THERMAL DRAG BETWEEN TWO COUPLED QUANTUM WIRES

Master Thesis
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Abstract

In the last two-three decades the study of quantum wires created a lot of interest, in particular for what concerns their transport properties. Different theoretical methods, along with an incredible development of experimental techniques, have provided a lot of information on these peculiar low dimensional systems. In this framework, Coulomb drag effect in coupled systems has been intensively studied recently; drag measurement are now a standard experimental technique to study a quite different variety of physical properties and several different setups were considered so far.

In this thesis we study the thermal drag between two coupled quantum wires, with the coupling provided by a Coulombian interaction. This topic is very interesting for both its theoretical and experimental implications: it could help to explain, not only thermal, but also thermoelectric effects in systems exhibiting drag. Furthermore the study of thermal drag will nicely complement the information provided by the “conventional” Coulomb drag for electrical transport. It will then provide new and powerful experimental methods to study drag and could have important technological application in reducing the dissipation of heat. In the system we consider, a thermal gradient is applied to the first wire and, thanks to the coupling between the density fluctuations in the two wires, a drag thermal current is induced in the second wire. We derive an expression for the transresistivity (proportionality coefficient between thermal bias and current) valid under general conditions and then we assume to be in a low temperature regime, in order to exploit bosonization formalism and find a Wiedemann-Franz-like law, which allows to connect our result with the much more studied Coulomb drag case. Then we analyse the contribution of forward scattering and backward scattering to the drag, finding out that the first one is dominant. The analytical results obtained using bosonization will be finally complemented by numerical simulations based on the MPS formalism. In the simulations we are able to detect a stationary thermal current dragged by the Coulomb coupling between the wires and reproduce the predicted dependence of the current on the interaction strength, quadratic for small couplings. On extending the simulations range to stronger couplings, we find also the quartic corrections to the thermal drag.
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The study of transport phenomena has always raised a lot of interest in many areas of sciences. In condensed matter the study of heat or charge transfer has been a key problem over the last century. In this context quantum nanowires, which will be considered in this thesis, are arguably at present one among the most intensively studied phenomena in the very productive research field of transport in nanostructures.

The reasons behind the intense and continuous interest in transport phenomena are several, one of them being the practical uses that such effects have in the everyday life. On the other hand, understanding this topic has its intrinsic theoretical value and can help to enrich our knowledge, not only of the out-of-equilibrium physics, but also of the equilibrium features, to which transport properties are strictly related in the linear response regime.

One of the most recent trends of contemporary physics is the study of systems of low dimensionality, going from 3D bulk systems to 0-dimensional quantum dots and passing through 2D layers and 1D quantum wires: indeed, in this thesis we will analyse several transport properties in solid-state 1D wires.

One of the reasons of this interest towards the low-dimension physics, is the increasing importance of quantum coherence and electron-electron interaction, compared to kinetic effects, in systems with reduced dimensionality. These characteristics pave the way to new and intriguing phenomena, where quantum mechanics plays a central role: new phases of matter can be observed, along with a plethora of phase transitions and unusual transport properties, without any classical analogue.

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All this activity is triggered by an intense experimental work; in particular the huge boost in the field of nanotechnology greatly contributed to raise the interest towards this topic. It is now possible to fabricate, in a very precise and controlled way, a lot of devices living in 2D, 1D or even 0D; these devices have a characteristic scale of the order of the nanometer and quantum effects are, therefore, very relevant in determining their properties. This development of more and more sophisticated techniques of miniaturization, has made it possible to create a “laboratory” where it is possible to manipulate, almost at will, the nanostructures, performing accurate measurements of their transport properties.

On the theoretical side, accounting of (possibly very strong) interactions among the electrons makes the analysis of these phenomena a demanding challenge.

In the present thesis, we focus on the study of transport in quantum wires, which can be described, with good approximation, as one dimensional systems of interacting electrons. In 1D the effect of interactions is dramatic, leading at low energy to a regime, known as the Luttinger liquid, where the low-lying excitations are sound-like bosonic modes. This is very different from the Fermi-Landau liquid, which emerges in 2D and 3D. In the Luttinger liquid regime, the low energy excitations are bosons with linear dispersion; the only effect of an interaction is to renormalize the parameters that describe the Luttinger liquid (Refs. [1] and [2]). Despite the decisive role of interactions in one-dimensional systems, there are a number of analytical and numerical techniques (bosonization, density matrix renormalization group, MPS formalism, etcetera) that allow to treat in an elegant and effective way even strong interactions.

A Luttinger liquid is not just an abstract theoretical model; actually, it has been observed during experiments in 1D wires at low temperatures, as reported, for example, in Refs. [3] and [4].

Quantum wires in the Luttinger liquid regime have been studied mostly through transport measurements, in particular in the linear response regime; in this limit, in fact, one is able to extract through these measurements precious information on the equilibrium properties (see Ref. [5] for more details). An important step forward in understanding the transport properties of Luttinger liquids was carried out by Kane and Fisher (Refs. [6] and [7]), who addressed electric transport in a quantum wire; in their articles they considered both the case of a simple quantum wire and the case of a wire with an impurity, deriving an expression for electrical conductance and for current-voltage characteristic at zero temperature.

Together with its electrical counterpart, thermal transport is generating a continuously increasing interest for both its theoretical and practical implications. It is desirable to prevent the heating of a system, in particular now that our technological devices are getting smaller and smaller, and the understanding of how thermal transport works is very useful for this purpose.

The study of thermal properties in Luttinger liquids was carried out in several papers, such as [8] and [9]: in Ref. [8] the thermal conductivity is calculated in a situation where the two ends of a single wire are held at different temperatures and there is an impurity in the middle of the wire, while in Ref. [9] the same problem is studied analysing an inhomogeneous wire. A more recent paper (Ref. [10]) studied...
the relation between thermal and electric transport in inhomogeneous Luttinger liquids. The work discussed in this article is related to the Franz-Wiedemann law, i.e. a quite general law stating the proportionality between the electric transport coefficient and the thermal one for the currents associated to electrons; it does not take into account the heat transport related to phonons, which we are not considering in this thesis.

Over the past two decades a very interesting setup, made by two coupled quantum wires (see Figure 1), has been studied, proving to be at the base of a novel and qualitatively rich physics.

The two wires are usually coupled together by an interwire Coulomb interaction (but also interactions with different properties can be considered). This interaction couples together the density fluctuations of the two wires and creates the peculiar effect of a drag between them: if an electric field is applied to the first wire, an induced electric current begins to flow in the second one. This particular effect, often called “Coulomb drag”, has been originally studied in two dimensions by Zheng and MacDonald in Ref. [11]. Their work, however, provides important technical tools (like memory function formalism, see Ref. [12]) useful also in 1D. In their article a formula is derived for the proportionality coefficient between the current in one layer and the electric field in the other layer; in this expression, the dependence on the coupling and on the density fluctuations of the two wires is manifest. In Refs. [13] and [14], Coulomb drag in 1D quantum wires is studied and compatible results are derived; the two papers are interesting because of the methods employed and of the good analysis made on the various contributions to the drag, that derive from forward scattering (which is a process with small momentum transfer) or backward scattering (which scatters particles between the two end of the Fermi sphere).

Drag measurements are now widely performed and the predictions made in Refs. [13] and [14] have been tested. See, for example, Refs. [15], [16] and [17].

The main purpose of this thesis is to analyse drag effects also on the thermal properties, extending the previous studies to the case where thermal currents may be induced between coupled wires. We aim to present a study on “thermal drag”, characterizing the so-called thermal transresistivity, namely the proportionality coefficient between the thermal gradient in the first wire and the induced thermal

Figure 1: Diagram of the system: it is possible to see the two parallel quantum wires, along with the coupling region, which is represented as a red-coloured zone.

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current in the second wire. Thermal drag has never been studied in literature so far; we therefore need to set the necessary formalism to this aim.

We therefore think that it is natural, and also very interesting, to seek for possible new features emerging when thermal drag is analysed, making therefore a direct comparison with the most studied case of Coulomb drag.

The system studied in this thesis is made of two quantum wires of spinless electrons and the coupling is provided by a Coulomb interaction. As in the case of the Coulomb drag, the interwire interaction couples the density fluctuations in the two wires, leading to a peculiar correlation between the currents flowing in the conductors. Referring to our particular system, we will see later how the thermal drag exhibits effects that are absent in the electric case. One of these effects is the presence of an additional induced thermal current between the two wires; this is a direct consequence of the fact that a thermal current is ultimately an energy flow and the coupling allows energy exchange between the wires, but not charge exchange.

We will find many analogies with the Coulomb drag, especially studying the thermal drag between Luttinger liquids. We will first characterize the transresistivity in a general framework, deriving an analytic formula which is remarkably valid in any regime of temperature. Then we will confine our analysis to a regime in which the Luttinger liquid theory holds, finding a sort of Wiedemann-Franz law also for drag.

Finally, we will run a first set of numerical simulations, using a computational protocol which rely on the formalism of MPS (see Ref. [18]), which is a recent and promising method used to study 1D systems. Although this technique presents some limits, it is a powerful tool and we will use it as a complementary method to study the thermal drag; indeed, this computational work will provide us some additional insight in the physics involved, since it permits to study our system even outside of the Luttinger liquid regime.

This thesis could be a good starting point for further studies on more and more complicated systems that exhibit thermal drag, like an array of parallel quantum wires where a cascade drag effect could appear. It can even be a useful basis to study the same setup including the spin and analysing, not only the spin-charge separation of the excitations, but also the corrections to the drag arising from additional terms in the hamiltonian, which couple these two types of excitations.

Furthermore, a good understanding of electric and thermal drag, based on the Wiedemann-Franz law analogue that we obtained, could enhance the knowledge of thermoelectric behaviour also in the case of drag between two wires; a very direct application could be the calculation of induced thermoelectric currents.

We should mention, of course, that the study of thermal drag in one dimension could be exploited to understand this physical effect also in two dimensions, or even in systems exhibiting a dimensional crossover, such as a system made of a two dimensional and a one dimensional conductor.

On the other hand, a physical phenomenon like thermal drag (in 1D or in 2D) is interesting not only on a theoretical level, but also on a practical one, since it could have a lot of relevant direct applications. A first example is the case of dissipative systems where the dispersion of heat assumes a central role and the thermal drag effect could be used to induce a heat flow and, definitely, a temperature gradient in
a controllable way; a second manner to apply these theoretical concepts, could be to employ them in order to “recycle” the heat lost by quantum engines that were doing a mechanical work. The fabrication of a device able to transform a fraction of dissipated heat (that would get lost) into a “less degraded” form of energy, would be a great step forward for technology in general and for nanotechnology in particular. Thermal drag (together with Coulomb drag) could be employed to measure the induced current and, once known the laws that rule these phenomena, to calculate the coupling between the wires, thus providing a method to measure indirectly the interwire interaction in various conditions.

This thesis is organized as follows.

In Chapter 1, some background information about the quantum wires will be given, showing that they can be described, at first sight, as one dimensional systems and, then, as Luttinger liquids; afterwards the bosonization procedure will be explained and applied. In Chapter 2, we will review the state of the art about the Coulomb drag and discuss the most relevant works related to this thesis, describing the system setup and the methods used. Chapters 3 and 4 contain the original part of the work; in the former we will examine the thermal drag between two quantum wires of spinless electrons, deriving an expression for the transresistivity. In order to do so, two different approaches can be followed, respectively those outlined in Refs. [11] and [13]; we will use the first one and then give a review of the second in the Appendix F. Initially, a result valid for any range of temperature will be obtained and, then, we will make some approximations, in order to derive a Franz-Wiedemann law for drag resistivity; this expression will be analysed, presenting its most relevant features. Chapter 4 will be devoted to the exposition of the numerical simulations made, describing the protocol and the various parameters used during each simulation and presenting plots and tables that show the results obtained.

We finally present the conclusions of the thesis, summarizing what we have found and which directions could be taken to extend our comprehension of thermal drag.
In this Chapter we review the theory of quantum wires and show, in section 1.1, that they can be equivalently considered as one dimensional systems of correlated electrons. Then, the peculiar properties of one dimensional systems are examined, explaining the Luttinger liquid regime and how the bosonization technique can be applied to study 1D systems. For simplicity, we will consider spinless electrons, whereas the spin will be treated in Appendix A.3. We will largely follow the first chapters of the book by Giamarchi, Quantum physics in one dimension (Ref. [1]), focusing mainly on the qualitative features rather than on the formalism behind this theory.

Only in Appendix A a glossary of the bosonization tools will be given, along with the relevant formulas and the equations that we will need in the next chapters.

1.1. Overview

What is a quantum wire? Let us begin our presentation by trying to answer to this question.

A quantum wire is a nanoscale system, where, roughly speaking, electrons are confined to move in one dimension, the motion in the two perpendicular directions being quantum mechanically frozen out. This confinement is usually achieved through a potential that traps the electrons and forces them to move in this particular direction.
It useful at this stage to give a brief description of the experimental methods employed to fabricate quantum wires and then turn our attention to their properties. The techniques used to realize quantum wires are divided into two fundamental categories: top-down and bottom-up approaches (see Ref. [19]). These two methods differ mainly because the former starts from a scale greater than the quantum wire scale, while the latter method does the opposite; this is schematically shown in Figure 1.1:

Loosely speaking, the top-down method begins the fabrication of a quantum wire starting from a sample material that has a scale of millimeters and “carving” it with an electron beam or with litographic techniques employing radiation going from the extreme ultraviolet to the X-ray spectrum.

On the other hand, the bottom-up approach starts from a sample of dimensions between 2 nm and 10 nm and builds up the wanted structure; self assembly of structures is often smartly used during the fabrication. The process proceeds through controlled chemical reactions and biological-inspired procedures; the principal limitation of this technique is that it can produce only some naturally formed and well-defined structure, whereas a litographic process allows to manipulate the final form of the structure in a great variety of ways.

An example of self assembled structure is the carbon nanotube, which is approximately a layer of graphene rolled up to form a cylindrical structure, where electrons can move mostly along the axis of this cylinder (see Figure 1.2 and Ref. [20]).

A carbon nanotube provides a very good example of how the confinement of electrons allows to describe a quantum wire in terms of a purely 1D system. In fact, thanks to the cylindrical symmetry, the tangential momentum in the plane perpen-
diccular to the cylinder’s axis is quantized and the energy gap between the ground state of perpendicular motion and the first excited state is approximately 1 eV; this is huge energy and the electrons are all in the ground state of the perpendicular motion, having only one degree of freedom left: the motion along the axis direction.

Figure 1.2: Illustration of a carbon nanotube (left panel) and actual image of a nanotube between two electrodes (right panel). Picture from [19]

The previous statement can be generalized to situations where the confinement of the electrons is achieved through an external potential, rather than through the structure of the quantum wire. We will show, thanks to the following simple argument, the effects of a confinement on the physics of the system.

Let us consider a generic system, where the quantum wire extends along the $x$ direction and the electrons are subject to a potential (whose origin is not specified) of the form

$$V(y, z) = V_1(y)V_2(z)$$

where $V_i$ is a potential sharply peaked around zero and very deep (like a delta function or a hard-wall potential).

The energy of an electron will be the sum of the energy related to the motion along $x$ axis and of the energy related to the transverse mode (inside the $y - z$ plane). If the potential has a typical extension of $a$, then the electron wavefunction in the plane $y - z$ will be exponentially suppressed for $|z| > a$ or for $|y| > a$ and will oscillate inside the region where the potential acts. Thus, the motion inside the transverse plane is quantized and the modes have wavelength proportional to $a$, so that energy gap between a transverse mode and the next goes like $1/a^2$: the narrower is the potential, the wider is the gap.

When the gap is much larger than all the other relevant energy scales, as it happens when it is of the order of eV, the electrons occupy the ground state of transverse modes and can be considered, with a very good approximation, as moving in one dimension. Thus, in order to have a system that looks as one dimensional as possible, we just need to use a sufficiently narrow confining potential.

Clearly, there is much more to say about quantum wires and their fabrication, but this would go beyond the purposes of this thesis; we are therefore going to analyse the physics of electrons living in one dimension and subject to an interaction between
1.2. Interacting electrons in one dimension

As discussed in the previous section, quantum wires can be approximately described as one dimensional systems. We now want to show their essential features and analyse what makes them so different from interacting systems of higher dimensionality. Since we are going to study only spinless electrons in Chapter 3, the spin will now be ignored in our treatment, unless otherwise specified.

In the second quantization formalism, a system of electrons can be described by a fermionic field $\psi(x)$; this field obeys canonical anticommutation rules:

$$\{\psi(x), \psi^\dagger(x')\} = \psi(x)\psi^\dagger(x') + \psi^\dagger(x')\psi(x) = \delta(x - x')$$
$$\{\psi(x), \psi(x')\} = 0$$

here, and throughout the whole thesis, is assumed $\hbar = 1$.

The hamiltonian of a free electron gas is given by

$$H = \int dx \mathcal{H}(x) = \int dx \frac{1}{2m} \left( \frac{\partial}{\partial x} \psi^\dagger(x) \psi(x) \right)$$

The hamiltonian can be diagonalized by mean of creation and annihilation fermionic operators $a_k$ and $a_k^\dagger$, defined by

$$\psi(x) = \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} a_k$$

where $\Omega$ is the volume of the system considered (in one dimension $\Omega = L$). The free electrons hamiltonian becomes

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k$$

while the density of the system is given by

$$\rho(x) = \psi^\dagger(x)\psi(x) = \frac{1}{\Omega} \sum_k \epsilon^{iqx} a_k^\dagger a_{k+q}$$

(1.2.1)

Because of the Pauli exclusion principle, it is known that two electrons cannot occupy the same state: then, the ground state of the previous equation contains all the states within the Fermi sphere (it is actually a sphere in 3D, while it’s a circle in 2D and a segment in 1D), which includes all the states whose wavevector modulus is smaller than the Fermi wavevector $k_F$; the boundary of the Fermi sphere is called the Fermi surface and the energy corresponding to $k_F$ is the Fermi energy $E_F = k_F^2/2m$ (often referred to as Fermi level). In one dimension this means that the allowed states have $-k_F < k < k_F$ and the Fermi surface is made of two points $k = k_F$ and $k = -k_F$. From the Fermi energy can also be defined the Fermi temperature $T_F$ as $E_F = k_B T_F$, where $k_B$ is the Boltzmann constant.

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Figure 1.3: Plot of the occupation number $n_k$ versus $k$ for a free electron gas (a) and for interacting electrons (b). Picture from [1].

A plot of the occupation number $n(k)$ is shown in Figure 1.3, left panel (a).

What does it change if we add an interaction among the electrons? The hamiltonian becomes

$$H = \frac{1}{2m} \int dx \partial_x \psi^\dagger(x) \partial_x \psi(x) + \frac{1}{2} \int dx dx' \psi^\dagger(x) \psi^\dagger(x') V(x' - x) \psi(x') \psi(x) \quad (1.2.2)$$

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k + \frac{1}{2\Omega} \sum_{k,k',q} a_{k+q}^\dagger a_{k'-q}^\dagger V(q) a_k a_{k'} \quad (1.2.3)$$

The scattering process associated to the interaction is represented in Figure 1.4.

Figure 1.4: Schematic scattering process produced by the interaction (here the spin is ignored).

Qualitatively, for dimensions higher than one ($d = 2$ and $d = 3$), the main change introduced by the interaction is that the elementary particles are no more the "bare" electrons, but electrons “dressed” with particle-hole excitations; this is the same concept of electrodynamics where a photon dresses itself with virtual electron-positron pairs. These new objects, called quasiparticles, have fermionic nature and are almost free, so that the occupation has still a jump at the Fermi momentum but...
1.2. Interacting electrons in one dimension

with a amplitude smaller than 1 (Figure 1.3, right panel (b)). The quasiparticles are excitations that behave as free electrons excitations with an effective mass \( m^* \) and have a lifetime that diverges near the Fermi surface, so that they are well defined excitations (i.e. they are almost hamiltonian eigenstates).

In one dimension, the effect of interaction is far more dramatic and the change is so drastic that perturbation theory fails completely. Qualitatively, in higher dimensions an electron can propagate without perturbing too much the other electrons; this is not possible in one dimension where every electron has to “push” its neighbours in order to move; this leads to the existence of only collective excitations.

In a more formal way, we can see that, if we tried a time-dependent perturbation theory (with frequency \( \omega \)) in the interaction strength, the density-density response function would diverge. This susceptibility expresses the response of the system to an external perturbation coupling with the density and is given by (see Ref. [21])

\[
\chi(q, \omega) = -i \int_0^\infty dt \int_{-\infty}^\infty dx e^{-iqx+i\omega t} \langle [\rho(x,t), \rho(0,0)] \rangle
\]

where \( \xi(k) = E_k - \mu = k^2/2m - \mu \). Here \( \mu \) is the chemical potential and \( f_{FD} \) is the Fermi-Dirac distribution function at temperature \( T \).

Our target is to calculate the real part of \( \chi \) for \( \omega \to 0 \). If we are on the Fermi surface (so that \( \xi(k) = 0 \)) and we can find a wavevector \( Q \) such that both \( \xi(k) \) and \( \xi(k+Q) \) are zero, then we have a singularity; the wider is the region of \( k \) for which this property holds, the stronger is the singularity.

In 1D we can linearize the dispersion relation around the Fermi energy and prove that \( \xi(k + 2k_F) = -\xi(k) \) (see, for more details on the derivation, the Chapter 1 of Ref. [1]); this is called nesting property and, substituted in the susceptibility, leads to

\[
\text{Re} \chi(Q, \omega = 0) = -\frac{1}{L} \sum_k \tanh(\beta \xi(k)/2) \frac{2\xi(k)}{2\xi} = -\int d\xi N(\xi) \frac{\tanh(\beta \xi/2)}{2\xi}
\]

Roughly speaking, in one dimension there is a logarithmic singularity regularized by the temperature (\( E \) is an ultraviolet cutoff); the essential difference is that, for higher dimensions, the nesting property is valid only for isolated points of the Fermi surface, while in 1D it holds all over the Fermi surface. This singular behaviour of the susceptibility tells us that the perturbation theory cannot be applied and thus the ground state and the excitation spectrum in the presence of an interaction are radically different from the ground state of the free electrons gas.

An observation that can help to find a way out to this problem is the study of the particle-hole excitations; they are excitations created when an electron below the Fermi level jumps above, leaving a hole within the Fermi sphere. Since we are destroying a particle with momentum \( k \) and creating one with momentum \( k + q \), the excitation has a well defined momentum \( q \). On the other hand the energy \( E_k(q) \) of
1. Quantum wires

the excitation depends in general from both $k$ and $q$; in the non interacting case it is equal to

$$E_k(q) = \xi(k + q) - \xi(k) = \frac{k}{m}q + \frac{q^2}{2m}$$

If we are interested in low temperatures behaviour ($T \ll T_F$) we can look just at the low-energy excitation, restricting ourselves in a narrow space around the Fermi energy, namely $k \approx k_F$ and $q \ll k_F$. In this limit, $E_k(q)$ is the sum of two contributions: an average energy of excitation independent of $k$:

$$E(q) = \frac{k_F}{m}q = v_F q$$

and a small energy width quadratic in $q$:

$$\delta E(q) = \frac{q^2}{2m}$$

The absence of any dependence of the excitation energy on $k$ is a peculiarity of the 1D case, since in higher dimension one has also an extra dependence on the direction of $k$.

From this simple calculation, it can be deduced that, for low energy excitation, the energy depends only on the momentum of the excitation, while the ratio between the energy width and the energy itself goes to zero; it implies that these are well defined excitations with an infinite lifetime at the Fermi energy. These excitations have a bosonic character, since they originate from the destruction and the creation of an electron.

Therefore, from a system of interacting one-dimensional electrons, emerges a bosonic behaviour for the excitations; this is an incredible property, peculiar of 1D systems. This bosonic behaviour is at the base of the bosonization method, which tries to describe the Hamiltonian and the fermionic operators in terms of bosonic operators.

The particle-hole excitations can be re-expressed in terms of density fluctuations (which are collective excitations):

$$\rho^\dagger(q) = \sum_k a_k^\dagger a_k$$

The bosonization method starts from here, treating the density fluctuations as bosonic objects, which, nevertheless, are not well defined excitations unless we are very close to the Fermi energy.

It is therefore very convenient to focus our attention on the low energy limit, considering an hamiltonian for which the boson excitations are almost exact eigenstates.

In this limit, it is natural to make the linear spectrum approximation, linearizing the kinetic hamiltonian around the Fermi surface.

$$H = \sum_{k,r=R,L} v_F (r k - k_F) a_{r,k}^\dagger a_{r,k}$$

(1.2.4)
1.2. Interacting electrons in one dimension

This is the Hamiltonian of the Tomonaga-Luttinger model. We are forced to introduce two species of electrons (right movers and left movers) to describe the two points of the Fermi surface.

Here $r$ labels the right and left moving electrons; when used in mathematical expressions $r$ assumes the value $r = +1$ for $r = R$ right movers and $r = -1$ for $r = L$ left movers.

![Figure 1.5: Fermi sea with a parabolic spectrum (a) and Dirac sea with a linear spectrum (b). The grey coloured region represents the filled states. Picture from [1].](image)

The linearization of the spectrum has the effect of replacing the Fermi sea, where all states with energy between zero and the chemical potential are filled, with a Dirac sea where there is an infinite number of filled states with negative energy ($\xi_{r,k} < 0$). For more details, see Figure 1.5.

Also the fermionic field can be linearized, retaining only the contributions of a narrow momentum band around the Fermi surface:

$$
\psi(x) \approx \frac{1}{\sqrt{L}} \sum_{k \sim -k_F} e^{ikx} c_k + \frac{1}{\sqrt{L}} \sum_{k \sim k_F} e^{ikx} c_k = \psi_L(x) + \psi_R(x)
$$

Since we are looking only at what happens nearby the Fermi surface, the same procedure can be applied to the interaction, dividing the scattering processes into three categories, that are shown in Figure 1.6. These categories can be found using the interaction term in equation (1.2.2) and seeing how many $\psi_L$ or $\psi_R$ appear.

The spin is taken into account in order to show the difference between spinless electrons and electrons with spin.

The process shown on the left ($V_4$ type) couples two electrons on the same side of the Fermi surface and is responsible for a small momentum exchange (forward scattering). $V_4$ scattering involves operators of the type $\psi^\dagger_L(x) \psi^\dagger_R(x') \psi_R(x') \psi_R(x)$ or the same with $R \rightarrow L$.

The process in the middle of Figure 1.6 couples two electrons on opposite sides of the Fermi surface, giving them again a small momentum exchange; it involves different types of electrons, including terms like $\psi^\dagger_L(x) \psi^\dagger_L(x') \psi_L(x') \psi_R(x)$.

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*Thermal drag between two coupled quantum wires*
1. Quantum wires

Finally also the process on the right involves electrons on opposite sides of the Fermi surface, but this time the interaction exchange them, giving origin to a $2k_F$ momentum exchange (backscattering). $V_1$ is associated to terms mixing right and left movers, like $\psi_R^\dagger(x)\psi_L^\dagger(x')\psi_R(x')\psi_L(x)$.

For spinless electrons the processes $V_2$ and $V_1$ coincide, since the outgoing particles are indistinguishable and one can exchange them. This is not true for electrons with spin, where the spin forbids this exchange, and so $V_1$ and $V_2$ are two distinct processes; on the other hand, for interaction that are not too short-ranged, the $V_1$ processes are negligible.

In the following of this thesis we will deal with spinless electrons, so we will analyse particularly this case; in Appendix A.3 we will briefly show how the theory is generalized to include the spin.

In section 1.3, on the other hand, we will apply the bosonization, first to the free spinless electron case, and, then, we will see how to treat the interaction.

1.3. Bosonization of spinless electrons

We have already seen in the previous section how the linearized spectrum looks like and how the density fluctuations can be written.

Since $\rho(x)$ is real, we have $\rho^\dagger(q) = \rho(-q)$; $\rho$ is a bosonic operator that creates an excitation with well defined momentum $q$ and it can be written as a linear combination of some appropriate bosonic operators $b_k$ and $b_k^\dagger$. This substitution has the advantage of changing the interaction into an operator quadratic in the bosonic language, instead of an operator containing four fermionic fields.

The possibility of writing the density in terms of canonical boson operators depends on its commutation rules; we will prove that the presence of a Dirac field with all the negative energy states filled ensures the proper commutation rules for the density.

We consider the density of left and right moving electrons, which can be written in terms of the left and right fermion fields previously introduces. Because of the...
characteristics of the Dirac sea, we actually have to work with normally ordered operators, in order to avoid the infinities arising from an expectation value on the vacuum state:

\[ \rho_r(x) := \psi_r^\dagger(x) \psi_r(x) : \]

The Fourier component of the density presents a slight subtlety when the momentum is \( p = 0 \), because there is a non zero expectation value on the vacuum state:

\[ \rho_r^\dagger(p) := \sum_k \langle 0 | a_{r,k}^\dagger a_{r,k} | 0 \rangle (p \neq 0) \]

\[ = N_r \sum_k \left[ a_{r,k}^\dagger a_{r,k} - \langle 0 | a_{r,k}^\dagger a_{r,k} | 0 \rangle \right] (p = 0) \]

Here \( |0\rangle \) is the vacuum state of density fluctuations and correspond to the ground state (also called Dirac sea) of the hamiltonian in equation (1.2.4); it is the state where all the negative energy states are occupied. From this moment we will imply that an operator is normal ordered and will omit the \( : : \) notation.

With these premises, we can prove that \([\rho_r^\dagger(p), \rho_r^\dagger(p')] = 0\) and that

\[ [\rho_r^\dagger(p), \rho_r^\dagger(-p')] = \sum_k \left( \langle 0 | a_{r,k+p}^\dagger a_{r,k} | 0 \rangle - \langle 0 | a_{r,k}^\dagger a_{r,k-p} | 0 \rangle \right) \]

The momentum \( k \) is quantized and, with periodic boundary conditions, we have \( k = 2\pi n/L \). Moreover \( \langle 0 | a_{r,k_1}^\dagger a_{r,k_2} | 0 \rangle = 1 \) if the state is occupied and zero otherwise. We can see that, if \( r = R \) and \( p > 0 \), the second term in the sum includes more occupied states than the first one, exactly \( pL/2\pi \); on the other hand, if \( r = L \), the second term includes less occupied states in the sum. This difference arises from the fact that the occupied states of the Dirac sea extends towards infinite negative energies. If \( p < 0 \) the previous reasoning has to be reversed and there is a sign change; mathematically we have:

\[ [\rho_r^\dagger(p), \rho_r^\dagger(-p')] = -\delta_{rr'}\delta_{pp'} \frac{rpL}{2\pi} \]  

Equation (1.3.2) states that the density has commutation relations analogous to those of bosonic operators. Step by step, we are slowly approaching our goal: writing the density fluctuations in terms of canonical boson operators.

Now, we observe that, if we apply the operator \( \rho_R^\dagger(p) \) to the Dirac sea \( |0\rangle \), we obtain 0 for \( p < 0 \) (i.e. \( \rho_R^\dagger(p < 0)|0\rangle = 0 \)). In fact, \( \rho_R^\dagger(p < 0) \) would create a right moving excitation with negative momentum, which means that we would have to destroy an electron outside of the Dirac sea and add another one into the Dirac sea: this is clearly impossible, so no such excitation can exist. Analogously, if we apply \( \rho_L^\dagger(p) \) to the ground state we get 0 for \( p > 0 \) (i.e. \( \rho_L^\dagger(p > 0)|0\rangle = 0 \)).

These considerations tell us what is the exact decomposition of \( \rho_L^\dagger \) and \( \rho_R^\dagger \) in terms of new boson operators \( b^\dagger \) and \( b \) that have the Dirac sea as vacuum state; moreover equation (1.3.2) tells us how to rescale \( b^\dagger \) and \( b \) in order to ensure the
proper commutation rule between them. In the end we obtain

\[ b_p^\dagger = \left( \frac{2\pi}{L|p|} \right)^{1/2} (Y(p)p_R^\dagger(p) + Y(-p)p_L^\dagger(-p)) \]

\[ b_p = \left( \frac{2\pi}{L|p|} \right)^{1/2} (Y(p)p_R^\dagger(-p) + Y(-p)p_L^\dagger(-p)) \]

with \( Y \) the Heavyside step function, which assumes the value 0 for \( p < 0 \) and the value 1 for \( p > 0 \).

From equations (1.3.2) and (1.3.3), it is immediate to verify that these operators satisfy the canonical commutation relation \([b_k, b_{k'}^\dagger] = \delta_{k,k'}\).

This new basis of operators can also be used to re-write also the hamiltonian in this way. Starting from the free electrons hamiltonian, \( H \) is commuted with \( b_{p_0} \) for \( p_0 > 0 \) and with \( b_{p_0}^\dagger \) for \( p_0 < 0 \), deducing that the hamiltonian must be of the form

\[ H \approx \sum_{p \neq 0} v_F|p|b_p^\dagger b_p \]

Repeating the same procedure also with the fermionic field, it can be found

\[ [\rho^\dagger_r(p), \psi_r(x)] = -e^{ipx} \psi_r(x) \]

and so it could be tempting to write

\[ \psi_r(x) \approx e^{\sum_p e^{ipx}\rho^\dagger_r(-p)\left(\frac{2\pi}{pL}\right)} \]

but it would be quite unprecise, since the left hand side would change the number of electrons by one, while the right hand side would conserve that number, since it is made only of density fluctuations. We have therefore to introduce two fermionic operators \( U_R \) and \( U_L \) that change the number of electrons; they are usually called Klein factors and are such that \( U_R^\dagger \) adds one electron of species \( r \) and \( U_r \) commutes with the boson operators. Therefore, we have

\[ \psi_r(x) = U_r e^{\sum_p e^{ipx}\rho^\dagger_r(-p)\left(\frac{2\pi}{pL}\right)} \]

Also two bosonic field \( \phi(x) \) and \( \theta(x) \) can be introduced

\[ \phi(x) = -(N_R + N_L)\frac{\sqrt{\pi}x}{L} - \frac{i\sqrt{\pi}}{L} \sum_{p \neq 0} \frac{1}{p} e^{-\alpha|p|/2-ixp} (\rho_R^\dagger(p) + \rho_L^\dagger(p)) \]

\[ \theta(x) = (N_R - N_L)\frac{\sqrt{\pi}x}{L} + \frac{i\sqrt{\pi}}{L} \sum_{p \neq 0} \frac{1}{p} e^{-\alpha|p|/2-ixp} (\rho_R^\dagger(p) - \rho_L^\dagger(p)) \]

\( \alpha \) is a cut-off introduced to regularize the theory: its inverse \( \Lambda = 1/\alpha \) mimics a finite bandwidth and prevents the momentum from becoming too large and assuming values where the linear approximation is no longer valid. The cut-off \( \Lambda \) can be estimated as follows: the real dispersion equation (fixing the zero at the Fermi energy) is \((k_F + \Lambda)^2 - k_F^2)/2m = v_F\Lambda + \Lambda^2/2m\) which, in the linear regime, is

\[ \text{Thermal drag between two coupled quantum wires} \]
approximated with $v_F \Lambda$; in order for the approximation to be good, i.e. the difference between the two energies to be small, it must hold $\Lambda^2/2m \ll v_F \Lambda$ or $\Lambda \ll 2k_F$.

Formally, the limit $\alpha \to 0$ should be taken: this is a problem, because we should consequently take the limit $\Lambda \to \infty$; nonetheless, values like $\Lambda \approx 0.1 \div 0.4k_F$ are a good compromise between respecting the validity of linear dispersion approximation and taking the limit $\alpha \to 0$.

We can now write the final form of fermionic and bosonic field (taking also the limit $L \to \infty$)

$$\psi_r(x) = U_r \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} e^{i r k_F x} e^{-i \sqrt{\pi}(r\phi(x) - \theta(x))} \quad (1.3.5)$$

$$\phi(x) = -\frac{i}{L} \sum_p \left( \frac{L|p|}{2} \right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2-|p|} (b_p^\dagger + b_{-p}) \quad (1.3.6)$$

$$\theta(x) = \frac{i}{L} \sum_p \left( \frac{L|p|}{2} \right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2-|p|} (b_p^\dagger - b_{-p}) \quad (1.3.7)$$

From equations (1.3.6) and (1.3.7) it is manifest that the bosonic fields are hermitian.

In addition, it is not too complicate to verify that $\phi(x_1)$ and $\phi(x_2)$ commute, as well as $\theta(x_1)$ and $\theta(x_2)$; moreover

$$[\phi(x_1), \theta(x_2)] = i \frac{2}{L} \text{sign}(x_2 - x_1) \quad (1.3.8)$$

Therefore we see that $\phi$ and $\nabla \theta$ satisfy the canonical commutation rule for boson fields and the conjugate momentum to the field $\phi$, $P(x) = \nabla \theta(x)$, can be defined.

Furthermore the fields can be connected with the densities of right and left movers, as we will see in the Appendix A in equation (A.1.2).

Finally the free electrons hamiltonian can be written

$$H = \frac{v_F}{2} \int dx [P(x)^2 + (\nabla \phi(x))^2] \quad (1.3.9)$$

The hamiltonian is purely quadratic in the boson operators and, therefore, makes the calculations very easy to perform.

Now we want to treat interactions like $H = \int dx V(x - x') \rho(x) \rho(x')$. We apply another time the linearization of dispersion around Fermi energy, writing the density in terms of the fermion fields of right and left movers and classifying the scattering processes as in Figure 1.6.

Since we are dealing with spinless electrons, the processes that we have to consider are the first two of Figure (1.5). The first process involves only right or left movers at the same time; it can be written in the boson language using (A.1.2):

$$\frac{V_4}{2} \psi_R^\dagger(x) \psi_R(x) \psi_R^\dagger(x) \psi_R(x) = \frac{V_4}{2} \rho_R(x) \rho_R(x) = \frac{V_4}{2} \frac{1}{4\pi} (\nabla \phi - \nabla \theta)^2$$

Obviously there is another $V_4$ process which involves terms with $R \to L$. Using equation (A.1.2) we see that, in order to take into account this change, it is enough to make the change $\phi - \theta \to \phi + \theta$.
Therefore, the total contribution of $V_4$ processes to the hamiltonian is
\[ H_4 = \frac{V_4}{4\pi} \int dx [(\nabla \phi)^2 + (\nabla \theta)^2] \]

Analogously, the $V_2$ processes mix density of left and right movers and their contribution to the hamiltonian can be written as
\[ H_2 = V_2 \psi_R^\dagger(x)\psi_R(x)\psi_L^\dagger(x)\psi_L(x) = \frac{V_2}{4\pi}[(\nabla \phi)^2 - (\nabla \theta)^2] \]

Adding these two terms to the kinetic hamiltonian of equation (1.3.9), the total hamiltonian becomes
\[ H = \frac{v}{2} \int dx \left( gP(x)^2 + \frac{1}{g} (\nabla \phi(x))^2 \right) \tag{1.3.10} \]

where we have introduced the Luttinger parameters $v$ and $g$: $v$ is a renormalized Fermi velocity and indicates roughly how fast the excitations travel; instead $g$ is an index of the interaction strength, so that the interaction is repulsive for $g < 1$, attractive for $g > 1$ and null for $g = 1$. They depend on the Fermi velocity of the system and on the strength of the interaction:
\[ v = v_F \left( 1 + \frac{V_4}{2\pi v_F} - \frac{V_2}{2\pi v_F} \right) \]
\[ g = g_F \left( 1 + \frac{V_4}{2\pi v_F} + \frac{V_2}{2\pi v_F} \right) \]
\[ v = v_F \left( (1 + y_4/2)^2 - (y_2/2)^2 \right)^{1/2} \]
\[ g = \left( 1 + y_4/2 - y_2/2 \right) \left( 1 + y_4/2 + y_2/2 \right)^{1/2} \tag{1.3.11} \]

with $y = V/(\pi v_F)$.

Some considerations are in order at this stage. In the spinless case we have shown that for low energy excitations only the processes involving the Fermi surface are relevant; furthermore the bosonized hamiltonian (which was quadratic in the free case) remains quadratic except for a renormalization of the Fermi velocity and of the relative weight associated to the two boson fields.

This is a quite interesting result, since it allows to treat a great variety of interactions, simply by changing the parameters of the hamiltonian and retaining its very simple quadratic form.

The bosonic operators $b$ and $b^\dagger$ previously defined no more diagonalize the hamiltonian; indeed we can define new bosonic operator in terms of which $H$ assumes a number conserving form:
\[ \phi(x) = -\frac{i}{L} \sum_p \left( \frac{L|p|g}{2} \right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2} e^{-ipx} (b_p^\dagger + b_{-p}) \tag{1.3.13} \]
\[ \theta(x) = \frac{i}{L} \sum_p \left( \frac{L|p|}{2g} \right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2} e^{-ipx} (b_p - b_{-p}) \tag{1.3.14} \]
\[ H = v \sum_k |k|b_k^\dagger b_k \tag{1.3.15} \]
The Hamiltonian in equation (1.3.15) can be used to calculate the temporal evolution of the relevant operators, as we will see in Appendix A.

So far we have analysed the problem within the linear spectrum approximation, where bosonization works very well; we would like to continue to use the bosonization framework, but consider also a non perfectly linear dispersion.

This can be done by including into the bosonized Hamiltonian some terms which are cubic or quartic in the boson fields and account for the band curvature effects (see for example Ref. [22]). These terms lead to a scattering involving three or four excitations, suggesting therefore the presence of a finite lifetime of the excitations, along with decay processes.

Things get more complicated when we consider spin. As we will see in Appendix A, the pure linear Hamiltonian separates the degree of freedom relative to spin and charge density; nevertheless, the correcting terms include also mixed products of spin fields and charge fields, leading to scattering processes involving both spin and charge excitations, breaking the spin-charge separation and allowing for the decay of excitations of one type into excitations of different types.

Thermal drag between two coupled quantum wires
Chapter 2

Coulomb drag between coupled quantum wires

In this chapter we will introduce the main concepts associated to what is known as Coulomb drag, focusing, in particular, on the one-dimensional case.

Throughout the whole chapter we will always indicate the electric current with the symbol $J$.

As anticipated in the Introduction, the study of electric transport properties is often and widely used to understand the microscopic dynamics and the equilibrium properties of a many-body system. In fact, within the linear response regime, the experimental results of transport measurements depend on the properties of the unperturbed system.

Usually such measurements are carried out by applying a voltage across a conductor and measuring the current flowing in it or, vice versa, by driving a current and measuring the voltage drop.

This is only the simplest experimental setup that can be realized and, in principle, more complicated configurations can be considered; a relevant example is due to Pogrebinskii (1977, Ref. [23]), who first suggested the possibility of analysing a system which involves two distinct but closely spaced coupled conductors.

In this situation, the phenomenon of Coulomb drag emerges: an electric current $J_1$ flowing in the first conductor induces a current (or, for an open circuit, a voltage drop) in the second conductor. Intuitively, the charge carriers flowing in the first conductor (or a voltage applied to it) drag the carriers in the second conductor through Coulomb interaction.

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This drag effect can be observed in systems of various dimensionality, such as coupled quantum wires (see for example Figure 2.1) and systems made of bidimensional layers; this second example is the original setup proposed by Pogrebinskii.

As explained in Ref. [24], the phenomenon of drag arises because the carriers of the two conductors are subject to a "mutual friction", i.e. there are scattering processes due to the interaction between the two conductors, which are characterized by the exchange of both momentum and energy.

It is interesting to observe that Coulomb drag between two electron systems is ultimately caused by fluctuations (or inhomogeneities) of the charge density in the two conductors. For example, an infinite bidimensional layer with a uniformly distributed electric charge creates a uniform electric field perpendicular to the layer itself and does not exert any lateral force upon the carriers in the other layer and thus it cannot drag them; an analogous statement is true also for wires.

Since the drag properties are related to those of the interwire coupling, it is very natural that the first experimental works on Coulomb drag were employed to make quantitative measurement of the interaction strength between subsystems in various devices; some examples are given by coupled two-dimensional and three-dimensional electron systems in AlGaAs/GaAs heterostructures (Ref. [25]) and 2D electron systems in AlGaAs/GaAs double quantum wells (Ref. [26]). Drag between 3D systems was also numerically simulated in Jacoboni and Price (1988, Ref. [27]).

In the last two decades drag measurements have become a standard experimental tool in condensed matter physics and have been used to investigate a huge amount of physical properties, such as: properties of electron-electron scattering in low-density 2D electron systems (Ref. [28]); quantum coherence of electrons (Ref. [29]); Luttinger liquid effects and one-dimensional sub-bands in quasi 1D wires (Ref. [14]).

Furthermore, the interconductor interaction and its transport properties have been studied also in hybrid devices made of a quantum wire and a quantum dot (Ref. [30]) or a superconductor (SC) film and a 2D electron gas (2DEG), see Ref. [31].

The progresses made in the experimental field were matched by all the theoretical works, which were initially dedicated to the study of Coulomb drag in various systems, such as 1D systems coupled to conductors of arbitrary dimensionality (Ref. [32]) and 1D wires coupled between them (Ref. [33]). Moreover a lot of work was done in order to understand the drag between two degenerate 2DEGs (Ref. [34]), because of the successful experiments in AlGaAs/GaAs double quantum wells.

Successively, the theoretical studies were generalized and Coulomb drag between two 2DEGs was extended to dilute 2D hole systems (Ref. [35]). Besides this, the Coulomb drag between composite electrons was generalized to include also a phonon-mediated interaction (Ref. [36]) and drag between non-Fermi-Liquid phases was studied, comprising Luttinger liquids, Wigner crystals and strongly localized electrons (Refs. [37] and [38]).

Further developments and generalization of the drag concepts include interlayer Seebeck effect, spin drag and Coulomb drag in ballistic quantum wires and coupled 2D-1D systems (Refs. [39] and [40]).

Until now, we have presented an overview of the main contributions to the study
of Coulomb drag. We now describe the system in which we are interested and where we want to study the Coulomb drag; it is sketched in Figure 2.1: it comprises two quantum wires of length $L$ interacting in a small region of length $L_I$. The coupling is provided only by the Coulomb interaction between the wires and has an Hamiltonian of the form

$$H_{\text{int}} = \int dx dy U_{12}(x-y)\rho_1(x)\rho_2(y)$$

(2.1)

This Hamiltonian commutes with the number of particles on each wire; it implies that there can be no particle flow between the two wires and therefore no interwire electric current.

The electric current can be calculated by mean of a continuity equation, which derives from the charge conservation: $\partial_x J_i(x,t) + e\partial_t \rho_i(x,t) = 0$.

Exploiting the charge conservation on each wire, the currents can be calculated at the end of the wires, with the following relations holding: $J_1(L/2) = J_1(-L/2)$ and $J_2(L/2) = J_2(-L/2)$; this is shown in Figure 2.1.

\[
\begin{array}{c}
J_1(-L/2) & \rightarrow & J_1(L/2) \\
U_{12} & \downarrow & \\
V/2 & \rightarrow & -V/2
\end{array}
\]

Figure 2.1: Sketch of the system of coupled quantum wires, with the interaction region, the electric currents and the applied potential.

If a voltage is applied between the two ends of wire 2, two effects appear: the flow of a current $J_2$ and the flow of an induced current $J_1$ in the first wire.

This induced current is a consequence of the coupling: since the excitations in wire 1 are coupled to the excitations in wire 2, when the latter move, the first experience a drag effect which leads to the insurgence of an induced current.

In the next sections we show the efforts made in literature to calculate the transconductivity (or the transresistivity), i.e. the coefficient of proportionality between $J_1$ and the potential $V$ (or viceversa).

We are going to present the approaches developed in three different articles, discussing the main differences and similarities between the various methods and results obtained.

In section 2.1, we examine the work in Ref. [13], describing the employed method of analysis and making a comparison with the results found for 2D Coulomb drag in Ref. [11] and discussed in section 2.2. Afterwards, in section 2.3 we explain the procedure proposed by Flensberg in Ref. [14] to study the drag between Luttinger liquids and make a comparison we the method of Ref. [13].

\section{The dynamical approach}

In this section we will present and explain the results obtained in Ref. [13]. In
this paper the authors study the drag in the system showed in Figure 2.2, which is equivalent to the one in Figure 2.1:

\[ r = -\lim_{J_1 \to 0} \frac{e^2}{2\pi\hbar L} \frac{1}{dJ_1} \left( \frac{dV_2}{dJ_1} \right) \]  \hspace{1cm} (2.1.1)

(it is normalized to the single wire resistivity). \( r \) will be calculated in the limit of small couplings, retaining only the leading order; for this reason, it is necessary to calculate \( dV_2/dJ_1 \).

In the Luttinger liquid model the only source of drag between the wires can be the interwire backscattering (associated to a large momentum transfer); accounting also for non linearity in the electron spectrum allows to observe also a forward scattering contribution (associated to a small momentum transfer) to the drag. In the article discussed is showed that \( r = r_0 + r_{2k_F} \) and \( r_0 \) dominates in a wide range of temperatures.

The Hamiltonian of the system is given by

\[ H = H_1 + H_2 + H_{12} \]

with \( H_i \) given by equation (1.2.2) and \( H_{12} \) given by equation (2.1).

The electrons in wire 2 are subject to a force given by the interwire coupling. This force arises from the density fluctuations present in wire 1; in fact a homogeneous wire cannot a generate field longitudinal to itself and, thus, influence the second wire. The force density reads

\[ \mathcal{F}_2 = \int dx \frac{dU_{12}(x-y)}{dx} \langle \rho_1(x)\rho_2(y) \rangle \]
Since electrons in wire 2 are in equilibrium, this force must be exactly balanced by the electric force: the balance of forces is

\[ e n_2 \mathcal{E}_2 + \mathcal{F}_2 = 0 \]

where \( \mathcal{E}_2 = -V_2/L \) is the electric field and \( n_2 \) is the electron density in wire 2. The correlation function can be expanded up to the first order in the coupling, yielding

\[ \frac{V_2}{L} = \frac{1}{e n_2} \int \frac{dk d\omega}{(2\pi)^2} k U_{12}^2(k) \tilde{S}_1(k, \omega) \tilde{S}_2(-k, -\omega) \]

where \( \tilde{S}_1(k, \omega) = \int dx dt e^{i\omega t - ikx} \langle \rho_i(x, t) \rho_i(0, 0) \rangle \) are the dynamic structure factors, which are calculated in the presence of a finite current \( J_1 \) (the tilde superscript indicates this difference from the equilibrium structure factor, that will be defined in section 2.3).

In wire 2 the dynamic structure factor assumes its equilibrium value \( \tilde{S}_2 = S_2 \), while in wire 1 the electronic subsystem is moving with a drift velocity given by \( v_d = J_1/en_1 \) and the dynamic structure factor can be obtained with a Galileian transformation: \( \tilde{S}_1(k, \omega) = S_1(k, \omega - kv_d) \).

Then, the dynamic structure factor is expanded in powers of \( v_d \), showing that the leading order in the coupling is the linear one. At this point, the spectral function is defined and related to \( S_i \) through the fluctuation dissipation theorem:

\[ 2A_i(k, \omega) = \int dx dt e^{i\omega t - ikx} \langle [\rho_i(x, t), \rho_i(0, 0)] \rangle \]

\[ S_i(k, \omega) = \frac{2A_i(k, \omega)}{1 - e^{-\omega/T}} \]

here \( \rho \) is the equilibrium density. The properties of \( A_i \) are exploited to write

\[ r = \frac{1}{4\pi^3 n_1 n_2 T} \int_0^\infty dk \int_0^\infty d\omega k^2 U_{12}^2(k) \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2T)} \]  \hspace{1cm} (2.1.2)

One can rewrite equation (2.1.2) to obtain the transresistivity

\[ \rho_{21}^{EE} = V_2/J_1 \]

\[ \rho_{21}^{EE} = \frac{1}{2\pi^2 e^2 n_1 n_2 T} \int_0^\infty dk \int_0^\infty d\omega k^2 U_{12}^2(k) \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2T)} \]  \hspace{1cm} (2.1.3)

After calculating the drag resistivity, the equation (2.1.2) is analysed in the limit of zero backscattering and for various temperature regimes. Since backscattering is neglected, the major contribution to \( r \) comes from small momentum and energies (compared to the Fermi level).

Moreover, if free electrons are considered, the \( A_i \) are sharply peaked around \( \omega = v_i k \), with a width of the peak estimated as

\[ \delta \omega(k, T) = \max\{k^2/m, kT/k_F\} \]

Furthermore, the exact f-sum rule for the spectral function permits to estimate the height of the peak \( A_i \sim k/2\delta \omega \). In the end, the \( A_i \) are therefore approximated as step-functions of width \( \delta \omega \) and height \( \frac{k}{2\delta \omega} \).
If the difference $\delta v$ between the velocities is small compared to the velocities themselves, then the substitution $\omega = v k$ can be made inside the hyperbolic sine, calculating separately the integral in $\omega$ of what is left. Then, it is possible to write:

$$r = \frac{1}{4 \pi^3 n_1 n_2 T} \int_0^\infty \frac{dkk^2 U_{12}^2(k)}{\sinh^2(v_F k / 2T)} \alpha(k, T)$$

$$\alpha(k, T) = \int_0^\infty d\omega A_1(k, \omega) A_2(k, \omega)$$

**Figure 2.3**: Plot of the temperature dependence of $r$ between identical wires. $T^*$ marks the crossover between backscattering domination and small momentum transfer domination ($T > T^*$). Picture from [13].

For identical wires it can be proved that $\alpha(k, T) \approx k^2 / 4 \delta \omega(k, T)$. In the problem there are two scales: the temperature $T$ and the scale characterizing the $k$ dependence of the coupling $T_0 = v_F k_0$. It is found that for identical wires

$$r \propto \begin{cases} 
T^2 & \text{if } T \ll T_0 \\
\text{constant} & \text{if } T_0 \ll T \ll T_F \\
T^{-3/2} & \text{if } T \gg T_F
\end{cases}$$

This behaviour can be observed in Figure 2.3.

For the case of different wires, the difference between the velocities introduces a new scale $T_1 = k_F \delta v$. For $T \ll T_1$, the authors find a sort of activation law for the drag resistivity:

$$r \propto \frac{T}{T_1} e^{-T_1 / T} \quad T \ll T_1$$

Therefore the drag exhibits a sharp fall for low temperatures. This behaviour is a consequence of the step function approximation made for the $A_i$ and is too dramatic in the case of interacting electrons, when the $A_i$ always overlap. In this case the drag resistivity has a power law correction at small temperatures $\delta r \propto T^5$.

The width $\delta \omega$ has been calculated in analytically using the Calogero-Sutherland model (that is characterized by a very specific interaction potential); this model is employed to present an example of the effects of an intrawire interaction. The width

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*Thermal drag between two coupled quantum wires*
at zero temperature changes only for a numerical factor, leading to the following proportionality

\[ r \propto c_g T^2 \quad \text{with} \quad c_g \propto \frac{g^6}{1 + g} \]

Therefore, the presence of intrawire interactions changes the numerical prefactor, in the case of identical wires, and leads to a small correction, only for very low temperatures, in the case of different wires.

### 2.2. The memory function approach

In this section we briefly review the work presented in Ref. [11]; in this article is studied the Coulomb drag between bidimensional layers, deriving a formula similar to equation (2.1.3) by mean of a different method.

Although the work of Ref. [11] refers to the 2D case, the parallelisms with the equation (2.1.3) are manifest and the generalization to the 1D case of the method here reported does not present particular difficulties.

The system studied in the article is the same proposed by Pogrebinskii (Ref. [23]) and showed in Figure 2.4.

Figure 2.4: Schematic picture showing setup proposed by Pogrebinskii to study the Coulomb drag in 2D. Picture from [24].

Contrary to section 2.1 and in analogy with the procedure discussed in section 2.3, no fine tuning of the voltage drop on a layer is made, whereas the starting point is a Kubo-like expression for conductivity, which is expressed as a 2 × 2 matrix and
2.2. The memory function approach

does not need any force balance equation to be derived:

\[
\sigma_{ij}(\omega) = \frac{\beta}{\nu} \int_0^\infty dt e^{i\omega t} \frac{1}{\beta} \int_0^\beta d\lambda \langle \hat{J}_i(t), \hat{J}_j(i\lambda) \rangle
\]

where \( \nu \) is the area of the systems and \( J \) indicates the integral of the current over the plane. The indices \( i \) and \( j \) label the two bidimensional layer and can refer to the left one (L) or to the right one (R), so that the transconductivity is given by \( \sigma_{LR} \).

The memory function formalism is applied in order to transform the formula of transconductivity into an expression for the transresistivity. The key point of the procedure is to express the transconductivity as the Laplace transform of an appropriate scalar product, which is defined as

\[
C_{AB}(t) \equiv \langle \hat{A}(t), \hat{B} \rangle \equiv \beta^{-1} \int_0^\beta d\lambda \langle \hat{A}_1(t), \hat{B}(i\lambda) \rangle
\]

\( C_{AB} \) acts as a scalar product over a certain Hilbert space, whose vectors are the observables of the system. Then, \( \sigma_{ij} \) is proportional to the Laplace transform of this scalar product and can be inverted with some formal manipulations, carried out in the previously defined Hilbert space. All the calculations lead to the expression written below

\[
\rho_{ij}(z) = \chi_{ii}^{-1} \chi_{jj}^{-1} \frac{\beta}{\nu} \int_0^\infty dt e^{izt} \langle \hat{J}_i, \hat{J}_j(-t) \rangle
\]

where \( \chi_{ij} = \frac{ne^2}{m} \delta_{ij} \) is the inner product of two currents evaluated at the same time.

In the previous manipulations there has been an “exchange” between a dependence on a current-current correlator in the transconductivity and a current derivative-current derivative dependence in the transresistivity. The advantage of this expression is that the time derivative of \( J \) can be related to the force experienced by the electrons:

\[
\dot{J}_i = -e/m F_i
\]

and this force depends on the coupling potential \( U_e(q) \) and on the electron densities \( \rho_i(q) \):

\[
F_{R(L)} = \pm i \sum_q \vec{q} \rho_L(\vec{q}) \rho_R(-\vec{q}) U_e(q)
\]

In the absence of interlayer coupling, the forces are uncorrelated, so the transresistivity is zero.

Summarizing all these considerations, the result is

\[
\rho_{LR}(z) = \frac{\beta}{2n_L n_R e^2 \nu^3} \times \sum_q q^2 \int_0^\infty e^{izt} dt |U_e(q)|^2 \langle \hat{A}(t), \hat{A}_1^\dagger(0) \rangle_0
\]

with \( \hat{A}(t) = \rho_L(-\vec{q},t) \rho_R(\vec{q},t) \); the subscript 0 indicates that the correlator has to be evaluated in the absence of interaction (since we are already considering the leading order in interaction).
Intuitively, the transresistivity calculation has been reduced to the calculation of two density-density correlation functions over decoupled wires. This really looks like the result found in Ref. [13] and exposed in section 2.1.

The analysis can be continued by working with a representation of exact eigenstates, in the limit \( z = 0 \), and inserting the expression for the dynamic structure factor of layer \( i \)

\[
S_i(\vec{q}, \omega) \equiv \frac{1}{\nu} \sum_{n,m} \exp\left(-\beta E_n\right) |\langle n | \rho_i(\vec{q}) | m \rangle|^2 \times \delta[\omega - (E_m - E_n)/\hbar]
\]

related to the retarded density-density response function \( \chi_i(\vec{q}, \omega) \) by the fluctuation-dissipation theorem:

\[
S_i(q, \omega) = \frac{\hbar}{1 - e^{-\hbar \omega \beta}} \text{Im} \chi_i(q, \omega)
\]

With these definitions the transresistivity reads

\[
\rho_{LR} = \frac{\hbar^2 \beta}{\pi n \nu L_{W} \nu} \times \sum_q q^2 |U_e(q)|^2 \int_0^\infty d\omega \frac{\text{Im} \chi_R(q, \omega) \text{Im} \chi_L(q, \omega)}{e^{\beta \hbar \omega} + e^{-\beta \hbar \omega} - 2} \tag{2.2.1}
\]

It can be noticed that \( \text{Im} \chi_i(q, \omega) = 2A_i(k, \omega) \) and that equation (2.2.1) looks a lot like equation (2.1.3). Apparently, there is a general law for the Coulomb drag which does not depend on the dimensionality of the system.

Finally for clean systems, a \( T^2 \) dependence is found for the transresistivity.

2.3. The approach by Flensberg

We now want to describe an alternative approach, used by Flensberg in Ref. [14] to study the Coulomb drag between two coupled Luttinger liquids.

The article studies the Coulomb drag in a system like the one in Figure 2.5: the wires have equal length \( L_W \) and the interaction region is long \( L_I \).

![Figure 2.5: Sketch of the system. Picture from [14].](image)

The problem analysed is identical to that of section 2.1, but the method employed is different. In fact, no fine tuned voltage is applied to a wire to set to zero the
current in that wire; indeed, the transconductivity (instead of the transresistivity) is calculated starting from a Kubo like formula, which does not require any force balance equation involving an electric potential drop, as discussed also in section 2.2. The two articles derive a formula for two different quantities, which are related, since one can be obtained from the other by simply inverting a matrix. Another difference is that in Ref. [14] the bosonization is applied from the very beginning, studying therefore the Coulomb drag directly for low temperatures, while in Ref. [13] also the limit of high temperatures is addressed.

The single wire Hamiltonian is similar to that in equation (1.3.10):

\[
H_{0i} = \frac{v_i}{2} \int dx \left( [P_i(x)]^2 + \frac{1}{g_i^2} [\partial_x \phi_i(x)]^2 \right)
\]

and the bosonic fields are defined with a minus sign in front with respect to those of equations (1.3.6) and (1.3.7).

The coupling Hamiltonian is analogous to that of equation (2.1), but retains a more general coupling form:

\[
H_{\text{int}} = \int dxdy U_{12}(x,y) \rho_1(x) \rho_2(y)
\]

It can be seen from equation (A.1.5) that the density reads

\[
\rho(x,t) = -\frac{1}{\sqrt{\pi}} \partial_x \phi(x,t) + \frac{1}{\pi \alpha} \cos[2k_F x - 2\sqrt{\pi} \phi(x,t)]
\]

It has two major contributions: the one arising from the boson field, which is a slow varying term, and the one coming from the cosine, which oscillate much more rapidly, thanks to the \(k_F\) in the argument. Therefore, the product of two densities will exhibit three types of terms: a product of slow oscillating terms only, a product of fast oscillating terms only and a product of mixed terms.

The article discusses how the first and the third type of terms do not influence the Coulomb drag and drops them, retaining therefore only the backscattering contribution (on the contrary of [13], where it is neglected). Moreover, it makes the approximation of being at an energy scale much smaller than \(v_F/L_I\) so that the spatial dependence of the \(\phi\) in the backscattering term can be neglected, yielding, in the end, the following expression for the interaction Hamiltonian:

\[
H_{\text{int}} = \frac{1}{2\pi^2 \alpha^2} \int dxdy U_{12}(x,y) \times \{ \cos \left[ 2(k_F x - k_F y) + 2\sqrt{\pi}(\phi_1(0) - \phi_2(0)) \right] \\
+ \cos \left[ 2(k_F x + k_F y) + 2\sqrt{\pi}(\phi_1(0) + \phi_2(0)) \right] \}
\]

At this point, it is observed that the coupling depends only on the sum or the difference of the boson fields (calculated for \(x = 0\)) and a unitary transformation is carried out:

\[
\Phi = \phi_1 + \phi_2, \quad P = (P_1 + P_2)/2
\]

\[
\Theta = \phi_1 - \phi_2, \quad \Pi = (P_1 - P_2)/2
\]
2. Coulomb drag between coupled quantum wires

Exploiting this trick, Flensberg finds $H = H_0 + H' + H_{int}$, where

$$H_0 = \frac{\bar{v}}{2} \int dx \left( |P(x)|^2 + \frac{1}{g^2} [\partial_x \Phi(x)]^2 + |\Pi(x)|^2 + \frac{1}{g^2} [\partial_x \Theta(x)]^2 \right)$$

$$H' = \bar{g} \int dx \left[ aP(x)\Pi(x) + \frac{b}{g^2} \partial_x \Phi(x) \partial_x \Theta(x) \right]$$

Here $\bar{v}$ and $\bar{g}$ are averages (properly weighted), of the two wires Luttinger parameters, while $a$ and $b$ account for the differences between the wires; they are zero in the case of identical wires.

The Hamiltonian is thus reduced to that of two wires, each one with an impurity of different strength, coupled by an interaction $H'$.

This interaction depends on the difference between the parameters of the two wires and so it could be treated perturbatively if the wires were slightly different.

Using equation (A.1.6), the electric current operator is expressed as $j_i = \frac{\bar{v} F P_i}{\sqrt{\pi}} = -\partial_t \phi_j(x,t)x + \frac{b}{\bar{g}^2} \partial_x \Phi(x) \partial_x \Theta(x)$ and then the Kubo formula is applied to get the transconductivity:

$$G_{21}(\omega) = \frac{i \omega e^2}{4\pi} [D^r_\Phi(x,x';\omega) - D^r_\Theta(x,x';\omega)]$$

where the retarded Green’s functions have been employed:

$$D^r_\Phi(t - t') = -i \theta(t - t') [\Phi(x,t'), \Phi(x',t)]$$

For identical wires $H' = 0$ and so the Hamiltonian factorizes; it is easy to show that the transconductance is:

$$G_{21} = \frac{1}{4} \left( G_{Lutt}(V_1, 2g) - G_{Lutt}(V_2, 2g) \right)$$

$$V_{1,2} = \frac{D}{2\pi v_F} \int dxdy U_{12}(x,y)e^{2ik_F(x+y)}$$

Here $D = v_F/\alpha$ parameterizes the cut-off of the bosonization theory.

In other words, the transconductivity can be calculated by simply computing the conductivity for a single wire with a doubled Luttinger interaction parameter $2g$ and with an impurity, whose scattering amplitude is given by $V_i$.

The results of Ref. [14] for identical wires in the low temperature regimes follow:

- For $g > 1/2$, $G_{21} \sim T^{4g-2}$; for non interacting wire the drag is quadratic in $T$.
- For $g < 1/2$, $G_{21} \sim T^{1/g-2}$.
- For $g = 1/2$, the problem maps to that of two Fermi liquids and can be solved with perturbative renormalization theory (Ref. [41]) or with Bethe Ansatz (Ref. [42]), obtaining

$$G_{21} = \frac{e^2}{4\pi} \frac{(|W_2|^2 - |W_1|^2) \gamma}{[1 + |W_1|^2 \gamma][1 + |W_2|^2 \gamma]}$$

where $t = T/D$, $W_i = V_i/v_F$ and $\gamma = 4g - 2$. 

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Figure 2.6: Plot of the transconductance for a short interaction region \((k_F L_I = 1)\) in the left panel and a long interaction region \((k_F L_I = 20)\) in the right panel; the solid (dashed) line is for \(t = 0.1\) \((0.001)\). Picture from [14].

The results are summarized in Figure 2.6, where is plotted the transconductance as a function of \(g\) for \(T/D = 0.1\) (solid line) and for \(T/D = 0.001\) (dashed line) and for two values of the interaction range, which are \(k_F L_I = 1\) for the left panel and \(k_F L_I = 20\) for the right panel. It can be noticed that in the first regime (short interaction regions) \(G_{21}\) exhibits a peak around \(g = 0.5\), whose position depends on the temperature; on the other hand, for long interaction regions, \(G_{21}\) shows a flat behaviour for low \(g\), with a wider plateau for lower temperatures.

Then, the case of different wires is treated, writing the action and integrating out the field at \(x \neq 0\). The impurities terms are the same, while the kinetic part of the Hamiltonian is

\[ S_0 = \frac{1}{\beta} \sum_{\omega_n} \frac{|\omega_n|}{g} \left( \frac{\Phi_\omega}{\Theta_\omega} \right)^* \begin{pmatrix} k_+ & k_- \\ k_- & k_+ \end{pmatrix} \begin{pmatrix} \Phi_\omega \\ \Theta_\omega \end{pmatrix} \]

where \(k_\pm = \frac{1}{2}(v_2 + v_1)(g_1 \pm g_2)^2/[(g_1^2 v_2 + g_2^2 v_1)]^{1/2}\). If the two wires have the same velocities but different interaction parameters, the treatment made for identical wires is still valid, while for a general solution a renormalization group calculation is needed; it shows that for \(g_1 + g_2 < 1\) the drag is strongly enhanced.

The \(T^2\) dependence of transconductivity for free electrons is compatible with the \(T^2\) dependence at low temperatures of transresistivity in section 2.1; in fact, for free electrons is irrelevant whether the backscattering or the forward scattering contribution is considered; moreover, at the leading order in the coupling the transresistivity can be written as

\[ \rho_{12} \approx \frac{G_{21}}{G_{11} G_{22}} \]

and, since the single wire conductivity is independent of temperature (see Refs. [6] and [7]), the temperature dependence of transresistivity and transconductivity is the
This discussion implies that two quite different approaches to the problem of Coulomb drag, with different limits of validity, converge to the same temperature dependence when the parameters studied are the same. Each method has its strong points: the procedure of Ref. [14] allows to make a more detailed study of the dependence of transconductivity on the intrawire interaction, while the approach in Ref. [13] is valid for a wider range of temperatures.
Chapter 3

Thermal drag

In this chapter we present our original contribution to the study of thermal drag between two coupled quantum wires of spinless electrons. We calculate the proportionality coefficient between the temperature gradient in the first wire $\nabla T_1$ and the induced thermal current in the second wire $J^T_2$, when there is no current flow $J^T_1$ in the first wire. The calculation is made in the linear response regime and the coefficient is called transresistivity ($\rho_{TT}^{12}$); when $J^T_1 = 0$ its defining equation is $\nabla T_1 = \rho_{TT}^{12} J^T_2$.

The interwire coupling arises solely from the electron-electron Coulomb interaction; the calculations will be made in the limit of weak coupling, retaining only the leading order, and for an approximately homogeneous intrawire interaction.

Using a matricial equation, the relation between the thermal biases and the thermal currents can be written in linear response as

\[
\begin{pmatrix}
-\nabla T_1 \\
-\nabla T_2
\end{pmatrix} = \begin{pmatrix}
\rho_{TT}^{11} & -\rho_{TT}^{12} \\
-\rho_{TT}^{21} & \rho_{TT}^{22}
\end{pmatrix} \cdot \begin{pmatrix}
J^T_1 \\
J^T_2
\end{pmatrix}
\]

(3.1)

\[
\begin{pmatrix}
J^T_1 \\
J^T_2
\end{pmatrix} = \begin{pmatrix}
\sigma_{TT}^{11} & \sigma_{TT}^{12} \\
\sigma_{TT}^{21} & \sigma_{TT}^{22}
\end{pmatrix} \cdot \begin{pmatrix}
-\nabla T_1 \\
-\nabla T_2
\end{pmatrix}
\]

(3.2)

the minus sign in front of $\rho_{TT}^{12}$ has been introduced in order to be coherent with the definition previously made. The resistivity matrix is defined by equation (3.1) and is the inverse of the conductivity matrix, defined in the second equation (3.2); the coefficient $\sigma_{TT}^{21}$ will be called transconductivity.
In what follows we are going to define the problem and to compute the thermal transresistivity.

In section 3.1 we discuss in details the definition of thermal currents since the situation is not as simple as in the case of electrical drag; in section 3.2 we present a brief summary of the results obtained in this chapter.

In section 3.3, a formula for the transresistivity is derived; this can be done by using two different approaches, applied respectively in the articles [13] and [11] to the Coulomb drag case.

The first method (that we will call dynamical method, as in section 2.1) relies on a sort of force-balance equation, relating the temperature gradient to density-density correlation functions, which are evaluated out of equilibrium; expanding to the first order in the current, an expression for the transresistivity can be then directly derived. This method is well explained in Appendix F, to which the reader can refer for more details.

The second approach extends to the thermal case the memory function method described in Ref. [11] and section 2.2. As already seen in Chapter 2, this method starts from a Kubo-like formula for the thermal conductivity matrix and then inverts it using the memory function formalism and obtaining the resistivity matrix. The transresistivity is proportional to a correlator between $J^T_1$ and $J^T_2$ time derivatives, that can be expressed in terms of the coupling between the wires. After having derived a general expression for the transresistivity, we specialize to the case of low temperatures and, using a property valid in the linear dispersion regime, we express the transresistivity as the integral of the density-density correlation functions of the decoupled wires, yielding a Wiedemann-Franz law between thermal and electric transresistivity.

Even though both these methods yield the same result for transresistivity, they are discussed because each one has its own qualities: for example the method of Ref. [11] employs a more formal and general approach to the problem, while the method of Ref. [13] uses a more physical derivation, which can be easily extended to the thermoelectric case.

In section 3.4, the correlation functions appearing in transresistivity are calculated, while in section 3.5 and 3.6 the various contributions to the drag are analysed, plotting their behaviour for some particular cases. Finally, in section 3.7, we generalize the results of the previous sections to the case of thermoelectric coefficients, by employing (as anticipated) the method outlined in Appendix F.

### 3.1. System overview and definition of thermal currents

The system is sketched in Figure 1 and reported also in Figure 3.1: it is composed by two parallel quantum wires of length $L$ interacting in a region of length $L_I \ll L$.

The thermal currents do not behave exactly as their electrical counterparts and the case of thermal drag requires a bit more attention for what concerns their defi-
nition.

The i-th wire is governed by the Hamiltonian $H_i$, which accounts for the free-electrons dispersion and for the intrawire interaction; it is written in terms of an Hamiltonian density $H_i = \int dx \psi_i^\dagger(x) \psi_i(x)$, whose expression can be derived from equation (1.2.2)

$$H_i(x) = \frac{1}{2m} \partial_x \psi_i^\dagger(x) \partial_x \psi_i(x) + \frac{1}{2} \int dx' \psi_i^\dagger(x) \psi_i^\dagger(x') V(x' - x) \psi_i(x') \psi_i(x) \quad (3.1.1)$$

here $x = 0$ indicates the center of the wire.

In the low energy limit the i-th wire can be equivalently described using the Luttinger liquid Hamiltonian:

$$H_i(x) = \frac{v}{2} [g P(x)^2 + 1/g (\partial_x \phi(x))^2]$$

The Hamiltonian describing the coupling between the wires is the one introduced in equation (2.1):

$$H_{int} = \int dx dy U_{12}(x - y) \rho_1(x) \rho_2(y)$$

proportional to the densities $\rho_i$ of the i-th wire; here $x$ and $y$ refer respectively to the wire 1 and 2. This coupling decays very rapidly to zero outside the interaction region.

The total Hamiltonian of the system is therefore given by

$$H = H_1 + H_2 + H_{int}$$

In the case of Coulomb drag, the total Hamiltonian commutes with the charge on each wire, so that there cannot be an electric current between the two wires. Let us now specifically consider the thermal currents: the previous statement is not true for the thermal drag, since the coupling does not conserve the energy on each wire and enables an interwire energy transfer. In general, there will be a current flowing between the two wires unless they have equal energy.

In addition, if there is a temperature difference between the extremes of wire 1, there will be a current flowing in the wire 2 as well as a current in wire 1 (see Figure 3.1). Since there can be heat transfer between the two wires, the thermal currents at the two extremes of a wire are not forced to be equal and thus one has to carefully consider what current to use, in order to properly describe the drag.

Figure 3.1: Schematic picture of the system and of the thermal currents. Here the superscript $T$ of the thermal currents has been dropped.
At this scope, two longitudinal currents (one for each wire), that flow throughout one wire, can be defined, along with a transverse current, that flows between the two wires. A convenient definition for the longitudinal current is the average of the currents evaluated at the two extremes of the wires, i.e. very far from the region of interaction:

\[
J_{T_i}^{\parallel} = \lim_{x \to +\infty} \frac{J_{T_i}^1(x) + J_{T_i}^2(-x)}{2} \rightarrow \frac{J_{T_i}^1(L/2) + J_{T_i}^2(-L/2)}{2} \tag{3.1.2}
\]

where is implicitly assumed that the limit is however calculated for \(L_I \ll x < L/2\).

The transverse current is defined as the difference between the current evaluated at the extremes (it is assumed positive if it flows from wire 1 to wire 2):

\[
J_{T_i}^{\perp} = -\lim_{x \to +\infty} (J_{T_i}^1(x) - J_{T_i}^2(-x)) = -\int \partial_x J_{T_i}^1(x) dx = \int \partial_x J_{T_i}^2(x) dx \tag{3.1.3}
\]

where the integrals are performed on the whole wires.

In other words, the longitudinal current represents the energy per time flowing from one end to the other of the same wire, while the transverse current is the energy per time flowing from wire 1 to wire 2.

For convention the currents are considered positive when they flow towards greater \(x\).

Until now we have not discussed how to calculate the thermal currents. This will be done by writing the continuity equation, which is the natural starting point for every attempt to compute any current. Since we will need also a formula for the electric current, a quite general treatment will be presented.

The thermal current has already been indicated as \(J_T\), while the notation used for the electric current is \(J_E\). Using, firstly, the conservation of particles number and energy and, then, translating it into the conservation of charge and excitation energy, the divergence of a current can be related to the time derivative of density or energy density, writing the following continuity equations:

\[
\partial_x J_E(x) + e \partial_t \rho(x) = 0 \quad \partial_x J_T + \partial_t \tilde{H}(x) = 0
\]

where \(\tilde{H}(x) = H(x) - \mu \rho(x)\) is the excitations energy density and \(\mu\) is the chemical potential.

From the second continuity equation written above, it follows that

\[
\partial_x J_{T_2}^1(x) = i \int dx'[\tilde{H}_2(x), H_2(x')] + i[\tilde{H}_2(x), H_{int}]
\]

The first term originates the so-called “decoupled” thermal current \(J_{T_2}^{01}\) (calculated in the absence of interwire coupling), while the latter contributes to an “interaction” current \(J_{T_2}^{int}\), so that \(J_{T_2}^2 = J_{T_2}^{01} + J_{T_2}^{int}\).

The decoupled current is constant throughout the whole wire, while the interaction current is generally not constant, but is not negligible only in the region of interaction. Under the assumption that the region of interaction is much smaller than the dimension \(L\) of the wires (and does not scale with \(L\)), we can neglect the

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interaction current and use only the decoupled current to calculate the $J_{2\parallel}$, approximating it with the average on the whole wire. In other words, the decoupled current is averaged on the right half of the wire to estimate the current at $x = +\infty$ and an analogous calculation is done for the current at $x = -\infty$:

$$J_{\parallel}^{T} = \frac{1}{L} \int dx J_{0\parallel}^{T}(x)$$ (3.1.4)

The calculation of longitudinal currents in a system of coupled wires has somehow been reduced to the calculation of the currents on each wire and without the coupling. The continuity equations written above can now be combined with equations (1.2.1) and (3.1.1) in order to find the decoupled current (see also Ref. [43]):

$$J_{0}^{E}(x) = \frac{e}{2m} (\psi^\dagger(x) \partial_x \psi(x) - \partial_x \psi^\dagger(x) \psi(x))$$ (3.1.5)

$$J_{0}^{T}(x) = -\frac{1}{2m} (\partial_x \psi^\dagger(x)(\partial_t - i\mu)\psi(x) + (\partial_t + i\mu)\psi^\dagger(x)\partial_x \psi(x))$$ (3.1.6)

where the time derivative of the field can be easily expressed from the Schrodinger equation

$$i\partial_t \psi(x) = -\frac{\partial^2 \psi(x)}{2m} + \int dx' \psi^\dagger(x')V(x' - x)\psi(x')\psi(x)$$

Equations (3.1.5) and (3.1.6) can also be used to write the longitudinal currents in a “nice” form, employing the creation and annihilation operators of Chapter 1. Exploiting the definition in equation (3.1.4), we find

$$J_{\parallel}^{E} = \frac{1}{L} \sum_k \frac{k}{m} a_k^\dagger a_k$$ (3.1.7)

$$J_{\parallel}^{T} = \frac{1}{2} \sum_k \frac{k}{m} \left( \frac{k^2}{2m} - \mu \right) a_k^\dagger a_k + \frac{1}{L^2} \sum_{k,k',q} \frac{k + k'}{2m} a_k^\dagger a_{k'}^\dagger a_{k+q} a_{k'-q} V(q) a_k a_{k'}$$ (3.1.8)

For the expressions of electric and thermal currents in the bosonization framework, the reader is referred to Appendix A.

In the following, we will exclusively consider thermal currents (except for section 3.6) and, thus, the index $T$ can be dropped, indicating without ambiguity the thermal current with $J_{\parallel}$; in addition, we drop also the subscript 0 from the decoupled currents $J_{01}$ and $J_{02}$, since we will always deal with currents calculated for a decoupled wire and it is sufficient to indicate them with $J_{1}$ and $J_{2}$.

### 3.2. Summary of results

In order to help the reader through in understanding this Chapter, in this section we summarize the results that will be later derived.

As discussed in section 3.1, the thermal drag phenomenon emerges between two coupled quantum wires, inducing a thermal current in a wire when a thermal gradient is applied to the other wire. Beside the drag current, also an interwire interaction...
current appears when there is an energy difference between the wires. The longitudinal currents are used to characterize the drag.

In section 3.3, we will focus on the calculation of a Kubo-like formula for the transconductivity, involving an expectation value of two longitudinal currents:

\[ \sigma_{21}^{TT} \propto \lim_{s \to 0} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int_0^\beta d\beta' \langle J_1(\beta) J_2(\beta') \rangle \]

This will be done by appropriately modifying the procedure proposed by Luttinger in Ref. [5].

Then the memory function formalism (from Refs. [12] and [11]) will be applied in order to invert the conductivity matrix of equation (3.2) and obtain the resistivity matrix of equation (3.1); in other words, the matrix of current-current correlators will be inverted, expressing it as a matrix of correlators between the time derivatives of the currents:

\[ \rho_{12}^{TT} \propto -\lim_{s \to 0} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int_0^\beta d\beta' \langle \dot{J}_2(-t) \dot{J}_1(\beta') \rangle \]

After having derived this expression for the transresistivity, we will write the current’s time derivative in terms of the interwire coupling \( U_{12} \), getting an expression at the second order in \( U_{12} \) (which is the leading order) and dependent on the product of two particular correlators, each one containing twice the fluctuation density and once the thermal current. These correlators can be expressed, in a very general way, as four points correlators, i.e. expectation value containing the product of four fermion operators. Thus, a formula valid in any regime of temperatures, but rather difficult to employ at a more practical level, will be found in equation (3.3.19).

To continue our analysis, we will make the approximation of low temperatures \( T \ll T_F \) and use the Luttinger liquid model and the bosonization. In this regime we will find a more manageable formula for the correlators considered, expressing them in terms of a density-density expectation value (see equations (3.3.20) and (3.3.21)); this result is similar to that obtainable studying the electric current, but is valid only for low enough temperatures.

Interestingly, we will write the thermal transresistivity as

\[ \rho_{12}^{TT} = \frac{9}{2\pi^2} \frac{v_1 v_2}{k_B^5 T^4} \int_0^\infty d\omega \int_0^\infty dk |U_{12}(k)|^2 k^2 A_1(k,\omega) A_2(k,\omega) \frac{\sinh^2(\omega/2k_B T)}{\omega^2} \]

where the \( v_i \) are the Luttinger velocities and the \( A_i \) are the spectral functions of equation (3.3.25). Comparing this equation with the electric transresistivity \( \rho_{12}^{EE} \), we will derive a Franz-Wiedemann law for the drag phenomena.

\[ \frac{\rho_{12}^{EE}}{\rho_{12}^{TT}} = \frac{\pi^2}{9} k_B^5 T^3 \frac{1}{e^2 n_1 n_2 v_1 v_2} \]

Here \( n_i \) is the average density of wire \( i \) defined in section 2.1.

Equipped with an easily manipulable formula, all the density-density correlation functions needed will be calculated, distinguishing between forward scattering terms (with a small momentum transfer) and back-scattering contributions (associated to
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We will analyse separately the various contributions, helping us with plots, numerical results and various approximations, and conclude that in a wide range of temperatures the forward scattering is the dominant term.

Several different regimes of temperature will be studied, including both the cases of identical and different wires. Two temperature scales naturally emerge: $T_0$ associated to the characteristic wavevector of the coupling and $T_1 < T_0$ associated to the difference between the Luttinger velocities of the two wires.

The transresistivity behaviour will be analysed for the three ranges $T \ll T_1$, $T_1 \ll T \ll T_0$ and $T_0 \ll T$. Furthermore, using a particular form for the coupling, we will plot the transresistivity expression, as long as the transconductivity one, in order to underline all their essential features. The results that will be found are: the transresistivity $\rho_{12}$ behaves like $\sim T$ for $T \ll T_1$, decreases like $\sim 1/T$ for $T_1 \ll T \ll T_0$, and then has a trend proportional to $\sim 1/T^3$ for $T_0 \ll T$; on the other hand the transconductivity $\sigma_{21}$ initially grows as $\sim T^3$, then undergoes a crossover and behaves like $\sim T$ and, finally, after a peak decreases like $\sim 1/T$.

Finally, we will extend our calculations to the thermoelectric coefficients, basing our derivation on the results of Appendix F.

3.3. Calculation of the transresistivity

In this section thermal drag is studied by using the Kubo formalism to obtain the transconductivity and then applying the memory function formalism to get the transresistivity.

3.3.1. Calculation of thermal transport coefficients

Let us now calculate the thermal transport coefficients following the formalism presented by Luttinger in Ref. [5].

The thermal transconductivity can be written as a Kubo formula, involving a correlation between $J_{1\parallel}$ and $J_{2\parallel}$, as can be done for the electric case (see section 2.3). While in the electric case it is straightforward to write the perturbation Hamiltonian due to the introduction of an electric field, it is not so immediate in the thermal case.

The method proposed in Ref. [5] to treat a perturbation created by a thermal gradient is to introduce an additional auxiliary scalar field $\eta(x)$, which couples with the excitations Hamiltonian density $\tilde{H}(x)$, producing a perturbed Hamiltonian $H_T$:

$$H_T = H + \int \eta(x)\tilde{H}(x)dx$$  \hspace{1cm} (3.3.1)

here $H$ represents the unperturbed Hamiltonian.

The perturbation due to $\eta(x)$ adds to that produced by the thermal gradient field, so that the expectation values of the operators are modified by both the fields. It can be proved (see Ref. [5]) that, when no current flows in the system, the Fourier
components of these two quantities must satisfy

\[ \eta_q = \frac{\beta_q}{\beta} = -\frac{T_q}{T} \]

In other words the field \( \eta \) compensates somehow for the thermal gradient. Let us now suppose that we want to calculate the expectation value of a generic current (indicated \( J \)), which is zero at the equilibrium. As previously stated, this expectation value depends on both \( \eta \) and \( \nabla T \):

\[ \langle J \rangle = -K \frac{\nabla T}{T} - \tilde{K} \nabla \eta \]

here \( K \) and \( \tilde{K} \) are generic transport coefficients and the equation holds in linear response.

The previous expression can also be rewritten in Fourier components:

\[ \langle J \rangle_q = -iqK \frac{T_q}{T} - iq\tilde{K} \eta_q \]

When this no current flows in the system it is possible to write

\[ iqK \frac{T_q}{T} + iq\tilde{K} \eta_q = 0 \]

and using the equilibrium condition for \( \eta \) previously discussed, one finds

\[ K = \tilde{K} \]

This result suggests us that a thermal transport coefficients can be determined by simply calculating the linear response to a perturbation \( \eta \) which couples to the Hamiltonian as in equation (3.3.1) and in absence of currents satisfies

\[ \eta(x) = \frac{\beta(x)}{\beta} \]

where \( \beta = 1/k_B T \) and \( \beta(x) \) represents the perturbation to the total inverse temperature, which is given by \( \beta_T(x) = \beta + \beta(x) \).

This conclusion is also compatible with the a less formal and more intuitive reasoning: when calculating statistical averages and partition functions, one uses the product \( \beta(H - \mu N) = \int dx \beta(\mathcal{H}(x) - \mu \rho(x)) \), with \( N \) the total number of particles; when the temperature is locally perturbed one needs to take into account this variation, calculating

\[ \int dx \beta_T(x)(\mathcal{H}(x) - \mu \rho(x)) = \beta(H - \mu N) + \int dx \beta(x) \mathcal{H}(x) \]

so that, when calculating statistical expectation values, a varying temperature is equivalent to a perturbation in the Hamiltonian.

Let us now turn back to the Luttinger’s procedure, considering the perturbed Hamiltonian for the two interacting wires; since we apply a temperature gradient to wire 1, the auxiliary field \( \eta(x) \) couples with \( \mathcal{H}_1(x) \) only.

\[ H_T = H_1 + H_2 + H_{\text{int}} + e^{st} \int \eta(x) \mathcal{H}_1(x) dx \]

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\( e^{st} \) represents an adiabatic switching of the perturbation, which is zero for \( t = -\infty \).

The perturbation in the Hamiltonian will produce a perturbation in the density matrix. It can be proved that, in the linear regime, the perturbed density matrix is given by the unperturbed density matrix plus a correction

\[
\rho_T = \rho + f e^{st}
\]

where \( \tilde{\mathcal{H}}_1(x, -t - i\beta') \) is calculated applying a time evolution to \( \mathcal{H}_1(x) \) in the Heisenberg picture:

\[
\tilde{\mathcal{H}}_1(x, -t - i\beta') = e^{iH(-t-i\beta')} \mathcal{H}_1(x) e^{-iH(-t-i\beta')}
\]

Summarising, the correction to the density matrix is proportional to the time derivative of the perturbation evolved in both real and imaginary time.

In order to get rid of the time derivative, the continuity equation can be used:

\[
\partial_t \tilde{\mathcal{H}}_1(x, -t - i\beta') = -\partial_x J_1(x, -t - i\beta')
\]

and the equation for \( f \) becomes

\[
f = \rho \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \int \eta(x) \partial_x J_1(x, -t - i\beta') dx
\]

(3.3.2)

The temperature gradient in wire 1 can have two types of contributions: one constant throughout the wire, that introduces a net temperature difference between the two wires, and one that has a zero average over the wire, introducing only a temperature gradient on wire 1. Therefore \( T_1(x) = T + \Delta T_1 + \delta T_1(x) \) so that \( \eta(x) = \Delta \eta + \delta \eta(x) \); here \( \Delta T_1 \) and \( \Delta \eta \) are constant throughout the wire, while \( \delta T_1(x) \) and \( \delta \eta(x) \) have zero average over the wire.

The integration in equation (3.3.2) can be performed directly for the first term and by parts for the second one.

\[
\int dx \eta(x) \partial_x J_1(x, -t - i\beta') = \lim_{x\to\pm\infty} (J_1(x, -t - i\beta') - J_1(-x, -t - i\beta')) \Delta \eta
\]

\[
- \int J_{01}(x, -t - i\beta') \nabla \delta \eta(x) dx
\]

\[
\int \eta(x) \partial_x J_1(x, -t - i\beta') dx = -J_\perp (-t - i\beta') \Delta \eta - \int J_1(x, -t - i\beta') \nabla \eta(x) dx
\]

since \( \nabla \delta \eta(x) = \nabla \eta(x) \). An analysis of the Fourier components yields \( \nabla \eta(x) = (\nabla \eta)_q e^{iqx} \) and the integral becomes

\[
\int \eta(x) \partial_x J_1(x, -t - i\beta') dx = -J_\perp (-t - i\beta') \Delta \eta - J_1(-t - i\beta') - q (\nabla \eta)_q
\]

The linear response is considered in the so-called rapid case limit where \( q \) approaches zero faster than \( s \). In this regime, equation (3.1.4) is applied

\[
\int \eta(x) \partial_x J_1(x, -t - i\beta') dx = -J_\perp (-t - i\beta') \Delta \eta - \int dx J_1(x, -t - i\beta') (\nabla \eta)_0 =
\]
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\[
-J_\perp (-t - i\beta') \Delta \eta - L J_{1\parallel} (-t - i\beta') (\nabla \eta)_{\eta=0}
\]

At this point, the current expectation value is calculated as the trace of \(\rho T J\). Since at the equilibrium the current is zero, in the trace just the correction to the density matrix has to be considered; the contribution arising from the \(\eta(x)\) perturbations is

\[
\langle J_{2\parallel}(\eta) \rangle = \text{Tr}(f J_{2\parallel}) = - \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{\perp}(-t - i\beta') J_{2\parallel} \rangle \Delta \eta - \\
- L \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{1\parallel}(-t - i\beta') J_{2\parallel} \rangle \nabla \eta
\]

\[
\langle J_{\perp}(\eta) \rangle = - \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{\perp}(-t - i\beta') J_{\perp} \rangle \Delta \eta - \\
- L \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{1\parallel}(-t - i\beta') J_{\perp} \rangle \nabla \eta
\]

This expression allows us to write the expression for the transport coefficients due to \(\eta\) and, consequently, also those due to \(\nabla T_1\): the response of the system to the perturbation \(\Delta \eta\) is equal to that due to \(\Delta T_1\) and, analogously, the response to \(\nabla \eta\) is the same of that due to \(\nabla T_1\)

\[
\langle J_{2\parallel} \rangle = - \sigma_{2\perp}^{TT} \Delta T_1 - \sigma_{21}^{TT} \nabla T_1
\]

\[
\langle J_{\perp} \rangle = - \sigma_{1\perp}^{TT} \Delta T_1 - \sigma_{11}^{TT} \nabla T_1
\]

The transconductivity (i.e. the proportionality coefficient between the induced current in wire 2 and the temperature gradient in wire 1) and the interconductance (i.e. the proportionality coefficient between the interwire current and \(\Delta T_1\)) have been defined:

\[
\sigma_{21}^{TT} = \frac{L}{T} \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{1\parallel}(-t - i\beta') J_{2\parallel} \rangle
\]

\[
\sigma_{11}^{TT} = \frac{1}{T} \int_0^\infty dt e^{-st} \int_0^\beta d\beta' \langle J_{\perp}(-t - i\beta') J_{\perp} \rangle
\]

Note that the dimensions of interconductance and transconductivity are different since they refer to different fields (respectively a temperature difference and a temperature gradient). We have written also the expression for the interconductance for completeness, but we will not use it, since our interest focuses on the transconductivity; furthermore, we will not consider the coefficients \(\sigma_{22}^{TT}\) and \(\sigma_{12}^{TT}\), to which we are not interested.

Using the procedure given in Ref. [21], equations (3.3.3) and (3.3.4) can be expressed also as

\[
\sigma_{21}^{TT} = \lim_{\omega \to 0} \frac{L}{\omega T} \int_0^\infty dt e^{i\omega t} \langle [J_{2\parallel}(t), J_{1\parallel}] \rangle
\]

\[
\sigma_{11}^{TT} = \lim_{\omega \to 0} \frac{1}{\omega T} \int_0^\infty dt e^{i\omega t} \langle [J_{\perp}(t), J_{\perp}] \rangle
\]

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From equation (3.3.3), $\sigma_{T1}^{T2}$ can be derived with the exchange $1 \leftrightarrow 2$, while the expressions for $\sigma_{11}^{TT}$ and $\sigma_{22}^{TT}$ are well known in literature (see Ref. [5]); it is therefore straightforward to reconstruct the conductivity matrix of equation (3.2).

### 3.3.2. Application of the memory function formalism

We now want to apply the memory function formalism (outlined in Refs. [11] and [12]) to equation (3.3.3) in order to obtain the transresistivity. This method permits to invert the conductivity matrix and calculate the resistivity matrix; we will first discuss the framework in which the formalism has to be applied and then we will anticipate the final result, giving the proof successively.

First of all, the invariance of the expectation value for temporal translations is used, writing equation (3.3.3) in the form

$$\sigma_{21}^{T2} = \frac{L\beta^2}{k_B} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int_0^\beta d\beta' \langle J_{11} | (-t) J_{22}^{(i\beta')} \rangle$$

where $\beta = 1/(k_B T)$ has also been used. Let us define the following function

$$C_{ij}(t) \equiv \frac{1}{\beta} \int_0^\beta d\beta' \langle J_{i}^{(i\beta')} | (-t) J_{j}^{(i\beta')} \rangle$$

(3.3.6)

here the indices $i$ and $j$ refer to the two wires. $C_{ij}(t)$ can be regarded as a particular scalar product between operators, which is defined as:

$$\langle A | B \rangle \equiv \frac{1}{\beta} \int_0^\beta d\beta' \langle A^\dagger e^{-\beta' H} B e^{\beta' H} \rangle$$

(3.3.7)

Due to the hermitianity of the current operator the following relation holds $C_{ij}(t) = \langle J_{i}^{(i\beta')} | (-t) J_{j}\rangle$; since $C_{ij}(t = 0)$ is a correlation function between two currents at the same time, it is not affected by the interwire coupling and is therefore zero when $i \neq j$, so that $C_{ij}(t = 0) = \delta_{ij} C_{ii}(t = 0)$.

This scalar product acts on the Hilbert space whose vectors are the observables of the system considered; it can be proven that this Hilbert space is complete and that the scalar product is well defined, linear and definite positive.

Using the memory function formalism, equation (3.3.5) can be transformed into the following equation

$$\rho_{T1}^{T2} = -\frac{L\beta^2}{k_B} \chi_{i1}^{-1} \chi_{22}^{-1} \lim_{s \to 0} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int_0^\beta d\beta' \langle J_{22}^{(i\beta')} | (-t) J_{11}^{(i\beta')} \rangle$$

(3.3.8)

with

$$\chi_{ii} = \frac{L\beta^2}{k_B} C_{ii}(t = 0)$$

(3.3.9)

The proof of this statement follows below; during this demonstration we will omit the "||" label from the currents for book keeping reasons. The reader not interested can go directly to section 3.3.3.

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A set of “super-operators” acting on the vectors of the previous Hilbert space can be defined; in general they act on an operator $A$ yielding another operator $B$. This is the case of the following super-projector:

$$P \equiv \sum_k \frac{1}{|J_k|^2} |J_k\rangle\langle J_k| = 1 - Q \quad (3.3.10)$$

where $k = 1, 2$. It projects onto the Hilbert subspace spanned by the currents $J_1$ and $J_2$; in fact, $P|J_i\rangle = |J_i\rangle$. Note that $P|J_i\rangle = 0$, since for time reversal invariance we have $\langle J_j|J_i\rangle = 0$. Furthermore $\langle J_1|J_2\rangle = 0$, so that $\{J_1, J_2\}$ is an orthogonal base; all these considerations suggest us that the two currents form a bidimensional Hilbert subspace and that their time derivatives are orthogonal to this subspace.

We also define the Liouville super-operator $\mathcal{L}$ whose action is $\mathcal{L}|A\rangle = \{[H, A]\}$; thus the temporal evolved of an operator is given by $|A(t)\rangle = e^{t\mathcal{L}}|A\rangle$; from the definition it follows $|\dot{A}\rangle = i\mathcal{L}|A\rangle$.

Operating the substitution $s \rightarrow -iz$, equation (3.3.5) becomes

$$\sigma_{21}^{TT} = \lim_{z \rightarrow 0} \frac{L\beta^2}{k_B} \int_0^\infty dt e^{izt} \frac{1}{\beta} \int_0^\beta \frac{d\beta'}{\beta'} \langle J_1\|(-t)J_2\|i\beta'\rangle = \lim_{z \rightarrow 0} \frac{L\beta^2}{k_B} C_{12}(z)$$

$$\rho_{12}^{TT} = -[(\sigma^{TT})^{-1}]_{12} = -\lim_{z \rightarrow 0} \frac{k_B}{L\beta^2} [C^{-1}(z)]_{21} \quad (3.3.11)$$

where $C_{ij}(z)$ is the Laplace transform of $C_{ij}(t)$ (see Ref. [12])

$$C_{ij}(z) = \int_0^\infty dt e^{izt} C_{ij}(t) = \langle J_i\| \frac{i}{z + \mathcal{L}} |J_j\rangle$$

Here $\frac{i}{z + \mathcal{L}}$ is a super-operator defined as the inverse of $-i(z\mathbb{1} + \mathcal{L})$ and $[C^{-1}(z)]_{ij}$ is the $ij$ element of the the matrix $[C^{-1}(z)]$. Summarizing, the elements of the conductivity matrix are related to the elements of the matrix $C(z)$, which can be inverted, allowing the calculation of the resistivity matrix and thus of $\rho_{12}^{TT}$.

A series of formal manipulation is now performed to obtain the inverse of $C(z)$: breaking the superoperator $\mathcal{L}$ as $\mathcal{L}P + \mathcal{L}Q$, we can write the previous fraction as the sum of two terms. The first one will be proportional to $C_{ij}(t = 0)$, while we will try to manipulate the second one and write it in terms of known quantities.

Starting the calculations, one gets:

$$\frac{i}{z + \mathcal{L}} = \frac{i}{z + \mathcal{L}P + \mathcal{L}Q} = \frac{i}{z + \mathcal{L}Q} \frac{1}{z + \mathcal{L}P} \frac{i}{z + \mathcal{L}}$$

$$C_{ij}(z) = \langle J_i\| \frac{i}{z + \mathcal{L}Q} |J_j\rangle - \langle J_i\| \frac{1}{z + \mathcal{L}Q} \mathcal{L}P \frac{i}{z + \mathcal{L}} |J_j\rangle \quad (3.3.12)$$

The first term in equation (3.3.12) is simply

$$\langle J_i\| \frac{i}{z} |J_j\rangle = \frac{i}{z} C_{ij}(t = 0)$$

because $\frac{1}{z + \mathcal{L}Q}$ can be written as a formal series of powers of $\mathcal{L}Q$; on the other hand, except for the zeroth order, all the terms contain $Q$, which applied to $J_i$ is zero for definition.
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Let us proceed by analysing the second term in equation (3.3.12):

\[ \mathcal{P} \frac{i}{\varepsilon + \mathcal{L}} |J_j\rangle = \sum_k \frac{|J_k\rangle}{\langle J_k|J_k\rangle} \frac{i}{\varepsilon + \mathcal{L}} |J_j\rangle = \sum_k C_{kk}^{-1}(t = 0) |J_k\rangle C_{kj}(z) \]

Here \( C_{kk}^{-1}(t = 0) \) is the inverse of the element of matrix \( C_{kk}(t = 0) \) and not the \( kk \) matrix element of \( C^{-1}(t = 0) \). The last formula implies

\[ C_{ij}(z) = \frac{i}{\varepsilon} C_{ij}(t = 0) - \sum_k C_{kk}^{-1}(t = 0) \langle J_i| \frac{1}{\varepsilon + \mathcal{Q}} \mathcal{L} |J_k\rangle C_{kj}(z) \]  

(3.3.13)

Finally, only the matrix element in the second sum of equation (3.3.13) has to be calculated

\[ \langle J_i| \frac{1}{\varepsilon + \mathcal{Q}} \mathcal{L} |J_k\rangle = \frac{1}{\varepsilon} \langle J_i| \mathcal{L} |J_k\rangle = \frac{1}{\varepsilon} \langle J_i| \mathcal{L} \frac{1}{\varepsilon + \mathcal{Q}} \mathcal{L} |J_k\rangle \]

\[ \langle J_i| \frac{1}{\varepsilon + \mathcal{Q}} \mathcal{L} |J_k\rangle = -\frac{i}{\varepsilon} \langle J_i| \dot{J}_k \rangle + \frac{i}{\varepsilon} \Omega_{ik}(z) = \frac{i}{\varepsilon} \Omega_{ik}(z) \]

\( \Omega_{ik}(z) \) can be written (see Ref. [12]):

\[ \Omega_{ik}(z) = \langle J_i| \mathcal{Q} \frac{i}{\varepsilon + \mathcal{Q}} \mathcal{L} |J_k\rangle = \langle J_i| \frac{i}{\varepsilon + \mathcal{Q}} \dot{J}_k \rangle \]  

(3.3.14)

where \( \mathcal{Q}|\dot{J}_k\rangle = |J_k\rangle \) has been used.

Summarizing all the equations written, we get

\[ C_{ij}(z) = \frac{i}{\varepsilon} C_{ij}(t = 0) - \frac{i}{\varepsilon} \sum_k C_{kk}^{-1}(t = 0) \Omega_{ik}(z) C_{kj}(z) \]

\[ \sum_k (z \delta_{ik} + i C_{kk}^{-1}(t = 0) \Omega_{ik}(z)) C_{kj}(z) = i C_{ij}(t = 0) \]

One can recognize that the left hand side of the previous equation is the element \( ij \) of a matrix product, equal to the element \( ij \) of a third matrix; therefore, this matricial expression can be inverted, deriving the expression desired

\[ [C^{-1}(z)]_{ij} = -i \sum_k C_{ik}^{-1}(t = 0) (z \delta_{kj} + i C_{jj}^{-1}(t = 0) \Omega_{kj}(z)) \]

Since \( C_{ij}(t = 0) \) is proportional to \( \delta_{ij} \), the previous expression can be simplified

\[ [C^{-1}(z)]_{ij} = -i C_{ii}^{-1}(t = 0) (z \delta_{ij} + i C_{jj}^{-1}(t = 0) \Omega_{ij}(z)) \]

At this point, we set \( i = 2 \) and \( j = 1 \) to calculate \( [C^{-1}(z)]_{21} \):

\[ [C^{-1}(z)]_{21} = C_{11}^{-1}(t = 0) C_{22}^{-1}(t = 0) \Omega_{21}(z) \]

from equations (3.3.9) and (3.3.14), it is found that

\[ \rho_{12}^{TT} = \lim_{z \to 0} \frac{k_B}{L_0^2} C_{11}^{-1} C_{22}^{-1} \Omega_{21}(z) \]
and finally equation (3.3.8) is obtained

$$\rho_{12}^{TT} = \frac{-L\beta^2}{k_B} \chi_1^{(1)} \chi_2^{(1)} \lim_{s \to 0} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int_0^\beta d\beta' \langle \dot{J}_{2\parallel}(t) \dot{J}_{1\parallel}(i\beta') \rangle$$

This concludes the proof of equation (3.3.8). A formal expression for the transresistivity has been derived in terms of a correlation between two time derivatives of currents. Again we can ask ourselves: what is the advantage of doing so? The answer is the same of section 2.3: the important gain is that the time derivative of the current can be easily related to the coupling, expliciting therefore the leading order in the interaction.

### 3.3.3. Final formula for the transresistivity

In this paragraph we will perform a series of manipulations and approximation in order to obtain a more manageable expression for the transresistivity. In the end we will find

$$\rho_{12}^{TT} = \frac{9}{2\pi^4} \frac{v_1 v_2}{k_B T^4} \int_0^\infty d\omega \int_0^\infty dk |U_{12}(k)|^2 k^2 \frac{A_1(k,\omega) A_2(k,\omega)}{\sinh^2(\omega/2k_B T)}$$

(3.3.15)

as anticipated also in section 3.2; all the derivation follows.

An expression for the current’s time derivative in function of the coupling is initially written, employing, successively, Fourier transform to simplify our formulas and make them more compact.

First of all, we want to find an expression for the time derivative of the current. To begin with, $\dot{J}_i = i[H, J_i]$ is employed, where $H = H_1 + H_2 + H_{int}$; $J_i$ commutes with $H_2$ by definition; an analogous argument applies to $J_2$. Let us show that

$$[H_i, J_i] = 0$$

where $i = 1, 2$ refers to the two wires.

Using equation (3.1.8) in the limit of a homogeneous interaction ($V(q) = V(0)\delta_{q,0}$) one obtains:

$$J_i = \frac{1}{L} \sum_k \frac{k^2}{m} \left( \frac{k^2}{2m} - \mu \right) a_{i,k} a_{i,k} + \frac{1}{L^2} \sum_{k,k'} \frac{k + k'}{2m} V_i(0) a_{i,k} a_{i,k'} a_{i,k} a_{i,k}$$

The formula $[a_{i,k} a_{k_2}, a_{k_3} \delta_{k_2,k_3}] = a_{i,k} a_{k_4} \delta_{k_2,k_3} - a_{i,k} a_{k_2} \delta_{k_1,k_4}$ can be used to see that

$$\sum_{k,k'} f_k g_{k'} [a_{i,k} a_{i,k'}, a_{i,k'} a_{i,k}] = 0$$

On the other hand, $[H_i, J_i]$ is made of sums like the previous one and thus

$$[H_i, J_i] = 0$$

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This result can also be retrieved in the Luttinger liquid regime, using all the bosonization tools.

It has been proven that the only non trivial commutator is that between the current and the interaction Hamiltonian; therefore, the current derivative is linear in the coupling:

\[
\dot{J}_1(i\beta') = i \int dx_1 dx_2 U_{12}(x_1 - x_2) [\rho_1(x_1, i\beta'), J_{1\parallel}(i\beta')] \rho_2(x_2, i\beta')
\]

\[
\dot{J}_2(-t) = i \int dy_1 dy_2 U_{12}(y_1 - y_2) \rho_1(y_1, -t) [\rho_2(y_1, -t), J_{2\parallel}(-t)]
\]

From now on the limit \(s \to 0\) is implied, although it is omitted in the formulas for book-keeping reasons. Let us substitute the two equations just written into equation (3.3.8) and obtain

\[
\rho_{12}^{TT} = \frac{L \beta^2}{k_B} \chi_{11}^{-1} \chi_{22}^{-1} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int d\beta' \int dx_1 dx_2 dy_1 dy_2 U_{12}(x_1 - x_2) U_{12}(y_1 - y_2) \\
\cdot \{ \langle \rho_1(x_1, i\beta'), J_{1\parallel}(i\beta') | \rho_1(y_1, -t) \rangle [\rho_2(x_2, i\beta'), \rho_2(y_2, -t), J_{2\parallel}(-t)] \}
\]

The correlation functions referred to different wires have been factorized; they can be written in the following form, by exploiting the temporal invariance of the expectation value and employing a Fourier transform:

\[
\langle \rho_1(y_1, -t) | \rho_1(x_1, i\beta'), J_{1\parallel}(i\beta') \rangle = \langle \rho_1(y_1 - x_1, -t - i\beta') | \rho_1(0, 0), J_{1\parallel}(0) \rangle = \\
= \frac{1}{L} \int \frac{dk d\omega}{(2\pi)^2} e^{-ik(x_1 - y_1) + i\omega t} e^{-\beta' \omega} \Sigma_i(k, \omega)
\]

\[
\langle \rho_2(y_2, -t) | \rho_2(x_2, i\beta') \rangle = \langle \rho_2(y_2 - x_2, -t - i\beta') | \rho_2(0, 0) \rangle = \\
= \frac{1}{L} \int \frac{dk d\omega}{(2\pi)^2} e^{-ik(x_2 - y_2) + i\omega t} e^{-\beta' \omega} \Sigma'_2(k, \omega)
\]

Two density-density-current correlation functions (called DDC functions) have been defined:

\[
\Sigma_i(k, \omega) = L \int dx dt e^{-ikx + i\omega t} \langle \rho_i(x, t) | \rho_i(0, 0), J_{1\parallel}(0) \rangle
\]

\[
\Sigma'_i(k, \omega) = L \int dx dt e^{-ikx + i\omega t} \langle [\rho_i(x, t), J_{1\parallel}(t)] \rho_i(0, 0) \rangle
\]

Substituting, the transresistivity reads

\[
\rho_{12}^{TT} = \frac{L \beta^2}{k_B} \chi_{11}^{-1} \chi_{22}^{-1} \int_0^\infty dt e^{-st} \frac{1}{\beta} \int d\beta' \int dx_1 dx_2 dy_1 dy_2 U_{12}(x_1 - x_2) U_{12}(y_1 - y_2) \\
\cdot \int \frac{dk_1 dk_2 d\omega_1 d\omega_2}{(2\pi)^4} \Sigma_i(k_1, \omega_1) \Sigma'_2(k_2, \omega_2) e^{ik_1(y_1 - x_1)} e^{ik_2(y_2 - x_2)} e^{i(\omega_1 + \omega_2)} e^{-\beta' \omega_1 + \omega_2}
\]

After having written the coupling in Fourier transform, several integrals yield a delta function as result, simplifying the expressions.

---

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The integral in $t$ yields a complex result, but we are interested in the real part only, which is
$$\lim_{s \to 0} \text{Re} \left( \int_0^\infty dt e^{-st} e^{i(\omega_1 + \omega_2)t} \right) = \pi \delta(\omega_1 + \omega_2)$$

Because of the delta function, the exponential containing the frequencies is constant in $\beta'$, so that the integrand $\frac{1}{\beta} \int_0^\beta d\beta'$ reduces to 1, giving

$$\rho_{12}^{TT} = -\frac{\chi_1^{-1} \chi_2^{-1}}{k_B T^2} \int \frac{d\omega}{4\pi} \int dx_1 dx_2 dy_1 dy_2 U_{12}(x_1 - x_2) U_{12}(y_1 - y_2) \cdot \int \frac{dk_1 dk_2}{(2\pi)^2} e^{-ik_1(x_1 - y_1)} e^{-ik_2(x_2 - y_2)} \Sigma_1(k_1, \omega) \Sigma'_2(k_2, -\omega)$$

here we have integrated in $\omega_2$ and replaced $\omega_1$ with $\omega$ for notation simplicity.

It can be useful to write the coupling in Fourier transform and make a massive integration in $x_1, x_2, y_1, y_2$ (the last integral yields a factor $L$), getting:

$$\rho_{12}^{TT} = -\frac{\chi_1^{-1} \chi_2^{-1}}{k_B T^2} \int \frac{d\omega}{4\pi} \int \frac{dk}{2\pi} U_{12}(k) U_{12}(-k) \Sigma_1(k, \omega) \Sigma'_2(-k, -\omega) \quad (3.3.17)$$

This is a quite remarkable result: we have factorized away the dependence on the coupling and reduced our problem to the calculation of two DDC correlation functions, $\Sigma$ and $\Sigma'$, evaluated for decoupled wires. Interestingly, they are not evaluated for independent values of $k$ and $\omega$, but in $(k, \omega)$ for the first wire and in $(-k, -\omega)$ for the second wire; this symmetry can be thought as the implementation of the conservation of both momentum and energy during an interwire scattering process.

In addition, the dependence on the coupling has been factorized, finding a $U^2$ proportionality; one can also notice that, since the coupling potential is real, $U(-k) = U^*(k)$.

The next step is the calculation of $[\rho(x), J_\parallel]$. This step is rather intricated and long, so we will simply write the result and postpone the derivation to Appendix B.

$$mL[\rho(x, t), J_\parallel(t)] = \psi^\dagger(x, t) \partial_x (\partial_t - i\mu) \psi(x, t) - (\partial_x (\partial_t - i\mu) \psi^\dagger(x, t)) \psi(x, t) \quad (3.3.18)$$

Equation (3.3.18) holds in the limit of a homogeneous intrawire interaction.

The calculation of $\langle \rho_i(x_2) | \rho_i(x_1, t), J_\parallel(t) \rangle$ has been reduced to the calculation of some four-point correlators:

$$\langle \rho(0, 0) | \rho(x, t), J_\parallel(t) \rangle = \frac{1}{mL} \langle \psi^\dagger(0, 0) \psi(0, 0) \psi^\dagger(x, t) \partial_x (\partial_t - i\mu) \psi(x, t) \rangle - \frac{1}{mL} \langle \psi^\dagger(0, 0) \psi(0, 0) (\partial_x (\partial_t + i\mu) \psi^\dagger(x, t)) \psi(x, t) \rangle \quad (3.3.19)$$

Equations (3.3.19), (3.3.17) and (3.3.16) allow to calculate the transresistivity for any value of temperature; at a more practical level, the interaction is still very hard to treat and a way to bypass this problem is to study the non interacting electrons limit (see Appendix E).
Interestingly, it makes sense to explore the Luttinger liquid limit, since in this approximation the calculations can be done far more easily, as we will soon verify.

From now on our problem will be treated under the approximation $T \ll T_F$; a pure quadratic Luttinger Hamiltonian (equation (1.3.10) will be used, ignoring, therefore, any correcting term that could cause backscattering but retaining the $2k_F$ oscillating term in the density. Furthermore, $V_2$ and $V_4$ processes (see section 1.2) that have the same amplitude will be considered, so that $v_F = v_g$, thanks to equation (1.11).

We do not show all the manipulations of equation (3.3.19), but write directly the result, referring to Appendix B for all the intermediate steps.

\begin{equation}
\langle \rho(x,t) | \rho(y,0), J_{\parallel}(0) \rangle = \frac{i v^2}{L} \partial_x \langle \rho(x-y, t) \rho(0,0) \rangle
\end{equation}

\begin{equation}
\langle [\rho(x,t), J_{\parallel}(t)] | \rho(0,0) \rangle = -\frac{i v^2}{L} \partial_x \langle \rho(x, t) \rho(0,0) \rangle
\end{equation}

If this very same method had been employed in the Coulomb drag, one would have found a result equal to the one written just above (apart from a different prefactor, of course), but true for any temperature regime.

This is strictly related to the form of the current: the electric one (see equations (3.1.5) and (3.1.7)) contains only one spatial derivative of the field, which is connected to the derivative of the density in the correlator, while the thermal current (equations (3.1.6) and (3.1.8)) contains both time and space derivatives; this can be seen also in the decomposition in operator, where the electric current is linear in the momentum, while the thermal one contains also cubic terms. On the contrary, as one can see from equation (A.1.8), the thermal current in the Luttinger liquid regime is linear in the momentum, justifying the only spatial derivative of equations (3.3.20) and (3.3.21).

At this point, only density-density correlation functions can be employed and the form factor $S_i(k, \omega)$ is introduced

\begin{equation}
S_i(k, \omega) = \int dx dt e^{-ikx+i\omega t} \langle \rho_i(x, t) \rho_i(0,0) \rangle
\end{equation}

\begin{equation}
\langle \rho_i(x, t) \rho_i(0,0) \rangle = \int \frac{dk d\omega}{(2\pi)^2} e^{ikx-i\omega t} S_i(k, \omega)
\end{equation}

The form factor can be easily related to the DDC functions $\Sigma$ and $\Sigma'$, obtaining

\begin{equation}
\Sigma(k, \omega) = ikv^2 S(k, \omega) \quad \quad \Sigma'(k, \omega) = -ikv^2 S(k, \omega)
\end{equation}

Equations (3.3.17) and (3.3.23) can be used to derive

\begin{equation}
\rho_{12}^{TT} = \frac{v_1^2 v_2^2}{k_B T^2} \chi_{11}^{-1} \chi_{22}^{-1} \int \frac{d\omega}{4\pi} \int \frac{dk}{2\pi} |U_{12}(k)|^2 k^2 S_1(k, \omega) S_2(-k, -\omega)
\end{equation}

The calculation of the transresistivity has been reduced to an integral involving the coupling potential and two single wire density-density correlation functions;
the effect of the interwire coupling has been separated from the effect of intrawire interaction. Following Refs. [13] and [11], the spectral function is defined as:

$$2A_i(k, \omega) = \int dxdte^{-ikx+i\omega t}\langle [\rho_i(x, t), \rho_i(0, 0)] \rangle$$ (3.3.25)

Thanks to the fluctuation-dissipation theorem, $A_i$ can be related to $S_i$:

$$S_i(k, \omega) = \frac{2A_i(k, \omega)}{1 - e^{-\omega/T}}$$ (3.3.26)

It can be easily verified that $A_i(-k, -\omega) = -A_i(k, \omega)$; in fact, it is sufficient to change variable in the integral, restoring the signs of $k$ and $\omega$ and changing those of $x$ and $t$, using the temporal invariance to go back to $x$ and $t$, and, then, change the order of the commutator, getting a minus sign. Employing all these equations and considerations, it is easy to obtain

$$S_1(k, \omega)S_2(-k, -\omega) = -4\frac{A_1(k, \omega)A_2(k, \omega)}{(1 - e^{-\omega/T})(1 - e^{\omega/T})} = \frac{A_1(k, \omega)A_2(k, \omega)}{\sinh^2(\omega/2T)}$$

So far we focused on the manipulation of the integral inside the transresistivity; let us now substitute the values of $\chi_{11}$ and $\chi_{22}$: Their calculation requires some attention and we will perform it in Appendix C; the final result reads:

$$\chi_{ii} = \frac{\pi}{3}v_1k^2B_T$$ (3.3.27)

Equations (3.3.24), (3.3.25) and (3.3.27) can be finally employed to obtain

$$\rho_{12}^{TT} = \frac{9}{8\pi^4}\frac{v_1v_2}{k_B^4T^4} \int d\omega \int dk|U_{12}(k)|^2k^2\frac{A_1(k, \omega)A_2(k, \omega)}{\sinh^2(\omega/2T)}$$

from this expression it is immediate to verify that the transresistivity is symmetric, i.e. $\rho_{21}^{TT} = \rho_{12}^{TT}$. Since the integrand is even in both $k$ and $\omega$, equation (3.3.15) can be finally retrieved:

$$\rho_{12}^{TT} = \frac{9}{2\pi^4}\frac{v_1v_2}{k_B^4T^4} \int_0^{\infty} d\omega \int_0^{\infty} dk|U_{12}(k)|^2k^2\frac{A_1(k, \omega)A_2(k, \omega)}{\sinh^2(\omega/2k_BT)}$$ (3.3.28)

Let us discuss again the physical meaning of the expression we derived above. We found that the transresistivity is quadratic in $U_{12}$ in the limit of weak coupling; it is also proportional to an integral of density-density correlation functions evaluated for decoupled wires at the equilibrium and weighted by an hyperbolic sine, which reflect the bosonic character of the excitations carrying the thermal current. The dependence on the temperature is not immediate to see, since it is hidden inside the integral in both the hyperbolic sine and the product of the two spectral functions: the purpose of the next sections is to analyse this dependence.

Moreover, as we will discuss in section 3.5, the dominant contribution to the transresistivity is positive, implying that $J_{2\parallel}$ and $\nabla T_1$ have the same verse. This is not surprising if one analyses qualitatively the situation of the system: the moving
3. Thermal drag

carriers in wire 2 tend to drag along the direction of their motion the carriers in wire 1; to keep them at rest a thermal gradient whose direction is the same of \( J_{2崖} \) must be applied to wire 1.

It can be noticed that equation (3.3.28) is identical to the formula for the electric transresistivity (Refs. [13] and [11] and equation (2.1.3)), except for the factor in front of the integral. This aspect can be thought as a consequence of the linear spectrum approximation made in the limit of low temperatures: in this limit the thermal current is linear in \( k \), exactly as the electric current (equation (3.1.7)).

Remarkably, a Wiedemann-Franz law has therefore been derived also for the drag between two coupled wires; in fact the electrical transresistivity \( \rho_{EE}^{12} \) is:

\[
\rho_{EE}^{12} = \frac{1}{2\pi^2 e^2 n_1 n_2 T} \int_0^\infty dk \int_0^\infty d\omega k^2 U_{12}^2(k) \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/2T)}
\]  (3.3.29)

and the two quantities in equations (3.3.28) and (3.3.29) are directly proportional, with a ratio given by

\[
\frac{\rho_{EE}^{12}}{\rho_{TT}^{12}} = \frac{\pi^2}{9} \frac{k_5 T^3}{e^2 n_1 n_2 v_2 v_2 \alpha}
\]  (3.3.30)

This ratio is proportional to \( T^3 \), in contrast to the Wiedemann-Franz law for conductivity (which exhibits a ratio proportional to \( T \)).

However, this Wiedemann-Franz law analogue holds only for low temperatures; it cannot said anything about the high energies regime, since it was not made an analysis of the four point correlator of equation (3.3.19).

Anyway, in the next sections the behaviour of transresistivity in the low temperatures regime will be studied.

3.4. Spectral function calculation

In the present section we proceed by deriving an analytic expression for the spectral function \( A(k,\omega) \) (the subscript has been dropped for book-keeping simplicity) in order to calculate the integral

\[
I = \int_0^\infty d\omega \int_0^\infty dk |U_{12}(k)|^2 k^2 A_1(k,\omega)A_2(k,\omega) \sinh^2(\omega/2k_B T)
\]  (3.4.1)

For the calculation of \( I \), we will separate the two principal contribution to \( A(k,\omega) \), deriving an expression for the forward scattering contribution and for the back-scattering contribution (associated to the fluctuations of wavevector \( 2k_F \) in the density and not to a backscattering term in the Hamiltonian), in order to determine if one of them is dominant and which. With these premises, the expression for the density in the bosonization framework is used (see equation (A.1.5)):

\[
\rho(x,t) = \frac{1}{\sqrt{\pi}} \partial_x \phi(x,t) + \frac{1}{\pi \alpha} \cos(2k_F x - 2\sqrt{\pi} \phi(x,t))
\]
For parity reasons (see the discussion before equation (B.2.3) in Appendix B.2 for more details), this becomes

\[
\langle [\rho(x, t), \rho(0, 0)] \rangle = \frac{1}{\pi} \langle [\partial_x \phi(x, t), \partial_{x'} \phi(x', 0)] \rangle |_{x'=0} + 
\]

\[
+ \frac{1}{\pi^2 \alpha^2} \langle [\cos[2k_Fx - 2\sqrt{\pi}\phi(x, t)], \cos[2\sqrt{\pi}\phi(0, 0)]] \rangle
\]

and because \( A(x, t) = \frac{1}{2} \langle [\rho(x, t), \rho(0, 0)] \rangle \) we obtain

\[
A(x, t) = A_{fs}(x, t) + A_{bs}(x, t)
\]

\[
A_{fs}(x, t) = \frac{1}{2\pi} \langle [\partial_x \phi(x, t), \partial_{x'} \phi(x', 0)] \rangle |_{x'=0}
\]

\[
A_{bs}(x, t) = \frac{1}{2\pi^2 \alpha^2} \langle [\cos[2k_Fx - 2\sqrt{\pi}\phi(x, t)], \cos[2\sqrt{\pi}\phi(0, 0)]] \rangle
\]

The first term describes the forward scattering contribution and can be calculated directly in an easy way; the second one describes a backward scattering term. We will treat them separately, since \( A_{bs} \) has a quite complicated expression, while \( A_{fs} \) will be found to be proportional to a delta function in \( k \) and \( \omega \).

### 3.4.1. Forward scattering term

Let us now demonstrate the last assertion made. \( A_{fs} \) is equivalent to

\[
\langle [\partial_x \phi(x, t), \partial_{x'} \phi(x', 0)] \rangle |_{x'=0} = \partial_x \partial_{x'} \langle [\phi(x, t), \phi(x', 0)] \rangle |_{x'=0}
\]

\[
= \partial_x \partial_{x'} \langle [\phi(x - x', t), \phi(0, 0)] \rangle |_{x'=0} = - \langle \partial_x^2 \phi(x, t), \phi(0, 0) \rangle
\]

We just need to calculate the commutator at different times. This can be done by evolving the operator \( \phi \) and decomposing it into creation and annihilation operators (see equation (A.1.3)):

\[
[\phi(x, t), \phi(0, 0)] = \sum_{k,k'} g \text{sign}(kk') e^{-\frac{|k + k'|}{2}} \left( e^{-ikx+iv|k|t} [b_k, b_{k'}] + e^{-ikx-iv|k|t} [b_k, b_{k'}] \right)
\]

\[
= \sum_k e^{-\alpha|k|} \frac{g}{2L} \frac{1}{|k|} \left( e^{ikx-iv|k|t} - e^{-ikx+iv|k|t} \right)
\]

Deriving with respect to \( x \) two times, one finds

\[
-\partial_x^2 \langle [\phi(x, t), \phi(0, 0)] \rangle = \sum_k e^{-\alpha|k|} \frac{g}{2L} |k| \left( e^{ikx-iv|k|t} - e^{-ikx+iv|k|t} \right)
\]

\[
A_{fs}(x, t) = \frac{g}{2} \frac{1}{(2\pi)^2} \int dk e^{-\alpha|k|} |k| \left( e^{ikx-iv|k|t} - e^{-ikx+iv|k|t} \right)
\]

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In Fourier transform, it can be obtained

\[
A_{fs}(k,\omega) = \frac{g}{2} \int \frac{dxdt}{(2\pi)^2} e^{-\alpha |k'|} |k'| e^{i\omega t - ik'x} [e^{ik'z - iv|k'|t} - e^{-ik'x + iv|k'|t}] =
\]

\[
= \frac{g}{2} \int dk' e^{-\alpha |k'|} |k'| [\delta(k - k')\delta(\omega - v|k'|) - \delta(k + k')\delta(\omega + v|k'|)] =
\]

\[
= \frac{gk}{2} e^{-\alpha |k|}[\delta(\omega - vk) - \delta(\omega + vk)]
\]

where the parity of the function for \(k \to -k\) has been used.

The limit \(\alpha \to 0\) can be safely taken; furthermore, \(A_{fs}(k,\omega)\) is integrated only for positive wavevectors and frequencies and since \(\omega + vk\) is always different from zero, it can be written

\[
A_{fs}(k,\omega) = \frac{gk}{2} \delta(\omega - vk)
\]

retrieving the result previously anticipated.

### 3.4.2. Backward scattering term

The calculation of the backward scattering is rather long and complicated and requires a series of technical tools, whose definition and use is postponed to Appendix D, writing here only the important results.

Two integral functions of real variables have to be introduced

\[
B_c(x, y) = \int_0^\infty dz \cos(zx) (\sinh z)^{-y}
\]

\[
B_s(x, y) = \int_0^\infty dz \sin(zx) (\sinh z)^{-y}
\]

The two special functions are plotted in Figure 3.2 and Figure 3.3. One verifies at once that they have opposite parity:

\[
B_s(-x, y) = -B_s(x, y) \quad \text{and} \quad B_c(-x, y) = B_c(x, y).
\]

The back-scattering term is

\[
A_{bs}(k,\omega) = \frac{\sin(\pi g)}{4\pi^2 v} \left( \frac{\pi \alpha}{\beta v} \right)^{2g-2} \cdot
\]

\[
\cdot \left[ B_c \left( \frac{\beta(\omega - v(k - 2k_F))}{2\pi}, g \right) B_s \left( \frac{\beta(\omega + v(k - 2k_F))}{2\pi}, g \right) +
\right.
\]

\[
+ B_s \left( \frac{\beta(\omega - v(k - 2k_F))}{2\pi}, g \right) B_c \left( \frac{\beta(\omega + v(k - 2k_F))}{2\pi}, g \right) +
\]

\[
+ B_c \left( \frac{\beta(\omega - v(k + 2k_F))}{2\pi}, g \right) B_s \left( \frac{\beta(\omega + v(k + 2k_F))}{2\pi}, g \right) +
\]

\[
+ B_s \left( \frac{\beta(\omega - v(k + 2k_F))}{2\pi}, g \right) B_c \left( \frac{\beta(\omega + v(k + 2k_F))}{2\pi}, g \right) \]

\[
\]
3.4. Spectral function calculation

Figure 3.2: Plot of $B_s(x)$ for various values of $y$.

Figure 3.3: Plot of $B_c(x)$ for various values of $y$.

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From equation (3.4.5) is manifest the connection between backward scattering and $2k_F$ momentum transfer processes.

### 3.5. Forward and backward scattering comparison

In this section we will compare the contribution of forward scattering and backward scattering to the transresistivity, trying to determine if there is a dominant term; during the analysis of the forward scattering we will refer in many occasions to Ref. [13], which presents some similarities with our derivation.

This section is split into three subsections in which are calculated the pure forward scattering contribution to the integral in equation (3.4.1) ($I_{ff}$), the mixed forward-backward scattering contribution ($I_{fb}$ and $I_{bf}$) and finally the pure backward scattering contribution ($I_{bb}$).

Before getting started, it is appropriate to specify the range of parameters in which we study the problem:

- the two wires have the same Fermi momentum $k_F$.
- the two wires have very similar Luttinger velocities $v_1$ and $v_2$: $\delta v = |v_2 - v_1| \ll v = \frac{v_1 + v_2}{2}$. Without loss of generality (since the integral $I$ in equation (3.4.1) is symmetric under the exchange $1 \leftrightarrow 2$) it can be assumed $v_2 > v_1$, so that $\delta v = v_2 - v_1$.
- the two wires have also similar Luttinger parameters $g_1$ and $g_2$: $\delta g = |g_2 - g_1| \ll g = \frac{g_1 + g_2}{2}$.

In the following the transresistivity will be calculated for several temperature regimes; in any case is implied that $T \ll T_F$, since we are operating in the range of validity of Luttinger liquid theory. The other relevant scales of temperature are:

- $T_0 \equiv v_F k_0 / k_B$ is associated to the typical scale $k_0$ over which the coupling $U_{12}(k)$ decays; $k_0$ is very small for long range interactions, while approaches infinity for point-like interactions. The coupling is a function of $k/k_0$ only, so that it is possible to write $U_{12}(k) = U_{12}(k/k_0)$.
- $T_1 \equiv k_F \delta v / k_B$ is associated to the difference of the Luttinger velocities between the two wires.
- $T_\alpha \equiv v_F / (\alpha k_B)$ is associated to the cut-off length $\alpha$ of the bosonization: $1/\alpha$ mimics a cut-off momentum to the bosonic fields and it must be of the order of the Fermi momentum or smaller, in order to ensure the validity of linear dispersion assumption (see section 1.3)

Furthermore $v_F = v g$ is assumed.

After the completion of our analysis, we will conclude that the pure forward scattering is the dominant term in the range of temperatures considered, yielding the following result for equation (3.4.1)
3.5.1. $I_{ff}$: pure forward scattering

The forward-forward term in the integral is

$$I_{ff} = \int_0^\infty d\omega \int_0^\infty dk k^2 |U_{12}(k)|^2 \frac{A_{fs1}(k,\omega) A_{fs2}(k,\omega)}{\sinh^2(\omega/2k_BT)}$$

This integral would contain the product of two delta functions (see equation (3.4.2)) which would lead to a divergence, in the case of identical wires, and to a zero transresistivity, in the case of different wires. This actually does not happen, since there are processes that broaden the correlation function.

These processes account for the finite lifetime of the excitations, whose decay is mainly caused by a band curvature effect or by a thermal effect. The first effect arises from the inaccuracy of our linear dispersion approximation, which does not perfectly fit the real dispersion curve (that is parabolic). The second effect describes the possibility of a thermal scattering of the excitations.

\[ A(k, \omega) \quad A(k, \omega) \]

Figure 3.4: Sketch of the approximated delta function in the case of step function approximation (left panel) and Lorentz function approximation (right panel).

A way to model this broadening is to approximate the delta function with a step function, centered in $vk$ and having width equal to $\delta \omega$ and height equal to...
1/δω \text{(Figure 3.4). This is the method used in Ref. [13], but it is a quite brutal approximation; in fact, by doing so, one introduces a discontinuity in the spectral function, which leads to a pathological behaviour in the case of different wires.}

A more suitable approximation could be to model the delta function with a Lorentz function:

\[ A_f s(k, \omega) = \frac{gk}{2} \frac{\delta \omega}{2\pi (\omega - vk)^2 + \delta \omega^2/4} \]

The choice of the function changes only a numerical prefactor for identical wires, while it influences also some qualitative features in the case of different wires.

The width δω is determined by the dominant effect between the band curvature broadening and the thermal broadening; from Refs. [13] and [44] it can be written

\[ \delta \omega = \max\{k^2/m^*, (k_B T)k/k_F\} \]

here \(m^*\) is not precisely the electron mass, since it is renormalized by the interaction; still, for not too high intrawire potentials the ratio \(m^*/m\) is of the order of unity.

Obviously, there will be a crossover regime where both the thermal and the band curvature broadening contribute in a similar measure to the total width; the range of this regime will be neglected, assuming that the width is given by

\[ \delta \omega \approx \begin{cases} k^2/m^* & \text{if } k > k_c \\ (k_B T)k/k_F & \text{if } k < k_c \end{cases} \]

where \(k_c = \frac{m^* (k_B T)}{k_F}\) is the threshold value of \(k\) that marks the two domains of \(\delta \omega\).

Now we will treat separately the case of identical and different wires; the qualitative behaviour that we will find is anticipated here for convenience

\[ \rho_{12}^{TT}(ff) \sim \begin{cases} 1/T & \text{if } T \ll T_0 \\ T_0^2/T^3 & \text{if } T_0 \ll T \ll T_F \end{cases} \]

for identical wires, while for different wires we have

\[ \rho_{12}^{TT}(ff) \sim \begin{cases} T/T_1^2 & \text{if } T \ll T_1 \\ 1/T & \text{if } T_1 \ll T \ll T_0 \\ T_0^2/T^3 & \text{if } T_0, T_1 \ll T \end{cases} \]

**Case I: identical wires.**

The following integral has to be calculated

\[ \alpha_{ff}(k, T) \equiv \int_0^\infty d\omega \frac{A_f s(k, \omega)}{\sinh^2(\omega/2k_B T)} \]

Using the step function, the integration is trivial to perform and yields

\[ \alpha_{ff}(k, T) = \frac{g^2 k^2}{4} \frac{1}{\delta \omega \sinh^2(vk/2k_B T)} \]
We can otherwise use the Lorentz function and employ the method of residues, after having extended the integration domain to the whole real axis. It can be done since the integrand is negligible on the negative axis; this claim is not exactly true for small \( k \lesssim \delta \omega \), since in that case the Lorentz function is not negligible also for \( \omega < 0 \); nevertheless, this region is very small and \( \alpha \) has to be weighted with \( k^2 \), so that this approximation is quite valid. In the end, a result similar to that of the step function is obtained:

\[
\alpha(k) = \frac{g^2k^2}{4\pi} \frac{1}{\delta \omega \sinh^2(vk/2k_B T)}
\]

In the next calculations, we will use the result from the Lorentz curve. Breaking the domain of integration in \( k \), \( I_{ff} \) can be written

\[
I_{ff} = \frac{g^2}{4\pi} \left( \int_0^{k_c} dk \frac{|U_{12}(k/k_0)|^2k_F k^3/(k_B T)}{\sinh^2(vk/2k_B T)} + \int_0^{\infty} dk \frac{m^* k^2 |U_{12}(k/k_0)|^2}{\sinh^2(vk/2k_B T)} \right)
\]

The temperature dependence can be qualitative analysed even without knowing the exact form of \( U_{12} \), simply by changing the integration variable to \( v k/k_B T \), defining \( a(g) \equiv \frac{m^* v}{2v_F} \) and using \( v_F = vg \):

\[
I_{ff} = \frac{m^* g^2 (k_B T)^3}{4\pi v^3} \left( \frac{1}{a(g)} \int_0^{a(g)} dx \frac{x^3 U_{12}^2(g T_0 x)}{\sinh^2(x/2)} + \int_0^{\infty} dx \frac{x^2 U_{12}^2(g T_0 x)}{\sinh^2(x/2)} \right)
\]

From the two previous equations, it can be argued that there are two relevant scales in the integrand: \( T \) and \( T_0 \); the one that dominates determines which function, between the coupling and the hyperbolic sine, varies faster in the integral.

For very low temperatures, i.e. \( T \ll T_0 \), the coupling can be considered constant, compared to the hyperbolic sine, and equal to \( U_{12}(0) \):

\[
I_{ff} = c(a(g)) \frac{m^* g^2 (k_B T)^3}{4\pi v^3} U_{12}(0)|^2 (k_B T)^3
\]

where the function \( c(a) \) (plotted in Figure 3.5) has been defined:

\[
c(a) = \frac{1}{a} \int_0^a dx \frac{x^3}{\sinh^2(x/2)} + \int_a^{\infty} dx \frac{x^2}{\sinh^2(x/2)}
\]

For values of the intrawire interaction not too high, \( a(g) \) is of the order of the unity and a good approximation for \( c(a(g)) \) is

\[
c(a(g)) \approx c(1) = \int_0^1 dx \frac{x^3}{\sinh^2(x/2)} + \int_1^{\infty} dx \frac{x^2}{\sinh^2(x/2)} \approx 11.2
\]

At higher temperatures (i.e. in the limit \( T_0 \ll T \ll T_F \)), the coupling cuts the integral well before the hyperbolic sine, precisely for values of \( x \) approximately equal to \( T_0/(gT) \ll 1, a(g) \). This implies that the argument of the sinh is very small, so it can be expanded, and that the second addend in equation (3.5.1) is negligible.
3. Thermal drag

The value of \( I_{ff} \) depends on the exact form of the coupling; anyway, it is \( T^2_0 T |U_{12}(0)|^2 \) times a numerical factor, which we are not very interested into.

Thus, we can finally sketch the qualitative aspect of the transresistivity as a function of \( T \) for the two regimes of temperature studied:

\[
\rho_{12}^{TT}(ff) \sim \begin{cases} 
\frac{1}{T} & \text{if } T \ll T_0 \\
\frac{T_0^2}{T^3} & \text{if } T_0 \ll T \ll T_F 
\end{cases}
\]

It must not be surprising that \( \rho_{12}^{TT} \) diverges as \( T \) goes to zero; in fact, this means that the thermal gradient must diverge to induce a finite current when \( T \to 0 \).

**Case II: different wires.**

Previously \( v_2 > v_1 \) has been assumed, with the definition \( \delta v = v_2 - v_1 \); under the condition \( \delta v \ll v \) the assumption \( \delta \omega_1 = \delta \omega_2 \) can be made.

Now, we briefly explain why the step function is a quite brutal approximation in the case of different wires. As can be noticed from Figure 3.6 (left panel), the two Afs overlap only if the condition \( k\delta v < \delta \omega \) is satisfied; in fact the length of the overlap region in the \( \omega \) domain is \( \delta \omega - k\delta v \).

Depending on which type of broadening dominates in \( \delta \omega \), the condition is

\[
\begin{cases} 
kgv < (kB)k_F/k_F & \text{if } k < k_c \\
kgv < k^2/m^* & \text{if } k > k_c \\
T > T_1 & \text{if } k < k_c \\
k > m^*\delta v = \frac{k_B T_1}{k_F/m^*} & \text{if } k > k_c 
\end{cases}
\]

Let us analyse carefully what this implies: for \( T > T_1 \) the two conditions are both satisfied so that there is an overlap and \( \alpha \neq 0 \); on the other hand if \( T < T_1 \)
the spectral functions never overlap for $k < k_c$, while in the region $k > k_c$ they overlap only for $k > \frac{k_B T_1}{\delta v k/m^*} = \frac{T_1}{T} k_c$. This implies that the integral starts from a well defined value and hence it has an exponential dependence from the temperature which results in a too drastic behaviour.

If the Lorentz function is used, the transresistivity exhibits a more regular behaviour, since the overlap between the spectral functions always exists, however small it may be (see Figure 3.6 right panel).

Extending again the integral to the whole real axis, $\alpha_{ff}$ becomes

$$\alpha_{ff}(k, T) \approx \frac{1}{\sinh^2(v k/2 k_B T)} \frac{g^2 k^2 \delta \omega^2}{4 \pi^2} \int_{-\infty}^{\infty} \frac{d\omega}{[(\omega - v_1 k)^2 + \delta \omega^2/4][\omega - v_2 k]^2 + \delta \omega^2/4}$$

where the relevant value of $k$ have already been substituted in the argument of hyperbolic sine.

The integrand has four simple poles; to calculate the integral we choose a contour passing in the upper half of the complex plane, so that only the residues in $v_1 k + i \delta \omega/2$ and $v_2 k + i \delta \omega/2$ are important:

$$\alpha(T) = \frac{1}{\sinh^2(v k/2 k_B T)} \frac{g^2 k^2 \delta \omega}{4 \pi k^2 \delta \omega^2 + \delta \omega^2}$$

Equations (3.4.1) becomes

$$I_{ff} = \frac{g^2}{4 \pi} \int_{0}^{\infty} \frac{\delta \omega k^4 |U_{12}(k)|^2}{(\delta \omega^2 k^2 + \delta \omega^2) \sinh^2(v k/2 k_B T)}$$

Now the integration domain is broken, depending on the value of $\delta \omega$; another
variable change is performed

\[
I_{ff} = \frac{m^* g^2 k_B^3 T^5}{4\pi v^3} \left( \int_0^{a(g)} \frac{U_{12}^2(g T_0 x)}{(a(g) T_1^2 + T^2) \sinh^2(x/2)} x^3 dx \right) + \int_{a(g)}^{\infty} \frac{U_{12}^2(g T_0 x) x^4 dx}{(a(g)^2 T_1^2 + x^2 T^2) \sinh^2(x/2)} \right)
\]

(3.5.2)

In the limit \( T_1, T_0 \ll T \ll T_F \), \( I_{ff} \sim T_0^2 T |U_{12}(0)|^2 \).

Figure 3.7: Plot \( \rho^{TT}_{12}(ff) \) renormalized to \( \frac{2m^* g^2 T^2 (0)}{\pi m^* v^3 k_B} \) for a step function (upper panel) and a Lorentz function (lower panel), for \( a(g) = 0.8, 1.0, 1.5 \) and for \( T \ll T_0 \).

Thermal drag between two coupled quantum wires
On the other hand if \( T \ll T_0 \), the coupling can be considered constant, obtaining

\[
I_{ff} = \frac{m^* g^2}{4\pi} |U_{12}(0)|^2 \frac{k_B^3 T^5}{v^3} c_2(a(g), T) \quad (3.5.3)
\]

\[
c_2(a, T) \equiv \frac{1}{(aT_1^2 + T^2)} \int_0^a \frac{x^3 dx}{\sinh^2(x/2)} + \int_a^\infty \frac{x^4 dx}{(a^2 T_1^2 + x^2 T^2) \sinh^2(x/2)} \quad (3.5.4)
\]

Equation (3.5.2) allows to calculate the forward scattering contribution to the transresistivity \( \rho_{T12}^{TT}(ff) \), whose plot (renormalized to \( \frac{9m^* g^2 U^2(0)T_1}{8\pi^2 \kappa T_1} \)) is shown in Figure 3.7 in the limit \( T \ll T_0 \) for three value of \( a(g) \) and for both a step function and Lorentz function approximation.

For intermediate values of \( T/T_1 \) the two functions have a similar behaviour, except for the angular point which is a peculiar (and quite unphysical) feature of the step function approximation, due to the discontinuities introduced in the spectral functions. For very low values of \( T \), the transresistivity drops much faster in the step function case, while for high temperatures the qualitative behaviour is the same but the numerical values are different in the two cases (due to the \( \pi \) factor of difference in \( \alpha_{ff} \) for identical wires).

The relevant limits of \( c_2(a, T) \) are: constant for \( T \ll T_1 \) and \( c_1(a)/T^2 \) for \( T \gg T_1 \). Therefore with a Lorentz function approximation, the same result of identical wires has been retrieved in the limit \( T \gg T_1 \), i.e. in a regime of temperatures where the difference between the wires is not relevant; furthermore, it has been verified that, in the opposite limit, the transresistivity behaves like a power law.

Summing up, the qualitative behaviour of \( \rho_{T12}^{TT}(ff) \) can be written

\[
\rho_{T12}^{TT}(ff) \sim \begin{cases} 
T/T_1^2 & \text{if } T \ll T_1 \\
1/T & \text{if } T_1 \ll T \ll T_0 \\
T_0^2/T^3 & \text{if } T_0, T_1 \ll T
\end{cases}
\]

\[
\ln(\rho_{T12}^{TT}) \sim \begin{cases} 
-\ln T & \text{if } T \ll T_1 \\
-3\ln T & \text{if } T_1 \ll T \ll T_0 \\
\ln T & \text{if } T_0, T_1 \ll T
\end{cases}
\]

Figure 3.8: Qualitative behaviour of \( \rho_{T12}^{TT}(ff) \) for \( \delta v = 0 \) (left) and \( \delta v \neq 0 \) (right).
A qualitative graphical representation is given in Figure 3.8. The essential feature noticeable is the presence of a peak in the different wires case; this peak appears because, for low $T$, the transresistivity grows, while it decreases when $T$ is far greater than $T_1$. It is indeed absent in the case of identical wires, since the transresistivity is always decreasing.

### 3.5.2. $I_{fb} + I_{bf}$: mixed forward-backward scattering

In this paragraph we want to analyse the behaviour of the mixed contributions to the transresistivity.

When calculating the integral

$$\alpha_{fb}(k,T) = \int_0^\infty d\omega \frac{A_{fs1}(k,\omega)A_{bs2}(k,\omega)}{\sinh^2(\omega/2k_BT)}$$

it can be argued that the integrand is relevant on a scale $\omega \sim k_BT \ll \nu k_F$ and the arguments of $B_s$ and $B_c$ can be expanded in powers of $\beta \omega$, obtaining

$$A_{bs}(k,\omega) = \frac{\sin(\pi g)}{4\pi^2 v} \left( \frac{\pi \alpha}{\beta v} \right)^{2g-2} \frac{\beta \omega}{\pi} \left[ B_{sc} \left( \frac{\beta v(k-2k_F)}{2\pi}, g \right) + B_{sc} \left( \frac{\beta v(k+2k_F)}{2\pi}, g \right) \right]$$

where

$$B_{sc}(x, y) = B_c(x, y) \partial_x B_s(x, y) - B_s(x, y) \partial_x B_c(x, y)$$

Equation (3.4.2) can be used, without any need to regularize the delta function with a finite width:

$$I_{fb} = \frac{g_1}{2} \int_0^\infty d\omega \int_0^\infty dk k^2 |U_{12}(k)|^2 \frac{A_{fs1}(k,\omega)A_{bs2}(k,\omega)}{\sinh^2(\omega/2k_BT)}$$

$$= \frac{g_1}{2} \int_0^\infty dk k^2 |U_{12}(k)|^2 \frac{k A_{bs2}(k,v_1 k)}{\sinh^2(v_1 k/2k_BT)}  \tag{3.5.5}$$

Again the integrand is relevant on a scale $k \sim k_BT/v_1 \ll k_F$, so it can be expanded to the first non zero order in $k$, getting

$$I_{fb} = \frac{g_1}{4\pi^3} \beta v_1 (\alpha k_F)^{2g-2} \bar{B}_{sc} \left( \frac{\beta v_2 k_F}{\pi}, g_2 \right) \int_0^\infty dk |U_{12}(k)|^2 k^4 \frac{1}{\sinh^2(v_1 k/2k_BT)}  \tag{3.5.5}$$

where the special function $\bar{B}_{sc}(x, y) = x^{2-2g} B_{sc}(x, y)$ has been defined. We observe that $B_{sc}(x, y)$ decays very rapidly (approximately exponentially) for arguments greater than 1 (see Figure 3.9) and the same assertion holds for $\bar{B}_{sc}(x, y)$.

The integral in equation (3.5.5) behaves like $T^5$ for $T \ll T_0$ and like $T^2$ for $T \gg T_0$ and, summing up, the transresistivity is qualitatively described by

$$\rho_{12}^{TT}(fb) \sim \begin{cases} 
  g_1 \sin(\pi g_2)(\alpha k_F)^{2g-2} \frac{1}{v_1^4} \bar{B}_{sc}(\beta v_2 k_F/\pi, g_2) & \text{if } T \ll T_0 \\
  g_1 \sin(\pi g_2)(\alpha k_F)^{2g-2} \frac{k_0^4}{T^3} \bar{B}_{sc}(\beta v_2 k_F/\pi, g_2) & \text{if } T \gg T_0
\end{cases}$$
In order to obtain $\rho_{12}^{TT}(bf)$ one just needs to exchange the indices 1 and 2 in $\nu$ and $g$.

At this point, it should be understood whether, in the various regimes of temperature introduced, one term between $\rho_{12}^{TT}(fb)$ and $\rho_{12}^{TT}(ff)$ dominates over the other. For values of $T$ low enough to be in the Luttinger limit, $\rho_{12}^{TT}(fb)$ is always very small compared to $\rho_{12}^{TT}(ff)$. In fact, their ratio is roughly

$$\frac{\rho_{12}^{TT}(fb)}{\rho_{12}^{TT}(ff)} \sim \begin{cases} \frac{T}{T_F} \cdot \hat{B}_{sc}(\beta v_2 k_F/\pi, g_2) & \text{if } T \ll T_0 \\ \frac{T_0}{T_F} \cdot \hat{B}_{sc}(\beta v_2 k_F/\pi, g_2) & \text{if } T \gg T_0 \end{cases}$$

The ratio $\rho_{12}^{TT}(fb)/\rho_{12}^{TT}(ff)$ is plotted in Figure 3.10 as a function of $T$; it has been calculated using reasonable values for $g_1 \approx g_2$ and for $\alpha k_F \approx 5$. It is negligible also at the Fermi temperature, and so the contribution of mixed terms in the transresistivity can be safely neglected in the following.

### 3.5.3. $I_{bb}$: pure backscattering

Here we study the pure backscattering contribution; after the previous paragraph we suspect that also this contribution is negligible. This is what we want to prove.
Again \( A_{bs} \) can be expanded in powers of \( \omega \), obtaining

\[
I_{bb} = \frac{\sin(\pi g_1) \sin(\pi g_2)}{16 \pi^6 v_1 v_2} \left( \frac{\pi \alpha}{\beta v_1} \right)^2 \left( \frac{\pi \alpha}{\beta v_2} \right)^2 \cdot \int_{0}^{\infty} dk k^2 |U_{12}(k)|^2 \int_{0}^{\infty} d\omega \frac{\beta^2 \omega^2}{\sinh^2(\omega/2k_B T)} \cdot \left[ B_{sc}\left( \frac{\beta v_1 (k - 2k_F)}{2\pi}, g_1 \right) + B_{sc}\left( \frac{\beta v_1 (k - 2k_F)}{2\pi}, g_1 \right) \right] \cdot \left[ B_{sc}\left( \frac{\beta v_2 (k + 2k_F)}{2\pi}, g_2 \right) + B_{sc}\left( \frac{\beta v_2 (k + 2k_F)}{2\pi}, g_2 \right) \right]
\]

Figure 3.10: Plot of the ratio between \( \rho_{bf} \) and \( \rho_{ff} \) (left) and of its logarithm (right) as a function of \( T/T_F \) for three different values of \( g \) and for \( \alpha k_F \approx 5 \).
3.6. Specific results and plots

The integral in $\omega$ is trivial and yields $\frac{4\pi^2}{T} (k_B T)$.

On the other hand, the integral in $k$ has to be cut off at a maximum wavevector which is the smallest between $k_0$ and $1/\alpha$, since the bosonization and consequently the expression for $A_{bs}$ is valid within this momentum range. In every case, the cut-off momentum is smaller than $k_F$ and the functions $B_{sc}$ can be expanded, finding

$$I_{bb} = \frac{\sin(\pi g_1) \sin(\pi g_2)}{3\pi^4 v_1 v_2 \beta} (\alpha k_F)^{2g_1 + 2g_2 - 4} \tilde{B}_{sc} \left( \frac{\beta v_1 k_F}{\pi}, g_1 \right) \cdot \tilde{B}_{sc} \left( \frac{\beta v_2 k_F}{\pi}, g_2 \right) \int_0^{k_{th}} dk k^2 |U_{12}(k)|^2$$

with $k_{th} = \min\{k_0, 1/\alpha\}$. Expanding the function $B_{sc}$ for large arguments, one gets

$$\rho_{12}^{TT} (bb) \sim \frac{\sin(\pi g_1) \sin(\pi g_2)}{T^3} (\alpha k_F)^{2g_1 + 2g_2 - 4} \tilde{B}_{sc} \left( \frac{\beta v_1 k_F}{\pi}, g_1 \right) \cdot \tilde{B}_{sc} \left( \frac{\beta v_2 k_F}{\pi}, g_2 \right) k_0^3 |U_{12}(0)|^2$$

for $k_0 \ll 1/\alpha$

$$\rho_{12}^{TT} (bb) \sim \frac{\sin(\pi g_1) \sin(\pi g_2)}{T^3} (\alpha k_F)^{2g_1 + 2g_2 - 7} \tilde{B}_{sc} \left( \frac{\beta v_1 k_F}{\pi}, g_1 \right) \cdot \tilde{B}_{sc} \left( \frac{\beta v_2 k_F}{\pi}, g_2 \right) k_0^3 |U_{12}(0)|^2$$

for $k_0 \gg 1/\alpha$

Since $\rho_{12}^{TT} (bb)$ is proportional to the product of two $\tilde{B}_{sc}$ functions, it is exponentially suppressed in comparison to $\rho_{12}^{TT} (ff)$ in the range of $T$ studied; we have proved that also this term can be safely neglected.

In conclusion, we have analysed the three contributions to the transresistivity, deriving that the dominant one is a positive term arising from the forward scattering and proving the statement made in section 3.3 about the positivity of the transresistivity.

3.6. Specific results and plots

In this section we will assume the interwire coupling to have a specific form and we will study the transresistivity in that case.

Let us first stress that only the forward scattering contribution is considered, because the backward scattering can be neglected in the low temperature regime ($T \ll T_F$), as discussed in the previous section.

Equations (3.3.28) and (3.5.2) provide:

$$\rho_{12}^{TT} = \frac{9m^* g^2 T}{8\pi^3 v} \left[ \frac{1}{(a(g) T_F^2 + T)^2} \int_0^{a(g)} \frac{U_{12}^2(g T x/T_0) x^3 dx}{\sinh^2(x/2)} + \int_{a(g)}^{\infty} \frac{U_{12}^2(g T x/T_0) x^4 dx}{(T_1^2 a(g)^2 + x^2 T^2) \sinh^2(x/2)} \right]$$

(3.6.1)

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*Thermal drag between two coupled quantum wires*
3. Thermal drag

Furthermore the approximation of a smooth intrawire potential with amplitude \( V_2 = V_4 = V \) has been made, so that
\[
v = \frac{v_F}{g} \quad \quad g = \left(1 + \frac{V}{\pi v_F}\right)^{-1/2}
\]

Let us now estimate the effective mass \( m^* \), using the formula given in Ref. [44]:
\[
\frac{1}{m^*} = \frac{v}{g} \frac{\partial}{\partial \mu} \left( v \sqrt{g} \right)
\]

At low temperatures it can be written \( \mu \approx \frac{1}{2} m v_F^2 \), so that
\[
\frac{1}{m^*} = \frac{v_F}{g^2} \frac{\partial}{\partial v_F} \left( \frac{v_F}{\sqrt{g}} \right) = \frac{1}{m} \frac{1}{g^2} \frac{\partial}{\partial v_F} \left( v_F \sqrt{g} \right)
\]

The expression for \( g \) is used, getting
\[
\frac{\partial}{\partial v_F} \frac{1}{\sqrt{g}} = \frac{g^2 - 1}{4v_F \sqrt{g}}
\]

and directly leading to
\[
\frac{1}{m^*} = \frac{1}{m} \left( \frac{3 + g^2}{4g^{3/2}} \right) \quad a(g) = \frac{4g^{3/2}}{3 + g^2} \quad (3.6.2)
\]

where also the value of the function \( a(g) \) introduced in section 3.5.1 has been reported.

Finally, substituting equation (3.6.2) into (3.6.1), it is possible to write:
\[
\rho_{TT}^{12} = \frac{9m}{8\pi^5 v_F} \frac{4g^{9/2}}{3 + g^2} \frac{T}{k_B^2 T_1^2} \left[ \frac{1}{a(g)(1 + (T/T_1)^2)} \int_0^{a(g)} U_{12}^2(gT x/T_0)x^3 dx \right. \\
\left. + \int_{a(g)}^\infty \frac{U_{12}^2(gT x/T_0)x^4 dx}{(a(g)^2 + x^2(T/T_1)^2) \sinh^2(x/2)} \right] \quad (3.6.3)
\]

Once known the form of the coupling, equation (3.6.3) can be studied numerically as a function of \( g, T/T_0 \) and \( T_1/T_0 \).

In order to plot some graphical representation of the transresistivity, a particular form for \( U_{12} \) will be chosen; although the quantitative behaviour of the plots depend on this choice, the qualitative features explained are far more general.

The following interaction is assumed
\[
U_{12}(k) = U_{12}(0)e^{-k/k_0}
\]

obtaining
\[
\rho_{TT}^{12} = \frac{mU_{12}^2(0)}{v_F k_B^2 T_0} \frac{9g^{9/2}}{2\pi^5 (3 + g^2)} \left[ \frac{1}{a(g)(1 + (\frac{T}{T_0})^2)} \int_0^{a(g)} e^{-2gT x/T_0} x^3 dx \right. \\
\left. + \int_{a(g)}^\infty \frac{e^{-2gT x/T_0} x^4 dx}{(a(g)^2 + x^2(\frac{T}{T_0})^2) \sinh^2(x/2)} \right]
\]

Summing up the observations made in section 3.5.1 and the numerical results obtained in this section, we can point out the essential features of the transresistivity:

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Figure 3.11: Plot of $\rho_{12} k_B^2 T_0/(mU_{12}(0))$ as a function of $T/T_0$ for three different values of $g$ and for $T_1/T_0 = 0.2$.

Figure 3.12: Plot of $\rho_{12} k_B^2 T_0/(mU_{12}(0))$ as a function of $T/T_0$ for three different values of $T_1/T_0$ and for $g = 1$.

- For $T_1 \neq 0$, $\rho_{12}$ exhibits three different behaviours separated by two crossovers: it initially grows like $T$, then shows a peak around $T_1$ (called the $T_1$ crossover).

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and decreases as $1/T$; finally, there is the $T_0$ crossover around $T_0/g$, where $\rho_{12}$ passes from a $1/T$ to a $1/T^3$ regime. This is shown in Figure 3.11 (for different values of $g$) and 3.12; unfortunately, the $T_0$ crossover is very hard to notice in a linear scale. In Figure 3.12 is also reproduced the dependence on three different values of $T_1$: the lower is this temperature, the higher is the transresistivity.

- For $T_1 = 0$, $\rho_{12}$ shows two behaviours and one crossover: it initially behaves like $1/T$ and then, after the $T_0$ crossover, its trend is $1/T^3$; see Figure 3.13. Therefore there is no peak: ideally we can think that the region of linear growth, present in the $T_1 \neq 0$ case, is pushed towards smaller and smaller temperatures as $T_1 \to 0$, until it disappears completely.

- The form of the coupling changes the quantitative behaviour of $\rho_{12}$ around the $T_0$ crossover, but not the limiting cases ($T \ll T_0$ and $T \gg T_0$).

- The value of $g$ influences the position of the $T_1$ peak through $a(g)$; nevertheless, this influence is not very evident. On the other hand $g$ changes the numerical value of $\rho$ (this can be seen in Figure 3.11), since the smaller is $g$ the smaller is the transresistivity; $g$ also influences the position of the $T_0$ crossover.

In the previous images the $T_0$ crossover is hard to notice, since it does not coincide with a peak; this problem can be solved plotting the behaviour of the transconductivity $\sigma^{TT}_{21}$. Proceeding like in section 2.2, we can invert the resistivity matrix at..
3.6. Specific results and plots

Figure 3.14: Plot of $\sigma_{12}v_F T_0^3/(mk^2 B U_{12}^2(0))$ as a function of $T/T_0$ for three different values of $T_1/T_0$ and $g = 1$.

Figure 3.15: Plot of $\sigma_{12}v_F T_0^3/(mk^2 B U_{12}^2(0))$ as a function of $T/T_0$ for three different values of $g$ and $T_1/T_0 = 0.1$.

the leading order in the coupling, carefully keeping trace of the minus signs and

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exploiting the symmetry $\rho_{21}^{TT} = \rho_{12}^{TT}$, the transconductivity is

$$\sigma_{21}^{TT} \approx \frac{\rho_{12}^{TT}}{\rho_{11}^{TT} \rho_{22}^{TT}} \approx \sigma_{11}^{TT} \sigma_{22}^{TT} \rho_{12}^{TT} = \frac{\pi^2}{36} k_B^2 T^2 \rho_{12}^{TT}$$  \hspace{1cm} (3.6.4)

where the expression for the single wire conductivity given in Ref. [8] has been used.

The three (two, for $T_1 = 0$) regimes of $\sigma_{21}$ are $T^3$, $T$, $1/T$ ($T$, $1/T$): thus, it must exhibit a peak at the $T_0$ crossover (that we will call $T_0$ peak), even for $T_1 = 0$.

The three regimes can be easily seen in Figure 3.14, where is plotted the behaviour of $\sigma_{21}$ for $g = 1$ and $T_1/T_0 = 0.1$, 0.3, 0.5: there is an initial cubic behaviour, followed by a linear trend and, after the $T_0$ peak, by a $1/T$ decrease. From this picture, it can also be deduced that the position of $T_0$ peak depends on $T_1$; this is not immediate to understand, but we can try observing that $T_1$ influences the denominator in the second integral of equation (3.6.3) and it can slightly change the position of the crossover.

Additionally, in Figure 3.15 and 3.16, is plotted the transconductivity as a function of $T/T_0$ for $g = 0.5$, 0.75, 0.8, 0.9, 1. In the panel at the right upper corner of Figure 3.16, there is a zoom of the region $T/T_0 < 1$, where it can be easily seen that the position of the $T_0$ peak depends on $g$.

Finally in Figure 3.17 are shown two different three-dimensional plots of $\sigma_{21}$. In the first is studied the dependence on $T/T_0$ and on $g$ (for values around the unity): it can be noticed that the $T_0$ peak is higher for greater $g$. In the second panel is studied the dependence on $T_1/T_0$ and $T/T_0$: the three regimes of $\sigma_{21}$ can be observed, along

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**Figure 3.16:** Plot of $\sigma_{12}^{TT} T_0^3/(mk_b^2 U_{12}^2(0))$ as a function of $T/T_0$ for three different values of $g$ and $T_1/T_0 = 0.1$. In the small panel on the right upper corner there is the zoomed region for $T/T_0 < 1$.

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*Thermal drag between two coupled quantum wires*
3.7. Generalization to thermoelectric effect

In this section we want to discuss the derivation of the thermoelectric coefficients for the drag; as it will be explained, the simplest way to prove the expression reported is the method described in Appendix F, to which we refer for the complete

with the fact that the $T_0$ peak occurs earlier for lower values of $T_1$.

3.7. Generalization to thermoelectric effect

In this section we want to discuss the derivation of the thermoelectric coefficients for the drag; as it will be explained, the simplest way to prove the expression reported is the method described in Appendix F, to which we refer for the complete

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3. Thermal drag

demonstration.

We know that thermal and electric transport are strictly related: the presence of an electric field $\mathcal{E}$ can generate a thermal current $J_T$ and, vice versa, a thermal gradient $\nabla T$ can give origin to an electric current $J_E$ (for convention simplicity the subscript $\parallel$ will be dropped from the current, but will be always considered the average of the current over a wire). This phenomenon is generally known as thermoelectric effect.

The previous statement is true not only for single wires, but applies also to our system; in analogy with equation (3.1), the most general situation is described by a matrix of transport coefficients:

$$
\begin{pmatrix}
\mathcal{E}_1 \\
-\nabla T_1 \\
\mathcal{E}_2 \\
-\nabla T_2
\end{pmatrix} = \begin{pmatrix}
\rho_{11}^{EE} & \rho_{12}^{ET} & -\rho_{12}^{EE} & -\rho_{12}^{ET} \\
-\rho_{21}^{EE} & \rho_{21}^{TT} & \rho_{21}^{EE} & -\rho_{21}^{TT} \\
-\rho_{21}^{TE} & \rho_{21}^{ET} & \rho_{21}^{TE} & \rho_{21}^{TT} \\
-\rho_{21}^{TE} & \rho_{21}^{ET} & \rho_{21}^{TE} & \rho_{21}^{TT}
\end{pmatrix} \cdot \begin{pmatrix}
J_E^1 \\
J_T \\
J_E^2 \\
J_T^2
\end{pmatrix}
$$

Alternatively the conductivity matrix can be considered, $\rho_{1i}^{EE}$ and $\rho_{1i}^{TT}$ are the single wire electric and thermal resistivity, while $\rho_{12}^{ET}$ and $\rho_{12}^{TE}$ account for thermoelectric effect in one wire: they are zero unless anharmonic interactions in the wire (due to the non perfect linearity of the dispersion) are considered (see Ref. [8]).

Since we want to study the thermoelectric effect within the drag framework, the matrices of interest are the 2x2 matrices at the right upper corner and at the left lower corner of the 4x4 resistivity matrix; thanks to the symmetry of the system, just the upper matrix can be considered:

$$
\begin{pmatrix}
\mathcal{E}_1 \\
-\nabla T_1
\end{pmatrix} = -\begin{pmatrix}
\rho_{1i}^{EE} & \rho_{12}^{ET} \\
-\rho_{21}^{EE} & \rho_{21}^{TT}
\end{pmatrix} \cdot \begin{pmatrix}
J_E^1 \\
J_T^1
\end{pmatrix} = -\rho_{12} \begin{pmatrix}
J_E^1 \\
J_T^1
\end{pmatrix}
$$

(3.7.1)

The thermal transresistivity (from equation (3.3.28)) is reported here for convenience:

$$
\rho_{12}^{TT} = \frac{9}{2\pi^4} \frac{v_1 v_2}{k_B T^4} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)}
$$

The electric transresistivity (from equation (2.1.3)) is:

$$
\rho_{12}^{EE} = \frac{1}{2\pi^2 e^2 n_1 n_2 k_B T} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)}
$$

An easy and fast way to derive the thermoelectric coefficients is to employ the dynamic approach used in Ref. [13] and explained in Appendix F. In fact, this method specifies the nature of a current (electric or thermal) only in some specific passages, that can be appropriately modified in order to account for the thermoelectric effect.

Now we just write the expression for (3.7.1), proving it in Appendix F.

$$
\rho_{12} = \int_0^\infty d\omega \int_0^\infty dk k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)} \left( \frac{1}{2\pi^4 v_1 n_1 n_2 k_B T} \frac{3}{2} \frac{v_2}{e^2 k_B T^2} \right)
$$

(3.7.2)
Exploiting the symmetry of the system we can find also

\[ \rho_{21} = \int_{0}^{\infty} d\omega \int_{0}^{\infty} dk k^2 |U_{12}(k)|^2 A_1(k,\omega) A_2(k,\omega) \frac{1}{2\pi^2 k_B T} \frac{\pi e n_1}{v_2} \frac{\pi e n_2 k_B^2 T^2}{v_1} \left( \frac{3}{\pi e n_1 k_B^2 T^2} \right)^{\frac{1}{2}} \left( \frac{3}{\pi e n_2 k_B^2 T^2} \right)^{\frac{1}{2}} \]

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Chapter 4

Numerical simulations

In this chapter we present the results obtained running numerical simulations for an out-of-equilibrium protocol equivalent to that of Chapter 3.

The employed computational technique relies on the formalism of Matrix Product States (MPS) to simulate the evolution of a one dimensional system of interacting spinless electrons. For more details, the reader is referred to Appendix G, where it is shown how the method implements the representation of quantum states and how it is possible to simulate both the real time and imaginary time evolution.

In the first section of this chapter we discuss briefly the model of 1D interacting electrons used during the computational work.

In section 4.2, we describe the protocol used to simulate the thermal drag effect and the relevant parameters employed.

Finally, in section 4.3, we show the results obtained.

4.1. Discretized Hamiltonian of 1D electrons

In order to exploit the computational power of calculators, it is quite natural to work with discrete systems. If we want to study a quantum wire of spinless electrons, we consider a chain of sites, each one containing up to one electron. The continuum limit can be retrieved by considering an infinite number of sites under the appropriate conditions; on the other hand, since the time of computation is finite, we will work for simplicity with a system with a properly chosen finite size.

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Once the size of the system is defined, the operators $a_i$ and $a_i^\dagger$ can be introduced; they, respectively, annihilate and create one electron on site $i$. They are canonical fermion operators and satisfy the standard anticommutation rules $\{a_i, a_j\} = 0$ and $\{a_i, a_j^\dagger\} = \delta_{i,j}$.

We now want to construct the Hamiltonian; the kinetic part can be described by introducing the possibility of a hopping between adjacent sites:

$$H_{\text{kinetic}} = -\frac{J}{2} \sum_i (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) \quad (4.1.1)$$

Here $J$ has the dimension of an energy and is a parameter related to the Fermi velocity of the system. If we diagonalize the previous expression by introducing the wavenumber $k$, an energy dispersion $\xi(k) = -J \cos k$ is found; in the continuum limit, this dispersion assumes the parabolic shape typical of free electron systems.

The simplest choice for the interaction part is to introduce a nearest neighbour interaction proportional to the densities of two adjacent sites; a contact interaction is indeed forbidden by the Pauli principle. The interaction strength can be tuned by introducing an anisotropy parameter $\Delta$:

$$H_{\text{int}} = J \Delta \sum_i \left( a_{i+1}^\dagger a_{i+1} - \frac{1}{2} \right) \left( a_i^\dagger a_i - \frac{1}{2} \right) \quad (4.1.2)$$

The total Hamiltonian is simply the sum of equations (4.1.1) and (4.1.2):

$$H = \sum_i h_i$$

$$h_i = -\frac{J}{2} (a_i^\dagger a_i + \text{h.c.}) + J \Delta \left( a_{i+1}^\dagger a_{i+1} - \frac{1}{2} \right) \left( a_i^\dagger a_i - \frac{1}{2} \right) \quad (4.1.3)$$

The model explained above is sketched in Figure 4.1

![Figure 4.1](image)

Figure 4.1: Schematic picture of the chain of interacting electrons. The two contributions to the Hamiltonian (hopping and nearest neighbour interaction) are showed.

$\Delta$ parameterizes the interaction strength: for $\Delta = 0$ we have free electrons, for $\Delta > 0$ the interaction is repulsive, while it is attractive for $\Delta < 0$. The peculiarity of this model is that there are two different ordered and gapped phases for $|\Delta| > 1$. For $\Delta < -1$ all the sites are either filled or empty; this is understandable even at a qualitative level, since the attraction between electrons becomes so strong that, once filled one site, also all the other sites tend to accept an electron. Instead, for $\Delta > 1$, the odd sites are filled and the even sites are empty (or viceversa); qualitatively, the
repulsion is so strong that there cannot be two electrons in adjacent sites. Actually, the phases just described are related to the ferromagnetic and antiferromagnetic order of a spin chain; in fact, our system is equivalent to the so-called spin-$\frac{1}{2}$ XXZ model, to which it is related by the Jordan-Wigner transformation. We refer the reader to Ref. [1] for more information on the mentioned model.

It can be proved that, for low energies and $|\Delta| < 1$, the Hamiltonian in equation (4.1.3) is equivalent to that of a Luttinger liquid and there are precise relations between the parameters of the two systems (see Ref. [1], chapter 6).

Until now, only the single wire Hamiltonian was analysed; we now want to consider two coupled wires. The Hamiltonian of each decoupled wire is given by equation (4.1.3), allowing different $J$ and $\Delta$ for the two wires.

The interwire coupling requires more care during its definition; we want an interaction proportional to the densities of both wires. A possible solution could be to consider a local coupling between the two corresponding sites of the wires, i.e.

$$H_{12} = \sum_i U_i \left( a_{i,1}^\dagger a_{i,1} - \frac{1}{2} \right) \left( a_{i,2}^\dagger a_{i,2} - \frac{1}{2} \right)$$

In other words, we are saying that the coupling is not negligible only between a site $i$ on wire 1 and the site $i$ on wire 2, decaying very rapidly if we instead consider the sites $i-1$ and $i+1$. The parameter $U_i$ is the coupling strength and its dependence on $i$ accounts for the possibility of a non uniform coupling.

4.2. Out-of-equilibrium protocol

The physical phenomena described in Chapter 3 are very hard to test in a computational framework; it is thus natural to approximate and simplify the situation, introducing a protocol that describes the out-of-equilibrium evolution of the problem and is still computationally tractable.

The protocol that we will introduce in section 4.2.1 presents two key differences with respect to that of Chapter 3. The first difference is that the protocol simulates the temporal evolution from an equilibrium situation to a non equilibrium steady state in a hamiltonian fashion. The second difference is that we are reducing the size of the system from $L \to \infty$ to a finite number of sites $N$; this brings into the simulation also boundary effects, that are due to the finite size and are absent in the situation of Chapter 3.

Despite these drawbacks, this protocol has already been considered in the literature to simulate various out-of-equilibrium phenomena, referring the reader, for example, to Refs. [45], [46], [47], [48] and [49].

4.2.1. Description of the protocol

In this subsection we discuss the features of the employed protocol; the description of how it is implemented in the numerical simulation is postponed to section 4.2.2.
4.2. Out-of-equilibrium protocol

We work with two identical wires (with a normalized $J = 1$) and express the various parameters and observables in terms of $J$: for example, $U$ is expressed in units of $J$, the time $t$ and the inverse temperature $\beta$ in units of $J^{-1}$ and the various currents $J$ in units of $J^2$.

We have to apply a temperature gradient to wire 1; in order to achieve this condition, at the beginning the wire is truncated into two halves, which are held at different temperatures. For simplicity $N$ is even, so that the left half comprehends the sites from 1 to $N/2$ and is held at an inverse temperature $\beta_L$, while the right half (from site $N/2 + 1$ to site $N$) is held at an inverse temperature $\beta_R$.

Therefore, for $t \leq 0$, the system is prepared into its initial thermal state, described by the following density matrix

$$\rho_1 = \frac{e^{-\beta_L H_L} \otimes e^{-\beta_R H_R}}{Z^{-1}}$$

(4.2.1)

where $Z$ is the partition function, which ensures the normalization of the density matrix, and $H_L = \sum_{i=1}^{N/2-1} h_i$ and $H_R = \sum_{i=N/2+1}^{N-1} h_i$ are the Hamiltonians of the two halves of wire 1. Here there is a subtlety to keep in mind: as defined in equation (4.1.3), $h_i$ indicates the Hamiltonian associated to the link between site $i$ and site $i+1$ and, in this case $i$ goes from 1 to $N-1$. Since the Hamiltonian associated to the link between site $N/2$ and site $N/2 + 1$ is removed, the left Hamiltonian comprises the $h_i$ from 1 to $N/2 - 1$, while the right one comprises the $h_i$ from $N/2 + 1$ to $N-1$.

Wire 2 is prepared with a uniform temperature, such that $\beta_2 = \frac{\beta_L + \beta_R}{2}$; we stress that this does not mean that the two wires have the same average temperatures (since we have averaged the inverse temperatures) nor they have the same mean energy (because the mean energy is not proportional to $\beta$), but it is however a good choice to prevent an interwire current.

During our study, we have considered two different protocols; in the first protocol one, wire 2 was not split into two halves like wire 1, since there was no need to do such truncation; on the other hand, after a few simulations we realized that there was a great energy difference between the two wires at $t = 0$: in fact wire 1 does not have the energy of the central link (i.e. the energy associated to the link between site $N/2$ and site $N/2 + 1$), while this energy is well present in wire 2. This produces a relative energy difference of approximately five percent, resulting in a significant interwire energy flow. This flow does not interfere too much with the drag effect, but is, however, an undesirable effect, which could introduce undesired physics. Considering the second protocol, we greatly reduce this effect by truncating also wire 2 into two identical halves during the phase of thermal preparation; this trick suppresses the interwire current of about three orders of magnitude.

The protocol used is the same and is shown in Figure 4.2, top panel.

At this point we describe how the real time evolution is carried out. At $t = 0$ the Hamiltonian is modified, reintroducing the link between sites $N/2$ and $N/2 + 1$ (for both wires), so that the Hamiltonian of the wires is $H_{1/2} = \sum_{i=1}^{N-1} h_{i,1/2}$. Furthermore a local coupling similar to that of equation (4.1.5) is introduced; the
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\[ t < 0 \]

\[ \beta_L, H_L \]

\[ \beta_R, H_R \]

\[ t > 0 \]

\[ \frac{\beta_L + \beta_R}{2}, H \]

\[ J_1 \parallel \]

\[ J_2 \parallel \]

local coupling

Figure 4.2: Schematic picture of the thermal state of the system for \( t \leq 0 \) (top panel) and of the time evolution of the system for \( t > 0 \) (bottom panel). In the upper panel, for each half of the wire are showed the inverse temperature and the Hamiltonian, while, in the lower one, the coupling region is coloured in blue and are shown the longitudinal currents in wire 1 and wire 2 (that are defined in section 4.3).

chosen parameters are \( U_{N/2} = U_{N/2+1} = U \) and \( U_i = 0 \) for the other values of \( i \):

\[
H_{12} = U \sum_{i=N/2,N/2+1} \left( a_{i,1}^{\dagger} a_{i,1} - \frac{1}{2} \right) \left( a_{i,2}^{\dagger} a_{i,2} - \frac{1}{2} \right) \tag{4.2.2}
\]

In other words, we use a local coupling acting only on the central sites, implementing, in this way, the request for a small coupling region in a computational language. Summing up, the temporal evolution is ruled by the Hamiltonian \( H = H_1 + H_2 + H_{12} \).

The bottom panel of Figure 4.2 shows schematically the time evolution situation of the system.

4.2.2. Numerical implementation

In this paragraph, we show how the protocol just described is implemented during a numerical simulation.

We work with two chains (wire 1 and wire 2), each one with open boundary conditions, and with the two respective auxiliary systems (ancilla 1 and ancilla 2).
Therefore the local Hilbert space (i.e. the Hilbert space for each site) has dimension equal to $16 = 2^4$. This dimension is greater than the Hilbert space dimension appearing in a simulation with a single wire (where it is $4 = 2^2$), for which the code employed has been tested up to now; this increase of the local Hilbert space dimension forces us to choose a smaller length for the two wires, which is $N = 20$.

At the beginning the protocol finds the maximally entangled state of the system wire 1+ancilla 1; the reason of this first step is that tracing out the degree of freedom of the ancilla, what is left is the thermal state at infinite temperature, or $\beta = 0$ (this is explained in more details in Appendix G). The search for the maximally entangled state is readily done by introducing an appropriate Hamiltonian, whose ground state is the desired state, and then by doing a variational search.

Afterwards, the thermal state is obtained by carrying out an imaginary time evolution on the maximally entangled of the system wire 1 + ancilla 1 and tracing out the auxiliary system; this is explained with more details in Appendix G.

This imaginary time evolution is carried out with a fourth order Trotter decomposition of the evolution operator and using imaginary time steps of $0.005\,\mathcal{J}^{-1}$. At every step, the state is described by a set of $D(\beta) \times D(\beta)$ matrices (namely the bond links, see also Appendix G), which account for the entanglement and the correlations of the system: the greater is $D(\beta)$, the greater is the accuracy with which we describe the correlations in the system; usually, $D(t)$ grows with time, slowing down the simulation and, when it reaches a maximum value $D$, the matrices are truncated, monitoring the error, which must be smaller than $10^{-10}$; furthermore, during the imaginary evolution, the maximum bond link dimension allowed is $D = 1000$.

For what concerns the preparation of thermal state of wire 2, the same protocol is used, employing the same Trotter decomposition, imaginary time step, maximum bond link dimension and error threshold.

The time evolution of the wires is carried out using as Hamiltonian $H_1 + H_2 + H_{12}$, while for the two ancillas is employed $-(H_1 + H_2 + H_{12})$; in fact, the time evolution of the ancilla is not relevant and can be arbitrarily chosen: in the literature (see Ref. [50]) is proved that the best computational choice is the previous one.

Since $N = 20$, the sites involved in the coupling are 10 and 11.

A fourth order Trotter decomposition is used again, but with a time step of $0.01\,\mathcal{J}^{-1}$. The error threshold is $10^{-10}$, while several maximum bond link dimensions can be employed: during our simulations, the most used was $D = 3000$; unfortunately, this choice greatly increases the computational times, forbidding to see a part of the relevant physical behaviour of the system: therefore, also the values $D = 2500$, 2000, 1500 and 1000 were successively tried, progressively simplifying the complexity of the simulations.

Summarizing, the numerical implementation of the protocol works as follow: we prepare the system of decoupled wires into a thermal state by mean of an imaginary time evolution; this preparation creates a temperature difference between the two halves of wire 1; at $t = 0$ we switch on the coupling between the central sites and let the system evolve.

Due to the small length of the wires $N = 20$, some undesirable boundary effect could be observed when the perturbation of the energy propagates from the center of the wires towards the ends; this is something that we do not want and so we can
consider only the results obtained before that the perturbation reaches the ends (or just only one end) of the wires. This is a control that we will have to make for every simulation.

The relevant observables are measured every $\delta t = 0.05 \mathcal{J}^{-1}$ during both imaginary time and real time evolution.

4.3. Results of simulations

In this section we describe briefly the methods employed to analyse the numerical data and report the plots obtained.

We will follow a sort of chronological order, firstly talking about the calculation of the currents and, then, explaining the results in the same order in which we obtained them.

4.3.1. Calculation of the currents

The ultimate target of this Chapter is the numerical simulation of the thermal drag effect, which will be done analysing the energy current: let us carefully explain how to calculate this quantity.

In section 4.2 we defined the Hamiltonian relative to the left half of wire 1 as $H_{1L} = \sum_{i=1}^{9} h_{i,1}$; an analogous definition can be made for the Hamiltonian relative to right half and to wire 2.

This definition is right during the imaginary time evolution ($t < 0$), but during the real time evolution for $t > 0$, the contribution of $h_{10,1}$ has to be considered, since the two central sites have been coupled at $t = 0$. This additional term contributes in equal parts to the left and to the right Hamiltonian, splitting into two halves; the same argument applies to wire 2.

Moreover, during the temporal evolution phase, also the energy deriving from the coupling has to be considered. It is

$$U\langle (a_{10,1}^\dagger a_{10,1}^\dagger - 1/2)(a_{10,2}^\dagger a_{10,2}^\dagger - 1/2) \rangle + U\langle (a_{11,1}^\dagger a_{11,1}^\dagger - 1/2)(a_{11,2}^\dagger a_{11,2}^\dagger - 1/2) \rangle$$

This energy is split between the two pairs of sites (10,10) and (11,11), where the notation $(,)$ indicates a site on wire 1 and a site on wire 2; this consideration tells us that the first addend in the previous equation contributes to both $H_{1L}$ and $H_{2L}$, while the second addend contributes to both $H_{1R}$ and $H_{2R}$; for symmetry reasons the contributions split into equal halves between the two wires.

Combining all the previous considerations, it is possible to write

$$H_{1L} = \sum_{i=1}^{9} h_{i,1} + \frac{1}{2} h_{10,1} + \frac{U}{2} (a_{10,1}^\dagger a_{10,1}^\dagger - 1/2)(a_{10,2}^\dagger a_{10,2}^\dagger - 1/2)$$

$$H_{1R} = \sum_{i=11}^{19} h_{i,1} + \frac{1}{2} h_{10,1} + \frac{U}{2} (a_{11,1}^\dagger a_{11,1}^\dagger - 1/2)(a_{11,2}^\dagger a_{11,2}^\dagger - 1/2)$$

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4.3. Results of simulations

and analogously for wire 2.

Once defined the correct Hamiltonians to use, let us address how to calculate the current: there are two similar ways to do it.

The first one is to calculate the current as the expectation value of the time derivative of the Hamiltonian, for example $J_{1L}(t) = \langle \dot{H}_{1L}(t) \rangle$; the derivative of the Hamiltonian is calculated through a commutator and yields a three point operator.

An alternative way to proceed is to calculate first the expectation value of the Hamiltonian and then perform an approximate discrete time derivative between two successive measurements in order to obtain the relative current.

We will follow this second method. Consequently, in order to calculate the expectation value of the Hamiltonians above, we need the measurements of five observables:

- $\langle a_{i,1}^\dagger a_{i,1} - 1/2 \rangle (a_{i,2}^\dagger a_{i,2} - 1/2)$: this correlation is needed in order to calculate the energy since it appears in the coupling Hamiltonian; note that, it is not necessary to calculate it throughout the whole wires but only for $i = 10, 11$.

- $\langle a_{i,1}^\dagger a_{i+1,1} - 1/2 \rangle (a_{i,2}^\dagger a_{i,2} - 1/2)$ and $\langle a_{i,2}^\dagger a_{i+1,2} - 1/2 \rangle (a_{i+1,2}^\dagger a_{i+1,2} - 1/2)$: these correlations are needed in order to calculate the intrawire interaction energies respectively for wire 1 and wire 2.

- $\langle a_{i+1,1}^\dagger a_{i,1} \rangle$ and $\langle a_{i+1,2}^\dagger a_{i,1} \rangle$: these correlations are needed in order to calculate the kinetic energies respectively for wire 1 and wire 2.

These observables are measured for every site and every $\delta t = 0.05 J^{-1}$. For this reason, the current at the time $t_j = j\delta t$ is defined as

$$J_{1L}(t_j) = \frac{E_{1L}(t_j) - E_{1L}(t_{j-1})}{\delta t}, \quad J_{1R}(t_j) = \frac{E_{1R}(t_j) - E_{1R}(t_{j-1})}{\delta t} \quad (4.3.1)$$

where $E_{i,L/R} = \langle H_{i,L/R} \rangle$ and $J_{1L}(t = 0) = J_{1R}(t = 0) = 0$; analogous definitions can be made for wire 2. In other words, $J_{1L}$ is the rate at which the the left half of wire 1 loses energy (and this explain the minus sign); instead, $J_{1R}$ indicates how much energy per time the right half of wire 1 gains.

So far, four currents have been defined: $J_{1L}$, $J_{1R}$, $J_{2L}$ and $J_{2R}$. Remarkably only three of them are independent, thanks to the conservation of total energy; following the derivation in section 3.1 we define the longitudinal current $J_{\parallel}$ as the average of $J_{iL}$ and $J_{iR}$ and the interwire current as $J_{\perp} = J_{2R} - J_{2L}$.

It can be argued that the longitudinal current in a wire is the net transfer rate of energy from the left half to the right half of the wire.

At this point an important observation has to be made: given these definitions, we are including also the contribution of the coupling into the calculation of the longitudinal currents; however, in Chapter 3 this contribution was ignored. The reason is that the coupling contribution is negligible for $L \to \infty$; here, on the other hand, we work with twenty sites and also the effect of the finite size of the system has to be considered.

---

**Thermal drag between two coupled quantum wires**
Having introduced all the tools used, we are now ready to calculate the physical quantities we need from the rough data.

4.3.2. Preliminary results in the absence of interwire coupling

In this paragraph, we discuss the information obtained from the first simulations.

The first simulations were carried out with $U = 0$ (i.e. decoupled wires), in order to verify that the protocol and the program written yield results compatible with those available in literature for a single wire (see for example Ref. [45]).

This comparison was performed plotting the temporal behaviour of the longitudinal current in wire 1, $J_{1\|}(t)$, and comparing its numerical value with that yielded by a simulation run using a single wire protocol. We anticipate that the numerical agreement between the two protocols is very good.

![Figure 4.3: Plot of $J_{1\|}(t)$ for $U = 0$, $\beta_L = 0.5 J^{-1}$, $\beta_R = 0.55 J^{-1}$ and $\Delta = 0.9$ (top), $\Delta = -0.9$ (bottom).](image)

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4.3. Results of simulations

Several simulations were run using as parameters $\beta_L = 0.5 \, J^{-1}$, $\beta_R = 0.55 \, J^{-1}$ and $\Delta = -0.9, -0.2$ and $0.9$; besides we also ran a simulation with more mismatched temperatures $\beta_L = 0.5 \, J^{-1}$, $\beta_R = 0.75 \, J^{-1}$ and $\Delta = 0.0$.

The plots in Figure 4.3 and 4.4 prove that the current exhibits the same qualitative behaviour even for different values of the parameters: it starts from zero and then grows and reaches a peak, whose position seems to slightly depend on $\Delta$, varying from $t = 0.6 \, J^{-1}$ for $\Delta = \pm 0.9$ to $t = 0.8 \, J^{-1}$ for $\Delta = -0.2$ and $0.0$; after this peak the current begins to decrease very gently, approaching a plateau for $t = t_{pl}$. Even $t_{pl}$ depends on the value of $\Delta$: it seems to range from $1.05 \, J^{-1}$ for $\Delta = -0.9$, to $1.3 \div 1.4 \, J^{-1}$ for $\Delta = -0.2$ and $0.0$.

![Image of Figure 4.4](image-url)

**Figure 4.4**: Plot of $J_{||}(t)$ for $U = 0$, $\beta_L = 0.5 \, J^{-1}$, $\beta_R = 0.55 \, J^{-1}$ and $\Delta = -0.2$ (top) and $U = 0$, $\beta_L = 0.5 \, J^{-1}$, $\beta_R = 0.75 \, J^{-1}$ and $\Delta = 0$ (bottom).

Comparing the two plots in Figure 4.4 it can be argued that the values of $J_{||}$ at the peak and at the plateau strongly depend on the temperature difference between the two ends of the wire.

Furthermore these values depend also on $\Delta$: they seem enhanced by a strong repulsive coupling, varying from $J_{||} \approx 0.03 \, J^2$ for $\Delta = -0.9$ to $J_{||} \approx 0.1 \, J^2$ for

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\( \Delta = 0.9 \), where \( J_{1\parallel} \) indicates the plateau value of the current.

In Figure 4.5 we show the energy profile of wire 1, plotting the energy associated to the 19 links between the 20 sites. The central peak at \( t = 0 \) corresponds to the missing link between the central sites, whose absence strongly increases the energy of the wire; we also notice that the energy profile of right half is slightly lower than that of left half, because of the different temperatures between the two ends. As \( t \) flows, we see that there is an energy transfer from the central link to the others, raising their energy; nevertheless, we continue to observe that the energy of right half is still a bit lower and this agrees with the fact that a current is still flowing also for \( t = 1.6 \mathcal{J}^{-1} \) (see Figure 4.3, top panel). Finally, it can be checked that the perturbation has not yet reached the ends of the wire and so we are not seeing any border effect.

Figure 4.5: Plot of the profile of energy in wire 1 for three different times; the energy pictured is actually the energy associated to the link between two sites. The parameters are \( U = 0 \), \( \beta_L = 0.5 \mathcal{J}^{-1} \), \( \beta_R = 0.55 \mathcal{J}^{-1} \) and \( \Delta = -0.2 \).

We do not show any plot of \( J_{2\parallel} \) and \( J_{\perp} \), since they are zero for \( U = 0 \) and the simulations yield values that are of order \( 10^{-7} \) and are generated by numerical errors.

Summing up, a comparison was made between the values of \( J_{1\parallel}(t) \) that we obtained and those obtainable applying the same protocol to a single wire with a greater number of sites (\( N = 80 \) or more) finding a good agreement between the numerical values.

4.3.3. Results in the presence of an interwire coupling

Having finished this check, various simulations with values of \( U \) different from zero were run, in order to see an induced current. Let us begin by considering the plot shown in Figure 4.6.

The first thing we notice is that the induced current is very small and it could be
4.3. Results of simulations

Figure 4.6: Plot of $J_{2L}(t)$ for $U = 0.1\mathcal{J}$, $\beta_L = 0.5\mathcal{J}^{-1}$, $\beta_R = 0.55\mathcal{J}^{-1}$ and $\Delta = 0.2$.

influenced by numerical errors cumulated during the simulation. The most natural choice was to increase the temperature difference in order to increase also the drag current: the temperature of the right half in wire 1 was set to $\beta_R = 0.75\mathcal{J}^{-1}$ and were run simulations for $U/\mathcal{J} = \pm0.005$, $\pm0.1$, $\pm0.2$ and $\pm0.3$ and $\Delta = 0$. This choice of the anisotropy parameter was made in order to speed up the processes, which are slowed down very much by the presence of an interaction.

In Figure 4.7 we can observe the plot comparing the drag current for $U = 0.1\mathcal{J}$ and $U = -0.1\mathcal{J}$. It can be noticed that the current amplitude seems to be increased by a factor 10 respect to the current in Figure 4.6; this amplification is compatible with a linear response regime, which would yield a factor 8.25 of difference; still, we have to keep in mind that for the temperature values just used, $T_L - T_R = 0.5T_R$ and, probably, this value is beyond the linear response regime; furthermore, this analysis is based on the peak values and not on the plateau values, which are the ones to consider.

Besides the amplification of the current, it can be noticed that the two plots for opposite $U$ are indistinguishable; the same assertion holds for the other values of $U$ tested. This symmetry between opposite values of $U$ is confirmed by the analysis of the numerical values of the current, which corresponds up to three or more significant digits.

This observation is the consequence of a very profound symmetry, present in the Hamiltonian used to simulate the system, and which forbid any dependence of the current on odd powers of $U$; this is what we are going to prove.

As explained in section 4.3.1, we calculate the currents by operating the discrete temporal derivative of quantities such as

$$E_{2L}^{(U)}(t) = \langle H_{2L}^{(U)}(t) \rangle = \text{Tr}(e^{-i(H_1+H_2+H_{12})t}\rho(0)e^{i(H_1+H_2+H_{12})t}H_{2L}^{(U)})$$

where the dependence on $U$ of the energy has been indicated with the superscript $(U)$; the density matrix at time $t = 0$ does not depend on $U$, since it is the result of

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Figure 4.7: Plot of $J_{||}(t)$ for $U = 0.1 \mathcal{J}$ (top panel) and $U = -0.1 \mathcal{J}$ (down panel), $\beta_L = 0.5 \mathcal{J}^{-1}$, $\beta_R = 0.75 \mathcal{J}^{-1}$ and $\Delta = 0$.

an imaginary time evolution with decoupled wires.

We now want to see what changes if the transformation $U \to -U$ is operated: first of all, we see that the coupling Hamiltonian changes sign $H_{12} \to -H_{12}$; indeed the single wire hamiltonians $H_1$ and $H_2$ remain the same; the most complicated transformation is that of $H^{(-U)}_{2L}$, since it goes into

$$H^{(-U)}_{2L} = \sum_{i=1}^{9} h_{i,2} + \frac{1}{2} h_{10,2} - \frac{U}{2} \langle (a^\dagger_{10,1} a_{10,1} - 1/2) (a^\dagger_{10,2} a_{10,2} - 1/2) \rangle$$

We now want to understand if there is a symmetry of the system that allows us to say that the temporal evolution is unchanged under the sign change of $U$. Indeed, such a symmetry exists and is implemented by the transformation:

$$a^\dagger_{i,2} \to (-1)^i a^\dagger_{i,2} \quad a_{i,2} \to (-1)^i a_{i,2}$$

Physically this transformation changes the sign of density fluctuations in wire 2. Let us call $V$ the operator which implements this transformation; then it has the

**Thermal drag between two coupled quantum wires**
following effects:

\[ V(a_{i+1,2} a_{i,2} + a_{i,2} a_{i+1,2}) V^{-1} = -a_{i+1,2} a_{i,2} - a_{i,2} a_{i+1,2} = a_{i+1,2} a_{i,2} + a_{i,2} a_{i+1,2} \]

\[ V\left(a_{i,2} a_{i,2} - \frac{1}{2}\right) V^{-1} = \left(a_{i,2} a_{i,2} - \frac{1}{2}\right) = \left(1 - a_{i,2} a_{i,2} - \frac{1}{2}\right) = -\left(a_{i,2} a_{i,2} - \frac{1}{2}\right) \]

this implies that \( V \) leaves unchanged all the \( h_{i,2} \), as well as the operators referring to wire 1; therefore \( \rho(0) \) is invariant and, from equations (4.1.3) and (4.1.4), it can be written

\[ VH_1 V^{-1} = H_1, \quad VH_1^{(-U)} V^{-1} = H_1^{(-U)}, \quad VH_2 V^{-1} = H_2, \quad VH_2 V^{-1} = -H_1 \]

In other words, the transformation \( V \) is equivalent to a sign change of \( U \).

The expectation value \( E_{1L}(t)^{(-U)} \) calculated for \( -U \) can be written as

\[ E_{2L}^{(-U)}(t) = \text{Tr}(e^{-i(U + H_2 - H_{12})t} \rho(0) e^{i(U + H_2 - H_{12})t} H_{2L}^{(-U)}) \]

if we apply the transformation \( V \), this value cannot change, so that

\[ E_{2L}^{(-U)}(t) = \text{Tr}(e^{-iV(H_1 + H_2 - H_{12})V^{-1}t} \rho(0) V^{-1} e^{iV(H_1 + H_2 - H_{12})V^{-1}t} V H_{2L}^{(-U)} V^{-1} = \text{Tr}(e^{-i(H_1 + H_2 + H_{12})t} \rho(0) e^{i(H_1 + H_2 + H_{12})t} H_{2L}^{(-U)}) = E_{2L}(t) \]

In conclusion, it has been proven that there is a symmetry of the system that leaves the energies and, consequently, the currents unchanged for a sign change of the coupling; this has the direct consequence that every current contains only even powers of \( U \). Since \( J_{\parallel} \) is zero when the coupling is absent, then its leading order for small coupling is \( U^2 \), as predicted by equation (3.3.17). We have therefore validated our result of section 3.2, extending it also to the case of non-linear response (the treatment made above does not depend at all from the temperature settings).

The purpose of the next subsections will be to construct a plot of the plateau value of \( J_{\parallel} \) versus \( U \).

### 4.3.4. Bond link dimension and accuracy of the simulations

We continue to analyse the temporal behaviour of \( J_{\parallel}(t) \); we can see it in Figure 4.7 and 4.8, where we show \( J_{\parallel}(t) \) for \( U/J = 0.005, 0.2 \) and 0.3 (for the sake of readability we do not report the plot for the corresponding negative values of \( U \).

The qualitative behaviour is approximately the same in all the case examined: initially the drag current is negative, growing in amplitude until a peak is reached (around \( t = 0.25, J^{-1} \)); then it begins to increase, passing through the zero (around \( t = 0.5 \div 0.6, J^{-1} \)) and reaching a positive peak (around \( t = 0.8 \div 0.9, J^{-1} \)); at this point the current decrease again, but this time much more gently. Then, the simulations stop, because the computational time required to continue was excessive and they were terminated.

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Figure 4.8: Plot of $J_2(t)$ for $U = 0.05 \mathcal{J}$ (top panel), $U = 0.2 \mathcal{J}$ (middle panel) and $U = 0.3 \mathcal{J}$ (down panel), $\beta_L = 0.5 \mathcal{J}^{-1}$, $\beta_R = 0.75 \mathcal{J}^{-1}$ and $\Delta = 0$.

The last trend of the current can be read as an indication of a possible steady state that the current is going to reach; unfortunately, we were not able to push the simulation until this plateau is reached. Various possibilities to reduce the thermal drag between two coupled quantum wires.
computational weight of the simulation can be examined: since decreasing the length of the wires is not an option because $N = 20$ is already a very small value, a possible solution is to try to reduce the maximum bond link dimension $D$, previously set to $D = 3000$ during the temporal evolution.

In fact, the computational time required for a time step is approximately proportional to $D(t)^2$, where $D(t)$ refers to the bond link dimension during that step. Therefore, truncating the maximum dimension, allows to reduce the computational times during the final steps of the simulation, when usually the matrices computed reach the maximum dimension allowed.

A great care is needed in reducing $D$, since important correlations between the states could be dropped in the calculation of the observables, obtaining, therefore, results with systematic and significant errors.

For this reason, we ran four simulations with parameters $U = 0.3 \mathcal{J}$, $\beta_L = 0.5 \mathcal{J}^{-1}$, $\beta_R = 0.75 \mathcal{J}^{-1}$, $\Delta = 0$ and $D = 2500, 2000, 1500, 1000$; the purpose of this simulations was to understand which values of $D$ permit to see the plateau of $J_{2\parallel}$ without affecting the data with huge errors.

In the plot of Figure 4.9 the behaviour of the current obtained for $D = 1500$ is reported as an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{current_plot}
\caption{Plot of $J_{2\parallel}(t)$ for $U = 0.3 \mathcal{J}$, $\beta_L = 0.5 \mathcal{J}^{-1}$, $\beta_R = 0.75 \mathcal{J}^{-1}$, $\Delta = 0$ and $D = 1500$.}
\end{figure}

It can be observed a plateau of the current, extending approximately from $t = 1.55 \mathcal{J}^{-1}$ to $t = 1.75 \mathcal{J}^{-1}$. It is also noticeable that the qualitative behaviour of the current is not changed by the reduction of $D$.

We did not limit ourselves to plot the current as a function of the time; for each value of $D$, the maximum time $t_{\max}$ reached by the simulation was measured. Furthermore, the values of the longitudinal current $J_{2\parallel}(t)$ yielded by the four additional simulations were compared with those of the “parent” simulation run with $D = 3000$; we called $t_*$ the first time at which a simulation yielded a result differing from that of the parent simulation for more than $10^{-8}$ (arguably the first simulations to be different are those with smaller values of $D$, as one can notice from Table 4.1).

Finally, we wanted to see how much is the error committed lowering $D$. For this
4. Numerical simulations

purpose the parameter $\delta_r J$ can be defined; it is, for a value $D = D_i$, the maximum relative difference between two values of $J_{2||}(t)$ evaluated at the same time, one for $D_i$ and the other for the value of $D$ immediately greater, i.e. $D_{i-1}$. In formulas, for $D = 1500$

$$\delta_r J = \max \left\{ \frac{J_{2||}(D = 1500, t) - J_{2||}(D = 2000, t)}{J_{2||}(D = 2000, t)}, \text{ for } t < t_{\text{max}}(D = 2000) \right\}$$

where clearly the time over which the comparison is made has to be smaller than the maximum time reached by the simulation for $D = 2000$. Indeed, the relative error is maximum exactly for this time, since the error grows with time. Plotting the relative error versus the time step for $D = 1000$, we noticed that, for a relative error smaller than 3%, it increases more than linearly but less than exponentially, while for greater values, it slows down its growth.

The results are summarized in the following table:

<table>
<thead>
<tr>
<th>$D$</th>
<th>$t_{\text{max}}$</th>
<th>$t_*$</th>
<th>$\delta_r J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3000</td>
<td>1.1 $J^{-1}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2500</td>
<td>1.3 $J^{-1}$</td>
<td>1.0 $J^{-1}$</td>
<td>0.008 %</td>
</tr>
<tr>
<td>2000</td>
<td>1.5 $J^{-1}$</td>
<td>0.9 $J^{-1}$</td>
<td>0.31 %</td>
</tr>
<tr>
<td>1500</td>
<td>1.7 $J^{-1}$</td>
<td>0.8 $J^{-1}$</td>
<td>2.1 %</td>
</tr>
<tr>
<td>1000</td>
<td>2.15 $J^{-1}$</td>
<td>0.7 $J^{-1}$</td>
<td>7.6 %</td>
</tr>
</tbody>
</table>

Table 4.1: Table summarizing the results obtained from the analysis of the simulation with different $D$.

Summing up, five different simulations with the same parameters were run, each time reducing the value of $D$ and extending the time reached by the simulation; it could be tempting to bond the results between themselves, extending the current behaviour in a particular time range with the results from the simulation with the successively lower value of $D$. The final product of this procedure is shown in Figure 4.10: for $t \leq 1.1 J^{-1}$ are reported the data from the simulation run with $D = 3000$; the red points represent the data added from the simulations with $D = 2500$, then $D = 2000$, afterwards $D = 1500$ and, finally, $D = 1000$.

This procedure creates an inaccuracy, which we tried to estimate with the parameter $\delta_r J$. This error is cumulative, in the sense that at $t = 1.7 J^{-1}$ the simulation with $D = 1000$ is a 7.6% less accurate than the one with $D = 1500$, which at $t = 1.5 J^{-1}$ is a 2.1% less accurate than the simulation with $D = 2000$, and so on. This inaccuracy can be noticed by the comparison between Figure 4.9 and Figure 4.10: even though in the last one the duration of the simulation is greater, the behaviour of the current is not as “natural” and smooth as in the former plot.

In conclusion, if an error smaller than 5% is acceptable, then $D = 1500$ is the best choice for the maximum bond link dimension; in fact, this value allows to reach the plateau of $J_{2||}$ within reasonable computational times and commits an error below the fixed threshold.

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Figure 4.10: Plot of $J_{2\parallel}(t)$ for $U = 0.3 \, J, \beta_L = 0.5 \, J^{-1}, \beta_R = 0.75 \, J^{-1}, \Delta = 0$. Various values of $D$ are employed: the blue line is constructed with the data from $D = 3000$, whereas the red points correspond to $D = 2500, 2000, 1500, 1000$.

4.3.5. Final results

In this subsection, the dependence of the plateau value of the drag current $\bar{J}_{2\parallel}$ on the coupling $U$ is analysed and plotted.

Given the conclusion of the last paragraph, we chose to run the simulation with $D = 1500$; furthermore the following values were employed: $\beta_L = 0.5 \, J^{-1}, \beta_R = 0.75 \, J^{-1}$ and $\Delta = 0$. The values of $U$ tested are $0, 0.05 \, J, 0.1 \, J, 0.15 \, J, 0.2 \, J, 0.25 \, J, 0.3 \, J, 0.4 \, J, 0.5 \, J, 0.75 \, J$ and $1.0 \, J$.

Figure 4.11: Plot of $J_{2\parallel}(t)$ for $U = 0.2 \, J, \beta_L = 0.5 \, J^{-1}, \beta_R = 0.75 \, J^{-1}, \Delta = 0$. The points of the plateau are highlined in red.

To determine $\bar{J}_{2\parallel}$, the procedure of Figure 4.11 was adopted: the points for which the current appeared to be approximately stationary were chosen and used.
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to calculate the average current. Nevertheless, this method has its intrinsic error, since the duration of the plateau is greatly limited by the growing computational cost of the simulation and the points on which the average is calculated are few.

The results obtained from $U = 0.05 \mathcal{J}$ were discarded, since the current assumes values that are too low and has a temporal behaviour that is qualitatively different from that observable for other values of $U$. This strange trend was attributed to cumulative rounding errors committed during the simulation, which could have influenced the value of the current, especially for large times.

The inaccuracy is not easy to determine, since we have to consider both the error arising from the finite value of $D$ and that arising from the calculation of $\bar{J}^2_\parallel$; a relative inaccuracy of 5% appears to be a quite larger and safe estimate.

The rough data (without error) are collected in Table 4.2.

<table>
<thead>
<tr>
<th>$U$ ($\mathcal{J}$)</th>
<th>$\bar{J}^2_\parallel$ ($\mathcal{J}^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.81 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>0.15</td>
<td>$4.8 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$8.8 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>0.25</td>
<td>$1.4 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$2.0 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$3.4 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$5.1 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>0.75</td>
<td>$9.2 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>1.0</td>
<td>$1.23 \cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.2: Table reporting the results obtained from the simulations.

Plotting the results, it can be immediately noticed that the quadratic dependence of $\bar{J}^2_\parallel$ is a very good approximation for $U \lesssim 0.5 \mathcal{J}$, but does not fit the data for $U \approx 1.0 \mathcal{J}$.

Thus, two different fits were performed: one was carried on with a parabola forced to pass through the origin ($\bar{J}^2_\parallel = a'U^2$) in the range $U = 0 \div 0.5 \mathcal{J}$; instead, a function including also a quartic term ($\bar{J}^2_\parallel = aU^2 + bU^4$) was used to fit all the available data. The agreement between these two curves is quite good, not only for small couplings, but also in the crossover region around $U \approx 0.5 \div 0.6 \mathcal{J}$; on the other hand, for couplings of the order of $\mathcal{J}$, the two fits greatly differ. This analysis is graphically represented in Figure 4.12.

The results of the fit are given in the following equations.

\[
\bar{J}^2_\parallel = 0.021 \cdot U^2 \quad \text{for } U \leq 0.5 \mathcal{J} \quad (4.3.2)
\]

\[
\bar{J}^2_\parallel = 0.022 \cdot U^2 - 0.01 \cdot U^4 \quad \text{for } U \leq 1.0 \mathcal{J} \quad (4.3.3)
\]

It can be argued that, for the precision standards we are adopting, in the limit of small couplings, carrying on a fit with a quadratic term only or including also a quartic correction is approximately equivalent, because the coefficients obtained differ by approximately 5%.

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Furthermore, it is noticeable that the quartic correction to the leading quadratic order is negative and suppresses the drag current at high coupling. This consideration suggests that employing a very strong coupling is not the best way to enhance the drag current; nevertheless, it cannot be excluded that, for even greater values of $U$, a positive correction going as $U^6$ could be relevant, increasing, therefore, the current.

To conclude our discussion, let us briefly summarize the results obtained: we ran various simulations using a widely tested out-of-equilibrium protocol and verified the good functioning of the program. The presence of a drag current, as predicted in Chapter 3, was noticed; adopting a threshold value for the error committed, we were able to observe a plateau in the current behaviour, measuring its value and plotting it as a function of $U$. The $U^2$ trend predicted in Chapter 3 for small couplings was verified and we tested the strong couplings regime, finding an additional $U^4$ term.
Conclusions

In these pages we want to summarize the results obtained in this thesis, reporting the key ideas and addressing the most interesting further developments.

We considered a system of two parallel quantum wires of spinless electrons coupled by a Coulombian interwire interaction, arguing that the phenomenon of thermal drag emerges: as in the case of Coulomb drag, the interaction couples the density fluctuations of the two wires and induces a thermal current in the second wire when a temperature gradient is applied to the first one. The coupling allows an energy flow between the two wires, leading to a novel effect absent in the case of Coulomb drag.

We studied the drag phenomenon by analysing the average of the thermal currents at the two ends of a wire.

In order to derive a Kubo-like formula for the transconductivity, we followed the idea of Luttinger (Ref. [5]) and simulated the effect of a thermal gradient introducing a scalar perturbation in the Hamiltonian.

Then the conductivity matrix was inverted to get the resistivity matrix and thus the transresistivity, applying the memory function formalism (Refs. [11] and [12]), which transforms a correlator between two currents into a correlator between the time derivatives of two currents.

The equation obtained was manipulated, getting a general formula for the transresistivity interestingly valid in any range of temperature. On the other hand, a further analysis of this expression was quite difficult to perform and we made the low energy approximation, deciding to specialize our work to the drag between two Luttinger liquids.
This approximation allowed us to employ the bosonization tools and simplify our formulas, finally writing the transresistivity in terms of two density-density correlation functions evaluated for decoupled wires. The dependence on the coupling, which is quadratic at the leading order, was separated from the intrawire interaction effect, manifesting the physical origin of the various contributions.

An expression directly proportional to that of electric transresistivity was obtained, deriving a Wiedemann-Franz law analogue for the drag phenomenon in the low temperatures limit: this is the main original contribution produced during this work. Physically, this Wiedemann-Franz analogue is a consequence of the low energy approximation: in fact, in this limit the thermal current assumes a form equivalent to that of electric current (they are both linear in the momentum) and the behaviour of the respective transresistivities is unsurprisingly similar.

To continue our analysis, we studied the contributions of forward and backward scattering terms to the transresistivity, concluding that, in a wide range of temperatures, the forward scattering is dominant.

Then we derived, in a qualitative fashion, the essential features of the transresistivity and transconductivity, indicating with $T_1$ the temperature scale associated to the difference between the Luttinger velocities of the two wires and with $T_0$ the scale related to the characteristic length of the coupling.

For $T_1 < T_0$ we found that the transresistivity has a peak around $T_1$, growing linearly before and decreasing after it; on the other hand, the transconductivity exhibits a peak around $T_0$, decaying towards zero for both $T \ll T_0$ and $T \gg T_0$. The presence of these peaks is a quite remarkable feature, since it is very easy to identify from the data of an experiment or a numerical simulation.

In the end, we applied the alternative method outlined in Appendix F (relying on a force balance equation rather than on the memory function formalism) to derive the thermoelectric coefficients of the drag phenomenon.

As a complement to our theoretical work, we performed a series of numerical simulations, using an out-of-equilibrium protocol equivalent to that of Chapter 3. The protocol relies on the formalism of MPS (see Refs. [18] and [45]) and simulates the temporal evolution of the current.

In order to run simulations that allow to observe the plateau reached by the current, we decided to lower the maximum bond link dimension to $D = 1500$, reducing the computational weight of the simulations and containing the error below a 5% threshold.

We focused on the dependence of the drag current on the coupling strength, validating the prediction of a quadratic behaviour for small couplings and extending the range of validity with the inclusion of a fourth order correction.

These are the results we produced; we now want to briefly discuss the possible topics that could be addressed in successive studies and what kind of experimental or numerical work can be carried out to study the problem treated.

We did not consider the spin and an immediate, but not easy, extension would be to include it in all the calculations; this is not an impossible task in the Luttinger liquid model, but the situations is much more complicated if one takes into account the non linearities in the dispersion relation, which introduce some correcting terms.
Levchenko (Ref. [22]) calculated the correction to the thermal conductivity of a single quantum wire due to these additional terms; an interesting work would be to do the same for thermal drag, considering the corrections to the transresistivity and extending the temperature range of validity of our results.

Another stimulating topic is the analysis of equation (3.3.17) in a wider range of temperatures. On the other hand, this theoretical work is not easy at all, since a general interaction between electrons makes the calculation very difficult to perform and some approximations may be needed in order to obtain even partial results.

A good alternative way to study a wide range of temperatures is to perform numerical simulation, which are usually easier to carry out for higher temperatures. For example, we only studied the dependence of the drag current on the interaction strength $U$ for precise values of $\Delta$, $T$ and $\nabla T$; a natural improvement is to test the dependence on $U$ for more values of the anisotropy parameter and of the temperatures.

We did our computational work operating with identical wires; clearly, also the situation with different wires is very interesting and worthy of being analysed; moreover, successive simulations could reproduce the temperature dependence of the transconductivity for both high and low temperatures, trying to verify the presence of the peak predicted by our results.

Interestingly, also a regime of non linear response could be studied through appropriate simulations.

There is also a lot of experimental work that can be done on thermal drag; an immediate and simple experiment could measure the current between two wires held at different temperatures, verifying this peculiarity of thermal drag.

Successive efforts could be made in order to measure the transresistivity and the transconductivity for various range of temperatures and verify if they exhibit the peaks predicted, while experiments conducted in an high temperature regime can be used even to predict the behaviour of the transresistivity in a situation that we did not analyse.

We think that these are the directions towards which the analysis of thermal drag should develop; we look forward for further studies on this interesting topic.
Appendix A

Bosonization formulary

In this Appendix we want to list all the formulas that has been used in Chapter 1 and Chapter 3. For a great part of them we will not give a formal proof; instead we will explain the method used to calculate the correlation functions.

Moreover, we will dedicate a section of this Appendix to the analysis of the bosonization of electrons with spin.

A.1. Boson fields, density and currents

The expressions of the boson fields can be written as

\[
\phi(x) = -\frac{i}{L} \sum_p \left( \frac{Lg}{2|p|} \right)^{1/2} \text{sign}(p) e^{-\alpha|p|/2} (e^{-ipx} b^\dagger_p - e^{ipx} b_p) \\
\theta(x) = \frac{i}{L} \sum_p \left( \frac{L}{2g|p|} \right)^{1/2} e^{-\alpha|p|/2} (e^{-ipx} b^\dagger_p - e^{ipx} b_p) \\
P(x) = \frac{1}{L} \sum_p \left( \frac{L|p|}{2g} \right)^{1/2} \text{sign}(p) e^{-\alpha|p|/2} (e^{-ipx} b^\dagger_p - e^{ipx} b_p)
\] (A.1.1)

As anticipated, the boson fields and the densities of left and right movers are strictly related:

\[
\nabla \phi(x) = -\sqrt{\pi} [\rho_R(x) + \rho_L(x)] \\
\nabla \theta(x) = \sqrt{\pi} [\rho_R(x) - \rho_L(x)]
\] (A.1.2)

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Additionally, equations (1.3.10) and (1.3.15) can be used to calculate the temporal evolution of the relevant operators

\[
\phi(x, t) = -\frac{i}{L} \sum_p \left( \frac{Lg}{2|p|} \right)^{1/2} \text{sign}(p)e^{-\alpha|p|/2}(e^{-ipx+iv|p|t}b_p^\dagger - e^{ipx-iv|p|t}b_p)
\]

\[
\theta(x, t) = \frac{i}{L} \sum_p \left( \frac{Lg}{2|p|} \right)^{1/2} e^{-\alpha|p|/2}(e^{-ipx+iv|p|t}b_p^\dagger - e^{ipx-iv|p|t}b_p)
\]

(A.1.3)

obtaining also

\[
\partial_t \phi(x, t) = vg\partial_x \theta(x, t) \quad \partial_t \theta(x, t) = \frac{v}{g} \partial_x \phi(x, t)
\]

(A.1.4)

The density can be written using equations (A.1.1) and (A.1.2):

\[
\rho(x, t) = -\frac{1}{\sqrt{\pi}} \partial_x \phi(x, t) + \frac{1}{2\pi\alpha} \left[ e^{2ik_fx}e^{-2\sqrt{\pi}\phi(x, t)} + \text{h.c.} \right]
\]

\[
= -\frac{1}{\sqrt{\pi}} \partial_x \phi(x, t) + \frac{1}{\pi\alpha} \cos[2k_Fx - 2\sqrt{\pi}\phi(x, t)]
\]

(A.1.5)

In all these formulas we are calculating the energy (responsible for the temporal evolution) with respect to the Fermi energy: an irrelevant phase $e^{i\mu t}$ has being omitted in the bosonic fields, as well as in $\psi_R$ and $\psi_L$. This phase is irrelevant for the calculation of physical observables, but will be crucial in Appendix B.2 during the calculation of quantities like $\partial_t \psi(x, t)$.

The bosonized expressions for the currents are obtainable from the continuity equations:

\[
J^E(x) = ev(\rho_R(x) - \rho_L(x)) = \frac{v}{\sqrt{\pi}}P(x)
\]

(A.1.6)

\[
J^T(x) = -\frac{v^2}{2} : \{\partial_x \phi(x), P(x)\} :
\]

(A.1.7)

If the average thermal current over the whole wire is desired, an integration over $x$ can be performed, using the symmetry properties of operators to drop the number non conserving terms and obtain

\[
J^T_\parallel = v^2 \sum_k k b_k^\dagger b_k
\]

(A.1.8)

### A.2. Path integral and correlation functions

Now we want to show how the path integrals (in imaginary time) can be used to calculate correlation functions; this method is well explained in Appendices A and C of Ref. [1].
First, it can be seen that the real time evolution of an operator reads like

\[ A(t) = e^{iHt} A e^{-iHt} \]

and also the evolution in an imaginary time \( \tau \) can be defined:

\[ A(\tau) = e^{H\tau} A e^{-H\tau} \]

The expectation value of an operator at temperature \( k_B T = 1/\beta \) is a trace

\[ \langle A \rangle = \frac{1}{Z} \text{Tr}[e^{-\beta H} A] \]

where \( Z = \text{Tr}[e^{-\beta H}] \) is the partition function. Using a functional integral, it can be written as:

\[ Z = \int \mathcal{D}\phi(x, \tau) \mathcal{D}\theta(x, \tau) e^{\int_0^\beta d\tau \int dx [i\partial_\tau \theta(x, \tau) \partial_x \phi(x, \tau) - H(\phi(x, \tau), \theta(x, \tau))] \]

where \( \int \mathcal{D}\phi(x, \tau) \) means that the integration is all over the possible functions \( \phi(x, \tau) \); only periodic function are allowed (\( \phi(x, \tau + \beta) = \phi(x, \tau) \)). Exploiting the potentiality of functional integrals, we can calculate the time ordered correlation functions:

\[ \langle T_\tau B(\tau) A(0) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\theta B(\phi(\tau)) A(\phi(0)) e^{\int_0^\beta d\tau \int dx [i\partial_\tau \theta(x, \tau) \partial_x \phi(x, \tau) - H(\phi(x, \tau), \theta(x, \tau))] \]

where the time ordering acts like

\[ T_\tau B(\tau) A(0) = Y(\tau_1 - \tau_2) A(\tau_1) B(\tau_2) + \epsilon Y(\tau_2 - \tau_1) B(\tau_2) A(\tau_1) \]

with \( Y \) the Heavyside function and \( \epsilon = 1 \) for boson like operators and \( \epsilon = -1 \) for fermion like operators.

We now want to apply this formalism to the following correlation function

\[ \langle [\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2)]^2 \rangle = \sum_{\mathbf{q}_1, \mathbf{q}_2} \frac{e^{-\frac{\alpha |\mathbf{q}_1| + |\mathbf{q}_2|}{2} (\beta L)^2}}{\langle \phi(\mathbf{q}_1) \phi(\mathbf{q}_2) \rangle} \langle e^{i\mathbf{q}_1 \cdot \mathbf{r}_1} e^{i\mathbf{q}_2 \cdot \mathbf{r}_2} \rangle \]

where

\[ \mathbf{r} = (x, v\tau) \quad \mathbf{q} = (k, \omega_n / v) \quad \mathbf{q} \cdot \mathbf{r} = kx - \omega_n \tau \]

and the field has been decomposed in Fourier components, not only over the momentum, but also over the frequency \( \omega_n \):

\[ \phi(r) = \frac{1}{\beta L} \sum_{\mathbf{q}} e^{-\alpha |\mathbf{q}| / 2} e^{i\mathbf{q} \cdot \mathbf{r}} \phi(\mathbf{q}) \]

It is sufficient to calculate the correlation between the Fourier components of the fields:

\[ \langle \phi(\mathbf{q}_1) \phi(\mathbf{q}_2) \rangle = \frac{1}{Z} \int \mathcal{D}\phi(x, \tau) \mathcal{D}\theta(x, \tau) \phi(\mathbf{q}_1) \phi(\mathbf{q}_2) e^{-S} \]

\[ -S = \int_0^\beta d\tau \int dx [i\nabla \theta(x, \tau) \partial_x \phi(x, \tau) - \frac{v}{2} (g(\nabla \theta)^2 + \frac{1}{g}(\nabla \phi)^2)] \]
We now write the action in the Fourier space, performing the integral and summing over $q$

$$-S = \frac{1}{\beta L} \sum_q \left[ -i\omega_n \phi(q)\theta(-q) - \frac{v}{2} k^2 \left( g\theta(q)\theta(-q) + \frac{1}{g} \phi(q)\phi(-q) \right) \right]$$

Since the expectation value is calculated for a function of $\phi$ only, the integral over $\theta$ can be performed by completing the square and changing variable $S = \frac{1}{\beta L} \sum_q \left[ \frac{v g^2 k}{2} \left( \theta(q) + i\omega_n \phi(q) \right) \theta(-q) \right] + \phi(q)\phi(-q) \left( \frac{v}{2g} k^2 + \frac{\omega_n^2}{2vg} \right)$

With the new variable $\tilde{\theta}(q) = \theta(q) + \frac{i\omega_n \phi(q)}{vgk}$ the action can be separated into $S = S_{\tilde{\theta}} + S_\phi$. The integration in $\tilde{\theta}$ yields $Z_{\tilde{\theta}}$, where $Z = Z_{\tilde{\theta}}Z_\phi$, with

$$Z_\phi = \int D\phi(x,\tau)e^{-S_\phi}$$

so that

$$\langle \phi(q_1)\phi(q_2) \rangle = \frac{1}{Z_\phi} \int D\phi(x,\tau)e^{-S_\phi}\phi(q_1)\phi(q_2)$$

Using the properties of gaussian integrals one gets

$$\langle \phi(q_1)\phi(q_2) \rangle = \frac{g\beta L \delta_{q_1,-q_2}}{\frac{v^2}{\pi} + v^2 k^2}$$

$$\langle \left[ \phi(r_1) - \phi(r_2) \right]^2 \rangle = \frac{1}{\beta} \sum_{\omega_n} \int \frac{dk}{\pi} e^{-\alpha k^2} \frac{g}{\frac{\omega_n^2}{v^2} + vk^2} \left( 1 - \cos(kx - \omega_n \tau) \right)$$

where $x = x_1 - x_2$ and $\tau = \tau_1 - \tau_2$.

To calculate the summation over $\omega_n$, the method outlined in Ref. [21] can be followed: to evaluate a summation of the type

$$-\frac{1}{\beta} \sum_n f(i\omega_n)$$

the residues of $f(z)$ have to be found:

$$-\frac{1}{\beta} \sum_n f(i\omega_n) = \sum_j r_j f_{BE}(z_j)$$

where $z_j$ is the location of a pole and $r_j$ is the corresponding residue, while $f_{BE}(z) = (e^{\beta z} - 1)^{-1}$ is the Bose-Einstein distribution function. Operating the substitution $\omega_n = -iz$, it is noticed that the residues are located in $z = \pm vk$ and thus:

$$\langle \left[ \phi(r_1) - \phi(r_2) \right]^2 \rangle = \int_0^{\infty} \frac{dk}{\pi} e^{-\alpha k^2} \frac{g}{k} \left( 1 + 2f_{BE}(vk) \right)(1 - \cos(kx) \cosh(vk\tau))$$
Deriving with respect to $\alpha$, one obtains
\[-\frac{\partial}{\partial \alpha} (|\phi(r_1) - \phi(r_2)|^2) = I = \frac{g}{\pi} \int_0^\infty dk e^{-\alpha k}(1 + 2f_{BE}(vk))(1 - \cos(kx) \cosh(vk\tau))\]

We use that $f_{BE}(z) = \sum_{n=1}^\infty e^{-\beta zn}$ to perform the integral.

Furthermore, in the limit of $\alpha \to 0$ only the significant terms can be retained, getting, after the integration in $\alpha$:
\[
\langle |\phi(r) - \phi(0)|^2 \rangle = \frac{g}{2\pi} \ln \left[ \frac{(x - iv\tau)(x + iv\tau)}{\alpha^2} \prod_{n=1}^\infty \left[ 1 + \frac{(x - iv\tau)^2}{(\beta vn)^2} \right] \left[ 1 + \frac{(x + iv\tau)^2}{(\beta vn)^2} \right] \right]
\]

Using the infinite product expansion sinh$(z) = z \prod_{n=1}^\infty (1 + z^2/(\pi^2 n^2))$, it is finally found
\[
\langle |\phi(r) - \phi(0)|^2 \rangle = \frac{g}{2\pi} \ln \left[ \beta^2 v^2 \sinh \left( \frac{\pi(x - iv\tau)}{\beta v} \right) \sinh \left( \frac{\pi(x + iv\tau)}{\beta v} \right) \right] \quad (A.2.1)
\]

This equation will be useful in the following during the calculation of density-density correlations.

Another interesting result that can be proved is
\[
\langle T_r \prod_j e^{i(A_j \phi(r_j) + B_j \theta(r_j))} \rangle = e^{-\frac{1}{2} \langle T_r | \sum_j (A_j \phi(r_j) + B_j \theta(r_j))|^2 \rangle} \quad (A.2.2)
\]
only if $\sum_j A_j = \sum_j B_j = 0$, whereas it is zero otherwise. In fact, since the action depends upon the derivative of $\phi$ and $\theta$, it is invariant under a transformation of the type $\phi \to \phi + \pi/(\sum_j A_j)$ or $\theta \to \theta + \pi/(\sum_j B_j)$. On the other hand one of these transformation has the effect
\[
T_r \prod_j e^{i(A_j \phi(r_j) + B_j \theta(r_j))} \to e^{i\pi T_r} \prod_j e^{i(A_j \phi(r_j) + B_j \theta(r_j))}
\]

Since we have to sum over all the configurations in the path integral, these two contributions sum to zero.

From this simple argument, it has also been demonstrated that
\[
\langle T_r \partial_{r_0} \phi(r_0) \prod_j e^{i(A_j \phi(r_j) + B_j \theta(r_j))} \rangle = 0 \quad (A.2.3)
\]
if $\sum_j A_j \neq 0$.

Equation (A.2.2) can be proved using again the functional integral.

From all these results, the density-density correlation function (the time ordering is implied) can be calculated; using (A.1.5) we obtain
\[
\langle \rho(r) \rho(0) \rangle = \frac{1}{\pi} \langle \nabla \phi(r) \nabla \phi(0) \rangle + \frac{1}{(2\pi \alpha)^2} [e^{-2ikFr} (e^{2i\sqrt{F} \phi(r)} e^{-2i\sqrt{F} \phi(0)}) + h.c.]
\]

The first term is the forward scattering term and is easily manageable (we will calculate it in section 3.3), while the last two terms are backward scattering terms which are more complicated to handle. We apply equation (A.2.2) to write
\[
\langle e^{2i\sqrt{F} \phi(r)} e^{-2i\sqrt{F} \phi(0)} \rangle = e^{-2\pi \langle |\phi(r) - \phi(0)|^2 \rangle}
\]

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then we substitute equation (A.2.1) to find

$$\langle e^{2i\sqrt{\pi}\phi(r)} e^{-2i\sqrt{\pi}\phi(0)} \rangle = \left(\frac{\pi^2 \alpha^2}{\beta^2 v^2}\right)^g \cdot \sinh \left(\frac{\pi(x - iv\tau)}{\beta v}\right) \sinh \left(\frac{\pi(x + iv\tau)}{\beta v}\right)^{-g}$$

(A.2.4)

### A.3. Bosonization of electrons with spin

We now want discuss how the bosonization applies to electrons with spin. This appendix is the natural extension of section 1.3.

A first modification to do in order to include the spin is to double the number of fields used, since we have now to deal with a doubled number of degrees of freedom; we introduce \((\phi_\uparrow, \theta_\uparrow, \phi_\downarrow, \theta_\downarrow)\) along with \(U_\uparrow, U_\downarrow\), where the subscripts \(\uparrow\) and \(\downarrow\) refer to the two possible values of spin.

The kinetic part of the Hamiltonian is trivially

$$H^0 = H^0_\uparrow + H^0_\downarrow$$

The interaction terms are far more complicated. To begin with, we have to distinguish between spin conserving processes (which we will label with an extra subscript \(\parallel\)) and scattering that flips the spin (labeled with \(\perp\)). The \(V_4\) and \(V_2\) processes have hamiltonians:

$$H_4 = \int dx \sum_{r=L,R} \sum_{\sigma=\uparrow,\downarrow} \left[ \frac{V_{4\parallel}}{2} \rho_{r,\sigma}(x) \rho_{r,\sigma}(x) + \frac{V_{4\perp}}{2} \rho_{r,\sigma}(x) \rho_{r,-\sigma}(x) \right]$$

$$H_2 = \int dx \sum_{\sigma=\uparrow,\downarrow} \left[ \frac{V_{2\parallel}}{2} \rho_{r,\sigma}(x) \rho_{L,\sigma}(x) + \frac{V_{2\perp}}{2} \rho_{r,\sigma}(x) \rho_{L,-\sigma}(x) \right]$$

These terms contain products like \(\partial_x \phi_\uparrow \partial_x \phi_\downarrow\), which are off diagonal in the spin basis. This can be solved with a straightforward diagonalization:

$$\phi_\rho(x) = \frac{1}{\sqrt{2}} [\phi_\uparrow(x) + \phi_\downarrow(x)]$$

$$\phi_\sigma(x) = \frac{1}{\sqrt{2}} [\phi_\uparrow(x) - \phi_\downarrow(x)]$$

and analogously for \(\theta\).

The fermionic field becomes

$$\psi_{r,\sigma}(x) = \frac{1}{\sqrt{2\pi\alpha}} U_{r,\sigma} e^{i r k_F x} e^{\frac{i}{\sqrt{2}} [r \phi_\rho(x) - \theta_\rho(x) + \sigma (r \phi_\sigma(x) - \theta_\sigma(x))]$$

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The kinetic energy thus becomes $H_0^0 + H_0^\sigma$ while the interacting terms are

$$H_4 = \frac{1}{4\pi} \int dx [V^\parallel_4 + V^\perp_4] \left[ (\partial_x \phi^\rho(x))^2 + (\partial_x \theta^\rho(x))^2 \right] +$$

$$+ \frac{1}{4\pi} \int dx [V^\parallel_4 - V^\perp_4] \left[ (\partial_x \phi^\sigma(x))^2 + (\partial_x \theta^\sigma(x))^2 \right]$$

$$H_2 = \frac{1}{4\pi} \int dx [V^\parallel_2 + V^\perp_2] \left[ (\partial_x \phi^\rho(x))^2 - (\partial_x \theta^\rho(x))^2 \right] +$$

$$+ \frac{1}{4\pi} \int dx [V^\parallel_2 - V^\perp_2] \left[ (\partial_x \phi^\sigma(x))^2 - (\partial_x \theta^\sigma(x))^2 \right]$$

We have seen in Figure 1.6 that is present also a process that exchanges fermions between the two ends of the Fermi surface. It can be written

$$H_1 = -V^\parallel_1 \int dx \sum_\sigma \rho_L,\sigma \rho_R,\sigma + \frac{2V^\perp_1}{(2\pi\alpha)^2} \int dx \cos(2\sqrt{2}\phi^\sigma(x))$$

Defining the Luttinger parameters also in the case of spin we have

$$v_\nu = v_F(1 + y^{4\nu}/2)^{-1/2}$$

$$g_\nu = \left[ 1 + 4y^{4\nu}/2 + y^{\nu}/2 \right]^{1/2}$$

$$V_\nu = V^\parallel_1 - V^\parallel_2; \pm V^\perp_2$$

$$V^\nu_4 = V^\parallel_4 \pm V^\perp_4$$

$$y_\nu = V_\nu/(\pi v_F)$$

where the upper sign refers to $\rho$.

$$H = \sum_{\nu=\rho,\sigma} H_\nu + \frac{2V^\perp_1}{(2\pi\alpha)^2} \int dx \cos(2\sqrt{2}\phi^\sigma(x))$$

with

$$H_\nu = \frac{v_\nu}{2} \int dx : g_\nu (\partial_x \theta^\nu(x))^2 + \frac{1}{g_\nu} (\partial_x \phi^\nu(x))^2 :$$

In the end, it can be observed that the introduction of spin breaks the pure quadratic form of the Hamiltonian, introducing a term dependent on the spin field.

We notice that the charge degrees of freedom are completely separated from the spin degrees; this is physically equivalent to say that an excitation factorizes into a collective charge excitation and into a collective spin excitation.

This is true until we remain in the linear dispersion regime; if we introduce correcting term to take into account the band curvature, things change a lot, as anticipated in section 1.3.
Appendix B

Calculation of commutators and linear approximation

In this appendix we want to prove two results: the former is equation (3.3.18), which expresses the density-current commutator in terms of only two fermion fields; the second one is given by equations (3.3.20) and (3.3.21), which gives the approximated expression of a four points correlation function in the Luttinger liquid regime.

B.1. Calculation of density-current commutator

In order to prove equation (3.3.18), we will use equations (1.2.1), (3.1.6), along with the commutation rule \( \{ \psi(x), \psi^\dagger(x') \} = \delta(x - x') \), which quite easily implies

\[
[\psi^\dagger(x)\psi(x), \psi^\dagger(x')] = \delta(x - x')\psi^\dagger(x) \\
[\psi^\dagger(x)\psi(x), \psi^\dagger(x')] = -\delta(x - x')\psi(x)
\]

Moreover we will need the expression for the field’s time derivative, given before equation (3.1.7) and reported also here for convenience:

\[
i\partial_t \psi(x) = -\frac{\partial^2 \psi(x)}{2m} + \int dx' \psi^\dagger(x')V(x' - x)\psi(x')\psi(x)
\]
With these tools at hands, the commutator can be calculated:

\[
[\rho(x), J_\parallel] = -\int \frac{dx'}{2mL} [\bar{\psi}(x')\psi(x), \partial_{x'}\bar{\psi}^\dagger(x') (\partial_t - i\mu)\psi(x') + (\partial_t + i\mu)\bar{\psi}^\dagger(x')\partial_{x'}\psi(x')]
\]

\[
[\rho(x), J_\parallel] = -\frac{1}{2mL} \int dx' \left\{ \partial_{x'}\delta(x - x')[\bar{\psi}(x') (\partial_t - i\mu)\psi(x') - (\partial_t + i\mu)\bar{\psi}^\dagger(x')\psi(x')] + \partial_{x'}\bar{\psi}^\dagger(x') \psi(x'), (\partial_t - i\mu)\psi(x')] + [\bar{\psi}^\dagger(x')\psi(x), (\partial_t + i\mu)\bar{\psi}^\dagger(x')\partial_{x'}\psi(x') \right\}
\]

The first two terms yield \( \frac{1}{2mL}[\bar{\psi}(x)\partial_t(x - i\mu)\psi(x) - (\partial_x\partial_t + i\mu)\bar{\psi}^\dagger(x))\psi(x)] \); for what concerns the others, it can be obtained:

\[
\int dx'\partial_{x'}\psi^\dagger(x')[\bar{\psi}^\dagger(x)\psi(x), (\partial_t - i\mu)\psi(x')] = \int dx'\partial_{x'}\psi^\dagger(x')(-i\frac{\partial^2_{x'}}{2m} + i\mu)\delta(x - x').
\]

\[
\cdot \psi(x) - i \int dx' \int dx'' \partial_{x'}\psi^\dagger(x')V(x'' - x') \delta(x - x'')\psi^\dagger(x'')\psi(x'') - \delta(x - x'')\psi^\dagger(x')V(x'' - x')\psi(x'')\psi(x') =
\]

\[
= \partial_x(-i\frac{\partial^2_{x}}{2m} + i\mu)\psi^\dagger(x))\psi(x) + i \int dx'' V(x'' - x)(\partial_{x'}\psi^\dagger(x))\psi^\dagger(x'')\psi(x'') =
\]

\[
= (\partial_x(\partial_t + i\mu)\psi^\dagger(x))\psi(x) + i \int dx'' V'(x'' - x)\psi^\dagger(x'')\psi(x'')\psi(x')
\]

With a constant intrawire interaction the last term vanishes. An analogous calculation can be carried out for the remaining term, finding

\[
mL[\rho(x), J_\parallel] = \psi^\dagger(x)\partial_{x'}(\partial_t - i\mu)\psi(x) - (\partial_x(\partial_t + i\mu)\psi^\dagger(x))\psi(x) \quad (B.1.1)
\]

This ends the proof of equation (3.3.18).

### B.2. Application of Luttinger liquid regime

In order to prove equations (3.3.20) and (3.3.21), we need to use the bosonized expression for the fermionic fields (see equation (1.3.5)). Interestingly the phase arising from the temporal evolution and dropped in equation (1.3.5) has to be considered here:

\[
\tilde{\psi}_{R/L}(x, t) = e^{\pm ik_F x} e^{-i\sqrt{\pi}(\pm\phi(x) - \theta(x))} e^{i\mu t}
\]

where we called \( \tilde{\psi} \) the field with the explicit temporal phase to distinguish it from that without \( e^{i\mu t} \); they are related by \( \tilde{\psi}_{R/L} = \psi_{R/L} e^{i\mu t} \). Interestingly, we always have to perform “covariant” derivatives, in the sense that we only have to calculate \( (\partial_t - i\mu)\tilde{\psi}(x, t)_{R/L} \) or \( (\partial_t + i\mu)\tilde{\psi}^\dagger(x, t)_{R/L} \). It can be verified that

\[
(\partial_t - i\mu)\tilde{\psi}(x, t)_{R/L} = \partial_t\psi_{R/L}(x, t)
\]

Therefore, it is equivalent whether one performs the covariant derivative of the field including the temporal phase or calculate the simple time derivative of the
field ignoring this additional phase. We will follow the second way because of book keeping reasons.

Another important observation to make is that in the linear regime the spatial derivative of $k_F x$ dominates with respect to the derivative of $\phi(x)$ and $\theta(x)$, since the latter vary with wavevector $k \ll k_F$.

Decomposing the fermionic field into the fields associated to left and right movers we find

$$
\psi^\dagger(x, t) \partial_x \partial_t \psi(x, t) = (\psi^\dagger_L(x, t) + \psi^\dagger_R(x, t)) \partial_x \partial_t (\psi_R(x, t) + \psi_L(x, t))
$$

In all the following formulas is implied that the fermionic and bosonic fields are all evaluated at the same time $t$ and position $x$, although we do not specify it. We use the following formulas:

$$
\begin{align*}
\partial_t \psi_{R/L} &= i \sqrt{\pi} (\pm \partial_t \phi + \partial_t \theta) \psi_{R/L} \\
\partial_t \psi^\dagger_{R/L} &= i \sqrt{\pi} (\pm \partial_t \phi - \partial_t \theta) \psi^\dagger_{R/L}
\end{align*}
$$

The product is separated:

$$
\psi^\dagger_R \partial_x \partial_t \psi_R + \psi^\dagger_L \partial_x \partial_t \psi_L = i k_F (\psi^\dagger_R \partial_t \psi_R - \psi^\dagger_L \partial_t \psi_L)
$$

$$
(\partial_x \partial_t \psi^\dagger_R) \psi_R + (\partial_x \partial_t \psi^\dagger_L) \psi_L = -i k_F ((\partial_t \psi^\dagger_R) \psi_R - (\partial_t \psi^\dagger_L) \psi_L)
$$

so that

$$
\psi^\dagger_R \partial_x \partial_t \psi_R + \psi^\dagger_L \partial_x \partial_t \psi_L - \text{h.c.} = i k_F \partial_t (\psi^\dagger_R \psi_R - \psi^\dagger_L \psi_L) = i k_F \partial_t (\rho_R - \rho_L) = i k_F \sqrt{\pi} \partial_t \partial_x \theta
$$

Using $v_F = v g$ and equations (A.1.2) and (A.1.4), it can be written

$$
\psi^\dagger_R \partial_x \partial_t \psi_R + \psi^\dagger_L \partial_x \partial_t \psi_L - \text{h.c.} = i m v g \sqrt{\pi} \partial_x \partial_x \phi = -i m v g \partial_x (\rho_R + \rho_L)
$$

For the mixed terms, equations (B.2.1) and (B.2.2) are used, finding

$$
\psi^\dagger_R \partial_x \partial_t \psi_L + \psi^\dagger_L \partial_x \partial_t \psi_R - \text{h.c.} =
\begin{align*}
&= i k_F ([\partial_t \psi^\dagger_R] \psi_L - \psi^\dagger_R \partial_t \psi_L + \psi^\dagger_L \partial_t \psi_R - (\partial_t \psi^\dagger_L) \psi_R] = \\
&= i k_F [2 (\partial_t \psi^\dagger_R) \psi_L - (\partial_t \psi^\dagger_L) \psi_R] + \partial_t (\psi^\dagger_L \psi_R - \psi^\dagger_R \psi_L)]
\end{align*}
$$

At this point the explicit form of the fermionic field is used

$$
\psi^\dagger_R \psi_R - \psi^\dagger_R \psi_L = \frac{2i}{2\pi \alpha} \sin[2k_F x - 2\phi] \\
\psi^\dagger_L \psi_R + \psi^\dagger_R \psi_L = \frac{2}{2\pi \alpha} \cos[2k_F x - 2\phi]
$$

verifying that

$$
(\partial_t \psi^\dagger_R) \psi_R - (\partial_t \psi^\dagger_L) \psi_R = \frac{2i \sqrt{\pi}}{2\pi \alpha} \partial_t \phi \cos[2k_F x - 2\sqrt{\pi} \phi] - \frac{2 \sqrt{\pi}}{2\pi \alpha} \partial_t \theta \cos[2k_F x - 2\sqrt{\pi} \phi]
$$

$$
\partial_t (\psi^\dagger_L \psi_R - \psi^\dagger_R \psi_L) = - \frac{4i \sqrt{\pi}}{2\pi \alpha} \partial_t \phi \cos[2k_F x - 2\sqrt{\pi} \phi]
$$

**Thermal drag between two coupled quantum wires**
ψ_R(x)∂_x ∂_t ψ_L(x) + ψ_L(x)∂_x ∂_t ψ_R(x) - h.c. = -\frac{2ik_F}{\pi \alpha} \sqrt{\pi} \partial_x \theta \sin[2k_Fx - 2\sqrt{\pi} \phi] = \\
= -\frac{2imv^2}{\pi \alpha} \sqrt{\pi} \partial_x \phi \sin[2k_Fx - 2\sqrt{\pi} \phi] = \\
= -\frac{imv^2}{\pi \alpha} \partial_x \cos[2k_Fx - 2\phi] + \frac{2imv^2k_F}{\pi \alpha} \sin[2k_Fx - 2\phi]

Recollecting all the pieces, it can be found

\[ [\rho(x,t), J_\parallel(t)] = -\frac{iv^2}{L} \partial_x \rho(x,t) + \frac{2iv^2k_F}{L\pi \alpha} \sin[2k_Fx - 2\phi(x,t)] \]

If we had started directly in the Luttinger liquid approximation, applying the bosonization framework, we would have obtained the same result. Now we want to go further in our analysis; our claim is

\[ \langle \rho(y)[\rho(x,t), J_\parallel(t)] \rangle = -\frac{iv^2}{L} \langle \rho(y) \partial_x \rho(x,t) \rangle \]

Therefore we just have to prove that

\[ \langle \rho(y) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] \rangle = \\
= \left( \frac{1}{\sqrt{\pi}} \partial_y \phi(y) + \frac{1}{\pi \alpha} \cos[2k_Fx - 2\sqrt{\pi} \phi(y)] \right) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] = 0 \]

The second term is zero for parity reasons. In fact, the field \( \phi \) contains each one a creation or an annihilation operator; the sine can be expanded in powers of \( \phi \) and contains only odd powers of \( \phi \) and thus only products with an odd number of \( b_k \) and \( b_k^\dagger \) appearing; similarly the cosine contains only even powers of operators. We would have to calculate an expectation value of a sum containing only products with an odd number of \( b_k \) or \( b_k^\dagger \); this value is obviously zero.

What is left is

\[ \langle \frac{1}{\sqrt{\pi}} \partial_y \phi(y) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] \rangle = 0 \]

Using equations (A.2.2) and (A.1.5), it can be proven that

\[ \langle e^{i2\sqrt{\pi} \phi(x,t)} \rangle = 0 \]

\[ \langle T \partial_y \phi(y) e^{i2\sqrt{\pi} \phi(x,t)} \rangle = 0 \]

\[ [\partial_y \phi(y), \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)]] \propto \cos[2k_Fx - 2\sqrt{\pi} \phi(x,t)] \]

With these information we see that

\[ \langle T \partial_y \phi(y) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] \rangle = \\
= Y(t) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] \partial_y \phi(y) + Y(-t) \partial_y \phi(y) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] = \\
= \partial_y \phi(y) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)] + Y(t) \sin[2k_Fx - 2\sqrt{\pi} \phi(x,t)], \partial_y \phi(y) \]

\[ \text{Thermal drag between two coupled quantum wires} \]
Here $Y$ indicates the Heavyside function.

\[
\frac{1}{\sqrt{\pi}} \partial_y \phi(y) \sin[2k_F x - 2\sqrt{\pi} \phi(x, t)] = \\
= T \frac{1}{\sqrt{\pi}} \partial_y \phi(y) \sin[2k_F x - 2\sqrt{\pi} \phi(x, t)] + Y(t) [\partial_y \phi(y), \sin[2k_F x - 2\sqrt{\pi} \phi(x, t)]]
\]

The expectation value is therefore zero using equation (B.2.3).

The same reasoning applies if we have a correlator of the commutator times the density.

So far we have proved that

\[
\langle \rho(x, t) [\rho(x, 0), J_{||}(0)] \rangle = -\frac{i\nu^2}{L} \langle \rho(x, t) \partial_y \rho(y, 0) \rangle = \frac{i\nu^2}{L} \partial_x \langle \rho(x - y, t) \rho(0, 0) \rangle
\]

and

\[
\langle [\rho(x, t), J_{||}(t)] \rho(0, 0) \rangle = -\frac{i\nu^2}{L} \partial_x \langle \rho(x, t) \rho(0, 0) \rangle
\]

which are exactly equations (3.3.20) and (3.3.21).
Appendix C

Calculation of $\chi_{ii}$

In this appendix, we want to derive an expression for $\chi_{ii}$; this involves an expectation value of two currents at the same time which, thus, should include also high energies; consequently we should check the validity of the Luttinger liquid regime.

We always consider a regime of low temperatures $T \ll T_F$; to validate the use of Luttinger liquid approximation, we calculate $\chi_{ii}$ for non-interacting electrons (making use of the Sommerfeld expansion) and then repeat the calculation in the bosonization framework. If the two results coincide in the limit of non-interacting electrons, then we could use the linear regime also for interacting electrons (always for low temperatures).

We will do the calculation first in the bosonization framework.

The expression for $J_{i\|}$ is (from equations (A.1.7) and (A.1.8)):

$$J_{i\|} = -\frac{v_i^2}{2L} \int dx \{ \partial_x \phi_i(x), P_i(x) \} = \frac{v_i^2}{L} \sum_k k \hat{b}_k^\dagger \hat{b}_k$$

Therefore we obtain from equations (3.3.6) and (3.3.9)

$$\chi_{ii} = \frac{L}{T} \frac{1}{\beta} \int_0^\beta d\beta' \langle J_{i\|} e^{-\beta' H} J_{i\|} e^{\beta' H} \rangle$$

A representation of exact eigenstates can be inserted into the previous equation:

$$\langle J_{i\|} e^{-\beta' H} J_{i\|} e^{\beta' H} \rangle = \sum_{n,m} \frac{e^{-\beta E_n}}{Z} \langle n | J_{i\|} | m \rangle e^{\beta(E_n - E_m)} \langle m | J_{i\|} | n \rangle$$

---

*Thermal drag between two coupled quantum wires*
Since \( \langle m|J_{||}|n \rangle \propto \delta_{m,n} \) one can obtain
\[
\chi_{ii} = \frac{L \beta}{T} \left[ \langle J_{||}|J_{||} \rangle \right] = \frac{L \beta}{T} \left( \frac{v_i^2}{L} \right)^2 \sum_{k,k'} kk' \langle b_k^\dagger b_{k'}^\dagger b_{k'} b_k \rangle
\]

From Ref. [51], the expectation value can be written as
\[
\langle b_k^\dagger b_{k'}^\dagger b_{k'} b_k \rangle = f_{BE}(k)f_{BE}(k')(1 + \delta_{k,k'}) + \delta_{k,k'}f_{BE}(k)
\]
where \( f_{BE}(k) \) is the Bose-Einstein distribution at temperature \( T \) for the momentum \( k \). Then, it can be obtained
\[
\chi_{ii} = \frac{L \beta}{T} \left( \frac{v_i^2}{L} \right)^2 \sum_{k,k'} kk' (f_{BE}(k)f_{BE}(k')(1 + \delta_{k,k'}) + \delta_{k,k'}f_{BE}(k)) = \frac{L \beta}{T} \left( \frac{v_i^2}{L} \right)^2 \sum_k k^2 f_{BE}(k)(1 + f_{BE}(k))
\]
where \( \sum_{k,k'} kk' f_{BE}(k)f_{BE}(k') \) has been eliminated, since it is zero for parity reasons.

Using \( f_{BE}(k) = \frac{1}{e^{\beta v_i k} - 1} \) and \( \sum_k = \frac{L}{2\pi} \int dk \), we obtain
\[
\chi_{ii} = \frac{L \beta}{T} \left( \frac{v_i^2}{L} \right)^2 \frac{L}{2\pi} \int dk k^2 \frac{e^{\beta v_i k}}{(e^{\beta v_i k} - 1)^2} = \frac{\beta}{T} \left( \frac{v_i^2}{L} \right)^2 \frac{1}{\pi} \int_0^\infty dk k^2 \frac{e^{\beta v_i k}}{(e^{\beta v_i k} - 1)^2}
\]

The last integral is easily calculated by partial integration
\[
\int_0^\infty dx x^2 \left( \frac{e^x}{e^x - 1} \right)^{-2} = \frac{\pi}{3} v_i k_B T
\]
and is
\[
\int_0^\infty dx x^2 \left( \frac{e^x}{e^x - 1} \right)^{-2} = \frac{\pi^2}{3} v_i^2 k_B^2 T
\]
so that we obtain
\[
\chi_{ii} = \frac{\pi}{3} v_i k_B T
\]

Now we will work without bosonizing the Hamiltonian but still in the free-electron regime, where the thermal current is
\[
J_{||} = \frac{1}{L} \sum_k \frac{k^2}{m} \left( \frac{k^2}{2m} - \mu \right) a_k^\dagger a_k
\]

The matrix element of the current is non zero only between two identical exact eigenstates, as previously proven also in the bosonization approximation. Therefore we find
\[
\chi_{ii} = \frac{L \beta}{T} \left( \langle J_i|J_i \rangle \right) = \frac{L \beta}{T} \frac{L^2}{L} \sum_{k,k'} \frac{k k'}{m^2} \left( \frac{k^2}{2m} - \mu \right) \left( \frac{k'^2}{2m} - \mu \right) \langle a_k^\dagger a_k a_{k'}^\dagger a_{k'} \rangle
\]

Using the formula given in Ref. [51], we find that
\[
\langle a_k^\dagger a_k a_{k'}^\dagger a_{k'} \rangle = f_{FD}(k)f_{FD}(k')(1 - \delta_{k,k'}) + f_{FD}(k)\delta_{k,k'}
\]

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C. Calculation of $\chi_{ii}$

For parity reasons, the sums with addends that are odd in $k$ are eliminated, giving

$$\chi_{ii} = \frac{L \beta T L 1}{2 \sum_k \frac{k^2}{m^2} \left( \frac{k^2}{2m} - \mu \right)^2 f_{FD}(k)(1 - f_{FD}(k))$$

where $E_k = k^2/2m$ and

$$f_{FD}(E)(1 - f_{FD}(E)) = -k_B T \frac{d}{dE} f_{FD}(E)$$

Now we transform the sum in an integral, integrate only on positive $k$ and then switch variable to $E$ ($dk = \sqrt{\frac{m}{2}} dE$), obtaining

$$\chi_{ii} = \frac{L \beta T L L 2}{2 \pi} \int_0^\infty dE \sqrt{\frac{2mE}{E}} \left( E - \mu \right)^2 (-k_B T \frac{d}{dE} f_{FD}(E))$$

This result is valid, for free electrons, in any range of temperatures; we now specialize to the case $T \ll T_F$. In this approximation the Fermi-Dirac distribution is almost a step-like function and its derivative is consequently very sharp, converging to a delta function for $T \to 0$; see Figure C.1.

Figure C.1: Plot of the Fermi-Dirac function and of its derivative (inside the box) for $T = 0$ and $T = 0.1T_F$; the delta function at $T = 0$ is schematized with an arrow.

The Sommerfeld expansion is applied at the second order:

$$\int dE G(E)(-\frac{d}{dE} f_{FD}(E)) \approx G(\mu) + \frac{\pi^2}{6} (k_B T)^2 G''(\mu)$$

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Here the function that has to be evaluated is $G(E) = \frac{\sqrt{2mE}}{m} (E - \mu)^2$, so that $G(\mu) = 0$ and $G''(\mu) = 2\frac{\sqrt{2m\mu}}{m}(k_B T)^3$

\[
\chi_{ii} = \frac{\beta}{T} \frac{1}{\pi} 2\frac{\sqrt{2m\mu}}{m}(k_B T)^3
\]

Since the leading order in powers of the temperature is needed, the value of $\mu$ has to be known. It can be easily calculated by requiring that the total number of particles is conserved:

\[
N = \sum_k f_{FD}(E_k) = \frac{L}{2\pi} \int \frac{dk}{e^{\beta(E_k-\mu)} + 1}
\]

At zero temperature, the chemical potential coincides with the Fermi energy $E_F = \frac{k_F^2}{2m}$, while the Fermi-Dirac distribution is a step-function, so that

\[
N = \frac{L}{2\pi} \int_{-k_F}^{k_F} dk = \frac{Lk_F}{\pi}
\]

At low temperature the Sommerfeld expansion is used again getting

\[
N \approx \frac{L}{2\pi} \sqrt{\frac{m}{2}} \int_0^{\infty} \frac{dE}{\sqrt{E}} \sqrt{\frac{m}{2}} \left( 2\sqrt{\mu} - \frac{1}{6} \frac{\pi^2 (k_B T)^2}{\mu^{3/2}} \right)
\]

\[
\mu(T) \approx E_F \left( 1 + \frac{\pi^2 (k_B T)^2}{12 E_F^2} \right)
\]

\[
(C.3)
\]

$\mu$ is approximately equal to the Fermi energy plus a second order temperature correction. Thus

\[
\chi_{ii} = \frac{\pi}{3} v_F k_B^2 T
\]

\[
(C.4)
\]

Equation (C.4) agrees with (C.1), since in the absence of interaction $v = v_F$. 

---

*Thermal drag between two coupled quantum wires*
Appendix D

Calculation of the backward scattering contribution

In this appendix we calculate the backward scattering term, proving the results reported in section 3.5.2. In section D.1 we give a brief review of the correlation functions formalism, which will be applied during the calculation of $A_b$ in section D.2.

D.1. Correlation functions

For two generic operators $B$ and $A$, the spectral correlation function is defined

$$\tilde{\chi}_{BA}(x,t) = \langle [B(x,t), A(0,0)] \rangle$$

(D.1.1)

Now the retarded correlation function for bosonic operators ($Y$ is the Heavyside function) is introduced

$$\chi^R_{BA}(x,t) = -iY(t) \langle [B(x,t), A(0,0)] \rangle$$

(D.1.2)

Always for boson operators the time ordered correlation can be defined function

$$\chi^T_{BA}(x,t) = -[Y(t) \langle B(x,t)A(0,0) \rangle + \theta(-t) \langle A(0,0)B(x,t) \rangle]$$

(D.1.3)

If $A$ and $B$ are hermitian it can be written

$$(\chi^R_{A^\dagger B})(-x,-t)^* = iY(-t)\left(\langle A^\dagger(-x,-t)B^\dagger(0,0) \rangle^* - \langle B^\dagger(0,0)A^\dagger(-x,-t) \rangle^* \right)$$

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\[ = iY(-t)\langle [B(x,t), A(0,0)] \rangle \]

Therefore, the following relation holds

\[-i\tilde{\chi}_{BA}(x,t) = \chi^R_{BA}(x,t) - (\chi^R_{AIB})(-x,-t) \]

and, in the case of \( B = A^\dagger \), one finds

\[-i\tilde{\chi}_{A^\dagger A}(x,t) = \chi^R_{A^\dagger A}(x,t) - (\chi^R_{A^\dagger A})(-x,-t) \]

Switching to Fourier transform, the previous equation becomes

\[-i\tilde{\chi}_{A^\dagger A}(k,\omega) = \int dxdt e^{-ikx+i\omega t}(\chi^R_{A^\dagger A}(x,t) - (\chi^R_{A^\dagger A})(-x,-t)) \]

The first term gives \( \chi^R_{A^\dagger A}(k,\omega) \), while the second can be written as

\[ \int dxdt e^{-ikx+i\omega t}(\chi^R_{A^\dagger A}(-x,-t)) = \int dxdt e^{ikx-i\omega t}(\chi^R_{A^\dagger A}(x,t)) \]

\[ = (\int dxdt e^{-ikx+i\omega t}\chi^R_{A^\dagger A}(x,t)) = \chi^R_{A^\dagger A}(k,\omega) \]

\[-i\tilde{\chi}_{A^\dagger A}(k,\omega) = \chi^R_{A^\dagger A}(k,\omega) - \chi^R_{A^\dagger A}(k,\omega) \]

\[ \tilde{\chi}_{A^\dagger A}(k,\omega) = -2\text{Im}\chi^R_{A^\dagger A}(k,\omega) \] (D.1.4)

As for retarded correlation function, it can written again

\[ (\chi^T_{A^\dagger B^\dagger}(-x,-t)) = -[Y(-t)(A^\dagger(-x,-t)B^\dagger(0,0))^* + Y(t)(B(0,0)A^\dagger(-x,-t))^*] = \]

\[ = -[Y(-t)B(x,t)A(0,0) + Y(t)(A(0,0)B(x,t))] \]

So that

\[ \chi^R_{BA}(t) = iY(t)[\chi^T_{BA}(x,t) - (\chi^T_{A^\dagger B^\dagger})(-x,-t)] \] (D.1.5)

If \( \chi^T_{A^\dagger A}(-x,-t) = \chi^T_{A^\dagger A}(x,t) \) then:

\[ \chi^R_{A^\dagger A}(x,t) = -2Y(t)\text{Im}\chi^T_{A^\dagger A}(x,t) \] (D.1.6)

These are the technical tools needed to calculate \( A_{bs}(k,\omega) \).

D.2. Application to backward scattering

Now we apply the concepts of the previous section to the calculation of the back-scattering term:

\[ A_{bs}(x,t) = \frac{1}{8\pi^2\alpha^2} \left[ e^{2ikF_x} \left( \left| e^{-2it\sqrt{\varphi}(x,t)} \right|^2 + \left| e^{2it\sqrt{\varphi}(x,t)} \right|^2 \right) + e^{-2ikF_x} \left( \left| e^{2it\sqrt{\varphi}(x,t)} \right|^2 + \left| e^{-2it\sqrt{\varphi}(x,t)} \right|^2 \right) \right] \]

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One just needs to calculate spectral correlation functions of this type:

$$\tilde{\chi}_{\pm\pm}(x, t) = \langle [e^{\pm 2i\sqrt{\pi}\phi(x,t)}, e^{\pm 2i\sqrt{\pi}\phi(0,0)}] \rangle$$

To do this, the time ordered correlation function can be calculated, using (D.1.5) to get the retarded function and finally applying (D.1.4) to obtain $$\tilde{\chi}_{\pm\pm}$$.

In order to derive the real time $$t$$ ordered function, we first calculate the imaginary time $$\tau$$ ordered function (which is an easy task using the functional integral technique); then an analytic continuation is performed by doing a Wick rotation: $$\tau = it + \epsilon \text{sign}(t)$$ (with $$\epsilon \to 0^+$$).

Starting from the imaginary time ordered function, which is simply a time ordered correlation function but with an imaginary time evolution (see Appendix A for more details), one has:

$$\chi_{\pm\pm}(x, \tau) = -\langle T_\tau e^{\pm 2i\sqrt{\pi}\phi(x,\tau)} e^{\pm 2i\sqrt{\pi}\phi(0,0)} \rangle$$

and, from equation (A.2.2), it can be seen that $$\chi_{++} = \chi_{--} = 0$$, so that

$$A_{bs}(k, \omega) = \frac{1}{8\pi^2 \alpha^2} \int dx dt e^{i\omega t - ikx} (e^{2ik_F x} \tilde{\chi}_{--}(x, t) + e^{-2ik_F x} \tilde{\chi}_{++}(x, t))$$

$$= \frac{\tilde{\chi}_{--}(k - 2k_F, \omega) + \tilde{\chi}_{++}(k + 2k_F, \omega)}{8\pi^2 \alpha^2} \tag{D.2.1}$$

Furthermore, from equation (A.2.4)

$$\chi_{+-}(x, \tau) = \chi_{-+}(x, \tau) = -\left( \frac{\pi \alpha}{\beta v} \right)^2 g \left( \frac{\sinh \left( \frac{\pi}{\beta} \left( x v - t + i \epsilon \text{sign}(t) \right) \right)}{\sinh \left( \frac{\pi}{\beta} \left( x v + t - i \epsilon \text{sign}(t) \right) \right)} \right)^g$$

From this equality follows that we just need to work on $$\chi_{+-}(x, \tau)$$. This is a correlation function of the kind $$\chi_{A^\dagger A}$$, with $$A = e^{-2i\sqrt{\pi}\phi}$$.

Operating the Wick rotation, the retarded correlation function is

$$\chi^R_{+-}(x, t) = -\left( \frac{\pi \alpha}{\beta v} \right)^2 g \frac{1}{\left( \frac{\sinh \left( \frac{\pi}{\beta} \left( x v - t + i \epsilon \text{sign}(t) \right) \right)}{\sinh \left( \frac{\pi}{\beta} \left( x v + t - i \epsilon \text{sign}(t) \right) \right)} \right)^g}$$

Since $$\chi^T_{+-}(x, t) = \chi^R_{+-}(-x, -t)$$, equation (D.1.6) can be applied; moreover, thanks to the Heavyside function, we can consider only positive $$t$$ and find

$$\chi^R_{+-}(x, t) = 2Y(t) \left( \frac{\pi \alpha}{\beta v} \right)^2 g \frac{1}{\left( \frac{\sinh \left( \frac{\pi}{\beta} \left( x v - t + i \epsilon \right) \right)}{\sinh \left( \frac{\pi}{\beta} \left( x v + t - i \epsilon \right) \right)} \right)^g}$$

The denominator can be written as $$e^{-g \ln(C)}$$ with $$C$$ given (up to the first order in $$\epsilon$$) by:

$$C = \sinh \left( \frac{\pi}{\beta} \left( x v - t + i \epsilon \right) \right) \sinh \left( \frac{\pi}{\beta} \left( x v + t - i \epsilon \right) \right)$$

$$= \sinh \left( \frac{\pi}{\beta} \left( x v + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( x v - t \right) \right) + i \epsilon \frac{\pi}{\beta} \sinh \left( \frac{2\pi t}{\beta} \right) \tag{D.2.2}$$

**Thermal drag between two coupled quantum wires**
D. Calculation of the backward scattering contribution

Since the logarithm of \( C \) has to be calculated, the first term contributes to the imaginary part if the product is negative; this is possible, and the contribution of the second term can be neglected, since it will be proportional to \( \epsilon \).

The product is negative when only one of the argument is negative; it means

\[
\sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} - t \right) \right) > 0, \quad \text{if } |x| > vt \\
\sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} - t \right) \right) < 0, \quad \text{if } |x| < vt
\]

Thus the imaginary part is non zero only in the region \( |x| < vt \); the exponent is

\[
-g \ln(C) = -g \ln \left| \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} - t \right) \right) \right| - ig \pi Y(vt - x) Y(vt + x)
\]

So that

\[
\chi_{+-}(x, t) = -Y(t) Y(vt - x) Y(vt + x) \left( \frac{\pi \alpha}{\beta v} \right)^2 2 \sin(\pi g) \cdot \left| \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} - t \right) \right) \right|^{-g}
\]

The Fourier transform is given by

\[
\chi_{+-}(k, \omega) = \int dx dt e^{i\omega t - ikx} \chi_{+-}(x, t)
\]

\[
\chi_{+-}(k, \omega) = -2 \left( \frac{\pi \alpha}{\beta v} \right)^2 \sin(\pi g) \int_0^\infty dt \int_{-vt}^{+vt} dx e^{i\omega t - ikx}. \cdot \left( \sinh \left( \frac{\pi}{\beta} \left( \frac{x}{v} + t \right) \right) \sinh \left( \frac{\pi}{\beta} \left( t - \frac{x}{v} \right) \right) \right)^{-g}
\]

Changing variable to \( \xi_+ = vt + x \) and \( \xi_- = vt - x \), the domain of integration includes \( \xi_+ > 0 \) and \( \xi_- > 0 \), while \( dxdt = \frac{d\xi_+ d\xi_-}{2v} \):

\[
\chi_{+-}(k, \omega) = -\frac{1}{v} \left( \frac{\pi \alpha}{\beta v} \right)^2 \sin(\pi g) \left[ \int_0^\infty d\xi_+ e^{i\xi_+ (\xi_- - k)} \sinh \left( \frac{\pi \xi_+}{\beta v} \right)^{-g} \right] \cdot \left[ \int_0^\infty d\xi_- e^{i\xi_- (\xi_+ + k)} \sinh \left( \frac{\pi \xi_-}{\beta v} \right)^{-g} \right]
\]

Employing equations (3.4.3) and (3.4.4), the previous integrals become

\[
\int_0^\infty d\xi_+ e^{i\xi_+ (\xi_- - k)} \sinh \left( \frac{\pi \xi_+}{\beta v} \right)^{-g} = \beta v \frac{\pi}{B_c} \left( \frac{\beta (\omega - vk)}{2\pi}, g \right) + i \frac{\beta v}{\pi} B_s \left( \frac{\beta (\omega - vk)}{2\pi}, g \right)
\]

\[
\int_0^\infty d\xi_- e^{i\xi_- (\xi_+ + k)} \sinh \left( \frac{\pi \xi_-}{\beta v} \right)^{-g} = \beta v \frac{\pi}{B_c} \left( \frac{\beta (\omega + vk)}{2\pi}, g \right) + i \frac{\beta v}{\pi} B_s \left( \frac{\beta (\omega + vk)}{2\pi}, g \right)
\]

---

Thermal drag between two coupled quantum wires
D. Calculation of the backward scattering contribution

and, using the previous equations, the back-scattering term is

\[ A_{bs}(k, \omega) = \frac{\sin(\pi g)}{4\pi^2 v} \left( \frac{\pi \alpha}{\beta v} \right)^{2g-2} \cdot \]

\[ \left[ B_c \left( \frac{\beta(\omega - v(k - 2k_F))}{2\pi}, g \right) B_s \left( \frac{\beta(\omega + v(k - 2k_F))}{2\pi}, g \right) + \right. \]

\[ + B_s \left( \frac{\beta(\omega - v(k - 2k_F))}{2\pi}, g \right) B_c \left( \frac{\beta(\omega + v(k - 2k_F))}{2\pi}, g \right) + \]

\[ + B_c \left( \frac{\beta(\omega - v(k + 2k_F))}{2\pi}, g \right) B_s \left( \frac{\beta(\omega + v(k + 2k_F))}{2\pi}, g \right) + \]

\[ + B_s \left( \frac{\beta(\omega - v(k + 2k_F))}{2\pi}, g \right) B_c \left( \frac{\beta(\omega + v(k + 2k_F))}{2\pi}, g \right) \]

as written in equation (3.4.5).

Thermal drag between two coupled quantum wires
In this appendix we want to get back to equation (3.3.17) and calculate the transresistivity in the free electron case. Making use of the fermion field expansion it is easy to see that

$$\Sigma'(x, t) = \sum_{k_1, k_2} e^{i(k_2 - k_1)x} e^{i(k_2^2 - k_1^2)t/2m} \left[ k_2 \left( \frac{k_2^2}{2m} - \mu \right) - k_1 \left( \frac{k_1^2}{2m} - \mu \right) \right] \langle a_{k_1}^\dagger a_{k_2} a_{k_3}^\dagger a_{k_4} \rangle$$

The formula given in Ref. [51] can be used

$$\langle a_{k_1}^\dagger a_{k_2} a_{k_3}^\dagger a_{k_4} \rangle = \delta_{k_1, k_2} \delta_{k_3, k_4} f_{k_1} f_{k_3} + \delta_{k_1, k_4} \delta_{k_2, k_3} f_{k_1} (1 - f_{k_3})$$

Since the addends are zero for $k_1 = k_2$, only the following expression survives

$$\Sigma'(x, t) = \sum_{k_1, k_2} e^{i(k_2 - k_1)x} e^{i(k_2^2 - k_1^2)t/2m} \left[ k_2 \left( \frac{k_2^2}{2m} - \mu \right) - k_1 \left( \frac{k_1^2}{2m} - \mu \right) \right] f_{k_1} (1 - f_{k_2})$$

A calculation of $\Sigma(x, t)$ proves that it has the same expression of $\Sigma'(x, t)$ with the exchange $k_1 \leftrightarrow k_2$, so that $\Sigma(x, t) = -\Sigma'(-x, -t)$

Therefore we can write

$$\Sigma'(k, \omega) = \frac{1}{mL^2} \sum_{k_1, k_2} \int dx dt e^{i(k_2 - k_1)x} e^{i(\omega + k_2^2 - k_1^2)t/2m} \cdot \left( \frac{k_2^2 - k_1^2}{2m} - \mu(k_2 - k_1) \right) f_{k_1} (1 - f_{k_2})$$

$$\Sigma(k, \omega) = -\Sigma'(-k, -\omega)$$

---

Thermal drag between two coupled quantum wires
Performing the integral over $x$ and $t$ and changing the summation in an integral, $\Sigma'(k, \omega)$ is equal to

$$
\int \frac{dk_1 dk_2}{m} \delta(k_2 - k_1 - k) \delta \left( \omega + k_2^2 - k_1^2 \right) \left[ \frac{k_2^2 - k_1^2}{2m} - \mu(k_2 - k_1) \right] f_{k_1}(1 - f_{k_2})
\int \frac{dk_1 k}{m} \delta \left( \frac{k_1 k}{m} - \omega + k^2 \right) \left[ \frac{k_1^2 + (k_1 + k)^2 + k_1(k + k_1)}{2m} - \mu \right] f_{k_1}(1 - f_{k + k_1})
\left[ \frac{(q - \frac{k}{2})^2 + (q + \frac{k}{2})^2 + (q - \frac{k}{2})(q + \frac{k}{2})}{2m} - \mu \right] f_{q - \frac{k}{2}}(1 - f_{q + \frac{k}{2}})
$$

where $q \equiv m\omega/k$ has been defined. At this point we look at equation (3.3.17), observing that an integration over $\omega$ and $k$ has to be made; a change of variables can be made:

$$
q_1 = \frac{m\omega}{k} - \frac{k}{2}, \quad q_2 = \frac{m\omega}{k} + \frac{k}{2},
\omega = k(q_1 + q_2), \quad k = q_2 - q_1
$$

obtaining

$$
\rho_{TT}^{12} = \frac{1}{8\pi^2 k_B T^2} \int dq_1 dq_2 \left| \frac{q^2 - q_1}{m} \right| U_{12}^2(q_2 - q_1) \left[ \frac{q_1^2 + q_2^2 + q_1q_2}{2m} - \mu_1 \right]. \quad (E.2)
\left[ \frac{q_1^2 + q_2^2 + q_1q_2}{2m} - \mu_2 \right] f_{q_1}^{(1)}(1 - f_{q_2}^{(1)}) f_{q_2}^{(2)}(1 - f_{q_2}^{(2)}) \quad (E.3)
$$

where the Fermi Dirac distribution is given by

$$
f_q^{(i)} = \left[ e^{\beta(q^2/2m - \mu_i)} + 1 \right]^{-1}
$$

Equation (E.2) allows to calculate the transresistivity of two quantum wires of free electrons for any temperature. Nevertheless, this expression is not very easy to manipulate, mainly because of the absolute value present in the integrand. Furthermore, we have to consider that the expressions (C.1) and (C.4) for $\chi_{ii}$ are valid in the limit of low temperatures, and so they should be recalculated before any substitution into equation (E.2).

For identical wires and in the particular limit $T_F \ll T \ll T_0$, something can be said on the qualitative behaviour of the transresistivity: a variable change can be made, verifying that the integral in equation (E.2) goes like $T^{7/2}$. Furthermore, a calculation of $\chi$ starting from equation (C.2) yields $\chi \sim T^{3/2}$, resulting therefore in $\rho_{TT}^{12} \sim T^{-3/2}$.

---

**Thermal drag between two coupled quantum wires**
Appendix F

Dynamic approach

In this appendix we show how to use the method given in Ref. [13] in order to derive, in an alternative fashion, the equation (3.3.28).

In Ref. [13] is used an equation of motion that relates the electric field to a coupling force between the wires. We will derive a similar relation relating the coupling force a generalized force due to the temperature gradient, but the proof is not straightforward as in the electric case and some care is needed.

The force expectation value is then evaluated to the second order in the coupling, deriving an expression for the thermal gradient in function of density-density correlation functions. The correlations are calculated out of equilibrium and must be expanded to the first order in the drift velocity of the excitation gas; this velocity is proportional to $J_2$, so that a proportionality coefficient between $\nabla T_1$ and $J_2$ can be evaluated.

Throughout all this Appendix we assume to be in the low energy limit ($T \ll T_F$).

We begin our analysis with the derivation of the force balance equation.

F.1. Force balance

The setup studied by this method is the very same of Figure 3.1; in analogy with the discussion of section 2.1, the thermal gradient $\nabla T_1$ is tuned to set $J_1 = 0$, so that the only thermal current flowing is $J_2$.

The equilibrium condition is written calculating the time derivative of the total momentum of electrons in the wire 1 (subject to a thermal gradient) and requiring that its expectation value is zero in a steady regime.
The Hamiltonian density is written from equation (1.2.3) using a decomposition in creation and annihilation operator
\[ H(x) = \frac{1}{L} \sum_{k_1,k_2} e^{i(k_2-k_1)x} \frac{k_1^2}{2m} a_{k_1}^\dagger a_{k_2} + \frac{1}{2L^2} \sum_{k_1,k_2,k_4,q} e^{i(q+k_4-k_1)x} a_{k_1}^\dagger a_{k_2}^\dagger V(q) a_{k_2+q} a_{k_4} \]

From equation (2.1) the coupling is
\[ H_{int} = \int dx dy U_{12}(x-y) \rho_1(x) \rho_2(y) \]
and the momentum operator is
\[ p_1 = \sum_k k a_{1,k}^\dagger a_{1,k} \]

Let us now evaluate \( \langle \dot{p}_1 \rangle = \langle i[H, p_1] \rangle \). The commutator gets contributions from \( H_1 \) and from the coupling:
\[ i[H, p_1] = i \int dx [H_1(x), p_1] + i \int dx dy U_{12}(x,y) [\rho_1(x), p_1] \rho_2(y) \]

Using \( [a_{k_1}^\dagger, a_{k_2}^\dagger a_{k_3}] = -a_{k_2}^\dagger \delta_{k_1,k_3} \) and \( [a_{k_1}, a_{k_2}^\dagger a_{k_3}] = a_{k_2} \delta_{k_1,k_2} \) it can be easily proven that
\[ i[\rho_1(x), p_1] = i \frac{1}{L} \sum_{k,k_1,k_2} ke^{i(k_2-k_1)x} [a_{1,k_1}^\dagger a_{1,k_2}, a_{1,k}^\dagger a_{1,k}] \]
\[ = \sum_{k,k_1,k_2} ike^{i(k_2-k_1)x} (a_{1,k_1}^\dagger a_{1,k}^\dagger a_{1,k_2}^\dagger a_{1,k} \delta_{k_1,k_2}) \]
\[ = \sum_{k,k_1,k_2} i(k_2-k_1) e^{i(k_2-k_1)x} a_{1,k_1}^\dagger a_{1,k_2} \]
\[ = \partial_x \rho_1(x) \]

In a similar way, we find
\[ i[H_1(x), p_1] = \partial_x H_1(x) \]
and thus
\[ \langle \dot{p}_1 \rangle = \langle \int dx \partial_x H_1(x) \rangle + \langle \int dx dy U_{12}(x-y) \partial_x \langle \rho_1(x) \rho_2(y) \rangle \rangle = 0 \]

The expectation value \( \langle \partial_x \rho_1(x) \rho_2(y) \rangle \) is zero at the zeroth order in the coupling, so we need to calculate it at the first order; in fact \( \rho_1(x) \) assumes a constant value and therefore \( \langle \partial_x \rho_1(x) \rangle = 0 \). For notation simplicity, we will implicitly assume that the expectation value has to be evaluated at the first order and perform a partial integration:
\[ \langle H_1(L/2) \rangle - \langle H_1(-L/2) \rangle - \int dx dy \frac{dU_{12}(x-y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle = 0 \] (F.1.1)

---

**Thermal drag between two coupled quantum wires**
The last term is the expression for the force between the wires, while the first two are related to the temperature gradient: in fact the expectation value of the Hamiltonian density is a function of the temperature, which has a spatial dependence; thus their difference will be a function of the temperature gradient. Supposing the temperature difference between the two ends of the wire to be negligible with respect to the average temperature $T$ of the wire (i.e. assuming a linear response regime), one finds

$$\langle H_1(T_1(L/2)) \rangle = \langle H_1(T) \rangle + \frac{\partial \langle H_1(T) \rangle}{\partial T} \Delta T_1(L/2)$$

$$\langle H_1(T_1(-L/2)) \rangle = \langle H_1(T) \rangle + \frac{\partial \langle H_1(T) \rangle}{\partial T} \Delta T_1(-L/2)$$

$$\langle H_1(L/2) \rangle - \langle H_1(-L/2) \rangle = \frac{\partial \langle H_1(T) \rangle}{\partial T} (\Delta T_1(L/2) - \Delta T_1(-L/2)) = L \frac{\partial \langle H_1(T) \rangle}{\partial T} \nabla T_1$$

$$L \frac{\partial \langle H_1(T) \rangle}{\partial T} \nabla T_1 = \int dx dy \frac{dU(x-y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle$$

(F.1.2)

**wire 1**

![Schematic picture of the forces acting on an electron in wire 1, which is subject to a thermal gradient. The thermal force and the force arising from the interwire coupling must balance.](image)

**wire 2**

The meaning of this equation is not really difficult to understand: when the wire is subject to a thermal gradient, a force of thermal origin appears, pushing the electrons in the direction opposite to that of thermal gradient; this force is balanced by the force arising from the interwire coupling. This is shown in Figure F.1.

To proceed on we need to calculate the expectation value of the Hamiltonian density.

**F.2. Calculation of $\langle H \rangle$**

At this point we put ourselves in the temperatures regime $T \ll T_F$. To calculate the expectation value of the Hamiltonian density, we will proceed as we did in "Thermal drag between two coupled quantum wires"
Appendix C for the χ: we will calculate such value in the non interacting electrons limit (for \( T \ll T_F \)) and in the l bosonization framework; we will then compare the two values to verify if the Luttinger liquid regime is equivalent to the parabolic dispersion regime.

The Hamiltonian density expectation value reads:

\[
\langle H \rangle = \frac{1}{L} \sum_{k_1,k_2} e^{i(k_2-k_1)x} \frac{k_1 k_2}{2m} f_{FD}(k_1) \delta_{k_1,k_2} = \frac{1}{L} \sum_k E_k f_{FD}(k) = \frac{1}{\pi} \int_0^\infty \frac{E_k dk}{e^{\beta(E_k-\mu)} + 1}
\]

At this point the value of the chemical potential has to be substituted, since it contains a temperature dependent correction. Using equation (C.3) one has

\[
\langle H \rangle = \frac{1}{\pi} \sqrt{\frac{m}{2}} \int_0^\infty \frac{dE}{E} e^{\beta(E-\mu)} + 1 = \frac{1}{\pi} \sqrt{\frac{m}{2}} \frac{\frac{2}{3} \mu^{3/2} + \frac{1}{2} \pi^2 (k_B T)^2}{\sqrt{\mu}}
\]

Thus the energy has a constant term plus a term quadratic in the temperature, which is the one we are interested into.

On the other hand, using the bosonization it can be retrieved:

\[
\langle H \rangle = \frac{1}{\pi} \sqrt{\frac{m}{2}} \int_0^\infty \frac{dE}{E} e^{\beta(E-\mu)} + 1 = \frac{1}{\pi} \sqrt{\frac{m}{2}} \frac{\mu^{3/2} + \frac{1}{6} \pi^2 (k_B T)^2}{\sqrt{\mu}}
\]

Thus the energy has a constant term plus a term quadratic in the temperature, which is the one we are interested into.

On the other hand, using the bosonization it can be retrieved:

\[
\langle H \rangle = \frac{1}{L} \sum_k v |k| \langle \hat{b}_k^\dagger \hat{b}_k \rangle = \frac{v}{2\pi} \int dk |k| f_{BE}(v |k|) = \frac{v}{\pi} \int_0^\infty \frac{dk}{e^{\beta|k|} - 1} = \frac{v}{6} (k_B T)^2
\]

Since \( v = v_F \) for non interacting electrons, \( \frac{\partial \langle H \rangle}{\partial T} \) yields the same result if calculated in the bosonization framework or not.

In the end it can be obtained

\[
\frac{\pi}{3} \frac{k_B^2 T}{v_1} \nabla T_1 = \frac{1}{L} \int dxdy \frac{dU_{12}(x-y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle
\]

F.3. Transresistivity formula

Now that we have an expression for the thermal gradient in function of the force, we can evaluate the force in term of the current. In order to do so, a series of manipulation is needed.

In the interaction representation the temporal evolved of a generic operator \( A \) is

\[
A(x,t) = e^{i(H_1+H_2)t} \mathcal{T} e^{i \int_{-\infty}^t H_{int}(t') dt'} A(x) \mathcal{T} e^{-i \int_0^t H_{int}(t') dt'} e^{-i(H_1+H_2)t}
\]

Thermal drag between two coupled quantum wires
Expanding up to the first order in the interaction, it can be found

\[ A(x, t) = A^I(x, t) + ie^{iH_0t} \frac{1}{\hbar} \int_0^t dt'[\mathcal{H}_\text{int}(t'), A(x)]e^{-iH_0t} \]

so that

\[ \langle \rho_1(x)\rho_2(y) \rangle = i \int_{-\infty}^0 dt \langle [\mathcal{H}_\text{int}(t), \rho_1(x, 0)\rho_2(y, 0)] \rangle \]

\[ \langle \rho_1(x)\rho_2(y) \rangle = i \int_{-\infty}^0 dt \int dx'dy' U_{12}(x' - y') \langle [\rho_1(x', t), \rho_1(x, 0)\rho_2(y', t)\rho_2(y, 0)] \rangle + \]

\[ + i \int_{-\infty}^0 dt \int dx'dy' U_{12}(x' - y') \langle [\rho_1(x', t)\rho_1(x, 0)\rho_2(y', t), \rho_2(y, 0)] \rangle \]

\[ \langle \rho_1(x)\rho_2(y) \rangle = i \int_{-\infty}^0 dt \int dx'dy' U_{12}(x' - y') \left( \langle [\rho_1(x' - x, t), \rho_1(0, 0)] \rangle \cdot \langle \rho_2(y' - y, t)\rho_2(0, 0) \rangle + \langle \rho_1(x' - x, t)\rho_1(0, 0) \rangle \langle [\rho_2(y' - y, t), \rho_2(0, 0)] \rangle \right) \]

The expectation values relative to the densities of the two wires have been separated. The dynamic structure factor is now introduced

\[ \langle \rho_1(x, t)\rho_1(0, 0) \rangle = \int \frac{dkd\omega}{(2\pi)^2} e^{ikx-i\omega t} \tilde{S}_1(k, \omega) \]  

(F.3.1)

Equation (F.3.1) differs from equation (3.3.22) because in the last equation the structure factor form is calculated at the equilibrium with no current flowing. Instead in equation (F.3.1), the density has to be evaluated for moving electrons.

Thus we find

\[ \langle [\rho_1(x, t), \rho_1(0, 0)] \rangle = \int \frac{dkd\omega}{(2\pi)^2} e^{ikx-i\omega t} (\tilde{S}_1(k, \omega) - \tilde{S}_1(-k, -\omega)) \]

and therefore

\[ \langle \rho_1(x)\rho_2(y) \rangle = i \int_{-\infty}^0 dt \int dx'dy' \frac{dk_1d\omega_1dk_2d\omega_2}{(2\pi)^4} U_{12}(x' - y')e^{ik_1(x' - x)}e^{-i\omega_1t}e^{ik_2(y' - y)} \cdot e^{-i\omega_2t} \left[ 2\tilde{S}_1(k_1, \omega_1)\tilde{S}_2(k_2, \omega_2) - \tilde{S}_1(-k_1, -\omega_1)\tilde{S}_2(k_2, \omega_2) - \tilde{S}_1(k_1, \omega_1)\tilde{S}_2(-k_2, -\omega_2) \right] \]

The procedure of section 3.3 is applied: the integration in \( t \) is performed, retaining only the real part and expanding in Fourier components the coupling potential

\[ \nabla T_1 = \frac{e_1}{k_B T} \frac{3}{\pi} \int \frac{dkd\omega}{8\pi^2} kU_{12}(k)U_{12}(-k)[2\tilde{S}_1(k, \omega)\tilde{S}_2(-k, -\omega) - \tilde{S}_1(-k, -\omega)\tilde{S}_2(-k, -\omega) - \tilde{S}_1(k, \omega)\tilde{S}_2(k, \omega)] \]
In this expression it can be easily verified that the second and the third term sum to zero, because, operating the change \( k \rightarrow -k \) and \( \omega \rightarrow -\omega \), the integrands are one the opposite of the other. Therefore

\[
\nabla T_1 = \frac{v_1}{k_B T} \frac{3}{4\pi^2} \int \frac{dkd\omega}{4\pi^2} k |U_{12}(k)|^2 \tilde{S}_1(k,\omega) \tilde{S}_2(-k,-\omega) \tag{F.3.2}
\]

Now we want to relate the dynamic structure factor to its equilibrium value \( S_i(k,\omega) \).

In the wire 1 there is no current flowing so \( S_1(k,\omega) = \tilde{S}_1(k,\omega) \). Indeed in the wire 2 there is a thermal current \( J_2 \) flowing, related to a drift velocity \( v_d \); in the reference frame moving along with the excitation gas, the structure factor assumes its equilibrium value.

In order to avoid confusion, we call \( \rho^{eq} \) the density at the equilibrium and have (from equation (3.3.22))

\[
S(k,\omega) = \int dxdt e^{-ikx+i\omega t} \langle \rho^{eq}(x,t), \rho^{eq}(0,0) \rangle
\]

To derive the relation between the density at the equilibrium and the density in presence of a current, suppose that at \( t = 0 \) a thermal gradient is applied to wire 1 and the electron fluids starts moving with velocity \( v_d \) in wire; the transient phase can be neglected if we consider a steady regime sufficiently long in time. At low temperatures, the scattering processes between the excitations are not important, so that the density moves in a rigid way, i.e. \( \rho(x+vdt,t) = \rho(x,0) \). Since at the equilibrium the density does not change for definition, then \( \rho(x,0) = \rho^{eq}(x,0) = \rho^{eq}(x,t) \). In conclusion, \( \rho(x,t) = \rho^{eq}(x-vdt,t) \), as shown in Figure F.2.

\[
J = 0
\]

\[
\rho^{eq}(x,t)
\]

\[
J \neq 0
\]

\[
\rho^{eq}(x,t) = \rho(x+vdt, t)
\]

Figure F.2: Sketch of the situation in a wire without current (top) and in a wire with current (bottom).
Consequently
\[ \hat{S}(k, \omega) = \int dx dt e^{-ikx+i\omega t} \langle \rho(x, t) \rho(0, 0) \rangle = \int dx dt e^{-ikx+i\omega t} \langle \rho^e(q(x-v_d t, t) \rho^e(0, 0) \rangle \]

Changing variable of integration, it can be obtained
\[ \hat{S}(k, \omega) = \int dx dt e^{-ikx-i kv_d t+i\omega t} \langle \rho^e(q(x, t) \rho^e(0, 0) \rangle = S(k, \omega - kv_d) \]
or
\[ \hat{S}_2(k, \omega) = S_2(k, \omega - kv_d) \]

From now on the densities are all considered at the equilibrium (in other words we are replacing \( \rho^e \) with \( \rho \) for book keeping reasons).

To relate \( v_d \) and \( J_2 \), we follow the Ref. [10]
\[ J_2 = \frac{\pi}{3} (k_B T)^2 \frac{v_d}{v^2} \]

here is the proof of this statement. To begin with, for a gas of excitation moving at velocity \( v_d \), the dispersion relation is \( E_k = v|k| - v_d k \). Using the bosonization formalism, the thermal current is
\[
J = \frac{1}{L} \sum_k \frac{v^2 k}{e^{\beta E_k} - 1} = \frac{v^2}{2\pi} \int_{-\infty}^{\infty} \frac{kdk}{e^{\beta(v|k| - v_d k)} - 1} \\
\int_{-\infty}^{\infty} \frac{kdk}{e^{\beta(v|k| - v_d k)} - 1} = \int_{0}^{\infty} \frac{kdk}{e^{\beta(v-v_d)k} - 1} + \int_{-\infty}^{0} \frac{kdk}{e^{\beta(-v-v_d)k} - 1} \\
= \int_{0}^{\infty} \frac{kdk}{e^{\beta(v-v_d)k} - 1} - \int_{0}^{\infty} \frac{kdk}{e^{\beta(v+v_d)k} - 1} \\
= \int_{0}^{\infty} \frac{x dx}{e^x - 1} (k_B T)^2 \left( \frac{1}{(v-v_d)^2} - \frac{1}{(v+v_d)^2} \right)
\]

Expanding up to the first order in \( v_d \), it can be obtained
\[ J = \frac{v^2}{2\pi} \frac{\pi^2}{6} (k_B T)^2 \frac{4v_d}{v^3} = \frac{\pi}{3} (k_B T)^2 \frac{v_d}{v} \]

from which it is straightforward to derive
\[ v_d = \frac{3}{\pi} \frac{v^2}{(k_B T)^2} J_2 \]

(F.3.3)

For small current values, the drift velocity will be small and the structure factor can be expanded to the first order
\[ \tilde{S}_2(-k, -\omega) = S_2(-k, -\omega + kv_d) \approx S_2(-k, -\omega) + kv_d \frac{\partial S_2(-k, -\omega)}{\partial \omega} \]

---

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We suspect that the term not dependent on $v_d$ is zero, i.e.

$$\int \frac{dk d\omega}{4\pi^2} k |U_{12}(k)|^2 S_1(k, \omega) S_2(-k, -\omega) = 0$$

In fact, by the fluctuation-dissipation theorem, we can write one more time $S(k, \omega) = 2A(k, \omega)/(1-e^{-\beta\omega})$. It is easy to see that $A(-k, -\omega) = A(k, \omega)$ and that

$$S_1(k, \omega) S_2(-k, -\omega) = \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)}$$

so that the integral of interest becomes

$$\int \frac{dk d\omega}{4\pi^2} k |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)}$$

The integrand is odd under the exchange $(k, \omega) \rightarrow (-k, -\omega)$, so we divide the plane of integration into four regions: I with $k > 0$ and $\omega > 0$, II with $k > 0$ and $\omega < 0$, III with $k < 0$ and $\omega > 0$ and IV with $k < 0$ and $\omega < 0$. We see that in the regions I+IV the integral is zero, as well as in II+III.

The integral considered is therefore null and we have to analyse the first order in $v_d$ of the expansion: equation (F.3.3) is used, along with the fact that for a real $\omega \neq 0$ and $\beta \omega > 0$ we can write one more time $A(-k, -\omega) = A(k, \omega)$ and $e^{-\beta\omega} \omega/2k_B T$.

$$\frac{\partial}{\partial \omega} A_2(-k, -\omega) = \frac{\partial}{\partial \omega} A_2(k, \omega) = \frac{e^{-\beta\omega/2}}{\sinh(\omega/2k_B T)}$$

Employing

$$S_1(k, \omega) = \frac{2A_1(k, \omega)}{1-e^{-\beta\omega}} = \frac{A_1(k, \omega) e^{\beta\omega/2}}{\sinh(\omega/2k_B T)}$$

a substitution into equation (F.3.4) can be made, finding

$$\rho_{12}^{TT} = \frac{v_1 v_2}{k_B^3 T^4} \frac{9}{\pi^2} \int \frac{dk d\omega}{4\pi^2} k^2 |U_{12}(k)|^2 \sinh^2(\omega/2k_B T) \left[ \frac{A_1(k, \omega) A_2(k, \omega)}{(1-e^{-\beta\omega})} - A_1(k, \omega) \frac{e^{-\beta\omega/2}}{\sinh(\omega/2k_B T)} \right]$$
Again the second term is odd for \((k, \omega) \rightarrow (-k, -\omega)\) and so its integral vanishes; what is left is
\[
\rho_{12}^{TT} = \frac{v_1 v_2}{k_B^2 T^4} \int dk d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{(1 - e^{-\beta \omega}) \sinh^2(\omega/2k_B T)}
\]
\[
= \frac{9 v_1 v_2}{2\pi^4 k_B^5 T^4} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)} \left( \frac{1}{1 - e^{-\beta \omega}} + \frac{1}{1 - e^{\beta \omega}} \right)
\]

It can be noticed that
\[
\frac{1}{1 - e^{-\beta \omega}} + \frac{1}{1 - e^{\beta \omega}} = \frac{1 - e^{\beta \omega} + 1 - e^{-\beta \omega}}{1 - e^{-\beta \omega} - e^{\beta \omega} + 1} = 1
\]

So the final result is
\[
\rho_{12}^{TT} = \frac{9 v_1 v_2}{2\pi^4 k_B^5 T^4} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k, \omega) A_2(k, \omega)}{\sinh^2(\omega/2k_B T)} \quad (F.3.5)
\]
this coincides exactly with equation (3.3.28), of which we put out an alternative derivation of equation (3.3.28).

Remarkably, the method just explained is particularly suitable to prove the generalization to the thermoelectric effect.

In fact, while employing this method we used the properties of the thermal current only twice: in the force balance equation and in the expression for \(v_d\):
\[
\pi \frac{k_B^2 T}{3} \nabla T_1 = \frac{1}{L} \int dxdy \frac{dU_{12}(x - y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle
\]
\[
v_d = \frac{3}{\pi} \frac{v_2}{(k_B T)^2} J_T^2
\]
the analogous equations for the electric case are
\[
-en_1 E_1 = \frac{1}{L} \int dxdy \frac{dU_{12}(x - y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle
\]
\[
v_d = \frac{1}{ne_2} J_E^2
\]

At a practical level, a wire can be subject to a force arising from a thermal or electric field, while into the wire is flowing a thermal or an electric current.

We introduce four coefficients to describe the proportionality of the force (or of the drift velocity) to the field (or current):
\[
\frac{1}{L} \int dxdy \frac{dU_{12}(x - y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle = -C_1^T (-\nabla T_1)
\]
or
\[
\frac{1}{L} \int dxdy \frac{dU_{12}(x - y)}{dx} \langle \rho_1(x) \rho_2(y) \rangle = -C_1^E E_1
\]

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and
\[ v_d = B^T_2 J^T_2 \quad \text{or} \quad v_d = B^E_2 J^E_2 \]
after a quick comparison with the previous equations, we see that \( C^E_1 = \frac{\pi}{\sqrt{3}} v_1 \), \( B^E_2 = \frac{1}{n_{e2}} \) and \( B^T_2 = \frac{3}{\pi} \frac{v_2}{(k_B T)^2} \).

Furthermore, the dependence of electric and thermal transresistivity on these coefficients can be written:

\[
\rho_{TT}^{12} = \frac{1}{2\pi^2 k_B T C^T_1} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

\[
\rho_{EE}^{12} = \frac{1}{2\pi^2 k_B T C^E_1} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

If we want to find the thermoelectric coefficients mixed coefficients have to be considered, i.e. one thermal and one electric.

For example, in the calculation of \( \rho_{ET}^{12} \) an electric field is applied to wire 1 and an induced thermal current is observed in wire 2: thus we have to use the coefficients relative to an electric force and to a thermal current, i.e.

\[
\rho_{ET}^{12} = \frac{1}{2\pi^2 k_B T C^T_1} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

Analogously for \( \rho_{TE}^{12} \) we consider a thermal force together with an electric current:

\[
\rho_{TE}^{12} = \frac{1}{2\pi^2 k_B T C^E_1} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

Expliciting the values of the coefficients, it can be written:

\[
\rho_{ET}^{12} = \frac{3}{2\pi^3 \frac{v_2}{n_{e1} k^3_{B T}}^3} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

\[
\rho_{TE}^{12} = \frac{3}{2\pi^3 \frac{v_1}{n_{e2} k^3_{B T}}^3} \int_0^\infty dk \int_0^\infty d\omega k^2 |U_{12}(k)|^2 \frac{A_1(k,\omega)A_2(k,\omega)}{\sinh^2(\omega/(2k_B T))}
\]

Thus, equation (3.7.2) has been recovered.
Appendix G

Matrix Product States

In this appendix we want to discuss the principal properties of the Matrix Product States (MPS) and how this method can be employed to represent quantum states (both pure and thermal) and simulate temporal evolution.

We will mainly give the key ideas, referring for all the calculations and the mathematical details to Ref. [18].

MPS is applied almost exclusively to one dimensional systems; in theory it is possible to employ this technique also for systems with higher dimensionality, but the method does not work well.

The system that we consider is a one dimensional chain where at each site is associated a local Hilbert space of dimension $d$; the chain contains $L$ sites, so that the dimension of the whole Hilbert space is $d^L$.

A generic pure quantum state describing the system can be written as

$$|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} c_{\sigma_1,\ldots,\sigma_L} |\sigma_1,\ldots,\sigma_L\rangle$$

where $\{\sigma_i\}$ represents a basis of the local Hilbert space on site $i$. The state is therefore a linear combination of the states spanning the total Hilbert space, and the coefficients of this linear combination are expressed by $c_{\sigma_1,\ldots,\sigma_L}$. To properly describe the state of the system $d^L$ coefficients should be specified and, therefore, any computation attempting to simulate the temporal evolution of the system (or even just to find its ground state) should deal with matrices of enormous dimension $d^L \times d^L$; already for $d=2$ this is a prohibitive task.

---

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It is thus natural to look for a method able to approximate the state of the system, lowering the computational time required and at the same time committing an error as small as possible.

The MPS formalism describes a quantum state with a product of matrices (often referred to as MPS state or simply MPS):

\[ |\psi\rangle = \sum_{\sigma_1, \ldots, \sigma_L} A^{\sigma_1} \cdot A^{\sigma_2} \cdot \ldots \cdot A^{\sigma_L} |\sigma_1, \ldots, \sigma_L\rangle \]

where

\[ A^{\sigma_1} \cdot A^{\sigma_2} \cdot \ldots \cdot A^{\sigma_L} \equiv \sum_{a_1, \ldots, a_{L-1}} A^{\sigma_1}_{a_1, a_1} A^{\sigma_2}_{a_1, a_2} \cdot \ldots \cdot A^{\sigma_L}_{a_{L-2}, a_{L-1}} A^{\sigma_L}_{a_{L-1}, a_{L-1}} \]

The \( A^{\sigma_i} \) are matrices of dimension \( D \times D \), where \( D \) is the bond link dimension; the first and the last matrices have only an index since we are considering open boundary conditions; if cyclic boundary conditions had been considered, we would have obtained two extremal matrices with two index.

In order to describe the system, the MPS method requires for each site \( d \) matrices (one for each possible value of \( \sigma_i \)) containing \( D^2 \) parameters and so the total number of parameters required is \( L \times d \times D^2 \).

Until now \( D \) has not been specified; to fully describe the system \( D \) should be really large and exponentially growing with \( L \).

Now the question is: can we choose a \( D \) such that the number of parameters is much less than \( d^L \) and, at the same time, make an acceptably small error in the description of the system? The answer is: yes, in some particular situations; the motivation is connected to the area law, but we will not talk about this topic and will just say that this choice can be made for the system that we consider in Chapter 4. Moreover, it can be proved that the error committed has an upper bound:

\[ |||\psi\rangle - |\psi\rangle_{\text{trunc}}||^2 \leq 2 \sum_{i=1}^{L} \epsilon_i(D) \]

The difference between the exact quantum state and the approximated (truncated) one is calculated with the norm induced by the standard scalar product; \( \epsilon_i(D) \) is the error made when approximating on site \( i \) with a matrix of dimension \( D \).

The important point is that we can choose a \( D \) that allows a good approximation of the state and does not scale exponentially with \( L \) (this is assured by the area law).

There is a very useful graphical representation of MPS states, which pictures them as a chain of sites linked together and with an outgoing line, representing an index that can be contracted with another MPS (to obtain a scalar product) or with an operator (to form another state); this representation is shown in Figure G.1 and G.2.

Now that we know how to treat the states, we can do the same also with operators: an operator is decomposed into an MPO (matrix product operator):

\[ \hat{O} = \sum_{\sigma_1, \ldots, \sigma_L} \sum_{\sigma_1', \ldots, \sigma_L'} W^{\sigma_1, \sigma_1'} \cdot W^{\sigma_2, \sigma_2'} \cdot \ldots \cdot W^{\sigma_L, \sigma_L'} |\sigma_1, \ldots, \sigma_L\rangle \langle \sigma_1, \ldots, \sigma_L| \]

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Here $W_{\sigma,\sigma'}$ is a matrix with $D^2$ parameters and for each site there are $d^2$ of them.

The MPO is an object that acts on an MPS and yields another MPS; thus, it does not break the good form into which the state of the system is written. Its form, as well as its action on an MPS, is represented in Figure G.3.

Once defined the MPOs, we can proceed on by discussing how to find the ground state of an Hamiltonian $H$; it can be done operating a variational procedure in the space of the MPS, looking for a state that minimizes the following functional

$$E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

The next step consists in understanding how to perform a temporal evolution with MPS: this is equivalent to the application of the operator $e^{-iHt}$ to the MPS.

Suppose that we can decompose the Hamiltonian as $H = \sum_i h_i$, where $h_i$ acts on the link between site $i$ and site $i+1$; then the time is divided into infinitesimal time steps, as $t = M\tau$ with $\tau \to 0$, and do a first-order Trotter decomposition of the evolution operator:

$$e^{-iH\tau} = e^{-ih_1\tau}e^{-ih_2\tau} \cdots e^{-ih_{L-1}\tau} + O(\tau^2)$$

---

*Thermal drag between two coupled quantum wires*
The previous expression contains an error due to the fact that the hamiltonians acting on two adjacent links does not commute; to reduce this inaccuracy an higher order decompositions can be made. Remarkably, all time evolutions on odd \( (e^{-iH_{\text{odd}}\tau}) \) links and even \( (e^{-iH_{\text{even}}\tau}) \) links commute and can be carried out at the same time.

The evolution operators can be represented with MPOs having maximum dimension \( d^2 \); therefore the bond link dimension increase from \( D \) to \( d^2 D \). If not truncated, there is an exponential growth of the bond link dimension.

As outlined in Ref. [18] the steps to evolve an MPS \( |\psi(t = 0)\rangle \) are the following:

- Application of the MPO relative to the odd links to \( |\psi(t = 0)\rangle \).
- Application of the MPO relative to the even links to \( e^{-iH_{\text{odd}}\tau}|\psi(t = 0)\rangle \).
- Compression of the MPS \( |\psi(t+\tau)\rangle = e^{-iH_{\text{even}}\tau}e^{-iH_{\text{odd}}\tau}|\psi(t = 0)\rangle \) from dimension \( d^2 D \) to dimension \( D \), monitoring the error committed. The truncation at every step can be substituted by setting a maximum bond link dimension for the time evolution process and truncating only when this dimension is reached.

A pictorial schematization is given in Figure G.4 and G.5:

The most popular Trotter decomposition goes to the fourth order and employs the following formula:

\[
e^{-iH\tau} = U(\tau_1)U(\tau_2)U(\tau_3)U(\tau_4)
\]

where

\[
U(\tau_i) = e^{-iH_{\text{odd}}\tau_i/2}e^{-iH_{\text{even}}\tau_i}e^{-iH_{\text{odd}}\tau_i/2}
\]

\[
\tau_1 = \tau_2 = \frac{\tau}{4 - 4^{1/3}} \quad \tau_3 = \tau - 2\tau_1 - 2\tau_2
\]

This construction was made for real time evolution, but it is still valid for an imaginary time evolution.

---

*Thermal drag between two coupled quantum wires*
Now that we know how to temporally evolve an MPS, we can finally look at the construction of thermal states. In fact, until this moment the application of MPS to pure states has been studied; in order to be able to work also with mixed states, the method known as purification will be used. In other words, we will write an arbitrary thermal state, described by the density matrix $\rho$, as the partial trace over an auxiliary Hilbert space of a pure state.

More formally, if we want to construct a thermal state over the Hilbert space $\mathbb{H}$, an auxiliary space $\mathbb{H}'$ is introduced. It can be proven that we can find a pure state $|\psi\rangle$ belonging to space $\mathbb{H} \otimes \mathbb{H}'$ and such that

$$\rho = Tr_{\mathbb{H}'} (|\psi\rangle \langle \psi|)$$

$\mathbb{H}'$ can be chosen equivalent to $\mathbb{H}$; therefore a mixed state can always be expressed in terms of MPS, by doubling the Hilbert space and finding the appropriate pure state, which can be written as an MPS. The auxiliary system is commonly called the ancilla.

---

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Suppose that we want to find the pure state associated to a thermal state at infinite temperature $\rho(\beta = 0) = \frac{1}{d}$. Then it is immediately verified that

$$|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_j |j\rangle_H |j\rangle_W$$

i.e. the desired state is the maximally entangled state between the system and the ancilla.

From this the density matrix at an arbitrary temperature can be easily written:

$$\rho(\beta) = Z(\beta)^{-1} e^{-\beta H} = Z(\beta)^{-1} e^{-\beta H/2} I e^{-\beta H/2}$$

Using that the identity matrix is equivalent to $Z(0) \rho(0)$ it can obtained

$$\rho(\beta) = Z(\beta)^{-1} e^{-\beta H} = (Z(0) / Z(\beta)) \text{Tr}_W \left[ e^{-\beta H/2} |\psi_0\rangle \langle \psi_0| e^{-\beta H/2} \right]$$

This equation tells us that all that we have to do is to evolve the maximally entangled state in imaginary time, in order to reach the state

$$|\psi_{\beta}\rangle = e^{-\beta H/2} |\psi_0\rangle$$

and then normalize it.

Summarizing, we have proved that a thermal state can be described introducing an auxiliary system (ancilla), finding the maximally entangled pure state of system+ancilla, evolving in imaginary time until $\beta/2$ and then performing a trace over the ancilla. We know how to implement all these operations in the MPS language.

All this formalism is shown in Figure G.6:

![Figure G.6: Representation of thermal state construction: one has to construct a ladder with two identical chains maximally entangled and then to perform an imaginary time evolution. Picture from [18].](image)
References


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