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Adiabatic spin cooling using high-spin Fermi gases

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Abstract. Spatial entropy redistribution plays a key role in adiabatic cooling of ultra-cold lattice gases. We show that high-spin fermions with a spatially variable quadratic Zeeman coupling may allow for the creation of an inner spin-1/2 core surrounded by high-spin wings. The latter are always more entropic than the core at high temperatures and, remarkably, at all temperatures in the presence of frustration. Combining thermodynamic Bethe Ansatz with local density approximation, we study the spatial entropy distribution for the particular case of one-dimensional spin-3/2 lattice fermions in the Mott phase. Interestingly, this spatially dependent entropy opens a possible path for an adiabatic cooling technique that, in contrast to previous proposals, would specifically target the spin degree of freedom. We discuss a possible realization of this adiabatic cooling, which may allow for a highly efficient entropy decrease in the spin-1/2 core and help access antiferromagnetic order in experiments on ultracold spinor fermions.

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1. Introduction

Ultra-cold atoms in optical lattices offer an extraordinary controllable scenario for the study of strongly correlated systems [1, 2], exemplified by the observation of the superfluid-to-Mott-insulator (MI) transition in ultra-cold bosons [3]. Remarkable progress has been made in lattice fermions as well, allowing for the precise analysis of the Fermi–Hubbard model, a key model in condensed-matter physics of particular relevance in the study of high-temperature superconductivity [4]. Exciting recent experiments have reported the realization of the metal-to-MI transition in two-component fermions [5, 6]. Due to super-exchange, the MI phase of spin-1/2 fermions is expected to exhibit a magnetic Néel (antiferromagnetic) ordering. Ongoing experiments are already very close to reaching the mean-field entropy per particle, s , for Néel ordering in a three-dimensional (3D) cubic lattice ($s/k_B = \ln 2$) [7]. However, quantum corrections reduce the critical s down to $s_N/k_B \simeq 0.35$ [8, 9].

Reaching such an extraordinarily low entropy constitutes a major challenge at present, requiring novel types of cooling specially designed for many-body systems in optical lattices [10]. A number of cooling proposals have recently been suggested [8, 11–17], most of them based on the redistribution of entropy within the trap, where certain regions act as entropy absorbers from the region of interest, i.e. an MI at the trap center.

Interestingly, spin degrees of freedom may be employed to design cooling techniques resembling adiabatic demagnetization cooling in solid-state physics [18]. In this method, a decrease in the strength of an externally applied magnetic field allows the magnetic domains of a given material to become disoriented. If the material is isolated, temperature drops as the disordered domains absorb thermal energy in order to perform their reorientation. In cold atoms, this technique was pioneered in Chromium experiments, where the spin-flip mechanism was provided by dipole–dipole interactions [19]. Recently, a novel demagnetization cooling mechanism was proposed for two-component fermions based on time-varying magnetic field gradients [20]. In that method, scalar domains are cooled by transferring particle–hole entropy into magnetic entropy in overlapping regions between the two components. Note, however, that gradient cooling does not address cooling of the spin degrees of freedom, contrary to the method discussed below.

In this paper, we study the spatial entropy distribution of multi-component spin- S fermions with an inhomogeneous quadratic Zeeman effect (QZE), and how this spatially dependent

entropy profile may be employed to design an adiabatic cooling method that specifically targets the reduction of spin entropy. We show, in particular, that an inhomogeneous QZE may lead to an effective pseudo-spin-1/2 core surrounded by spin- S fermions at the wings. We show that, remarkably, the spin- S wings act as entropy absorbers all the way to vanishing temperatures in the presence of frustration. We illustrate the idea with the specific example of 1D spin-3/2 lattice fermions in the Mott phase. Interestingly, the adiabatic growth of the spatially dependent QZE combined with spin redistribution via spin-changing collisions may open promising routes towards adiabatic cooling of the spin-1/2 core, which may in this way enter the antiferromagnetic spin coherent regime.

2. Entropy and frustration

In the following, we consider multi-component fermions loaded in an optical lattice. Increasing the number of spin components from 2 to N (effective spin $S = (N - 1)/2$) increases the capacity to allocate larger entropy per spin at high T by a factor of $\ln N/\ln 2$. This guarantees that at high T , well over the Néel ordering, spatial regions with a larger effective S act as entropy absorbers in the cooling process shown below. However, this simple argument does not apply at low T if the system acquires conventional Néel ordering⁴. The entropy of the Heisenberg antiferromagnet (HAF) in d dimensions scales for $T \ll T_N$ (Néel temperature, see footnote 4) as $s \sim S^{-d}$, and hence the entropy of a spin- S system decreases as compared to that of a spin-1/2 HAF. This is clear since larger spins attach more to the Néel direction, and hence the density of states is smaller at low T , leading to a lower entropy.

The situation is reversed in frustrated systems, which present a large degeneracy of classical ground states with many branches of soft excitations at low T . One arrives at the simplest frustrated large- S model starting from the $SU(N)$ symmetric Hubbard model

$$H = -t \sum_{m, \langle i, j \rangle} (c_{m,i}^\dagger c_{m,j} + \text{h.c.}) + U/2 \sum_i n_i^2, \quad (1)$$

where $m = (-S, \dots, S)$, $c_{m,i}$ annihilates fermions with spin m in the site i , $n_i = \sum_m n_{m,i} = \sum_m c_{m,i}^\dagger c_{m,i}$, and t and U are the hopping and interaction coupling constants, respectively. In the strong-coupling limit, $U \gg |t|$, and retaining one fermion per site, one can derive the effective permutation model in the second order of perturbation theory,

$$H_0 = J/2 \sum_{\langle i, j \rangle} P_{i,j}, \quad (2)$$

with $P_{i,j}$ being the permutation operator and $J = 4t^2/U$. This model is the $SU(N)$ generalization of the HAF. It is exactly solvable in 1D [26] and has $N - 1$ gapless spin modes, each dispersing at low momenta with velocity $2v_{1/2}/N$, where $v_{1/2} = \pi J/2$ is the spin wave velocity of the spin-1/2 HAF. At low T , the entropy is larger than that of spin-1/2 HAF, due to increasing fluctuations by ‘orbital’ degrees of freedom, following $s_S(T) = N(N - 1)\pi T/6v_{1/2}$ [26]. For equivalent 2D and 3D lattice models an increase of entropy with unbinding number of degrees of freedom is also expected at low T . There, the classical

⁴ In 1D and 2D, it must be understood as the characteristic temperature at which, e.g., the specific heat presents a peak.

ground state shows extensive degeneracy, and it is believed [27] that the Néel order does not get stabilized for $SU(N > 2)$ due to high frustration⁵.

Hence, due to frustration, with increasing S the Néel order is suppressed, leading to a larger entropy storage capacity at low T . Thus, if the Mott edges become frustrated while preserving $S = 1/2$ character in the central region, *one can use the frustrated edges as entropy absorbers all the way from high to extremely low T .*

3. One-dimensional spin-3/2 fermions in the Mott phase

In the following, we illustrate the possibilities provided by high-spin lattice fermions with the specific case of a balanced mixture of spin-3/2 fermions in a 1D optical lattice, in which the number of particles N_m with spin m satisfies $N_m = N_{-m}$. Interparticle interactions are characterized by the s-wave scattering lengths for channels with total spin 0 and 2, $a_{0,2}$. For s-wave interacting fermions, $S = 3/2$ is the lowest spin allowing for spin-changing collisions (which preserve the total magnetization but transfers atoms from $\pm 1/2$ into $\pm 3/2$ and vice versa). Due to the conserved magnetization the linear Zeeman effect does not play any role. However, the QZE, characterized by the externally controllable constant q , induces a finite chirality $\tau = \frac{1}{L}[(N_{3/2} + N_{-3/2}) - (N_{1/2} + N_{-1/2})]$.

For large enough interactions and at quarter filling (one fermion per site) the 1D system enters the MI regime, for which the ground state properties under QZE were studied in [29]. For large QZE the ground state is a pseudo spin-1/2 isotropic HAF. Reducing $q < q_{\text{cr}}$ ($q_{\text{cr}} = J \ln 2/2$ at $a_0 = a_2$) the system enters either a spin liquid phase (for $a_2 < a_0$) or a dimerized phase (for $a_2 > a_0$). For $a_0 \simeq a_2$ (the typical situation unless $a_{0,2}$ are externally modified), the gap of the dimerized phase is exceedingly small, and hence the system behaves in practice as a spin liquid down to extremely low temperatures. For $a_0 = a_2$, in the presence of a spatially variable QZE, the model Hamiltonian becomes $H = H_0 + \sum_i \mu_{m,i} n_{m,i}$, with H_0 given by equation (2) with $N = 4$. For homogeneous $\mu_{m,i}$, this model is exactly solvable and its thermodynamic properties may be calculated by means of thermodynamic Bethe Ansatz. We follow the method of [30], based on the self-consistent solution of 14 coupled integral equations, to obtain the corresponding free energy f . The chemical potentials for each component are $\mu_{m,i} = \mu + m^2 q_i$, where μ is the global chemical potential, and q_i denotes the QZE constant at site i . The entropy is then given by $s = -\partial f / \partial T$ and the chirality by $\tau = -\partial f / \partial q$.

As mentioned above, for $q > q_{\text{cr}}$ the system becomes pseudo-spin-1/2. Hence, the ratio between the entropy per spin for $q = 0$ (s_0) and that for $q > q_{\text{cr}}$ should follow the same dependence as $s_{3/2}(T)/s_{1/2}(T)$. We illustrate this point in figure 1, where we compare s at large $q = q_0 = 5J$ (s_c) to s_0 . Note that at large T , $s_0/s_c = 2$, whereas at low T , frustration leads to $s_0/s_c = 6$.

4. The spatially dependent quadratic Zeeman effect (QZE)

We consider at this point the entropy and chirality profiles for the case of a non-homogeneous QZE, which may be achieved by means of microwave or optical techniques [22–24]. We perform local QZE approximation (similar to the local density approximation standard in

⁵ Even if the Néel order eventually exists, due to the high frustration the ordering temperature will be much lower than the Néel temperature of the spin-1/2 antiferromagnet.

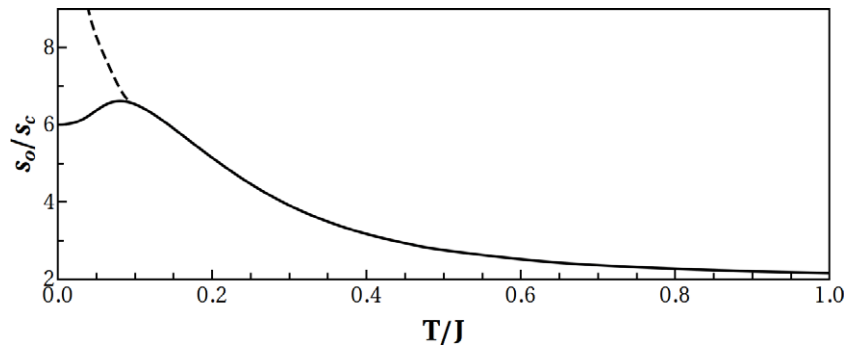


Figure 1. Ratio between the entropy per particle s_0 at $q = 0$ (continuous line) or at $q = \tilde{q}(T)$ (dashed line) ($\tilde{q}(T)$ is the QZE at which the entropy is largest at a given T) and the entropy s_c at high QZE ($q = q_0$) versus the temperature T in the lattice. Observe the crucial role of frustration at low T . If we were dealing with an unfrustrated $S = 3/2$ HAF, the curve would have arrived at low T at $\simeq 0.4$ [25] instead of 6.

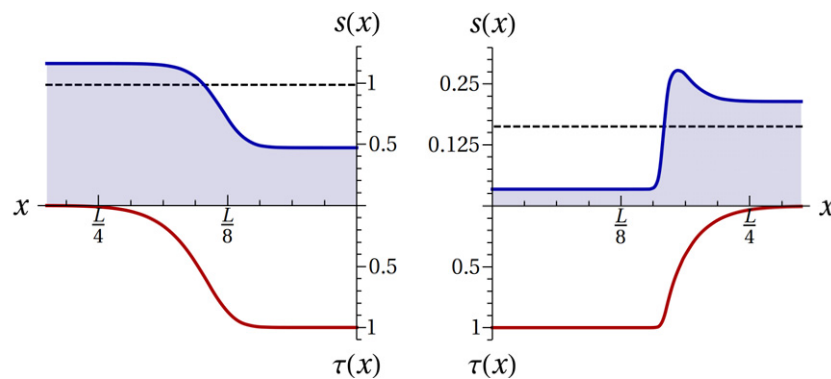


Figure 2. Entropy per particle (in k_B units) and chirality profiles for $L = 120$ atoms for a Gaussian QZE profile $q(x) = q_0 \exp(-100x^2/L^2)$, with $q_0 = 5J$. Dashed lines indicate the initial entropy prior to the switching of the inhomogeneous QZE profile. Before the lattice loading, $T_i/T_F = 0.1$ (left) and 0.016 (right). Note that for $q > q_{cr}$, $\tau \simeq 1$ and the system retains a spin-1/2 character. Note also how the entropy gain in the Mott core is larger for small T . The entropy bump for small T at $q = q_{cr}$ reflects the Van-Hove singularity at the bottom of the depleted Hubbard band.

trapped gases), i.e. we solve for the free energy at different positions \mathbf{x} by varying $q(\mathbf{x})$. The local QZE approximation demands a sufficiently slow variation of the QZE at the scale of the inter-site spacing. In this way we can evaluate the entropy profile inside the MI region. Note, finally, that although the calculation is performed for the case $a_0 = a_2$, the conclusions may be extended to the actual case where $a_{0,2}$ are slightly different, in which spin redistribution via spin-changing collisions occurs.

Figure 2 shows chirality and entropy profiles. Note that the entropy per site (i.e. per particle) is significantly larger at the Mott wings than at the center. Hence, if the total entropy is conserved in a process in which an initially spatially independent entropy is brought to the

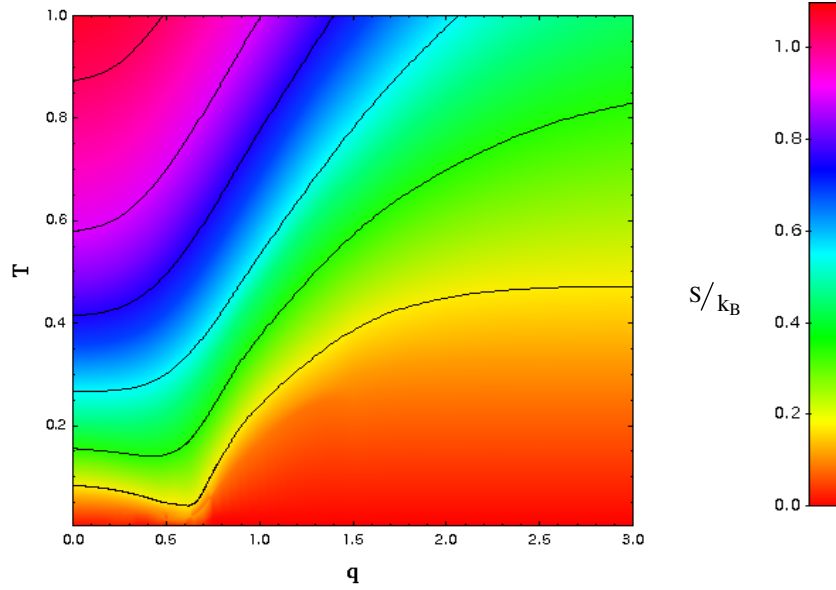


Figure 3. Sliding down the isentropic curves (solid curves) with reducing QZE induces a reduction of the temperature in the lattice at the second stage of the proposed adiabatic cooling scheme. Both T and q are measured in units of J .

profile of figure 2, the outer regions will remove entropy from the central core. This process is even more effective at low T . One may indeed estimate the entropy reduction (for $T \rightarrow 0$) at the Mott center for model (2) in 1D, when considering an initial uniform spin-1/2 system and a final step-like distribution with a spin-1/2 core and spin $S = (N - 1)/2$ wings:

$$\gamma \equiv s_i/s_c = [(1 - L_0/L)N(N - 1)/2 + L_0/L], \quad (3)$$

where L_0/L is the ratio of the number of sites in the spin-1/2 core to the total number of sites, and s_i (s_c) is the initial (final) entropy per particle at the center. The gain is hence much bigger than that at $T \gg T_N$, where one substitutes $N(N - 1)/2$ by $\ln N/\ln 2$ in equation (3).

5. Adiabatic cooling

The spatially dependent entropy distribution opens up interesting prospects for adiabatic spin cooling. A possible scheme would consist of three steps. In a first stage, a two-component balanced mixture is created at the lowest possible temperature T_i , being stabilized against spin-changing collisions [21] by means of a sufficiently large homogeneous QZE. In a second stage, a lattice is adiabatically grown and the homogeneous magnetic field is adiabatically decreased, allowing for spin-changing collisions throughout the sample. The drop in temperature in the lattice with the adiabatic decrease of the homogeneous magnetic field can be estimated from the isentropic curves in figure 3. However, to enter the spin coherent regime, local entropy should be reduced. This is achieved in the final step, which consists of slowly changing the QZE into a non-uniform profile by means of microwave or optical techniques [22–24], leading to the coexistence of a spin-1/2 HAF at the trap center and a spin- S spin liquid at the wings (figure 2).

Figure 4 shows, as a function of the initial T_i/T_F (T_F is the Fermi temperature of the original spin-1/2 prior to the lattice loading, see below), the central entropy per particle with

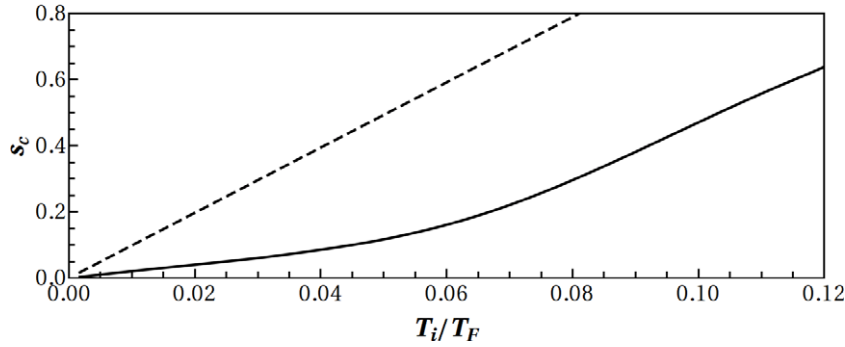


Figure 4. Entropy at the center s_c (in k_B units) as a function of the temperature before lattice loading, T_i/T_F , for the inhomogeneous $q(x)$ profile of figure 2 (solid) and for a homogeneous one $q = q_0$ (dashed), for which the whole system retains a spin-1/2 character. In order to associate a central s_c with a given T_i/T_F , we calculated for different T inside the lattice the entropy profiles and the total entropy $S_{\text{tot}}/L = \pi^2 k_B T_i/T_F$.

a large homogeneous QZE (dashed line) and with a Gaussian QZE (solid line). For the case considered, $s/k_B = 0.35$ is achieved for the homogeneous case at $T_i/T_F \simeq 0.035$, whereas for the inhomogeneous QZE profile it is reached at $T_i/T_F \simeq 0.09$, showing that an inhomogeneous QZE may allow for a large entropy reduction at the center as a result of the entropy excess at the storage wings. Note that for $T_i/T_F \simeq 0.1$ the central entropy may be reduced by more than a factor of 2 with respect to the entropy expected for the homogeneous spin-1/2 case.

Let us briefly comment on higher dimensional ($d = 2, 3$) systems. A simple estimate may be obtained from a spin-wave analysis when $a_2 > a_0$, where at least in the limit $a_2 \gg a_0$ Néel order sets in on bipartite lattices [31]; however, due to frustration the modulus of the Néel order parameter decreases with increasing spin. There is just one spin-wave mode at $q \rightarrow \infty$ and three spin-wave modes at $q = 0$. Taking the values of spin-wave velocities from [32], we obtain that at $T \rightarrow 0$ the factor γ is given by equation (3), albeit with $N(N-1)/2$ changed into $1 + 2[(a_2 + a_0)/(a_2 - a_0)]^d$, implying that *the cooling should be even more efficient for higher dimensions*. Note that the spin-wave analysis predicts γ to increase indefinitely when approaching $a_2 = a_0$, although spin-wave analysis becomes less reliable in the vicinity of that point [32].

6. Experimental feasibility

A possible experimental sequence may be devised for, e.g., ^{40}K . Prior to the lattice loading, a balanced mixture of, e.g., $F = 9/2$, $M_F = -5/2$ and $F = 9/2$, $M_F = -7/2$ states is prepared at the lowest possible T_i/T_F using standard techniques. At this stage a sufficiently large homogeneous magnetic field guarantees that the initial two-component mixture is stable against spin-changing collisions. The gas can be approximated with a good accuracy by a free Fermi gas and therefore the initial entropy per particle is given by $s = k_B \pi^2 T_i/T_F$ [28]. State of the art experiments may reach at this stage $T_i/T_F \approx 0.1$, corresponding to an entropy per particle of $s \approx k_B$. Once the gas is cooled down, a 3D lattice is grown and under proper conditions an MI with one particle per site develops at the trap center [5, 6]. The next step consists in slowly

lowering the magnetic field to allow for quasi-resonant spin-changing collisions [33] throughout the sample, leading to a redistributed population (between $F = 9/2$, $M_F = -3/2, -5/2, -7/2$ and $-9/2$)⁶ and a significant drop of T in the lattice. The temperature drop after this stage for the 1D case (assuming adiabaticity) is depicted in figure 4. Note that fermions with $S > 1/2$ suffer large three-body losses [34]. However, in our scheme, $S > 1/2$ is explored only at a later stage, when the system is already in the hard-core regime with one fermion per site; thus three-body losses are largely suppressed.

Finally, the $F = 9/2$, $M_F = -3/2$ and $F = 9/2$, $M_F = -9/2$ states may be slowly expelled from the trap center by the use of two Raman beam pairs. These beams couple $F = 9/2$, $M_F = -3/2$ to $F = 7/2$, $M_F = -1/2$ and $F = 9/2$, $M_F = -9/2$ to $F = 7/2$, $M_F = -7/2$. This $\Delta M_F = -1$ coupling may be realized with just three lasers. A slight blue detuning ensures that the resulting dressed states experience a repulsive potential. The effect on the $M_F = -7/2, -5/2$ states is largely suppressed by the ratio between Raman detuning and Zeeman splitting, which is kept on the order of q_{cr} . Furthermore, the Raman beams shift the resonance condition for spin-changing collisions in the center of the trap such that no more atoms in $M_F = -9/2$ and $M_F = -3/2$ are produced. The outer regions have then a larger effective spin and hence act as entropy absorbers, as discussed above. Note that, interestingly, the center of the Raman beams acts as a crystallization point for a slowly growing Néel state (if s_N is reached). In this way, the problem of approaching the ground state despite the presence of many metastable low-energy states may be circumvented.

Let us mention some final remarks. For the simplicity of our calculations, we have considered cooling only within an MI. In the presence of particle-hole excitations we expect an even greater cooling efficiency [6]. Note that the $S = 1/2$ region is deeper in the Mott region than in the $S > 1/2$ region, and hence the $S > 1/2$ region is expected to contain a larger particle/hole entropy. We stress also that the cooling efficiency is based on the assumption of adiabaticity. Spin-changing collisions are crucial in our cooling scheme. Although they are low energetic, they will typically be faster than super-exchange, and hence the adiabatic requirements of our model are comparable to those in other cooling schemes based on entropy relocations in optical lattices.

7. Conclusions

In summary, we studied a possible route for adiabatic spin cooling using high-spin lattice fermions. The process resembles demagnetization cooling, but the role of the magnetic field is played by a spatially dependent QZE, and spin flip is substituted by spin-changing collisions. The spatially dependent QZE leads to two distinct regions of different effective spins ($S = 1/2$ and $S > 1/2$). At high T the outer spin- S region acts as an entropy absorber simply due to the larger spin, whereas the same remains true at even very low T due to frustration. As a result, we showed that a significant reduction of the entropy (more pronounced at lower T and higher dimensions) of the spin-1/2 Mott central region can take place. Magnetic refrigeration using spatially variable QZE can hence significantly facilitate experimental realization of distinct magnetic ground states, and in particular Néel ordering in spin-1/2 fermions.

⁶ The process $(-3/2, -5/2) \rightarrow (-1/2, -7/2)$ is in principle possible, but the two-particle energies involved in it are different from the ones in the desired process $(-7/2, -5/2) \rightarrow (-9/2, -3/2)$. Hence, properly tuning the QZE [33] should lead to much lower superexchange spin-changing collision in the spurious channel.

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