\tau}$, $\mathrm{vol}(f)\leq\frac{2}{\log 1/\lambda}\sum_{e\in E_0}\ell(e)^d\left(\frac{1}{k}\right)^{|\tau|}$, therefore points have zero volume, and $\mathrm{vol}(\chi_{\dot{C}_\tau}) = \mathrm{vol}(\chi_{C_\tau})$, where $\dot{C}_\tau$ denotes the interior of C_τ. As a consequence, for the simple functions given by finite linear combinations of characteristic functions of cells or vertices,

$\mathrm{vol}(\varphi) = \frac{2}{\log 1/\lambda} \sum_{e \in E_0} \ell(e)^d \int \varphi \, dH_d$. Since continuous function are Riemann integrable w.r.t. such simple functions, the thesis follows. □

3. On the recovery of the Dirichlet energy

We propose the following expression for the energy form on a spectral triple (similar expressions were used in some previous papers [2, 16, 4]):

$$\mathcal{E}[a] = \mathrm{Tr}_\omega \, |[D,a]|^2 |D|^{-\delta},$$

or the residue form of the same formula:

$$\mathrm{Res}_{s=\delta} \, \mathrm{Tr}(|D|^{-s/2} |[D,f]|^2 \, |D|^{-s/2}).$$

This is motivated by the fact that, in noncommutative geometry, $\oint a = \mathrm{Tr}_\omega \, a|D|^{-\delta}$ describes the integral and that $[D,a]$ is a replacement for the gradient of a.

Hover, while for smooth manifolds δ coincides with the dimension, for singular structures such as fractals the metric dimension d is in general different from the energy dimension δ [4]. Moreover, elements with finite energy are not Lipschitz, namely $[D,a]$ is not bounded, but $|[D,a]|^2|D|^{-\delta}$ is in the domain of the Dixmier trace. Our main theorem here is the following.

Theorem 3.1 ([9]). *Let K be a nested fractal with scaling parameter λ and eigenvalue ρ for the eigenform, with the spectral triple described above. The formula* $\mathrm{Res}_{s=\delta} \, \mathrm{Tr}(|D|^{-s/2} |[D,f]|^2 \, |D|^{-s/2})$ *recovers a self-similar energy, with* $\delta = 2 - \frac{\log \rho}{\log \lambda}$.

The proof is a direct consequence of the results in [13] mentioned in Theorem 1.2.

3.1. An example of non uniqueness

When the fractal has a unique self-similar energy form, the Theorem above provides such unique form. We now discuss the case of the Vicsek fractal, where uniqueness does not hold. More precisely, there exists a unique energy form which is invariant under all the symmetries of the square. Removing such invariance request, self-similar energies for the Vicsek snowflake (up to a scalar multiple) are parametrized by the conductances $(1, 1, 1, 1, F, F^{-1})$, [11]. Such energies can be recovered with our approach via metric deformations, namely we consider 5 similitudes with scaling parameter $1/3$, whose fixed points coincide with the 4 vertices of a rhombus and with the center of the rhombus itself. This means that with our construction the 1-parameter family of energies for the Vicsek arise from a 1-parameter family of geometric deformations. We assume the side of the rhombus has length 1, and the smaller angle is $2\vartheta \le \pi/2$, so that the diagonals measure $2\sin\vartheta$ and $2\cos\vartheta$, and the ratio between the lengths of the diagonals is $\tan\vartheta$.

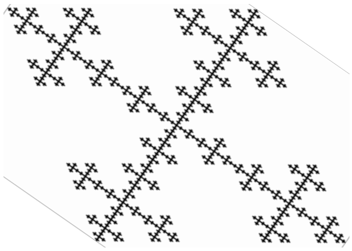

Figure 2. Rhomboidal Vicsek snowflake

Theorem 3.2. *Let K be the fractal described above, with the spectral triple as in Definition 2.3. Then the self-similar energy provided by Theorem 3.1 coincides (up to a multiple) with that associated with the conductances $(1,1,1,1,F,F^{-1})$, where $F=\dfrac{2+\sqrt{1+\tan^2\vartheta}}{2+\sqrt{1+\tan^{-2}\vartheta}}$.*

Proof. For the Vicsek we have $\lambda=1/3$, $\rho=1/3$. Therefore the constants c_e giving the energy $\mathcal{E}[f]=\sum_{e\in E_0}c_e|\langle f,e\rangle|^2$ are: $a=c_e=1$ for the sides of the rhombus, $c_e=g=(2\sin\vartheta)^{-1}=1/2\sqrt{1+\tan^{-2}\vartheta}$ for the longer diagonal, $c_e=f=(2\cos\vartheta)^{-1}=1/2\sqrt{1+\tan^2\vartheta}$ for the shorter one. According to a computation of De Cesaris [6], the constants (A,A,A,A,F,G) for the eigenform and the constants (a,a,a,a,f,g) giving rise to the same energy are related by

$$A=\frac{(a+f)(a+g)}{2a+f+g},\quad F=\frac{(a+f)^2}{2a+f+g},\quad G=\frac{(a+g)^2}{2a+f+g}.$$

Therefore the normalized F is

$$\frac{F}{A}=\frac{a+g}{a+f}=\frac{1+1/2\sqrt{1+\tan^2\vartheta}}{1+1/2\sqrt{1+\tan^{-2}\vartheta}}=\frac{2+\sqrt{1+\tan^2\vartheta}}{2+\sqrt{1+\tan^{-2}\vartheta}}.$$

□

4. On the recovery of the geodesic distance induced by the Euclidean structure

Let us recall that, for a given spectral triple (A,H,D), the distance between states on the C*-algebra A is given by [5]

$$d_D(\varphi,\psi)=\sup\{|\varphi(a)-\psi(a)|:a\in A,\|[D,a]\|\le 1\}.$$

When A is the algebra of continuous functions on a compact Hausdorff space K, we may take the states to be delta-functions, thus getting the distance between points:

$$d_D(x,y)=\sup\{|f(x)-f(y)|:f\in\mathcal{C}(K),\|[D,f]\|\le 1\}.$$

In many concrete cases, the seminorm $L(a)=\|[D,a]\|$ is a Lip-norm in the sense of Rieffel [14], namely it induces the weak*-topology on the state

space. When $A = \mathcal{C}(K)$, this property amounts to the fact that the distance d_D induces the original topology on K.
Assume now K to be a nested fractal in $\mathbb{R}^n$ with property

(A) for any pair x, y of points in K there exists a rectifiable curve in $\mathbb{R}^n$ joining x with y.

Set $d_{\text{geo}}(x, y) := \inf\{\ell(\gamma)|\gamma : [0, 1] \to K \text{ is rectifiable and } \gamma(0) = x, \gamma(1) = y\}$. We call such a distance the Euclidean geodesic distance on K.

Definition 4.1 (An essential Lip-norm for spectral triples). Let us consider the quotient map $p : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}$ to the Calkin algebra. Then, given a spectral triple $\mathcal{T} := (\mathcal{A}, \mathcal{H}, D)$, we consider the seminorm

$$L_{\text{ess}}(a) := \|p([D, a])\|, \qquad a \in \mathcal{A}. \tag{4.1}$$

The seminorm L_{ess} induces a distance d_{ess} on K: for $x, y \in K$,

$$d_{\text{ess}}(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{C}(K),\ L_{\text{ess}}(f) \leq 1\}. \tag{4.2}$$

Theorem 4.2. *The noncommutative distance d_{ess} on K coincides with the geodesic distance d_{geo} on K induced by the Euclidean structure in $\mathbb{R}^n$. If any edge of level n is the union of edges of level $n+1$, the Connes' distance d_D for the triple $\mathcal{T}$ coincides with the essential distance, hence with the geodesic distance.*

Remark 4.3. (a) Let us observe that for the standard triples on (possibily non-commutative) smooth manifolds the spectrum of $[D, a]$ has no non-essential parts, hence the seminorm L_{ess} coincides with the usual seminorm $\|[D, a]\|$.
(b) As mentioned at the beginning, our spectral triples are based here on the complete graph with vertices V_0. Therefore the last hypothesis of the Theorem is satisfied e.g. for the generalized Sierpinski triangles in the plane obtained by contractions of $1/p$, or by the higher-dimensional gaskets inscribed in n-simplices. However it is not satisfied for the poly-gaskets ($N > 3$), nor for the Lindstrom or Vicsek snowflakes. As a consequence, the use of the essential seminorm is necessary in order to recover the Euclidean geodesic distance on K with the discrete triples.

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Daniele Guido
Dipartimento di Matematica
Univ. Roma Tor Vergata
Via della Ricerca Scientifica
00133 Roma, Italy
e-mail: `guido@mat.uniroma2.it`

Tommaso Isola
Dipartimento di Matematica
Univ. Roma Tor Vergata
Via della Ricerca Scientifica
00133 Roma, Italy
e-mail: `isola@mat.uniroma2.it`