Time Optimal Control of a Dissipative Two-level Quantum System

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Anno accademico 2013/2014
Introduction

Richard P. Feynman, in a classical talk in 1959[^1] said: “I would like to describe a field, in which little has been done, but in which an enormous amount can be done in principle. This field is not quite the same as the others in that it will not tell us much of fundamental physics (in the sense of, “What are the strange particles?”) but it is more like solid-state physics in the sense that it might tell us much of great interest about the strange phenomena that occur in complex situations. Furthermore, a point that is most important is that it would have an enormous number of technical applications. What I want to talk about is the problem of manipulating and controlling things on a small scale.”.

Since the early days of Quantum Mechanics (QM), technological and theoretical advances have mutually influenced each other. To get a precise verification of the theoretical results of Quantum Mechanics and to obtain new insights about it, the ability in manipulating quantum systems is of fundamental importance. The quantum optimal control theory (QOCT) dates back to the 1960s, when the realization of the first lasers did stimulate the first attempts of examining and modifying quantum systems at the molecular scales, even though the best experimental successes became in the 1980s, with the development of femtosecond laser sources and of sophisticated pulse-shaping technologies [1].

Without referring to the particular quantum case, the optimal control theory (OCT) is a mathematical framework that deals with the so called control systems. A generic control system is a dynamical system subject to some external “agent” that is described by some parameter \( u \) (the “control”): if we are able to tune the control, the system can be “driven” in order to change its state.

The OCT has a long history which started with the calculus of variations, especially in the curve minimization problem. In this context optimizing the

[^1]: December 29th 1959. Annual meeting of the American Physical Society at the California Institute of Technology (Caltech)
control means to find a control law for $u$ which minimizes or maximizes some functional (or “cost”): for example the controlled system can be steered towards a final target following a path that minimises the used energy or the total time required. One of the most important results was in the works of Pontryagin in the 1950s [2] who formulated a maximum principle, the *Pontryagin maximum principle* (PMP), in a pseudo Hamiltonian form similar to that of the classical calculus of variations.

The classical control theory can be applied in a natural manner to Hamiltonian quantum systems: here the control can be a time dependent parameter contained in the Hamiltonian describing the system dynamics. In the early approaches most of the efforts were devoted to the control of closed quantum systems [3]. But ultimately, every physical system interacts with its surrounding environment [4] and the study and the control of the dynamics of open systems is thus of fundamental importance [5]. A typical phenomena involved in this context is the decoherence, the degradation of the quantum properties of the system, such as the superposition (coherence) between states. This dynamics is often well described in the Markovian approximation. In this case a given arbitrary initial quantum state left free to evolve typically relaxes towards a fixed point. The position of the fixed point in the “state space” depends both on the characteristic “quantum channel” describing the dissipation phenomena, and on the bath temperature, if the environment is in thermal equilibrium.

In this thesis we investigate the controlled (coherent, i.e. without feedback from measurements) dynamics of a dissipative two-level quantum system subject to *Generalized Amplitude Damping* (GAD) in the Markovian approximation. Our goal is to accelerate the relaxation of the system, starting from a state “more pure” than the fixed point. Roughly speaking, we want to speed up the *heating* of the system, when its initial state is *colder* than the bath. This problem may be of fundamental importance in quantum information and computation [6]. In particular, the challenge of the quantum computer requires, among other things, the design of devices based on the maintenance and the driving of coherent quantum states. Obtaining a good preparation of the system in a given state is then one of the main requirements for the implementation of the quantum computer and the simulation of complex systems, and the acceleration of the dynamics helps to limit errors.

The problem just described can be stated and solved within the framework of the *Time Optimal Control Theory* (TOCT), using the variational approach
based on the PMP, which turns out to be very powerful in a geometrical formulation for 2-D system, thanks to recent mathematical works \cite{7}. We would like to find the optimal control law for $u(t)$ (also called “optimal strategy”), and the corresponding optimal trajectory in the state space of the two-level system, in order to reach the target state (the fixed point) in an optimal minimum time $T_{opt}$. Furthermore, we tolerate that the quantum state arrives within a small distance $\epsilon$ from the target fixed point (an approximation useful in the experimental applications). We model the control by an electromagnetic field (e.g., a laser) in resonance with the energy difference $E_1 - E_0$ between the ground and the excited states of the system. We assume to be able to modulate the amplitude $u(t)$ of the electromagnetic field. We also make the physical hypothesis that the control is bounded (i.e. the laser has a maximum strength $u_{max} > 0$).

Similar problems have been studied in the literature recently. In particular in \cite{8}, for the same model we study here, the optimal strategy is found in the limit case of unconstrained control. In \cite{9,10,11,12,13} instead the authors study the optimal strategy problem for bounded, but constant control. Here we investigate how the scenario changes when the maximum value $u_{max}$ of the control at our disposal increases, starting from $u_{max} = 0$ (no control) until the control $u_{max}$ is arbitrary large, but however bounded.

The thesis is structured as follows.

In chapter 1 we briefly review the main properties of open systems. We introduce the Bloch sphere parameterization of the state space for a two level quantum system and the GAD channel and we describe the dynamics by means of the Markovian Master equation written in Lindblad form.

In chapter 2 we describe how to model mathematically a generic control problem. Then the PMP is stated in the particular case of the time minimum problem for an affine system. This is a very important dynamical system, well studied from a mathematical point of view and suitable for many physical contexts, in particular for the case of dissipative systems. Then we introduce a series of powerful tools valid for the systems in $\mathbb{R}^2$.

In chapter 3 we show how a quantum mechanical system can be studied as a controlled system, i.e. the way to embed the quantum dynamics in the framework of the control theory. In particular we review the basic concepts about the control theory as well as various methods used in QM in order to solve a large class of optimal control problems. Finally, we apply the mathematical tools introduced in the previous chapters to solve our physical problem. For low control strengths, thanks to to the regularity of the trajectories involved,
we are able to analytically find the optimal solution. Instead, for higher control strengths numerical methods are needed.

In chapter 4 we present concluding remarks, summarizing what has been achieved in this work and pointing out open problems and directions for possible future research.
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Chapter 1

Basic facts about open quantum systems

In this chapter we summarize some of the properties of open quantum systems. Every realistic physical system is coupled with other (more or less complex) systems and a perfect isolation is ultimately impossible (even though in some circumstances theoretical and experimental well motivated approximations allow to describe a system as closed). In the quantum case, the interaction with the environment takes on special features, since it causes the loss of the quantum properties (decoherence). To enlighten the peculiarity of open quantum systems, in section 1.1 we first briefly review the well known dynamics of closed quantum systems. In this case the state evolution is unitary and it is described by the Schrödinger equation (if the initial state is pure) or by the von Neumann-Liouville equation (for a mixed state). Respectively, the Hamiltonian and the Liouville superoperator play the role of the generator of the time-evolution unitary group. In section 1.2 we introduce the bipartite systems and the CPT maps (or quantum channels), which describe the most general discrete time evolution of an open system. We find that an open system does not follow a unitary dynamics and a differential equation as the von Neumann-Liouville is hard to find. However, at least in the Markovian approximation, the context is simple enough to find a generalization of this equation. In section 1.3 we find that the lack of memory of the environment allows a semigroup description of the dynamics and thus it is possible to find the Lindblad-Kossakovski Master Equation which is a first order differential equation and the generator of the CPT maps. Special emphasis is placed on the two level systems: in section 1.4 we describe the Bloch sphere representation and a particular dissipation channel, the Generalized Amplitude Damping.
1.1 Unitary evolution of closed systems

1.1.1 The Schrödinger equation

Let $\mathcal{H}$ be the Hilbert space describing a closed quantum system. The time evolution of a pure state $|\psi(t)\rangle \in \mathcal{H}$ is governed by the Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -iH(t)|\psi(t)\rangle,$$  \hspace{1cm} (1.1)

where $H(t)$ is the Hamiltonian of the system, in general time-dependent, and we have set $\hbar = 1$.

The solution of (1.1) can be given in term of a unitary time-evolution operator. If we set

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle,$$  \hspace{1cm} (1.2)

from (1.1) it follows that $U(t,t_0)$ must satisfy the equation

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t}U(t,t_0) = -iH(t)U(t,t_0), \\
U(t_0,t_0) = 1.
\end{array} \right.$$  \hspace{1cm} (1.3)

The solution of the last equation may be represented as a time-ordered exponential,

$$U(t,t_0) = T \exp \{-i \int_{t_0}^{t} ds H(s)\},$$  \hspace{1cm} (1.4)

whereas in the case of time-independent Hamiltonians we have $U(t,t_0) = \exp\{-iH(t-t_0)\}$.

To fix the terminology we usually call closed system a system for which there exists a Hamiltonian operator that is the generator of the dynamics. The definition includes the possibility of the presence of external fields (magnetic field, gravitational field etc.) and the Hamiltonian may be time-dependent.

An isolated system is a closed system whose Hamiltonian is time-independent.

1.1.2 The von Neumann-Liouville equation

In the following we will deal with open systems \cite{4,14}, whose behaviour is better understood via the use of the density matrix formalism which we now briefly review \cite{16}. 

1.1 Unitary evolution of closed systems

**Definition 1.2** The density matrix is the most general object which describes a quantum state: pure states and statistical mixtures. It is defined as:

\[ \rho = \sum_{\mu} p_{\mu} |\psi_{\mu}\rangle \langle \psi_{\mu}|, \]

where \( |\psi_{\mu}\rangle \) are not necessarily orthogonal and they describe the quantum state in the mixture, each one with probability \( p_{\mu} \). If the sum has only one element, \( \rho \) expresses a pure state. More generally it may be seen that \( \rho \) has the following properties:

(i) \( \rho \) is Hermitian: \( \rho = \rho^\dagger \);

(ii) \( \rho \) is positive: for any \( |\psi\rangle \), \( \langle \psi|\rho|\psi\rangle \geq 0 \);

(iii) \( \rho \) is normalized: \( \text{tr}[\rho] = 1 \), i.e. \( \sum_{\mu} p_{\mu} = 1 \);

(iv) The expectation value of any observable \( O \) is: \( \langle O \rangle = \text{tr}[O\rho] = \sum_{\mu} p_{\mu} \langle \psi_{\mu}|O|\psi_{\mu}\rangle \).

(v) \( \text{tr}[\rho^2] = 1 \) iff the state is pure. Otherwise \( 0 < \text{tr}[\rho^2] < 1 \).

Finally, we identify with \( \mathcal{D}(\mathcal{H}) \) the space of the density matrices. \( \mathcal{D}(\mathcal{H}) \) is convex, i.e. it is closed under convex linear combinations: given two density matrices \( \rho_1, \rho_2 \), the combination \( \rho(\lambda) = \lambda \rho_1 + (1 - \lambda) \rho_2 \) is a density matrix for any \( \lambda \in [0, 1] \).

Let the system be, at some initial time \( t_0 \), in a mixed state

\[ \rho(t_0) = \sum_{\mu} p_{\mu} |\psi_{\mu}(t_0)\rangle \langle \psi_{\mu}(t_0)|. \quad (1.5) \]

From (1.2) it follows that the density matrix at the time \( t \) is given by

\[ \rho(t) = \sum_{\mu} p_{\mu} U(t, t_0) |\psi_{\mu}(t_0)\rangle \langle \psi_{\mu}(t_0)| U^\dagger(t, t_0) = U(t, t_0) \rho(t_0) U^\dagger(t, t_0). \quad (1.6) \]

The analogous of the Schrödinger equation (1.1) for the density matrix is the **Liouville - von Neumann equation**

**Definition 1.3** (Liouville - von Neumann equation)

\[ \frac{d}{dt} \rho(t) = -i[H(t), \rho(t)]. \quad (1.7) \]
For future use, it is useful to look at the previous equation from a different point of view. We rewrite (1.6) in the form

$$\rho(t) = U_{(t,t_0)}\rho(t_0),$$  \hspace{1cm} (1.8)

where $U_{(t,t_0)}$ defines the one parameter unitary group of time evolutions. For fixed $t$, it maps the space $\mathcal{D}(\mathcal{H})$ of density matrix into itself, $U_{(t,t_0)} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$. We also set:

$$\frac{d}{dt}\rho(t) = \mathcal{L}(t)\rho(t),$$  \hspace{1cm} (1.9)

$\mathcal{L}(t)$ is the generator of the time evolution operator.

In close analogy to (1.4) the last equation leads to the following formal expression for the exponential map:

$$\rho(t) = T\exp\left\{-i\int_{t_0}^t ds\mathcal{L}(s)\right\}\rho(t_0).$$  \hspace{1cm} (1.10)

In the time-independent case we have $\rho(t) = \exp\{-i\mathcal{L}(t - t_0)\}\rho(t_0)$.

### 1.2 Bipartite systems, CPT maps and the Kraus representation theorem

Although a closed system follows a unitary dynamics, when we watch the evolution of some of its sub-parts the behaviour may be more complicated. In particular, although the whole system is initially in a pure state, and it remains pure under unitary evolution, this is not true for its sub-parts, in the sense that if initially a subsystem is in a pure state, after the global unitary evolution it may be in a mixed state.

In this section we describe the unitary evolution of a bipartite quantum system. We show how the dynamics of the subsystems can be described by a CPT map which can be represented by means of a set of operators (the Kraus operators) that act only on the Hilbert space of the subsystem in exam. Then we state the Kraus representation theorem.

Let us suppose that the initial state of the bipartite system is a tensor product of the form

$$\rho_{\text{tot}}(t_0) = \rho_S(t_0) \otimes |0\rangle_E \langle 0|,$$

relatively to an Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$.

We want to focus the attention on the sub-system $S$, viewed as an open system in a generic initial mixed state, and without loss of generality we can
1.2 Bipartite systems, CPT maps and the Kraus representation theorem

let the system $E$ be in a pure state\[1], this is always possible by means of a procedure called purification (for details see [6, 16]).

Because the entire system $S + E$ is closed, we know from the previous section that it evolves under the unitary evolution operator $U_{SE}(t,t_0)$. For ease of notation, from now on we set: $U_{SE}(t,t_0) \equiv U$, the initial density matrix for $S$, $\rho_S(t_0) \equiv \rho_S$, the final density matrix $\rho_S(t) \equiv \rho'_S$.

To find the final state of the system $S$ we take the partial trace over the degrees of freedom of the subsystem $E$ after the time evolution. The following representation is then a generalization of (1.6) and we call it unitary representation of the map $\Phi_t$:

**Definition 1.4 (Unitary Representation)**

\[
\rho_S \to \rho'_S \equiv \Phi_t(\rho_S) = tr_E[U(\rho_S \otimes |0\rangle_E \langle 0|_E)U] = \sum_{\mu} E\langle \mu|U|0\rangle_E \rho_S E \langle 0|U|\mu\rangle_E,
\] (1.11)

where $\{|\mu\rangle_E\}$ is an orthonormal basis for $\mathcal{H}_E$.

Then we can express the map in a more compact way, the so called Kraus Representation

**Definition 1.5 (Kraus Representation)**

\[
\Phi_t(\rho_S) = \sum_{\mu} M_\mu \rho_S M_\mu^\dagger,
\] (1.12)

where $E\langle \mu|U_{SE}|0\rangle_E \equiv M_\mu$ is an operator acting on $\mathcal{H}_S$.

**Remark 1.6** We have just derived the Kraus representation of $\Phi_t$ starting from the unitary representation (1.11), but the two representations are indeed equivalent, i.e. we can always find a unitary representation from a Kraus one. Another remark to do is that the Kraus representation is not unique [6].

We have just described a discrete-time evolution of $S$ in terms of a map $\Phi_t(\cdot)$ that connects the initial density matrix to its “output” counterpart. For a fixed final time $t > 0$ and a fixed $\rho_E$ the map $\Phi_t$ is a true linear map that takes density matrices to density matrices, $\Phi_t : \mathcal{D}(\mathcal{H}_S) \to \mathcal{D}(\mathcal{H}_S)$. To show

*The subscript $E$ and $S$ for the two parts of the whole system have been chosen because we will focus on the evolution of $S$, whereas $E$ may be thought as an environment (or a bath) that is a source of noise and decoherence for $S$. Indeed this distinction will be formal and well motivated when we will study the Markovian Master Equation, when $E$ has infinite degrees of freedom, but for now the difference between $S$ and $E$ is only conventional.
Basic facts about open quantum systems

this, note first that from the unitarity of $U_{SE}$ it follows that $\sum_\mu M_\mu M_\mu^\dagger = 1_S$. Then it is very simple to demonstrate that $\Phi_t$ preserves the properties of the density matrix [16], i.e.

(1) It preserves the Hermiticity: $\rho_S^\dagger(t) = \sum_\mu M_\mu \rho_S(t_0) M_\mu^\dagger = \rho_S(t)$.

(2) It preserves the trace: $tr_S[\rho_S(t)] = \sum_\mu tr_S[\rho_S M_\mu M_\mu^\dagger] = tr_S[\rho_S(t_0)] = 1$.

(3) It preserves the positivity: $\langle \psi | \rho_S(t) | \psi \rangle = \sum_\mu (\langle \psi | M_\mu \rangle \rho_S^\dagger(t_0) (M_\mu^\dagger | \psi \rangle) \geq 0$.

Summarizing, on the one hand we have shown that a unitary representation of a discrete-time evolution of a density matrix relative to $S$ can be expressed by means of an operator acting on $\mathcal{H}_S$, i.e. with a Kraus representation. This representation satisfies the properties (1-3). Unfortunately, the inverse is not always possible: a map satisfying (1-3) cannot have in general a Kraus representation. However, it is enough to add a further reasonable property to solve the problem. This additional assumption is a stronger version of (3)[4]

(3') $\Phi_t$ is completely positive: for all $\rho \in \mathcal{D}(\mathcal{H}_S \otimes \mathcal{H}_{aux})$, $[\Phi_t \otimes 1](\rho) \geq 0$.

This property is physically more relevant than positivity because we can never be certain that there is not a “hidden” system coupled to $S$, but the operation $[\Phi_t \otimes 1]$ acts locally on the first system, without influencing the second one.

Definition 1.7 (CPT maps) A linear map $\Phi_t : \mathcal{D}(\mathcal{H}_S) \rightarrow \mathcal{D}(\mathcal{H}_S)$ that satisfies the properties (1),(2),(3') is called a CPT map, or quantum channel. Such maps describe the most general discrete time evolution of a quantum system.

Now we can state a powerful and important theorem [4]:

Theorem 1.8 (Kraus representation theorem) A map $\Phi_t$ is CPT if and only if it has a Kraus representation, i.e. there exists a finite set of operators $\{M_\mu\}$, $M_\mu : \mathcal{H}_S \rightarrow \mathcal{H}_S$ such that:

$$\Phi_t(\rho_S) = \sum_\mu M_\mu \rho_S M_\mu^\dagger,$$

with the condition

$$\sum_\mu M_\mu M_\mu^\dagger = 1_S.$$
1.3 The Markovian Master Equation

1.2.1 Distance between states

It is useful to introduce a distance measure between quantum states, in the general case of a density matrix representation. The notion of distance gives an intuition about the distinguishability between different states. In the case of open quantum systems this notion has a “dynamic” usefulness. A non-unitary evolution, described by a CPT map, due to the interaction with an environment, reduces the distances and then we have less information about the system. This characteristic of the CPT maps is said contractivity [6]. To be more precise we introduce a possible definition of distance:

**Definition 1.9 (Trace distance)** Given two quantum states $\rho, \sigma$, their trace distance is defined as:

$$D(\rho, \sigma) \equiv \frac{1}{2} tr |\rho - \sigma|,$$

where the modulus $| \cdot |$ is defined by $|A| = \sqrt{A^\dagger A}$.

It is simple to prove [6] that a CPT map is contractive, whereas a unitary transform preserves the distance.

Besides the notion of distance, a counterpart is played by the fidelity:

**Definition 1.10 (Fidelity)** Given two quantum states $\rho, \sigma$, their fidelity is defined as:

$$F(\rho, \sigma) \equiv (\text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}).$$

(1.14)

The fidelity is maximal (equal to one) when the states are identical and minimal (zero) when the states are orthogonal.

In section 1.4 we show how a particular channel, the Generalized Amplitude Damping, acts contractively on the Bloch sphere (the state space of a two level quantum system).

1.3 The Markovian Master Equation

We have seen in section 1.1 that the unitary evolution of a closed system is described by a first order differential equation, the von Neumann-Liouville equation (1.7) which allows us to describe a continuous evolution of the system in its state space, while for a sub-system of a bigger quantum system in section 1.2 we have been able to give only a description of a fixed-final-time evolution by use of the theory of CPT maps. Furthermore for closed systems the Liouville super-operator is the generator of a unitary group of transformations: in the case of open systems we would like to have a generator for the CPT maps.
We will show that within a “Markovian” approximation, the CPT maps form a “semigroup” of transformations and it is possible to find the generator of such group, the Markovian Quantum Master Equation (ME in the following). For this reasons the ME approach is in a sense less general then that of the previous section, although it is more useful for our scopes.

We will show the ME equation in its more general form, the Lindblad-Kossakowski Master Equation. We won’t derive it, however we will emphasize the underlying hypothesis and its range of application.

Let us suppose to have a coupled quantum system $S + E$. $S$ is described by an $N$-dimensional Hilbert space $\mathcal{H}_S$, embedded in a very large environment $E$. $S$ is the *quantum open system* we will study. In the literature the environment is often called reservoir if it has infinite degrees of freedom. If it is also in thermal equilibrium it is called bath or heat bath.

At some initial time $t = 0$ let the state of $S + E$ be $\rho(0) = \rho_S(0) \otimes \rho_E(0)$.

Just to fix the ideas, we are free in the following to consider a situation in which the environment is a heat bath, with Hamiltonian $H_E$, at the inverse temperature $\beta$, described by a Gibbs state

$$\rho_E = \frac{\exp(-\beta H_E)}{\text{tr}_E[\exp(-\beta H_E)]}.$$ 

To introduce the semigroup hypothesis we start from equations (1.11,1.12). They describe the change of $\rho_S$. If we let the time $t > 0$ vary, these equations define a one-parameter family of CPT transformations $\{\Phi_t(\cdot) \mid t > 0\}$. Let us first introduce the following hypothesis:

(0) **Markovianity** Generally in a coupled system the effect of the interaction “flows” continuously, oscillating between the two parts. So the environment has a “memory” and its effect on $S$ depends not only on the instantaneous interaction, but also on earlier time interactions. This dynamics, though physically correct at a fundamental level, is quite involved and in many cases of interest a Markovian description is a very good approximation. In a Markovian open system the characteristic time scale $\tau_E$ over which the “time correlation function” of $E$ decays is much shorter than the characteristic dynamical time of $S$, $\tau_S$. Then there is a very quick remix of the degrees of freedom of the environment, which forgets the information that it acquired from $S$. The interaction is then local in time and there is no feedback from $E$ to $S$.

Markovianity is fundamental because it can be formalized via the *semigroup* properties:
1.3 The Markovian Master Equation

Definition 1.11 (Semigroup properties)

\[ \Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_2 + t_1}; \quad t_1, t_2 > 0, \]  
\[ \Phi_0 = 1. \]  

(1.15a)

(1.15b)

The lack of memory is characterized by the non existence of the inverse for the elements of the semigroup.

There are essentially two ways to derive the ME as a generator of the CPT maps: a macroscopic and a microscopic derivation.

Macroscopic derivation: the Lindblad-Kossakowsky Master Equation

The first derivation was introduced at the same time (1976) in [17] (for finite dimensional Hilbert spaces) and in [18] (for the general case of separable Hilbert spaces). From a physical point of view, it needs only the Markov hypothesis and it produces the most general form of the quantum dynamical semigroup generator.

Definition 1.12 (Lindblad-Kossakowski Master Equation)

\[ \frac{\partial}{\partial t} \rho_S(t) = -i[H(t), \rho_S(t)] + \sum_{\mu=1}^{N^2-1} \left( L_\mu \rho_S(t) L_\mu^\dagger - \frac{1}{2} \{ L_\mu^\dagger L_\mu, \rho_S(t) \} \right). \]  

(1.16)

On the right hand side of the previous equation we can recognize:

- A commutator, that expresses the unitary part of the evolution. \( H(t) \) cannot be in general identified with the free Hamiltonian of \( S \), because it may contain additional terms due to the coupling with the environment that cause a shift in the energy levels. For this reason, \( H(t) \) is often called in this context Lamb shift Hamiltonian.

- A dissipative part. It is a combination of operators \( L_\mu \) called Lindblad operators: they are a linear combination of elements of the basis of the Kraus operators and they depend on the particular dissipation channel.

\[ \mathcal{L} \rho_S(t) = \dot{\rho}_S(t) = \lim_{\epsilon \to 0^+} \frac{\rho(t + \epsilon) - \rho(t)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\Phi_{t+\epsilon}(\rho(0)) - \Phi_t(\rho(0))}{\epsilon} \]
according to the characteristics of the interaction between the system and the reservoir. $L_\mu$ can be chosen traceless.

Remark 1.13 We said that (1.16) is the most general form of the generator of the CPT semigroup. It means that, if we rewrite the ME as $\dot{\rho}(t) = \mathcal{L}[\rho(t)]$, we have the exponential map:

$$\Phi_t = \exp(\mathcal{L}t),$$

(1.17)

analogously to the equation (1.10) in the case of the unitary group for closed systems.

**Microscopic derivation**

To derive the ME for a concrete physical system the Markov approximation is not enough. On the other hand a more physical derivation - a microscopic derivation - is desirable. Actually, any other derivation must produce the same form as the (1.16). In particular we would like to find a ME starting from the Hamiltonian of the coupled system $S + E$:

$$H = H_S + H_E + H_I,$$

where on the right hand side we have respectively, the free Hamiltonian of the open system $S$, of the environment $E$ and the interaction Hamiltonian.

To produce a correct form of the ME that is also a generator of the CPT maps we need some further approximations in addition to Markovianity. Here we only review them (for a detailed microscopical derivation see [4])

1. **Weak-coupling limit or Born approximation** It consists in assuming that the interaction between the system and the reservoir is weak, so the influence of the system on the reservoir is small and we can make a perturbative treatment. In particular, the Born approximation is expressed by the fact that the state of the whole system remains factorized at any time and the state of the reservoir remains the same (e.g. a fixed-temperature thermal bath): $\rho(t) \approx \rho_S(t) \otimes \rho_B$.

2. **Rotating wave approximation** If we suppose that the dynamical time scale $\tau_S$ of the open system $S$ is large compared to its relaxation time $\tau_R$, then during the microscopic derivation of the ME we can eliminate the rapidly oscillating terms proportional to $\exp[i(\omega' - \omega)t]$ for $\omega \neq \omega'$, where $\omega$, $\omega'$ are characteristic frequencies of $S$. This is possible because $\tau_S$ is of order $|\omega' - \omega|^{-1}$ and then we can average out over them.
Remark 1.14 Note that the rotating wave approximation and the Markov approximation describe a coarse-graining of the time because the dynamical behaviour over times of the order of magnitude of the correlation times $\tau_R$ and $\tau_E$ are not resolved. A more general treatment of the theory of ME requires a non-Markovian approach.

1.4 The qubit case

In the language of quantum information theory, a two-level system is called qubit. The qubit is the basic unit of information in quantum computation. It is the simplest non trivial quantum system and the knowledge of its properties allows generalization and intuition about more complex systems. In this section we introduce the Bloch sphere parametrization of the qubit state space and a particular dissipation channel, the generalized amplitude damping. We show how this channel affects the geometry of the Bloch sphere.

1.4.1 The Bloch sphere

Because for a qubit the density matrix $\rho$ is a 2x2 Hermitian matrix, it can be expressed as a linear combination with real coefficients of the basis $\{1, \sigma_x, \sigma_y, \sigma_z\}$, where the $\sigma_i$ are the Pauli matrices $\dagger$. From the condition $\text{tr}[\rho] = 1$ it follows that any qubit density matrix has the representation:

Definition 1.15 (Bloch sphere representation) Let $\vec{r}$ be a 3D-vector: $\vec{r} \in \mathbb{R}^3$, $\vec{r} \equiv (r_1, r_2, r_3)$, $|\vec{r}| \leq 1$, and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$

\[
\rho(\vec{r}) = \frac{1}{2} \left( 1 + \vec{r} \cdot \vec{\sigma} \right) = \frac{1}{2} \begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}.
\]  

(1.18)

$\vec{r}$ is the only free parameter in the definition, and it completely describes any qubit-density matrix: there is a 1-1 correspondence between the density matrix and the points of the 3D sphere.

In particular pure states are located on the boundary of the sphere, where $|\vec{r}| = 1$, whereas mixed states are located inside the sphere. The center of the sphere corresponds to the maximally mixed state $\rho = \frac{1}{2} \mathbf{1}$.

\[\dagger\]

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
12 Basic facts about open quantum systems

1.4.2 The generalized amplitude damping channel

Among all possible quantum CPT channels, we study the generalized amplitude damping (GAD). This channel is simple to analyse and describes common energy dissipation processes, such as the spontaneous photon emission or the way in which a qubit approaches the equilibrium with its environment.

For simplicity we first describe in detail an amplitude damping channel. To fix the ideas, suppose to have an atomic two level system, where $|0\rangle_S$ is the ground state and $|1\rangle_S$ is the excited state, immersed in an environment (e.g., an electromagnetic field) initially in its vacuum state $|0\rangle_E$. Let $\gamma$ be the probability for the atom to emit a photon and to decay in the ground state (the spontaneous emission rate, or “damping probability”). If a photon is emitted, the environment has a transition from the vacuum (no photon) to the state $|1\rangle_E$ (one photon). This process is described by a unitary transformation:

$$|1\rangle_S|0\rangle_E \rightarrow \sqrt{\gamma}|0\rangle_S|1\rangle_E + \sqrt{1-\gamma}|1\rangle_S|0\rangle_E, \quad (1.19)$$

(if the system is initially in its ground state and the environment has no photons there is no transition: $|0\rangle_S|0\rangle_E \rightarrow |0\rangle_S|0\rangle_E$).

Applying directly the (1.11), we find the Kraus operator for the ADC:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}; \quad M_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (1.20)$$
From (1.12), the transformation of the elements of a density matrix under ADC is:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \rightarrow \Phi_{AD}(\rho) = \begin{pmatrix} \rho_{11} + \gamma \rho_{22} & \sqrt{1-\gamma} \rho_{12} \\ \sqrt{1-\gamma} \rho_{21} & (1-\gamma) \rho_{22} \end{pmatrix}, \quad \gamma \in [0,1].$$

(1.21)

It is useful to visualize the effect of the ADC on the Bloch sphere. The block vector transforms as:

$$(r_1, r_2, r_3) \rightarrow (r_1 \sqrt{1-\gamma}, r_2 \sqrt{1-\gamma}, \gamma + r_3 (1-\gamma)).$$

(1.22)

This transformation contracts the Bloch sphere towards an ellipsoid centered in $(0, 0, \gamma)$ (see figure 1.1). If we imagine to apply the transformation $n$ times, the element $\rho_{22}$ evolves to $\rho_{22}^n = e^{n \log(1-\gamma)} \rho_{22}$. We can think of this process as a continuum in time and since the diagonal density matrix elements are exactly the probabilities to find the system in the state $|0\rangle_S$ and $|1\rangle_S$, we can say that for $t \rightarrow \infty$ ($n \rightarrow \infty$) the system is driven to the ground state $|0\rangle_S$.

The meaning of contractivity for the CPT maps, introduced in section 1.2.1
Finally, the GAD channel describes the effect of dissipation in an environment at finite temperature, i.e. when the environment can either absorb or emit photons. A GAD performs the following transformation on the Bloch vector (see [6] for details):

\[(r_1, r_2, r_3) \rightarrow (r_1 \sqrt{1 - \gamma}, r_2 \sqrt{1 - \gamma}, \gamma(2p - 1) + r_3(1 - \gamma)), \quad (1.23)\]

where \(p\) is the probability that at the equilibrium, i.e. in the time limit \(t \to \infty\), the two level system is in the state \(|0\rangle\). Figure 1.2 shows the deformation of the Bloch sphere due to the application of a GAD channel: differently from the AD case, the fixed point of the dynamics, where the ellipsoid is centred, is \((0, 0, 2p - 1)\).
Chapter 2

Optimal control theory

A particular type of system design problem is the problem of “controlling”. For example, the engineer may be asked to design a device with certain response characteristics. The translation of the control-system design objectives into the mathematical language is called optimal control problem. The essential elements of this approach are: a mathematical model of the system to be controlled; a desired output of the system; a set of “admissible” input controls (possibly limited by physical constraints); a “cost functional” which measures the effectiveness of the protocol. A systematic study of optimal control theory dates back at least from the 1940s, for a review on the subject we refer the interested reader to e.g. [19, 20, 21].

In the following we shall give a brief presentation of the topic focusing on those aspects which are relevant to our physical problem. We start the chapter by introducing a generic control problem. Then in section 2.2 we focus on time optimality and we introduce the Pontryagin Maximum Principle (PMP), a first order necessary condition for optimality. Finally in section 2.3 we discuss the time optimal problem for 2-D manifolds introducing some of the mathematical tools that will be used in chapter 3 where we will study the optimal control of open quantum systems.

This chapter is rather formal. Principally we follow the construction of [7]. However, we will try to explain most of the statements in a simple language : in particular in section 2.3.4 we summarize all the results in a manner suitable for practical purposes. Furthermore in chapter 3 we shall apply the results to a physical model, thus the following abstract material will acquire a concrete form.
2.1 Control systems

A control system \[19, 20, 3, 7, 22\] can be defined, roughly speaking, as any system subject to some external agent (the control) that can be modulated. Mathematically speaking, a finite dimensional control system is a set of differential first order equations:

**Definition 2.1 (Control system)**

\[
\dot{x}(t) = f(x(t), u(t)) , \quad x \in M , \quad u \in U ,
\]

where \(M\) is a \(n\)-dimensional manifold, \(u\) is the control,\(^*\) and \(U\), the control space, can be any set.

Varying the control in time and space, the system can follow a number of trajectories \(x(t)\) lying on \(M\), defined at any point \(x\) by the tangent vector \(\dot{x}\).

From now on we restrict to the case \(M = \mathbb{R}^n\) and \(U = \mathbb{R}^m\).

A very useful point of view is to think a control system as a combination of assigned vector fields on the manifold \(M\).\(^†\) For example, the affine control system study in the following is defined as \(f(x, u) \equiv F(x) + uG(x)\), where \(F\) and \(G\) are vector fields.

**Definition 2.2 (Open and closed loop)** In the last definition \(u\) appears as function of time: this is the case of open loop, the only one that we consider. Another possible choice is the so called closed loop (or feedback), where \(u\) is function of the space variable \(x\) (examples of closed loop can be found in \([19, 20]\)).

**Definition 2.3 (Admissible (bounded) control and trajectories)** We indicate with \(U = \{u : [0, T] \to U\}\) the space of the admissible controls, determined by the constraints of the physical problem (e.g. the controls are bounded). \(X = \{x : [0, T] \to M\}\) is the space of trajectories on the manifold.

\(^*\)We are considering autonomous system and omit an explicit time dependence: since we have to treat a “Markovian” dissipative quantum system, limiting the study to autonomous system is physically well motivated (see section 1.3). A more general definition of control system in the case of non-autonomous system and a review of the applications of control theory to non-Markovian system can be found in \([5]\).

\(^†\)It may be useful to recall that a vector field is a map \(F : x \in M \mapsto F(x) \in T_x M\), where \(T_x M\) is the tangent space to \(M\) at \(x\). A vector field can be seen as an operator acting on real functions on \(M\) and in a local set of coordinates \(x = (x_1, x_2..., x_n)\) we have \(F(x) = \sum_{i=1}^n F_i \frac{\partial}{\partial x_i}\).
2.1 Control systems

our study to bounded controls, i.e. the space $U$ (the support of the control function) has boundaries. This is a reasonable condition from a physical point of view because it means that the control cannot exceed a given value (the maximum amplitude of a laser field, the gasoline flow in a car, the capacity of the fuel tank in a spacecraft etc.). Note that the final time $T$ is obviously the same for $u$ and $x$, but in general it is not fixed: for example, in the time minimum problem we want to minimize it.

**Definition 2.4 (Three examples of cost functionals)** The control problem is often formulated in three forms, each one characterized by a cost. The cost is a functional $J$ that we have to maximize (or minimize). It can be thought as the analogous of the action in the classical calculus of variation.

(i) **Mayer cost:** $J \equiv \phi(x(T), T)$.
This functional arises when there is a particular emphasis on the final state/time. It is often called final cost.

(ii) **Lagrange cost:** $J \equiv \int_0^T L(x(t), u(t))dt$.
This is typical for the minimization of a quantity that accumulates with time, e.g. the energy necessary to reach the target point. It is often called running cost. The minimum time problem can be easily formulate in this way with $L \equiv 1$ and then $J = T$.

(iii) **Bolza cost:** $J \equiv \phi(x(T), T) + \int_0^T L(x(t), u(t))dt$.
The Bolza problem is clearly a combination of Mayer and Lagrange.

Even though the three functionals appear different, they are equivalent: it is possible convert one to each other (for details and proofs, see [3]).

Now we are ready to define the **optimal control problem** in a general way.

**Definition 2.5 (Optimal control problem)** Given a cost functional $J : X \times U \rightarrow \mathbb{R}$, the optimal control problem consists in finding the function (trajectory) $x \in X$ and the control $u \in U$ that minimizes (maximizes) the cost functional, and that satisfies the boundary conditions $x(0) \in M_0$, $x(T) \in M_T$, $M_0$, $M_T \subset M$ and the constraint (2.1). The manifolds $M_0$ and $M_T$ are called respectively source and target.

This formulation of the control problem is very general. It allows to tackle a number of physical situations, depending on the following choices: the final state is fixed or free, the dimensions of the source and of the target manifolds, the kind of cost (Mayer, Lagrange, Bolza), the source and/or the target can depend on time, the set of admissible controls (e.g. bounded or unbounded).
2.2 Time optimal control: the Pontryagin Maximum Principle

The Pontryagin Maximum Principle (PMP) \[2\] is a generalization of the Hamiltonian formulation of the classical calculus of variations. It gives a set of necessary conditions that a trajectory must satisfy to maximize the cost functional. In particular the PMP gives a definition of “Hamiltonian” more general than the classical one because on the Hamiltonian depending on the control. The PMP also gives a generalization of the Hamilton equations and introduce a maximization condition for the Hamiltonian. This formulation is said to be “pseudo-Hamiltonian”.

The PMP may assume various forms depending on: the form of the control system (2.1), the kind of source and target, the control cost (minimum energy, minimum time etc.). For more general statements see [2, 23].

We state the PMP directly in a particular form suitable for our purposes: (i) the final time is free, (ii) the source is a point and the target a (smooth) submanifold of $M$ of any dimension, (iii) the functional is of Lagrange type (see definition 2.4), (iv) the source and the target do not depend on time. Furthermore we assume that a set of regularity conditions on $L$, $\phi$ and $f$ are satisfied (for details see [7]). In other words, we deal with the time minimum problem for an affine system on $\mathbb{R}^2$. This is a very important dynamical system, well studied from a mathematical point of view and suitable for many physical contexts, in particular for the case of simple dissipative systems.

Definition 2.6 (Affine control system) Consider a control system on $\mathbb{R}^2$, for bounded control $\|u(t)\| \leq 1$.

$$\dot{x}(t) = F[x(t)] + u(t)G[x(t)] , \quad x(t) \in \mathbb{R}^2 , \quad |u(t)| \leq 1. \quad (2.2)$$

$F$, $G$ are vector fields on $\mathbb{R}^2$. $F$ is a drift term: it determines the free evolution $u(t) \equiv 0$ of the dynamics (e.g. the dissipation).

Let $F(x_0) = 0$: $x_0$ is the fixed point of the free dynamics ($\dot{x}(t) = 0$). The problem considered is to reach an arbitrary point of the plane in minimum time, starting from $x_0$. All propositions that we state in this chapter assume this\footnote{\textsuperscript{2}The PMP is a theorem, but traditionally it is called “principle”. In the original works of Pontryagin the theorem is formulated as a minimum principle. Nowadays it is preferred to formulate the principle as a “maximum” condition.}

\footnote{\textsuperscript{3}When we will consider our quantum problem we will relax the hypothesis that the bound on the control is $[-1, 1]$ and we will deal with a control with generic bounds $[-\bar{u}, \bar{u}]$, $\bar{u} \in \mathbb{R}$. For now we may see this choice as a convenient normalization.}
hypothesis, which guarantees local controllability. Furthermore we choose $x_0 = 0$. Nevertheless our application consists in an inverse problem, i.e. stabilizing in minimum time all the points of $\mathbb{R}^2$ to $x_0$, but all properties that we are going to study are still valid in this case (see [7]).

**Definition 2.7 (Cost)** Let us consider a Lagrange problem (def 2.4). For the minimum time problem the natural way to define the cost is to set the Lagrangian $L = 1$. In this way, $J = T$. We have to minimize the final time $T$.

**Definition 2.8 (Hamiltonian)** Define for every $(x, p, u) \in \mathbb{R}^2 \times (\mathbb{R}^2)^* \times [-1, 1]$ the pseudo Hamiltonian$^*$:

$$
\mathcal{H}(x, \lambda, u) = \lambda \cdot (F(x) + uG(x)),
$$

and:

$$
H(x(t), \lambda(t)) = \max_u \{ \mathcal{H}(x, \lambda, u) : u(t) \in [-1, 1] \}.
$$

**Theorem 2.9 (Pontryagin Maximum Principle)** If $\gamma : [0, a] \to \mathbb{R}^2$ is a time optimal trajectory corresponding to a control $u : [0, a] \to [-1, 1]$, then there exist a covector $\lambda$ along $\gamma$ that is a Lipschitz function$^\|$ $\lambda : [0, a] \to (\mathbb{R}^2)^*$ never vanishing, and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in \text{Dom}(\gamma)$:

(i) $\dot{\lambda}(t) = -\lambda(t) : (\nabla F + u(t)\nabla G)(\gamma(t))$,

(ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$,

(iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$.

The dot “$:\cdot$” is the scalar product: since $TM = \mathbb{R}^n$, then the dual space $T^*M = (\mathbb{R}^n)^*$ and the scalar product is simply the matrix product (row vector per column vector).

Condition (i) and the definition of Hamiltonian allow to write the dynamics of the system in a “pseudo-Hamiltonian” form (pseudo because with respect

---

$^*$More rigorously the pseudo Hamiltonian should be defined as $\mathcal{H}(x, \lambda, \lambda_0, u) = \lambda \cdot f(x, u) + \lambda_0 L(x, u)$, which enlightens the analogy with the classical form of the Hamiltonian $H \equiv p\dot{q} - L(p, q)$. For the minimum time problem the Hamiltonian doesn’t include the cost factor $\lambda_0 L$ that in this case is just $\lambda_0$.

$^\|$Let $(X, d_X), (Y, d_Y)$ be two metric spaces, with $d_X$ the metric of the space $X$. A function $g : X \to Y$ is said Lipschitz continuous if there exists a constant $K \geq 0$ such that for all $x_1, x_2 \in X$, $d_Y(g(x_1), g(x_2)) \leq K d_X(x_1, x_2)$. A Lipschitz function is ever continuous. The inverse is not true in general.
to the classical Hamilton equation, here there is a dependence on the control):

\[
\dot{x}(t) = \frac{\partial H}{\partial \lambda}(x(t), \lambda(t), u(t)) \tag{2.4}
\]

\[
\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(x(t), \lambda(t), u(t)). \tag{2.5}
\]

**Definition 2.10 (The optimal synthesis)** The PMP gives only a necessary condition for optimality: the optimal trajectory and the corresponding control law that globally minimize the cost, must be searched among all the extremal trajectories. One generally proceeds step by step (in chapter 3 we shall be more “concrete”): (1) use the PMP to study the properties of the optimal trajectories; (2) select a finite number of candidate globally optimal trajectories; (3) construct a synthesis, formed of extremal trajectories with some regularity properties; (4) among the trajectories in the synthesis find the optimal trajectory.

Another crucial point is that the PMP gives only an existence criteria for the covector \( \lambda \). In general \( \lambda \) is not an “impulse”. To find the dynamics of \( \lambda(t) \) it is necessary to solve the Cauchy problem for all possible initial conditions. This is a formidable task, which in general is possible to be solved only using numerical methods.

### 2.2.1 Controllability, reachable set and accessibility

Before facing an optimal control problem it is obviously necessary to verify that the problem allows solutions. For example, in a minimum time problem, we must be certain that the target state is reachable using a given class of admissible controls. This is the problem of controllability[24][25] and is separate from the problem of minimization of the cost functional, of finding the optimal trajectory, etc.

**Definition 2.11 (Controllability)** The system (2.1) is said controllable if for any source \( x_0 \) and for any target \( x_f \) in \( M \), there exists an admissible control \( u(t) \) defined on some time interval \([0, T]\) such that the system (2.1) reaches the target in time \( T \).

The controllability is a very strong property and is hard to check. A related weaker property but often much easier to prove is the accessibility or local controllability. To define accessibility we first need the notion of reachable set:

**Definition 2.12 (Reachable set)** We call reachable set from a point \( x_0 \)
within a time $T > 0$ the set:

$$\mathcal{R}_{x_0}(T) \equiv \{ x \in M : \text{there exist } t \in [0, T] \text{ and a trajectory } \gamma : [0, T] \to M \text{ of } (2.1) \text{ such that } \gamma(0) = x_0, \gamma(T) = x \}. $$

**Definition 2.13 (Accessibility or local controllability)** The system (2.1) is said to be accessible from $x_0$ if for every $T > 0$ the set $\mathcal{R}_{x_0}(T)$ is a neighborhood of $x_0$.

Note that the controllability problem is equivalent to check that $\mathcal{R}_{x_0}(\infty) = M$. Thus we can see the accessibility problem as the local counterpart of the controllability problem.

### 2.2.2 Calculus of variation and OCT

In the chapter’s introduction we mentioned the fact that OCT is a generalization of the calculus of variations for nonholonomic systems. This is a technical issue, systematically treated for example in [24, 26] and sketched in [27]. Without going into the details we note that the classical calculus of variation [28] can be expressed as a Lagrange problem, with (i) the dynamics $\dot{x} = u$, (ii) a holonomic constraint (2.1) and where the source and the target are respectively defined by $x(0) = x_0$ and $x(T) = x_T$.

But the greatest difference emerges when trying to give an Hamiltonian formulation of the control problem. For example in the typical case it is in general impossible to obtain a Hamiltonian function from a Lagrangian since it is impossible to make a Legendre transformation [27].

### 2.3 Time optimal control on 2-D manifolds

Now that we have the PMP at our disposal, we can use it to characterize the extremal trajectories in an increasingly better way. In the case of affine systems on a 2-D manifold (that for us is simply $\mathbb{R}^2$) there exists a set of useful tools that give a series of conditions about, for example, the value of the control at any time and the possibility of comparing two extremal trajectories to select the optimal one.

**In the literature [28] a constraint on the position (and possibly time) is said to be holonomic. It can be eliminated by restricting the problem to a submanifold $M' \subset M$. A nonholonomic constraint is a constraint on the velocities that can not be reduced to an holonomic constraint.**
2.3.1 The switching function

For the affine systems, the basic and typical tool that gives us the first hints about the dynamics is the switching function \([19, 20, 7, 26]\).

**Definition 2.14 (Switching function)** Let \((\gamma, \lambda) : [0, \tau] \rightarrow \mathbb{R}^2 \times (\mathbb{R}^2)^*\) be an extremal pair. The corresponding switching function (SF) is defined as

\[
\phi(t) \equiv \lambda(t) \cdot G(\gamma(t)).
\]  

(2.6)

**Remark 2.15 (Properties of the switching function)** From the PMP it follows immediately that the SF describes when the control switches from 1 to \(-1\) and vice versa. This is because for an extremal pair \((\gamma, \lambda)\):

1. if \(\phi(t) \neq 0\) for some \(t \in [0, \tau]\), from theorem 2.9 (iii) (the maximization condition of the Hamiltonian) it follows that \(\gamma\) corresponds to a constant control \(u = \text{sign}(\phi)\): in fact \(u\) must be of maximum value (in modulus) to maximize the second part of the pseudo Hamiltonian, the only part on which the control acts, and its sign depends on the sign of \(\phi(t)\), in such a way that \(u \phi(t) > 0\).

2. If \(\phi\) has an isolated zero at \(t\), \(\dot{\phi}(t)\) is strictly greater (resp. smaller) than zero: then there exist \(\epsilon > 0\) such that \(\gamma\) corresponds to constant control \(u = -1\) on \((t - \epsilon, t)\) and control \(u = +1\) on \((t, t + \epsilon)\) (resp. \(u = +1\) and \(u = -1\)).

The situation becomes less trivial when \(\phi(t) = 0\) on an interval \([c, d] \subset [0, \tau], c \neq d\). In this case we have to study the so called singular trajectories (an extensive study of the singular trajectories in control theory can be found in \([26]\)). In the fig.2.1 an example of switching function that gives the three types of control.

To be more concise in the following, let us give some definitions, typical in the context of control theory.

**Definition 2.16** Let \(u : [0, \tau] \rightarrow [-1, +1]\) be a control

- **bang control**: \(u\) is a bang control in an interval \([a, b] \subseteq [0, \tau]\) if it is constantly equal to \(\pm 1\) on \([a, b]\).

\[\text{††To be more rigorous (for details see \([7]\)), from the Lipschitzianity of \(\lambda\) and the smoothness of \(G\) it follows that \(\phi\) is regular enough to permit that the maximization of the control at its boundary values, \(\pm 1\), is a property that is valid in an open interval \([t - \epsilon, t + \epsilon]\), for some appropriate \(\epsilon\). Those hypothesis of regularity also allow to prove many other properties reviewed along the chapter.}\]
2.3 Time optimal control on 2-D manifolds

Figure 2.1: A possible shape of a switching function and the relative controls. (Figure extracted from \[7\])

- **X and Y trajectories/fields:** Define the vector fields:
  \[
  \hat{X} \equiv F - G, \quad \hat{Y} \equiv F + G.
  \]  
  (2.7)

We call \(X\)-trajectory an extremal trajectory \(\gamma(t)\) corresponding to \(u = -1\) and \(Y\)-trajectory an extremal trajectory \(\gamma(t)\) corresponding to \(u = +1\).

- **non-regular time:** \(t \in [0, \tau]\) is said non-regular if \(\phi(t) = 0\).

- **concatenation:** if \(u_1 : [a, b] \to [-1, +1]\) and \(u_2 : [b, c] \to [-1, +1]\) are two controls, their concatenation \(u_2 \ast u_1\) is the control:
  \[
  u_2 \ast u_1(t) \equiv \begin{cases} 
  u_1(t) & \text{for } t \in [a, b] \\
  u_2(t) & \text{for } t \in [b, c];
  \end{cases}
  \]

and similar definitions follow for the corresponding trajectories \(\gamma(t)\).

- **bang-bang control:** a concatenation of trajectories (and respective controls) of type \(X \ast Y, Y \ast X, X \ast Y \ast X\) and so on is a bang-bang trajectory (resp. control).

- **switching time:** \(t\) is a switching time if \(\phi(t) = 0\) and in \(t\) the trajectory passes from \(X\) to \(Y\) or vice versa. Switching times are particular types of non-regular times.

- **singular trajectory:** an extremal trajectory \(\gamma\) is said singular on an interval \([c, d]\), and we indicate it by \(Z\), if \(\phi(t) = 0\) on \([c, d]\).
2.3.2 Singular sets

For the study of the singular trajectories the following two functions, constructed from the fields $F$ and $G$, are fundamental.

**Definition 2.17** Let $F_1$, $F_2$ and $G_1$, $G_2$ be the components resp. of $F$ and $G$. Let $[\cdot, \cdot]$ be the Lie bracket\[1\] for each $x \in \mathbb{R}^2$

\[
\Delta_A(x) \equiv \det(F(x), G(x)) = F_1(x)G_2(x) - F_2(x)G_1(x), \quad (2.8)
\]

\[
\Delta_B(x) \equiv \det(G(x), [F, G](x)) = G_1(x)[F, G]_2(x) - G_2(x)[F, G]_1(x). \quad (2.9)
\]

In the following we will be interested in the sets of point for which these functions become zero,

\[
C \equiv \Delta_A(0)^{-1} \equiv \{x \in \mathbb{R}^2 : \Delta_A(x) = 0\}, \quad (2.10)
\]

\[
S \equiv \Delta_B(0)^{-1} \equiv \{x \in \mathbb{R}^2 : \Delta_B(x) = 0\}. \quad (2.11)
\]

Typically these sets divide the plane into regions in which the behaviour of the trajectories is quite easy to study, and the controls are bang or bang-bang. What happens on the singular sets is instead matter of a involved study.

The function $\Delta_A(x)$ is useful for studying abnormal extremals (extremals for which $\lambda_0 = 0$). This last concept goes beyond the scope of this thesis (for a detailed description of abnormal extremals and their use, see [7, 26]). $\Delta_B(x)$ is instead useful for detecting singular trajectories and we shall study and use it in details.

The functions (2.8, 2.9) and their sets of zeroes (2.10, 2.11) have also an important physical meaning, which become more clear when we will apply them to our quantum system in the next chapters.

**The set $\Delta_B$**

The role of the function $\Delta_B$ naturally emerges from the PMP [7].

Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2 \times (\mathbb{R}^2)^*$ be an extremal pair and $\phi(t) = \lambda(t) \cdot G(\gamma(t))$ the corresponding switching function. Let us limit the study to a small neighborhood of the origin, where for hypothesis $F(0) = 0$. Suppose that $\gamma$ has a control $u \neq \pm 1$ on some interval $I \subset [0, \tau]$ of positive measure (in particular $|u| < 1$): it is the simple case for which a maximization condition is done by $\frac{\partial H}{\partial u} = 0$. From the PMP (theorem 2.9 (ii)), since for an extremal $H = \lambda \cdot (F + uG) = 0$ we have (by continuity, near the origin): $\phi = 0$ on $I$

\[\text{Given two vector fields } X, Y, \text{ the Lie bracket is the vector field defined by } [X, Y] \equiv \nabla Y \cdot X - \nabla X \cdot Y.\]
and thus $\dot{\phi} = 0$ on $I$. Now

$$
\dot{\phi} = \frac{d}{dl} \phi = \frac{d}{dl} (\lambda \cdot G(\gamma)) = \dot{\lambda} \cdot G + \lambda \cdot (\nabla G \cdot \dot{\gamma}) \\
= (-\lambda \cdot \nabla (F + uG) + \lambda \cdot (\nabla G \cdot (F + uG))) \\
= \lambda \cdot \nabla G \cdot (F + uG) - \nabla (F + uG) \cdot G = \\
= \lambda \cdot [(\nabla G \cdot F + u \nabla G \cdot G - \nabla F \cdot G - u \nabla G \cdot G)] = \lambda \cdot [F, G].
$$

Where we have simply applied the derivation rules. Hence $0 = \phi(I) = \dot{\phi}(I)$ means $0 = \lambda \cdot G = \lambda \cdot [F, G]$. Since $\lambda \neq 0$ a.e. we have that $G$ and $[F, G]$ must be parallel.

Summarizing, we have found that a necessary condition to have a singular control (near to the origin) is that $G$ and $[F, G]$ are parallel, i.e. just $\Delta_B(x) = 0.$ Conversely, to avoid this situation and than to have a bang-bang control it is sufficient that these two fields are not parallel.

It’s important to point out the chain of implication that we have made: starting from the assumption $F(0) = 0$ and $|u| < 1$ on a time-interval $I \subset [0, \tau]$, it follows from PMP that $\phi = 0$ and $\dot{\phi} = 0$ on $I$ and then, during the time $I$, the support of the trajectory is a singular set $S$.

The previous statement, even though deduced under particular hypothesis, can be generalized to obtain the following important theorem.

**Theorem 2.18** The support of singular trajectories is always contained in the set $S$.

Due to the importance of the singular set $S$ we would like to characterize it better. In the next section we will see that not every $S$ can be the support of optimal trajectories. To this scope we will introduce the concept of turnpike and antiturnpike arcs. But first let us state some other definitions and an important result relative to the sets made of ordinary points.

**Definition 2.19 (Ordinary and non-ordinary point)** A point $x \in \mathbb{R}^2$ is called an ordinary point if $x \notin S \cup C$. If $x$ is an ordinary point then it follows by the definition that the vector fields $F(x)$, $G(x)$ form a basis in $\mathbb{R}^2$, in fact they are not parallel and therefore linearly independent. Then there exist two functions $f(x)$, $g(x)$ such that $[F, G](x) = f(x)F(x) + g(x)G(x)$. On the contrary, $x$ is a non-ordinary point if $x \in S \cup C$.

The function $f$ plays a fundamental role. It can be easily seen (for details
see [7]) that, for any ordinary point \( x \), the following relation holds:

\[
f(x) = -\frac{\Delta_B(x)}{\Delta_A(x)}.
\] (2.12)

As anticipated, on the sets of ordinary points we can have a simple characterization of the optimal trajectories [7]:

**Theorem 2.20** Let \( \Omega \in \mathbb{R}^2 \) be an open set of ordinary points. Then all extremal trajectories \( \gamma \) in \( \Omega \) are bang-bang with at most one switching.

**Remark 2.21** It is important to emphasize that to say “at most” in the above theorem means that an extremal trajectory throughout \( \Omega \) can be only of one of the following types: \( X, Y, X \ast Y, Y \ast X \).

**Turnpike and antiturnpike arcs**

We are now interested in what happens at points \( x \) that are non-ordinary. The question we ask is “what happens when a bang extremal trajectory reaches a singular set?” We limit our study to the case \( \Delta_B(x) = 0 \) and \( \Delta_A(x) \neq 0 \). The case \( \Delta_A(x) = 0 \) is related to the study of abnormal extremals and, as we have just said, we are not interested about them.

The first remark to make is that theorem 2.18 is referred to a general singular extremal trajectory whose only property is that its SF is zero for a given time interval. The theorem doesn’t say either that a singular trajectory is optimal or that this kind of trajectory is “admissible” (in a sense that we are going to specify).

In fact there is a variety of possibilities for the behaviour of the singular trajectories. First of all we have to determine the value of the singular control, that must lie between the boundary values \( \pm 1 \): we will find an exact formula for this value, but in principle it is possible that the result is a value greater than 1 and hence not admissible for hypothesis. On the other hand, even if the control is admissible, the corresponding trajectory may not be an extremal. This last possibility is related to the geometrical “essence” of the set \( S \) and not to the value of the control: we will see that the sets \( S \) may be of two types, **turnpike** or **antiturnpike**, and only in the first case a trajectory lying on it can be optimal.

Before defining the turnpike and antiturnpike arcs we have to be more precise about the nature of the set \( S \). Now we give some topological definition but we avoid to be too formal (see [7] for a more rigorous statement).
Definition 2.22 (Isolated non-ordinary arc) A non ordinary arc $S$ is a connected one-dimensional submanifold of $\mathbb{R}^2$ with the property that every $x \in S$ is a non-ordinary point. A non-ordinary arc is said to be isolated if there exists a set $\Omega$ that contains $S$ and is divided by $S$ in exactly two connected components. Furthermore $\Omega \setminus S$ is made of only ordinary points.

Remark 2.23 In our model in the next chapter it can be easily seen that the sets $S = \Delta_B(0)^{-1}$ tipically are isolated non-ordinary arcs. Shortly, for us, a non-ordinary arc is a continuous one-dimensional curve made of the points solution of the equation $\Delta_B(x) = 0$ and which divides $\mathbb{R}^2$ (or a submanifold of it in the case of the projection of the Bloch-ball onto a plane) in two components.

Now we are ready to define the turnpike and antiturnpike arc. The following definition \cite{[7, 22]} is very important because it is “constructive”.

Definition 2.24 (Turnpike and antiturnpike arc) A turnpike (resp. antiturnpike) is an isolated non-ordinary arc $S$ that satisfies the following condition:

(T1) For every $x \in S$, $\Delta_B(x) = 0$ and $\Delta_A(x) \neq 0$.

(T2) For every $x \in S$ the vector fields $\hat{X}(x)$, $\hat{Y}(x)$ in definition \cite{2.7} are not tangent to $S$ and point to opposite sides of $S$.

(T3) Let $\Omega$ be an open set in definition \cite{2.22}. If we label as $\Omega_+$ the connected component of $\Omega \setminus S$ towards which the vector fields $\hat{Y}$ points and as $\Omega_-$ the component of $\Omega \setminus S$ pointed by $\hat{X}$ then the function $f$ in definition \cite{2.12} satisfies:

$$f(x) > 0 \text{ (resp. } f(x) < 0 \text{ ) on } \Omega_+,$$
$$f(x) < 0 \text{ (resp. } f(x) > 0 \text{ ) on } \Omega_-.$$

Remark 2.25 It is important to remark that all the above conditions T1, T2, T3 must be satisfied in order to characterize $S$ as a turnpike or an antiturnpike. But it is indeed possible that $S$ is neither turnpike nor antiturnpike: though we will not encounter this situation, this is typical in many applications (see for example \cite{9} where the dissipation channel describes amplitude damping and dephasing).

In the following proposition we prove the formula that determines the values of singular control:
Proposition 2.26 (Value of the singular control) Let $S$ be a turnpike or an antiturnpike. Let $x : [c, d] \to \mathbb{R}^2$ be a trajectory (not necessarily extremal) such that $x(c) \in S$. Then $x(t) \in S$ for every $t \in [c, d]$ iff $x$ corresponds to the singular control (often called feedback control):

$$u_s(x) = -\frac{\nabla \Delta_B(x) \cdot F(x)}{\nabla \Delta_B(x) \cdot G(x)}$$

(2.13)

Proof. Assume that $x([c, d]) \subset S$ and let $u_s$ be the corresponding control. Then $\dot{x}(t) = F(x(t)) + u_s(t)G(x(t))$ for a.e. $t$. From $\Delta_B(x(t)) = 0$ for a.e. $t$ we have

$$0 = \frac{d}{dt} \Delta_B(x(t)) = \nabla \Delta_B(x(t)) \cdot \dot{x}(t) = \nabla \Delta_B(x(t)) \cdot (F(x(t)) + u_s(t)G(x(t)))$$

Definition 2.27 (Regular turnpike and antiturnpike) We say that a turnpike or an antiturnpike is regular if $|u_s(x)| < 1$ for every $x \in S$.

Finally we can state the results that link the singular sets to the singular extremal trajectories.

Lemma 2.28 (Extremal trajectories over a singular set) Let $\gamma : [0, \bar{t}] \to \mathbb{R}^2$ be an extremal such that $\gamma(\bar{t}) = x$, $x \in S$ (turnpike or antiturnpike), and $\phi(t) = 0$. Moreover let $\gamma' : [0, t'] \to \mathbb{R}^2$, $t' > \bar{t}$ a trajectory such that:

- $\gamma'|_{[0, \bar{t}]} = \gamma$.
- $\gamma'|_{[\bar{t}, t']} \subset S$.

Then $\gamma'$ is extremal. Moreover if $\phi'$ is the SF of $\gamma'$, $\phi'|_{[\bar{t}, t']} \equiv 0$.

This theorem means that a trajectory that starts extremal, remains extremal also if it switches to lie on a singular set. This is true for both turnpike and antiturnpike. But the crucial point is that only in the turnpike case the trajectory can be optimal. In fact the following theorem holds.

Theorem 2.29 (Optimality of the turnpike)

(i) Let $S$ be an antiturnpike and $\gamma : [0, \tau] \to \mathbb{R}^2$ an extremal trajectory such that, for $[c, d] \subset [0, \tau]$, $\gamma([c, d]) \subset S$. Then $\gamma$ is not optimal.

(ii) Let $\gamma : [0, \tau] \to \mathbb{R}^2$ an optimal trajectory that is singular for some interval $[c, d] \subset [0, \tau]$. Then $\gamma([c, d])$ is contained in a regular turnpike.
The above theorems characterize the support of a singular trajectories and help us to construct an optimal syntheses. We have learned that if there are turnpikes and antiturnpikes:

1. if an extremal $Y$ (resp. $X$) trajectory reaches an antiturnpike at some point $x$ it necessarily goes beyond in the same state.

2. If the extremal trajectory reaches a turnpike it may follow the turnpike with a control (2.13) (if $u_s < 1$) or go beyond in the same “state” with control $+1$ (resp. $-1$).

We emphasize once again that in the second case the better choice among the two possibilities depends on the global comparison of the total time required by the trajectories.

### 2.3.3 The clock form

We have avoided to report the proof of the theorem 2.29 that can be found in [7]. However this proof consists ultimately in the introduction of a tool that has an independent importance in the comparison of two extremal trajectories. This tool is the so called clock form and it is also introduced in [26, 9, 22, 12].

**Definition 2.30 (The clock form)** The clock form $\alpha$ is a one-form [28, 29] which satisfies the conditions:

$$\begin{align*}
\alpha(F) &= 1, \\
\alpha(G) &= 0.
\end{align*}$$

(2.14)

From the one-form $\alpha$ one can define the two-form $d\alpha$ which is expressed in terms of the function $\Delta_A(x)$ and $\Delta_B(x)$:

$$d\alpha = \left(\frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2}\right) dx_1 \wedge dx_2 = \frac{\Delta_B(x)}{[\Delta_A(x)]^2} dx_1 \wedge dx_2 \equiv g(x_1, x_2) dx_1 \wedge dx_2. \quad (2.15)$$

where $x_1, x_2$ are the coordinates of $\mathbb{R}^2$ and $\alpha_{1,2}$ are the two components of the one-form: $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2$. One sees that $g(x_1, x_2)$ is zero on $S$ and has constant sign in the region delimited by $S$.

The clock form allows to compare the time taken by two different extremal trajectories starting and ending at the same points, though the presence of $[\Delta_A(x)]^2$ at the denominator prevents to use it for trajectories that cross the set $C$. 
To show the properties of the clock form let us integrate $\alpha$ along a trajectory $\gamma$:

$$
\int_{\gamma} \alpha = \int_{0}^{T} \alpha(\dot{x}) dt = \int_{0}^{T} \alpha(F) dt = T.
$$

The first equality follows because we take the value of the one-form on the tangent vector of the trajectory. This is defined by the velocity $\dot{x}(t)$. The second equality follows directly by (2.6) and (2.14).

Let us now take two paths $\gamma_1$ and $\gamma_2$ starting and ending at the same points and of duration respectively $T_1$ and $T_2$.

The Stoke’s theorem allow us to write:

$$
T_1 - T_2 = \int_{\gamma_1 \cup \gamma_2^{-1}} \alpha = \int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \int_{\sigma} d\alpha,
$$

where $\sigma$ is the surface delimited by the closed contour $\gamma_1 \cup \gamma_2^{-1}$.

Figure 2.2 shows a typical example of use of the clock form. The shape of the sets $S$ and $C$ is similar to the one which we will use for our model in chapter 3. The time taken by the paths $\gamma_1$ is shorter than the one taken by $\gamma_2$ since the two paths belong to a region with $g < 0$.

2.3.4 Tools for finding optimal extremals

Until now we have introduced many definitions and results that give a good description of the behaviour of extremal trajectories. The selection of the extremals is important because among them we can find the optimal one (this is the essence of the PMP). Recall that our mathematical problem is to reach a generic point in a minimum time. Various theorems assure that this problem has a solution for the affine systems in $\mathbb{R}^2$. Without loss of generality we can express the problem saying that we want to reach any point starting from the origin. The origin is defined as the point $x_0$ such that $F(x_0) = 0$.

We know that a trajectory can leave the origin in three ways, i.e. it can be a $X$, a $Y$ or a singular trajectory. But a trajectory can be singular only if its support is a non-ordinary set $S$. The switching function tells us when
a trajectory switches and when it can stay on a singular set with a singular control. Outside of the singular sets, the trajectory can be bang or bang-bang, with at most one switching.

At a given time a bang trajectory could reach a singular set: it must go beyond if the set is an antiturnpike, whereas it can continue on the singular set if the set is a turnpike.

The main point is to select the global optimal concatenation of trajectories. The clock form is a helpful tool, but only when the trajectories under study don’t cross a singular set $C$, otherwise a numerical comparison is required.

Finally, to avoid misunderstanding, it is important to remark that the switching function has been used fundamentally as a theoretical tool, but in general it is impractical to directly use it for a simple reason: in most cases we don’t know it, since we do not know the dynamics of the covector $\lambda(t)$. In fact the PMP assures only that $\lambda(t)$ exists and that the vector $(\lambda, \lambda_0) \neq 0$. In order to have an explicit solution $\lambda(t)$, we have to solve the equation $\dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t) \nabla G)(\gamma(t))$ for all the possible initial conditions.

The situation could seem very complicated, because we don’t know even when there can be switches. Indeed in $\mathbb{R}^2$ the problem has always a solution: this is guaranteed by the algorithm described in [7].

An alternative method is to introduce the so called $\theta$-function. It has the advantage not to depend on $\lambda(t)$. Even though it doesn’t give an exact collocation of switches, it gives us strong conditions about the possible times when they can happen.

### 2.3.5 The $\theta$-fuction

A full definition of $\theta(t)$ can be found in [7]. Physical applications of $\theta$, relative to a control of a two-level open system can be found in [9, 10].

The function $\theta(t)$ allows us to obtain information about the switching time without the need of knowing the covector, but only by means of geometrical reasoning. $\theta(t)$ is useful also because its shape tells us the last time for which a trajectory can be extremal if it doesn’t switch.

In this section we will motivate the necessity of $\theta(t)$ with a constructive derivation. Then, without proof, we will state a formula that links the stationary points of $\theta(t)$ with the zeroes of $\Delta_B(x(t))$. Finally, making use of some figures, we will illustrate the ways to use $\theta(t)$.

In the following we will refer to an extremal trajectory $\gamma(t)$. Let us proceed step by step.
• Deriving equation (2.6) with respect to time we obtain the adjoint equation:
\[ \dot{v} = (\nabla F + u \nabla G) \cdot v, \quad u = \pm 1, \] (2.16)
where \( v = \frac{d}{dt} x(t) \) is a tangent vector: \( v \in T_{x(t)} \mathbb{R}^2 \), which “lives” in the same space of the vector fields \( G \) and \( F \). Note that equation (2.16) is very similar to the “variational equation” describing the dynamics of the covector \( \lambda(t) \) in the PMP (theorem 2.9(i)): \( \dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t) \nabla G)(\gamma(t)) \).

• A property of \( v \) is that \( \lambda \cdot v \) is a constant. In fact, from definitions:
\[ \frac{d}{dt} (\lambda \cdot v) = \dot{\lambda} \cdot v + \lambda \cdot \dot{v} = 0. \]

• Let us consider the Cauchy problem for \( v(t) \):
\[ \begin{align*}
\dot{v}(t) &= (\nabla F + u(t) \nabla G)(\gamma(t)) \cdot v(t), \\
v(\bar{t}) &= G(\gamma(\bar{t})).
\end{align*} \] (2.17)
We indicate its solution as \( v(t) = v(G(\gamma(\bar{t})), \bar{t}; t) \) to underline that this is the solution of the Cauchy problem at time \( t \), with initial condition \( G(\gamma(\bar{t})) \) at time \( \bar{t} \).

• Let \( \phi(t) = \lambda(t) \cdot G(t) \) be the SF. From the fact that \( \lambda \cdot v \) is constant and since \( G(\gamma(t)) = v(G(\gamma(t)), t; t) \), we can write the following relation, holding for all \( t \):
\[ \phi(t) = \lambda(t) \cdot G(\gamma(t)) = \lambda(0) \cdot v(G(\gamma(t)), t; 0). \] (2.18)
Now let us focus on the vector \( v \) as a function of the initial-condition time \( \bar{t} = 0 \): \( \bar{v}(t) \equiv v(G(\gamma(t)), t; 0) \). In other words, the dynamics of the solution of (2.17) is propagated backwards in time to the origin, via the adjoint equation. We can see \( \bar{v}(t) \) as a vector rotating in time around the origin of axis \( x_1, x_2 \).

• In particular we are interested in comparing the vector \( \bar{v}(t) \) with its initial value \( \bar{v}(0) = \bar{v}(G(\gamma(0)), 0; 0) = G(\gamma(0)) \).
Furthermore note that in general \( \phi(t) = \lambda(t) \cdot G(\gamma(t)) \neq \lambda(0) \cdot G(\gamma(0)) = \phi(0) \). The equality can be true accidentally or if \( t \) and 0 are switching times.

We then introduce the \( \theta \)-function:
Definition 2.31 ($\theta(t)$ function)

\[
\theta : \text{Dom}(\gamma(t)) \rightarrow [-\pi, \pi],
\]
\[
\theta(t) \equiv \text{arg}(\vec{v}(0), \vec{v}(t)),
\]

where the angle \text{arg} is measured counterclockwise.

\[
\text{Figure 2.3: } \phi \text{ and } \theta \text{ for } G(\gamma(0)) = (0, 1). \text{ Figure extracted from [7]}
\]

Before illustrating the use of $\theta(t)$ we state an important relation between the $\theta(t)$-function and the function $\Delta_B(x(t))$ (for proof see [7]):

\[
\text{sign}(\theta(t)) = \text{sign}(\Delta_B(\gamma(t))).
\] (2.21)

This equality is very useful to check the monotonicity of $\theta(t)$, its maxima and minima and the presence of plateaus in its shape. It will be especially useful in our numerical algorithm of chapter 3.

To illustrate the usefulness of $\theta(t)$, let us suppose that $\gamma$ is initially a $Y$ trajectory (the case for which $u = -1$ is similar) and let $t_1$ be the first time at which $\phi \equiv 0$: $t_1$ is a switching point. We would like to know when another switch can happen.

For a better understanding we refer to figure 2.3. Note that $\theta(0) = 0$. If $G(\gamma(t_1))$ is different from zero, it follows that $\lambda(0)$ and $v(G(\gamma(t_1)), t_1; 0)$ are orthogonal. Let $\theta(t_1) \in [-\pi/2, \pi/2]$ be the value of $\theta$ at time $t_1$, then:

- At time $t_2$ we have $\phi(t_2) = 0$ iff $\theta(t_2) = \theta(t_1) \pm n\pi, \; n \in \mathbb{N}$
• \( \phi(t) > 0 \) for every \( t \) on some interval \((a, b)\) if for every \( t \in (a, b) \):

\[
\begin{align*}
\theta(t) &\in (\theta(t_1), \theta(t_1) + \pi) \text{ if } \theta(t_1) < 0, \\
\theta(t) &\in (\theta(t_1) - \pi, \theta(t_1)) \text{ if } \theta(t_1) > 0.
\end{align*}
\] (2.22)

Instead we have \( \phi(t) < 0 \) if:

\[
\begin{align*}
\theta(t) &\in (\theta(t_1) + \pi, \theta(t_1) - 2\pi) \text{ if } \theta(t_1) < 0, \\
\theta(t) &\in (\theta(t_1), \theta(t_1) + \pi) \text{ if } \theta(t_1) > 0.
\end{align*}
\] (2.23)

Summarizing we get the following relation between \( \theta(t) \) and \( \phi(t) \):

• for a bang-bang trajectory a switching happens when \( \phi \) changes sign; alternatively when \( \theta(t_2) \) crosses the values \( \theta(t_1) \pm n\pi \).

• If \( \gamma(t) \) is a singular arc for \( t \in [a, b] \) then: \( \phi(t) = \dot{\phi}(t) = 0 \) whereas \( \theta(t) = \theta(t_1) \pm n\pi \) and \( \dot{\theta}(t) = 0 \) for all \( t \).

The necessity to find the zeroes and the stationary points of \( \theta(t) \) will be clear from the following properties [7].

**Definition 2.32 (Properties of \( \theta \))** We define \( \theta^+ \) as a \( \theta \) function related to a trajectory starting with control \( u = 1 \) (an \( Y \) trajectory).
2.3 Time optimal control on 2-D manifolds

- $t_f^+ = \min \{ t \in [0, \tau] ; |\theta^+(s_1) - \theta^+(s_2)| = \pi \text{ for some } s_1, s_2 \in [0, \tau] \}$, with $t_f^- = \tau$ if $|\theta^+(s_1) - \theta^+(s_2)| < \pi$ for all $s_1, s_2 \in [0, \tau]$.

**Remark 2.33** $t_f^+$ is the last time at which an $Y$ trajectory is extremal, so we have $t_{optimal} \leq t_f^+$.

- We select the times $t_i^+, t_i^-$ as the “increasingly large” local maximum increasingly small local minimum of $\theta^+$ respectively (see figure 2.4) and by induction we define:

\[
\begin{align*}
t_0^+ &= t_0^- = 0; \\
t_1^+ &= \inf \{ t > 0 : \theta^+ \text{ has a local max. at } t, \theta^+(t) > 0 \}; \\
t_i^+ &= \inf \{ t > t_{i-1}^+ : \theta^+ \text{ has a local max. at } t, \theta^+(t) > \theta^+(t_{i-1}^+) \}; \\
t_i^- &= \inf \{ t > 0 : \theta^+ \text{ has a local min. at } t, \theta^+(t) < 0 \}; \\
t_i^- &= \inf \{ t > t_{i-1}^- : \theta^+ \text{ has a local min. at } t, \theta^+(t) < \theta^+(t_{i-1}^-) \}; \\
s_i^+ &= \left\{ \begin{array}{ll}
\max \{ t \in [t_{i-1}^+, t_i^+] : \theta^+(t) = \theta^+(t_{i-1}^+) \text{ if } t_{i-1}^+ \text{ is defined } \}, \\
\max \{ t \in [t_{i-1}^+, t_i^+] : \theta^+(t) = \theta^+(t_i^-) \text{ otherwise } \};
\end{array} \right. \\
s_i^- &= \left\{ \begin{array}{ll}
\max \{ t \in [t_{i-1}^-, t_i^-] : \theta^+(t) = \theta^+(t_{i-1}^-) \text{ if } t_{i-1}^- \text{ is defined } \}, \\
\max \{ t \in [t_{i-1}^-, t_i^-] : \theta^+(t) = \theta^+(t_i^-) \text{ otherwise } \};
\end{array} \right.
\]

Finally we can state the following proposition that motivate the last definition and tell us in which time interval a switch can happen:

**Proposition 2.34** *(Switches and $\theta$ function)* if $t^*$ lies in one of the open interval $(s_i^+, t_i^+)$ or $(s_i^-, t_i^-)$ then there exist an extremal control of the form:

\[
u(t) = \begin{cases}
1 & \text{ if } t \in [0, t^*), \\
-1 & \text{ if } t \in [t^*, t^* + \epsilon).
\end{cases}
\]
for some $\epsilon > 0$. On the other hand, no extremal control of the form (2.24) if $t^*$ is not contained in any one of the closed intervals $[s_i^+, t_i^+]$ or $[s_i^-, t_i^-]$.

**Remark 2.35** Summarizing, in order to determine the time interval when switches can happen we use the $\theta$-function. In general the $\theta$-function is derived numerically. A detailed study of $\theta(t)$ requires the knowledge of the exact position of its zeroes and its stationary point. For this purpose is useful the relation (2.21), which tell us that $\theta(t)$ is stationary in the correspondence of the zeroes of $\Delta_B(\gamma(t))$. It is important to note that $\theta(t)$ is useful only to predict when the first switch can happen. It does not predict the complete evolution of an optimal trajectory. Once we decide that an ordinary trajectory switches, the $\theta(t)$ ceases to be useful.
Chapter 3

Quantum control of open quantum systems

The Quantum optimal control theory (QOCT) lies at the heart of the modern experimental and theoretical develop of the QM. The design of complex quantum systems requires some form of control and QOCT allows us to match the stringent requirements needed to developments quantum technologies, quantum protocols, and to improve their performances. We refer principally to the research field of the quantum information and computation \cite{6,16}.

In order to understand how the OCT can be embedded in the quantum formalism, we recall the chapter \ref{chapter1}, where in particular we have seen that the dynamics of the quantum systems follows a first-order linear differential equation. Then we can identify two prototypical forms of optimal quantum control: state-selective or state-independent. In the first case, widely treated in this thesis, we want to steer the quantum system from an initial state to a target state by acting on the generator of the dynamics, e.g. assuming the Hamiltonian as a function of a time-dependent control parameter $u(t)$, i.e. $H = H(u(t))$ \cite{3,30}. This case is important for example in the “preparation” of a quantum system. Alternatively, in the state independent control we are interested in steering the evolution operator to a given quantum gate \footnote{The quantum gates are the building blocks of quantum circuits \cite{6}, like the classical logic gates.} for example minimizing the time necessary to perform this unitary operation.

All the above kinds of control are said unitary, since they are implemented by means of unitary operations. Other control strategies use non-unitary controls or measurements, e.g. the quantum Zeno effect.
Focusing on the open quantum system, the first way to prevent decoherence (see the introduction) is obviously isolating the quantum devices. Indeed a perfect isolation is impossible. Time optimal control can be used to make the quantum process of our interest faster than decoherence time scale, limiting the occurrence of quantum errors [31].

From a theoretical point of view, referring to the time minimum problem, studies in closed [32, 33] and open [34] quantum systems have determined upper bounds on the speed of the quantum evolution (the “Quantum Speed Limit”) and control theory can be applied to find these limits [35]. Instead open loop and closed loop control techniques have been used in [36] to simulate the dynamics equation of open quantum systems.

Now we face our particular problem. In chapter 2, for the case of time optimal control, we have seen that at least for affine systems in the 2-dimensional case several mathematical tools that allow one to identify the optimal strategies have been developed. Here we will exploit these machineries to investigate the controlled (coherent, open loop) dynamics of a dissipative two-level quantum system (qubit) subject to GAD in the Markovian approximation. The control is modelled by an electromagnetic field (e.g., a laser) in resonance with the energy difference $E_1 - E_0$ between the excited and the ground states of the qubit and whose amplitude $u(t)$ is assumed to be externally modulated. We also make the physical hypothesis that the control is bounded (i.e. the laser has a maximum strength $u_{\text{max}} > 0$). To avoid misunderstanding we remark that a 2-level quantum system in general is not mappable to a 2-D manifold (a faithful parametrization requiring a 3-D manifold, see e.g. section 1.4). Nonetheless one can still reduce to the study of trajectories lying on a plane by properly constraining the control parameters. For the case we are considering, as we shall see later, this is obtained by working with a real control field (in particular the real part of the Rabi frequencies of the laser-qubit interaction). Our goal is to accelerate the relaxation of the system, starting from a state “more pure” than the fixed point associated with the bare open quantum system evolution. We tolerate that the quantum state arrives within a small distance from the target fixed point and, under this approximation, we find the best time-evolution of the control input.

In the literature, only recently this problem has been faced (see [8, 9, 13].

However, it is important to remark that generic quantum errors may happen also because the quantum gates cannot be implemented with perfect accuracy; the effects of small imperfections in the gates will accumulate, eventually leading to a serious failure in the computation. Time OCT must be combined with the quantum error correction.
In [8], the same model is studied without the hypothesis of bounded control: the value of $u(t)$ can be only infinite (i.e. unconstrained, corresponding to instantaneous unitary rotations in the Bloch ball) or $u(t) = 0$ (pure relaxation). In [9, 13, 10, 12] instead the authors study the optimal strategy problem for bounded, but constant control.

Keeping in mind these two limit paradigmatic cases, here we investigate all the intermediate cases, when the control strength $u_{\text{max}}$ at our disposal increases, starting from $u_{\text{max}} = 0$ (no control) until the control $u_{\text{max}}$ is arbitrarily large, but however bounded. We find four different control strategies, each one optimal in some specific range of the control $u_{\text{max}}$. In particular, we examine quantitatively the trend of the total optimal time as a function of $u_{\text{max}}$. Notably, we find that for low control strengths, the free evolution, $u(t) \equiv 0$, is always faster than the controlled dynamics. Instead, for control strengths higher than a critical value, increasing the control strength reduces the total optimal time.

The chapter is organized as follows. In section 3.1 we introduce the model. In section 3.2 we state the problem in the language of the time optimal control for the affine control system. In sections 3.3, 3.4, 3.5 we analyse the ordinary trajectories and the singular sets of the model. In particular we show that the set $C$ is the set of the limit points of the ordinary trajectories for $t \to \infty$. The set $S$ is instead divided in two subsets with different characteristics: where it is turnpike, it can support optimal trajectories, whereas where it is antiturnpike optimal trajectories cannot stay over it (see section 2.3.2). We find that there is only one turnpike, corresponding to the $z$-axis, where the value of the (singular) control must be null (pure relaxation). In section 3.6 we find the optimal synthesis for gradually increased control strengths. Different strategies are tested. In section 3.6.1 we find the optimal strategy for low values of the control: until a critical value of $u_{\text{max}}$ the free relaxation is always faster than the controlled dynamics. We are able to prove this property using the clock form and pure analytical reasoning. In section 3.6.1 we select the various possible switch-strategies for increased values of the control. Among the strategies (a priori an infinite number) satisfying the Pontryagin principle, only four can be optimal. We use the $\theta$ function to determine the time intervals where the switches are possible and adopt numerical analysis to find the best switching sequence for each strategy. Finally, in chapter 3.6.5 we compare the total times for each strategy to determine the global optimal one for any value of $u_{\text{max}}$. 

[10, 12, 37, 11].
3.1 The master equation

Following the derivation of chapter 1 we describe the dynamical evolution of our two-level system model in term of the following Lindblad master equation (ME):

\[ \frac{\partial}{\partial t} \rho(t) = -i[H_S(t), \rho(t)] + \mathcal{L}_D(\rho(t)), \]  

(3.1)

Where \( H_S(t) = H_0 + H_c(t) \) represent the two level system Hamiltonian including a constant free “drift” term \( H_0 \) and a time dependent part \( H_c(t) \) describing the external control, while \( \mathcal{L}_D \) is the Lindblad superoperator describing the dissipative dynamics induced by the Markovian interaction with the environment. Hereafter we set \( \hbar = 1 \) for simplicity.

More specifically, setting \(|0\rangle\) and \(|1\rangle\) respectively the ground and the excited state of the system:

\[ |0\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

we write

\[ H_0 \equiv \frac{1}{2} \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} = \frac{1}{2} \omega \sigma_z. \]

The control Hamiltonian instead is expressed as a dipole interaction term \(-\vec{D} \cdot \vec{E}_c(t)\) between the dipole operator \( \vec{D} \) of the two-level system and a classical coherent electromagnetic field (e.g. a laser) \( \vec{E}_c(t) = \vec{\epsilon}_c e^{i\omega t} + \vec{\epsilon}_c^* e^{-i\omega t} \) so that,

\[ H_c(t) \equiv -\frac{1}{2} u e^{-i\omega t} \sigma_+ - \frac{1}{2} u^* e^{i\omega t} \sigma_- = -\frac{1}{2} \begin{pmatrix} 0 & u e^{-i\omega t} \\ u^* e^{i\omega t} & 0 \end{pmatrix}, \]

where \( \sigma_\pm \equiv \frac{1}{2}(\sigma_x \pm i \sigma_y) \) and where \( u = u_1 + iu_2 \) is proportional to the complex Rabi frequency: \( u \equiv 2\vec{D} \cdot \vec{\epsilon}_c \). The real quantities \( u_1 \) and \( u_2 \) play the role of “control parameters”: acting on the amplitude \( \vec{\epsilon}_c \) of the e.m. field, we can change them.

The non unitary part \( \mathcal{L}_D \) describes a GAD channel (see section 1.4.2) defined by the expression:

\[ \mathcal{L}_D(\rho) \equiv \sum_{\mu=1}^2 \left( L_\mu \rho L_\mu^\dagger - \frac{1}{2} \{ L_\mu^\dagger L_\mu, \rho \} \right). \]  

(3.2)

\[ ^1 \text{In matrix form:} \]

\[ \sigma_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

where \( \sigma_+ \) is responsible for the dipole transition \(|0\rangle \rightarrow |1\rangle\), whereas \( \sigma_- \) causes \(|1\rangle \rightarrow |0\rangle\).
3.1 The master equation

whose Lindblad operators are given by:

\[ L_1 \equiv \sqrt{\gamma_{01}} \sigma_+, \quad L_2 \equiv \sqrt{\gamma_{10}} \sigma_- . \]  

The coefficients \( \gamma_{01} \) and \( \gamma_{10} \) are respectively the excitation and relaxation rates for the qubit. They can be expressed via the decoherence rate \( \gamma \) and the inverse temperature of the bath \( \beta = 1/T \):

\[ \gamma_{01} \equiv \frac{\gamma}{e^\beta - 1}, \quad \gamma_{10} \equiv \frac{\gamma e^\beta}{e^\beta - 1} . \]

Moving into the interaction picture and introducing the total decay rate \( \Gamma \)

\[ \Gamma \equiv \frac{1}{2}(\gamma_{01} + \gamma_{10}) , \]

we can then rewrite the ME (see appendix A for the detailed derivation) as:

\[
\begin{cases}
\dot{x} = \frac{1}{2} \gamma \frac{x}{z_f} + u_2 z , \\
\dot{y} = \frac{1}{2} \gamma \frac{y}{z_f} + u_1 z , \\
\dot{z} = -\gamma + \gamma \frac{z_f}{z_f} - u_1 y - u_2 x ,
\end{cases}
\]  

where the parameters \( x, y \) and \( z \) are:

\[ x \equiv 2 \Re(\rho_{01}) ; \quad y \equiv -2 \Im(\rho_{01}) ; \quad z \equiv \rho_{00} - \rho_{11} , \]

and where the parameter \( z_f \) is the Bloch \( z \)-component of the fixed point (FP) \( \rho_{FP} \) of the dissipative dynamics of the system, obtained by solving the equation \( \dot{\rho} = 0 \) for \( u = 0 \), i.e.:

\[
y_f \equiv 0 , \\
z_f \equiv -\frac{\gamma}{2\Gamma} = -\frac{(e^\beta - 1)}{(e^\beta + 1)} .
\]

The equations (3.4) can be further simplified by rescaling them by a factor \( \gamma \). Specifically, redefining the controls as \( u_i \rightarrow u_i/\gamma \equiv u_i \) and resetting the time as \( t \rightarrow \gamma t \equiv t \), the time derivatives become \( \dot{x} = \partial_x / \partial t \rightarrow \partial_x / \partial \gamma t \equiv \dot{x} \) (analogous relations for \( y \) and \( z \)). We finally can rewrite (3.4) as an 3-D affine control system of the type:

\[
\dot{\vec{\mu}} = F_0(\vec{\mu}) + \sum_{i=1,2} u_i(t) G_i(\vec{\mu}) ,
\]

where \( \vec{\mu} \equiv (x, y, z) \in M = \mathbb{R}^3 \), whereas \( u_i \in U = \mathbb{R}^2 \) (\( i = 1, 2 \) and):

\[
F_0 \equiv \begin{pmatrix}
\frac{1}{2} x \\
\frac{1}{2} y \\
-1 + \frac{z}{z_f}
\end{pmatrix} ; \quad G_1 \equiv \begin{pmatrix}
0 \\
z \\
-y
\end{pmatrix} ; \quad G_2 \equiv \begin{pmatrix}
z \\
0 \\
-x
\end{pmatrix} .
\]  

(3.7)
3.2 Problem statement

The ME (3.4) is a system of linear differential first order equations. The solutions of (3.6) describe a family of 3-D trajectories lying on the Bloch ball, depending on two control parameters $u_1$ and $u_2$. Nevertheless our study will be restricted to the 2-D case and to only one control $u_1 \equiv u$ (the way to obtain this restriction will be clear in section 3.3). Thus the supports of the trajectories lie on the projection of the Bloch ball on a plane.

The control $u$ is a function of time, $u = u(t)$. Given the initial state of the open system, we can determine the time evolution of the control in order to drive the state along a required trajectory. Once the control law is found, one in principle can set up the experiment and no measurements have to be performed on the system in order to adjust the control: in the language of the control theory, this is an open loop problem.

The goal of this thesis is to investigate how the control $u$ can be modulated to increase the speed of the relaxation of the system towards the FP (3.5). This is a time optimal control problem. In particular we want to find the optimal trajectory and the corresponding control law that allow to reach the target in minimum time. We tolerate that the state arrives within a small trace distance $D = \frac{1}{2} \text{Tr} |ho - \rho_{FP}| = \epsilon$ from the target, with $\epsilon$ an arbitrary small positive constant fixed a priori (see section 1.2.1).

From equation (3.5) it is clear that the position of the FP depends upon the bath temperature $T = 1$. There are two extremal cases. For $T = 0$, $z_f = -1$ under the action of the “bare” dissipative dynamics the system tends to evolve towards the ground state (a pure state). For $T \to \infty$, instead $z_f = 0$: the system tends to collapse towards the maximally mixed state, the centre of the Bloch ball, whose density matrix is $\frac{1}{2} \mathbb{1}$. In order to treat a sufficiently general case, we choose a finite temperature, corresponding to $z_f = -0.5$. As initial state we choose a mixed state on the z-axis, below the fixed point $z_f$: $z_i = -0.75$, $y_i = 0$. The initial state is then “more pure” then the fixed point. Reaching the fixed point means to reach the thermal equilibrium between the open system and the external bath. This “thermodynamics” point of view suggest us to restate the problem in an alternative language: the greater the purity of the state, the lower its temperature. So, speeding up the relaxation

\footnote{Indeed the trajectory cannot be completely arbitrary. Restrictions to the shape of the trajectories are given by the accessibility and controllability of the system. In particular, given a starting point, the support of the trajectory will lie, at any time, inside the reachable set.}
of the open system means to accelerate the heating. On the other hand, if the initial point had a radius (in the Bloch ball) smaller than the fixed point, the relaxation would correspond to a cooling. Figure 3.1a shows a generic controlled trajectory that drives the initial state towards the FP.

In what follows we shall assume the control parameter $u$ to be bounded. Specifically we will allow it to take values in the interval $[-u_{max}, +u_{max}]$, with $u_{max} > 0$ being a fixed positive quantity. The latter corresponds to the maximum strength (amplitude) of the coherent e.m. field interacting with the two-level system. Our goal is to analyse how the optimal synthesis changes when $u_{max}$ varies, studying how the minimum time to reach the target depends on it.

This approach lies between two extreme cases previously studied. Sugny et al. in [9, 13, 10, 22], using the Pontryagin Maximum Principle, find the optimal syntheses for control bounded to a constant, fixed value $u_{max} = 1$ and for suitable dissipation parameters chosen in order to avoid complicated “pseudoperiodic” behaviours of the trajectories, i.e. to avoid the appearance of complex exponentials in the analytical expression of the trajectories (the solution of the control system for $u \neq 0$). We will see in the following that this situation is instead typical in our case, since for us the only free parameter is $u_{max}$, and for $u_{max} > 0.5$ the solutions become complex. In [11, 12, 37] a first approach with increasing control strength is studied. The opposite extremal case is treated in [8]. There, the hypothesis of bounded control is abandoned: the control is unconstrained and it can assume arbitrary large values. Under this hypothesis, while the purity of the state cannot be modified at will, its orientation in the Bloch ball can be rotated instantaneously. Then the optimal strategy to reach the fixed point is a concatenation of these rotations and of pure relaxations along the z-axis induced by the dissipation (see figure 3.1a).

This work is important because gives an analytic expression for finding a lower bound for the optimal time. In our case this formula can be rewritten as:

$$T_{opt}^{LB} = z_f \log \left( \frac{2z_f}{z_f + z_i} \right). \quad (3.8)$$

$T_{LB} = 0.1115$: it does not depend by $\epsilon$. Furthermore the optimal strategy found in [8] is a prototype for our optimal strategies when the control strength is very large (but however bounded).
3.3 The affine control system on a 2-D manifold

Examples of 3-D affine control systems analogous to the one of equation (3.6) have been treated in [22], where the particular case of infinite bath-temperature (the fixed point is the centre of the Bloch ball) is studied analytically for controls bounded to the value $u = 1$ and for particular values of the dissipation parameters. More general cases are also treated, but only by means of numerical simulations. In what follows we shall focus on a simplified scenario by setting $u_2 = 0$. Under this condition the problem is called single-input case (while the original one is typically called two-input case). It allows to reduce the problem to a 2-D manifold and then to use the methods reviewed in chapter 2. In fact for $u_2 = 0$ the first equation decouples and we can neglect it, because it can be separately solved: the solution of this equation is $x(t) = x(0)e^{\frac{i}{u}t}$, then choosing as a suitable initial condition $x(0) = 0$, we essentially reduce to
3.3 The affine control system on a 2-D manifold

Figure 3.2: Plot of the vector fields $F$ and $G$. $F$ pushes a given initial state towards the FP. The light-gray elliptical curves in fig 3.2a connect the states which take the same time to collapse on the FP following a free dynamics. $G$ acts on the state as a rotation (fig 3.2b).

study a control system in $\mathbb{R}^2$, whose dynamics is given by:

$$\begin{align*}
\dot{y} &= \frac{1}{2} \frac{y}{z_f} + uz \\
\dot{z} &= -1 + \frac{z}{z_f} - uy,
\end{align*}$$

(3.9)

where we have redefined $u_1 \to u$. Accordingly the system trajectories, instead of exploring the entire Bloch ball, are forced to lie on a plane (more precisely within a circle of unit radius) characterized by a fixed value of the $x$ coordinate. In other words there is a symmetry under rotations around the $z$-axis.

The dynamical equations given in (3.9) are the starting point for our analysis. They form a 2-D affine control system in the sense of definition 2.6 with vector fields:

$$F \equiv \begin{pmatrix} \frac{1}{2} \frac{y}{z_f} \\ -1 + \frac{z}{z_f} \end{pmatrix}; \quad G \equiv \begin{pmatrix} z \\ -y \end{pmatrix}.$$  

(3.10)

$F$ is a drift term: if the open system is left free to evolve ($u = 0$) it completely describes the state dynamics. We can imagine $F$ as a field of forces that push a given initial state towards the fixed point (figure 3.2a). On the other hand $G$ describes the controlled dynamics and its role is to contrast the dissipative dynamics. It acts on the state as a unitary transformation, a rotation around the origin of the Bloch ball: at any point of the plane, $G$ is orthoradial (figure 3.2b). Increasing the parameter $u$, the action of $G$ becomes stronger. For high
values of the control, the effect of $G$ becomes leading and the limit case $u \to \infty$ corresponds to a purely unitary dynamics: the dissipation effect are negligible and we can obtain instantaneous rotations of a state around the origin (as in the limit case treated in [8]). The rotation direction of the trajectories depends on the sign of $u$: clockwise for $u > 0$, counterclockwise for $u < 0$.

### 3.4 Behaviour of Trajectories

The Control System (3.9) is a inhomogeneous system of two linear first-order differential equations, with constant coefficients when $u$ is constant. It is solvable analytically for each value of $u$, given two initial conditions $y(0) \equiv y_0$, $z(0) \equiv z_0$. In particular for $u = 0$ the control system becomes $\dot{\mu} = F(\mu)$ and we obtain the expression for the free, or relaxation, trajectories, which define a family of “geodetic curves” (figure 3.2a). These are the trajectories followed by the state to reach the fixed point when the dynamics is not controlled.

**Definition 3.1** A trajectory is a solution of the first-order differential equation $\dot{x} = F + uG$. In the following a trajectory with $|u| = u_{\text{max}}$ is called ordinary. In particular, if $u = u_{\text{max}}$ the trajectories are identified with $Y$, instead if $u = -u_{\text{max}}$ the trajectories are called $X$. A trajectory with $|u| < u_{\text{max}}$ is called singular and we call it $Z$ trajectory.

We remark that the final point of a controlled trajectory in general is not the fixed point: for each value of $u$ the final point changes (in the section 3.5.1 we find the set of these final points). Then, in order to reach the FP, an optimal trajectory must be in general a concatenation of pieces of ordinary and singular trajectories.

#### 3.4.1 Periodic and pseudoperiodic solutions

We can rewrite (3.9) in the form $\dot{\mu} = \dot{C} + \dot{A}\mu$, where:

$$\dot{A} \equiv \begin{pmatrix} \frac{1}{z_f} & u \\ -u & \frac{1}{z_f} \end{pmatrix}, \quad \dot{C} \equiv \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad (3.11)$$

The study of the homogeneous part of this system show us the qualitative

---

*These names are due to the name of the sets where the trajectories are extremal. The ordinary trajectories are extremal on the set of ordinary points. The singular trajectories are extremal on the singular set. See section 2.3.2 in particular the definition 2.19. We will be more clear in section 3.5.*
3.4 Behaviour of Trajectories

Figure 3.3: Plots of ordinary aperiodic trajectories for $u \geq 0$. For further use we draw the set $S$ (the vertical dotted and the horizontal dot dashed lines) and set $C$ (the dashed ellipse). In fig 3.3a the free relaxation trajectory. In fig 3.3b an example of aperiodic trajectory.

behaviour of trajectories with respect to a change of $u$. The characteristic polynomial is:

$$Det(\hat{A} - \lambda) = \lambda^2 - \frac{3}{2z_f} \lambda + u^2 + \frac{1}{2z_f^2} = 0,$$

while the discriminant $\Delta$ is given by:

$$\Delta = 1 - 16u^2z_f^2. \quad (3.12)$$

We call *aperiodic* the solutions for $\Delta \geq 0 \Rightarrow |u| \leq \frac{1}{4z_f} = 0.5$, otherwise solutions are called *pseudoperiodic*. Aperiodic solutions show a regular behaviour in reaching the final point: the $y(t)$ and $z(t)$ “decay” with an exponential trend towards the final points. Instead a pseudoperiodic solution reaches the final point spiralling. Figure 3.3 and 3.4 show example of respectively aperiodic and pseudoperiodic ordinary trajectories In particular assuming the initial conditions $y(0) = 0$, $z(0) = z_i$, can be found:

$$y_u(t) = -\frac{2uz_f^2}{(1 + 2u^2z_f^2)} \left\{ e^{\frac{-\Delta t}{4z_f^2}} \left[ (3 - \frac{2z_i}{z_f}(1 + 2u^2z_f^2)) \right. \right.$$

$$\sinh(\frac{\sqrt{\Delta t}}{4z_f^2}) - \sqrt{\Delta} \cosh(\frac{\sqrt{\Delta t}}{4z_f^2}) \left. \right] + 1 \right\}, \quad (3.13)$$
Figure 3.4: Plots of ordinary pseudoperiodic trajectories for some $u \geq 0$. The spiral behaviour is clear. Note in particular fig 3.4b: $u \approx 5.2$ is the maximum control strength for which an ordinary trajectory does not cross the $z$-axis.

$$z_u(t) = -\frac{z_f}{(1 + 2u^2 z_f^2)} \left\{ \frac{\sqrt{\Delta t}}{\sqrt{\Delta}} \left[ (1 - \frac{z_i}{z_f}) + 2(4 - \frac{z_i}{z_f})u^2 z_f^2 \right] \right. \right.$$

$$\left. \left. \sinh\left(\frac{\sqrt{\Delta t}}{4z_f}\right) + \sqrt{\Delta} \left( 1 - \frac{z_i}{z_f} (1 + 2u^2 z_f^2) \cosh\left(\frac{\sqrt{\Delta t}}{4z_f}\right) \right) \right] - 1 \right\}, \quad (3.14)$$
which for \( u = 0 \) reduces to:
\[
\begin{align*}
y_0(t) &= 0, \\
z_0(t) &= z_f + e^{\frac{t}{T_f}}(z_i - z_f).
\end{align*}
\]

These special trajectories are important because, as we will see in the following, they constitute the building blocks of the optimal control solution.

### 3.5 The singular sets and the characterization of the Bloch ball

Following the construction presented in chapter 2 before starting with the study of the extremal trajectories, we have to characterize the Bloch ball by use of the vector fields \( F \) and \( G \). Consider hence the functions (see the definitions (2.8, 2.9) in chapter 2):
\[
\Delta_A(y, z) = -\frac{1}{2z_f} y^2 + z(1 - \frac{z}{z_f}), \quad (3.15)
\]
\[
\Delta_B(y, z) = y \left( \frac{z}{z_f} - 1 \right). \quad (3.16)
\]

We now identify two sets of points \((y, z)\). The set \( C \) formed by the zeroes of \( \Delta_A \) and the set \( S \) formed by the zeroes of \( \Delta_B \). They are called **singular sets**.

We note that \( C \) is an ellipse in the \( z < 0 \) half-plane of the Bloch-Ball (figure 3.5a), with vertical semi-axis of length \( |z_f|/2 \) and horizontal semiaxis of length \( |z_f|/\sqrt{2} \). It passes through the origin and the fixed point \( z_f \). On the contrary, the singular set \( S \) (figure 3.5b) is the union of two lines: \( y = 0 \) and \( z = z_f \).

#### 3.5.1 The singular set \( C \)

Now we show that the set \( C \) is related to the **speed of the purity change** and that it is the set of the limit points of the ordinary trajectories as \( t \to \infty \).

We know from the definition (1.2) that the purity of a quantum state is related to \( Tr[\rho^2(t)] \). In the particular case of a qubit, the purity is defined as the (euclidean) distance from the center of the Bloch ball. From (1.15), with \( x \) set to zero, we have that:
\[
2Tr[\rho^2(t)] - 1 = z^2 + y^2, \quad (3.17)
\]
The set $\Delta_A$ and the speed of purity change. Figure 3.5: Characterization of the Bloch ball with respect to the singular sets. In fig 3.5a the ellipse is the set $C$: inside the ellipse (white area) $\Delta_A$ is negative and the speed of the purity change increases. Outside $\Delta_A$ is positive and the speed of the purity change decreases. Darker colors correspond to more negative values of the speed. In fig 3.5b the vertical and the horizontal lines represent the set $S$. The vertical dotted line is a turnpike, the horizontal dot dashed line is an antiturnpike. Within the white areas $\Delta_B$ is positive ($g > 0$), whereas in the light gray areas $\Delta_B$ is negative ($g < 0$).

whose time derivative is hence given by:

$$\frac{d}{dt} \text{Tr}[\rho(t)^2] = z\dot{z} + y\dot{y} = \frac{1}{2z_f}y^2 - z + \frac{1}{z_f}z^2 = -\Delta_A(y, z).$$

Since the function $\Delta_A$ is negative inside $C$ and positive outside (figure 3.5a), we find that inside the ellipse $C$ the purity increases ($\Delta_A < 0$) with time, whereas outside $C$ the purity decreases ($\Delta_A > 0$) with time. In particular note that this speed does not depend on the control and then the control cannot locally compensate for the effect of the dissipation.

The points of $C$ are the limit points of the trajectories (3.13) and (3.14). In fact in the limit $t \to \infty$, for $u$ fixed, one finds that the system approaches the point:

$$y_\infty(u) \equiv \lim_{t \to \infty} y(t) = -\frac{2uz_f^2}{1 + 2u^2z_f^2}, \quad z_\infty(u) \equiv \lim_{t \to \infty} z(t) = -\frac{z_f}{1 + 2u^2z_f^2},$$

which cover the entire ellipse $C$ when varying $u$ from $-\infty$ to $+\infty$. In fact $y_\infty(u)$ varies continuously within $[\frac{z_f}{\sqrt{2}}, -\frac{z_f}{\sqrt{2}}]$ and reaches its maxima for $u = \pm \frac{1}{\sqrt{2}z_f}$.
3.5 The singular sets and the characterization of the Bloch ball

while \( z_\infty(u) \) varies in \([z_f, 0)\).

We recall that \( \mathcal{C} \) is the set of points where the vector fields \( F \) and \( G \) are parallel. This means that for each point \( \bar{\mu} \) there is a value \( u \) of the control such that \( F(\bar{\mu}) + uG(\bar{\mu}) = 0 \). In this case, the relaxation is blocked, and the state does not change. In a sense, the set \( \mathcal{C} \) extends the set of the fixed points.

3.5.2 The singular set \( S \). Turnpike and antiturnpike singular curves

The singular set \( S \) divides the \( y - z \) plane in four quadrants (figure 3.5b) of ordinary points (see definition 2.19). Recall that within each quadrant an extremal trajectory can be only bang or bang-bang with at most one switch. We enumerate the quadrants clockwise starting from the bottom left. Within each quadrant \( g = \Delta_B/|\Delta_A|^2 \) (see definition (2.15)) is constant.

We know from section 2.3.2 that the singular set \( S \) (for \( \Delta_A \neq 0 \)) can be the support of extremal trajectories, called singular. However an optimal singular trajectory can have support only on those parts of \( S \) that are turnpike, whereas the parts of \( S \) called antiturnpike cannot.

For the classification of \( S \) in turnpike and antiturnpike arcs we follow the constructive definition 2.24. Accordingly, we need to define two new vector fields \( \hat{X} \) and \( \hat{Y} \):

\[
\hat{X} \equiv F - uG = \begin{pmatrix} \frac{y}{z_f} - uz \\ -1 + \frac{z}{z_f} + uy \end{pmatrix}, \quad \hat{Y} \equiv F + uG = \begin{pmatrix} \frac{y}{z_f} + uz \\ -1 + \frac{z}{z_f} - uy \end{pmatrix},
\]

which represent the tangent vectors (velocity fields) of the ordinary trajectories, see section 3.4 and the definition 3.1. Note that, differently from definition 2.24 where \( u = \pm 1 \), here we define \( \hat{X} \) and \( \hat{Y} \) for arbitrary values of \( u > 0 \), since in principle by varying this parameter we can modify \( S \).

At first we study the signs of the function \( f(y, z) \equiv -\frac{\Delta_B}{\Delta_A} \) (see figure 3.6). Then we analyse separately the neighbourhood of the horizontal and the vertical lines that define \( S \) (see 3.16). Along the line \( y = 0 \) we have

\[
\hat{X}(y = 0, z) = \begin{pmatrix} -uz \\ -1 + \frac{z}{z_f} \end{pmatrix}, \quad \hat{Y}(y = 0, z) = \begin{pmatrix} uz \\ -1 + \frac{z}{z_f} \end{pmatrix}.
\]

Therefore, we note that the \( y \) component \( \hat{X}_y \) of \( \hat{X} \) is positive for \( z < 0 \), the \( y \) component \( \hat{Y}_y \) of \( \hat{Y} \) is positive for \( z > 0 \) and the \( z \) components \( \hat{X}_z \) and \( \hat{X}_z \) of both vector fields are positive for \( z < z_f \). In figure 3.7a the sets pointed by \( \hat{Y} \)
Figure 3.6: Vector fields $X$ and $Y$ for a control value $u = 1$ and division of the Bloch ball with respect to the sign of $f$. In light gray the areas where $f > 0$, in white the areas where $f < 0$. The blue and the red arrows represent respectively the vector fields $\hat{X}$ and $\hat{Y}$.

Figure 3.7: Definition of turnpike and antiturnpike. The vector fields $X$ and $Y$ are shown (arbitrarily zoomed) in a neighbourhood of the singular set $S$: in fig 3.7a for the set $y = 0$, in fig 3.7b for the set $z = z_f$. The sets of ordinary points pointed by $\hat{Y}$ are indicated with $\Omega^+$. The sets of ordinary points pointed by $\hat{X}$ are indicated with $\Omega^-$. Since $f > 0$ on $\Omega^+$ and $f < 0$ on $\Omega^-$, the entire $z$-axis is turnpike.

Analogously along $z = z_f$ we have:
\[ \dot{X}(y, z = z_f) = \left( \frac{y}{2z_f} - uz_f \right), \quad \dot{Y}(y, z_f) = \left( \frac{y}{2z_f} + uz_f \right). \]  

(3.20)

We obtain for the components \( \dot{X}_{y, z}(y, z = z_f) \) and \( \dot{Y}_{y, z}(y, z = z_f) \):

\[ \dot{X}_y > 0 \text{ for } y < 2uz_f^2; \quad \dot{X}_z > 0 \text{ for } y > 0; \]
\[ \dot{Y}_y > 0 \text{ for } y < -2uz_f^2; \quad \dot{Y}_z > 0 \text{ for } y < 0. \]  

(3.21)

Figure 3.7b shows the behaviour of \( \dot{X} \) and \( \dot{Y} \) along the line \( z = z_f \). Since \( f > 0 \) on \( \Omega_- \) and \( f < 0 \) on \( \Omega_+ \), the entire set \( z = z_f \) is antiturnpike.

Remark 3.2 Equation (3.21) shows a relationship between the vector fields \( \dot{X}_y, \dot{Y}_y \) and \( u \): the \( y \)-components of the fields change sign for \( \tilde{y} = \pm 2uz_f^2 \).

In figure 3.6 we show the vector fields for a control value \( u = 1 \), then \( |\tilde{y}| = 0.5 \) (vertical dashed lines). However, the singular set \( z = z_f \) still remains antiturnpike for all values of \( u \). In fact, for the set \( z = z_f \), the definitions of \( \Omega_\pm \) only depend on the \( z \)-components of \( X \) and \( Y \): these point respectively to the set where \( f < 0 \) and \( f > 0 \) for all values of \( u \). On the other hand, for the singular set \( y = 0 \), the vector fields \( \dot{X} \) and \( \dot{Y} \) do not depend on \( u \).

Finally, we calculate the value of the singular control for the turnpike line \( y = 0 \), i.e. the value that an extremal trajectory takes when its support is the turnpike. It can be shown from (2.13) and the definition of the vector fields \( F \) and \( G \) in (3.10) that:

\[ u_s(y, z) = -\frac{\nabla \Delta_B(y, z) \cdot F(y, z)}{\nabla \Delta_B(y, z) \cdot G(y, z)} = 0, \]  

(3.22)

In other words, the singular turnpike is a support for pure relaxation trajectories.

3.6 The time optimal syntheses

In this section, for each control strength between zero and an arbitrary large value, we study the optimal trajectory, i.e. the concatenation of extremal trajectories that allow us to reach the FP in minimum time. We allow to reach the target (the FP) with a small tolerable error. In other words we allow the optimal trajectory to reach a small closed curve encircling the FP. As a boundary we use a small circle of radius \( \epsilon \) around the fixed point, i.e. we admit to reach the fixed point up to a trace distance \( D = \epsilon \).
In our analysis we focus on those trajectories which are concatenations of “local” extremal trajectories satisfying the PMP (2.2). We recall that the latter are of two types: ordinary trajectories \((X, Y)\) for \(u = -u_{\text{max}}\) and \(u = u_{\text{max}}\) respectively or singular turnpikes (which in our case are obtained for \(u = 0\), see (3.22)). We call these concatenations “optimal synthesis”. In order to find the “global” optimal solution, a comparison of different concatenations is necessary, using either geometrical arguments (the clock form introduced in section 2.3.3) or numerical methods. A concatenations of extremal is constructed “switching” the control at a given suitable time, i.e. making a so called \textit{switch}. For “switch” (see section 2.3.1) we mean an instantaneous change of the control strengths. Only the following possibilities of switches (and the relative concatenations of extremal trajectories) are admitted:

1. In any quadrant of ordinary points we can have at most one switch from \(u_{\text{max}}\) to \(-u_{\text{max}}\), corresponding to a concatenation of trajectories of the type \(Y \ast X\)\(^{\text{1}}\) and vice versa switch from \(-u_{\text{max}}\) to \(u_{\text{max}}\), i.e. \(X \ast Y\) (hereafter we shall use the symbol \(\ast\) to indicates concatenation of trajectories).

2. When a \(Y\) or \(X\) trajectory reaches the turnpike switches are admitted from \(u_{\text{max}}\) or \(-u_{\text{max}}\) to a singular control \(u = 0\), corresponding to a concatenation of trajectories of the type \(Y \ast Z\) or \(X \ast Z\). On the other hand, a singular \(Z\) trajectory can leave the turnpike with a switch from \(u = 0\) to \(\pm u_{\text{max}}\), corresponding to a concatenation \(Z \ast Y\) or \(Z \ast X\).

Then, a locally optimal trajectory can be in general a concatenation (more or less involved) of the types \(X \ast Y \ast Z \ast X\), \(Z \ast Y \ast X \ast Y\), \(Y \ast Z \ast Y \ast Z\) and so on.

Specifically, we remember that in our study we choose the initial point to be located on the \(z\)-axis (the turnpike) below the FP. Then the starting local extremal trajectory can be any of the two possible choices, i.e. singular (with null control) or ordinary (with \(|u| = u_{\text{max}}\)). The case with \(u = u_{\text{max}}\) is an \(Y\) trajectory and goes in the first quadrant. Since in our system there is a symmetry under reflection with respect to the \(z\)-axis an ordinary \(X\) trajectory (with \(u = -u_{\text{max}}\)) is simply the mirror image of the \(Y\) with respect to the \(z\)-axis. For simplicity we discuss only initial \(Y\) trajectories in the first quadrant.

\(^{\text{1}}\)This notation has been introduced in the definition 2.16. In particular here we recall that the notation \(X \ast Y\) means that the \(X\) trajectory starts before that the \(Y\) trajectory, (similar for other possible concatenations).
As an example of strategies, let us consider that one shown in figure 3.14b: an $Y$ leaves the initial point, it goes through the first quadrant without switching, then it crosses the antiturnpike and reaches the second quadrant where, at a given suitable time, it switches to a $X$. Then, after some time, it reaches the turnpike and we make another switch of type $X*Z$. Finally the trajectory relaxes until the FP. The strategy just described is a concatenation $Y*X*Z$ and we will see in section 3.6.5 that it is a globally optimal strategy at least for some $u_{max}$.

Given $u_{max}$ we are able to find the globally optimal trajectory and then the optimal total time $T_{opt}(u_{max})$. In particular we are interested in determine whether or not by increasing $u_{max}$ we can always reach the target in a shorter time. In other words: given two values of $u_{max}$, say $u_1 < u_2$, is $T_{opt}(u_1) > T_{opt}(u_2)$? Are there any ranges of $u_{max}$ for which an increasing of the control does not speed up the dynamics? Answering to these questions means to find the optimal strategy for any value of $u_{max}$. A priori this task could seem hopeless. Luckily, we will see that the strategies to adopt are essentially only four, each one globally optimal in a range of control strengths related to the type of the ordinary trajectories: aperiodic, pseudoperiodic, intersecting or not the turnpike (see figures 3.3 and 3.4 for typical example of these cases).

In the following sections we analyse in detail these types of ordinary trajectories and for each of them (i.e. for each range of $u_{max}$ where these are defined) we find a set of reasonable concatenations of locally extremal trajectories which reach the target and that can be optimal. We begin from the study of the aperiodic case, $u_{max} < 0.5$. This case is, in turn, divided in two subcases: for low control strengths the selection of the optimal strategy is analytically solvable essentially using only the clock form introduced in section 2.3.3. For increasing control strengths we need a numerical analysis.

The pseudoperiodic case, $u_{max} > 0.5$, is also divided in two subcases, depending on whether or not the $Y$ trajectories intersect the turnpike and then have support in the third and fourth quadrant too. The need of this last subdivision will be more clear after the study of the $\theta$-function, a tool introduced in section 2.3.3 and useful to determine the time interval when switches can happen (see in particular definition 2.32, the proposition 2.34, the figure 2.4 and the remark 2.35).
3.6.1 Controls of low strength: pure relaxation is always more convenient

For control strengths in the aperiodic range \((u_{\text{max}} < 0.5)\) a typical ordinary \(Y\) trajectory and the corresponding \(\theta\)-function defined in section 2.3.5 are shown in figure 3.8 for \(u_{\text{max}} = 0.2\). In the aperiodic range, initially the \(Y\) trajectory lies on the first quadrant, where \(g < 0\). \(\theta(t)\) starts decreasing, passes through a minimum when the trajectory crosses the antiturnpike at the time \(t_1^{++}\) and then monotonically increases until an asymptotic value less than \(\pi\). From the properties of \(\theta(t)\) listed in the definition 2.32 and from the proposition 2.34 it follows that a switch can occur only in the first quadrant. However, a switch may not be necessary to reduce the time required for reaching the target. In fact we prove that until \(u = 0.5\) the pure relaxation is always more convenient.

Firstly, we consider the case (figure 3.9) in which the \(Y\) trajectory touches the trace distance circle around the target before reaching the horizontal antiturnpike \(z = z_f\) (we recall that the goal is to reach the fixed point from the initial point up to a given trace distance \(D = \epsilon\)). For the moment we do not consider the possibility of switches. We would like to compare the time taken

**Hereafter we follow the convention introduced in the definition 2.32 see also figure 2.4 for a general example.**
by these low-control trajectories $Y$ with the relaxation time. At the end of the section we show that switches in the first quadrant do not help to accelerate the dynamics.

Referring to the figure, the gray circle (arbitrarily zoomed) represents the locus of point where $D \leq \epsilon$. The red vertical line is the free relaxation trajectory, which ends at point $A$ on the trace distance circle. The black solid line is the $Y$ trajectory, which ends at point $B$ on the trace distance circle. Since both trajectories lie, at any time, in the same quadrant of ordinary points, it is possible to use the clock form. However, to apply this tool one needs a closed circuit made of trajectories, but in this case the two trajectories end at two different points $A$ and $B$. Specifically referring to figure 3.10, we want to compare the time $T_{1,r}$ (relaxation time needed to reach the point $A$) with the time $T_{1,u}$ (the time taken by the controlled trajectory to reach the target in the point $B$). To effectively include these two paths in a closed loop we make use of an unbounded, infinite control suitable to find a lower bound to the optimal time. An infinite control (see [8] and the discussion in section 3.2) yields an instantaneous (null time) unitary trajectory, that is a rotation around the centre of the Bloch ball. In figure 3.10 the infinite-control trajectory is indicated in blue solid line, connecting point $B$ to point $C$. Let $\tau > 0$ be the time taken by the relaxation trajectory to connect $A$ and $C$. Since $g < 0$ in the first
Figure 3.10: Same as figure 3.9 with the addition of the trajectory with infinite control (blue solid line). \( T_{1,r} \) and \( T_{1,u} \) are the times taken by the relaxation and the \( Y \) trajectories to reach respectively the points A and B. \( \tau \) is the time taken by the relaxation trajectory to reach the point C from the point A. The infinite control connects B to C in a null time.

The geometrical methods developed above also be used to prove that for ordinary trajectories, and for any \( u_{max} \), a switch in the first quadrant does not accelerate the dynamics and then is useless (even though allowed, as we have seen studying the \( \theta \)-function). In order to prove this assertion, we refer to figure 3.11 where we show the only two possible types of switching \( Y \times X \) trajectories in the first quadrant. In the first case (solid \( X_1 \) trajectory) the switching trajectory bifurcates at \( sw1 \) and it reaches the trace distance boundary around the FP at the point B before reaching the turnpike, thus we can apply the reasoning of the previous section using an unconstrained control. Instead, if the switching trajectory (indicated by a dashed \( X_2 \) trajectory)
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Figure 3.11: In this figure we show the two only possible types of switches in the first quadrant for a starting \( Y \) trajectory (see sec. 3.6.1). \( \text{sw1} \) and \( \text{sw2} \) are the switching points.

reaches the turnpike below the target, a direct use of the clock form proves the assertion.

Remark 3.3 The two cases just considered are the only possible. In fact it cannot happen that a \( X \) trajectory starting in the first quadrant reaches the antiturnpike \( z = z_f \) at a point of coordinate \( y < 0 \) before crossing the \( z \)-axis. In fact the vector field \( \hat{X} \), see (3.19), just describes the velocities of the \( X \) trajectories. In particular, referring to the figures 3.6 and 3.7, we can see that in the first quadrant, in a boundary of the antiturnpike, the \( X \) trajectories are forced to rotate counterclockwise and towards the part \( z < z_f \) of the \( z \)-axis.

Summarizing, until now we have proved that for low control strengths the free relaxation time is always shorter than any total time resulting by a strategy made of concatenations of \( Y \), \( X \) and \( Z \) trajectories, i.e. switches in the first (and fourth) quadrant are not necessary to speed up the free dynamics. In other words, the system does not need to be controlled in order to reach the FP in a minimum time. Furthermore, we have proved that, for each \( u_{\text{max}} \), switches \( Y \ast X \) cannot happen in the first quadrant and switches \( X \ast Y \) cannot happen in the fourth quadrant, respectively for \( Y \) and \( Y \) ordinary starting trajectories.

3.6.2 Strategies for increasing control strength

We have seen in section 3.4.1 that gradually increasing the strength of the control, the ordinary trajectories change properties. In section 3.5.1 we found
that for any $u_{\text{max}}$ these trajectories asymptotically ends, for $t \to \infty$, at different points of the set $C$. We also recall that ordinary trajectories (3.13,3.14) are aperiodic for $u_{\text{max}} \leq 0.5$, while for $u_{\text{max}} \geq 0.5$ they acquire a pseudoperiodic behaviour, spiralling around and towards their limit point on the set $C$.

Typical behaviours are shown in figures 3.3 and 3.4. For high $u_{\text{max}}$ the ordinary trajectories and their concatenations in general go through more than one quadrant and cross the set $C$: these are complications that prevent us to use the clock form, which has a singularity on $C$ (see (2.15)) and moreover cannot be used for closed trajectories lying in different quadrants. Then in this section we mainly use the $\theta$-functions, which we shall study in detail for each typical case of the ordinary $Y$ trajectories. This study is the main ingredient to construct a globally optimal synthesis, since $\theta(t)$ tells us where switches can happen.

**Study of the $\theta$ functions**

By studying the $Y$ trajectories and the corresponding $\theta(t)$ we find two typical behaviours: aperiodic and pseudoperiodic.

1. **The aperiodic case**, $u_{\text{max}} \leq 0.5$ has been largely studied in section 3.6.1 (see in particular the figure 3.8). We have deduced that switches can happen only in the first quadrant.

2. **The pseudoperiodic case**, $u_{\text{max}} > 0.5$, is characterized by the $\theta(t)$s shown in the figures 3.12 and 3.13. These two figure are exemplary of two distinct behaviours, depending on whether $Y$ intercepts or not the turnpike. We analyse these two case separately:

2A. $u \lesssim 5.2$, $Y$ does not cross the turnpike (figure 3.12). $\theta(t)$ starts decreasing, passes trough a minimum when the trajectory crosses the antiturnpike at the time $t_1^+$, then monotonically increases until its maximum value $\pi$. In general, for $u > 0.5$, differently from the aperiodic case, there is a time at which $\theta(t)$ reaches its maximum allowed valued $\pi$. After this time (see figure 3.12b) the $\theta(t)$ oscillates between the values $-\pi$ and $\pi$. But actually there are no stationary points other than $t_1^+$, since $\Delta \theta(t)$ has only one zero just in $t_1^+$. $t_f$ is the first time for which $\theta(t)$ increases of a value of $\pi$ (see figure 3.12c). In particular we have $|\theta(t_f) - \theta(t_1^+) = \pi$. $t_f$ corresponds to the first time when the trajectory crosses the set $C$, and it is the final time for which the trajectory is an extremal (see remark 2.33). Then, if the ordinary trajectory does not
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\[ t_1' + s_1' = t_0' + \]

reach the target within \( t_f \), a switch must occur before \( t_f \). However, as in the aperiodic case, from the theta we know that switches can happen only in the first quadrant.

\[ u \gtrsim 5.2. \] Y crosses the turnpike (figure 3.13). \( \theta(t) \) is oscillating (but not periodic) until \( t_f \). It starts decreasing, passes through a minimum when the trajectory crosses the antturnpike at the time \( t_1'^+ \) and then it increases until it reaches its first local maximum at \( t_1^+ \), corresponding

Figure 3.12: Fig 3.12a shows a trajectory in the pseudoperiodic range, \( u_{\text{max}} = 3 \), case (2A) (i.e. the trajectory does not cross the turnpike). In particular, in fig. 3.12b the \( \theta(t) \) is showed until a time \( t = 5 \). Fig. 3.12c is a zoom of fig. 3.12b for \( t < 1.6 \). \( t_f \) indicates the final time of extremality of the trajectory, i.e. the first time for which the \( \theta \) gains a \( \pi \) (in the grey area the trajectory cannot be extremal), then a switch must happen before \( t_f \). Fig. 3.12d shows the time interval where a switch can happen.

(a) ordinary trajectory for \( u = 3 \).

(b) \( \theta(t), \Delta_B(t) \) for \( u = 3 \).

(c) Extremality final time \( t_f \), \( u = 3 \).

(d) Zoom of \( \theta(t) \) for \( u = 3 \).
to the first time when it crosses the turnpike. The time $s_1^{+}$ is the first zero of $\theta(t)$. Switches can happen in the first quadrant, i.e. in the time interval $[0, t_1^{+}]$ or in the second quadrant but only in the time interval $[s_1^{+}, t_1^{+}]$. However switches in the latter interval do not correspond to an extremal concatenation, as we know from the proposition 2.34. Figure 3.13a is shows an example of trajectory with $u_{\text{max}} = 8$, figure 3.13b shows the corresponding $\theta(t)$: it is interesting to note that, differently from the case 2A, $t_f$ corresponds to the “third” intersection of $Y$ with the set $C$: in general, from a systematic study of various $Y$ we can note that $t_f$ is the first time when $Y$ crosses $C$ in the second quadrant leaving the remaining part of the spiral at negative values of the $y$ coordinate. This behaviour cannot be deduced directly by the shape of $\theta(t)$, it requires a simulation of the trajectory just as in figure 3.13a. Another example of the case 2B is shown in figures 3.13c and 3.13d for $u_{\text{max}} = 13$.

3.6.3 Four different strategies

The study of the typical $\theta(t)$ for different values of $u_{\text{max}}$ allows us to select a set of concatenations of extremal trajectories which is “reasonable” to include in our optimal synthesis. Other possibilities will be excluded on the basis that they are too complex (being composed by many pieces) to be efficient. We know that switches in the first quadrant are not useful. In the consecutive quadrants crossed by $Y$ the $\theta(t)$ tells us that we can have switches (in suitable time-intervals) but only for control $u_{\text{max}} > 5.2$ (case 2B). We start focusing our attention on three prototypical strategies that are well motivated by the $\theta(t)$. We will then use a fully numerically “random” search of the optimal trajectory without use of constraints coming from theoretical consideration on the $\theta(t)$ and guess the globally optimal strategy, in order to cover the whole range $u_{\text{max}} < 5.2$ (case 2A). Optimal trajectories have to be searched among these four strategies. For each strategy we compute the function $T_{\text{total}}(u_{\text{max}})$. In a further step we compare these total times.

**Strategy (1) $Y \ast Z$:** figure 3.14a. The ordinary $Y$ trajectory crosses the first, the second and eventually the third quadrant and then it relaxes towards the FP target once it has reached the turnpike for the second time. This strategy is completely fixed (for each $u_{\text{max}}$), in the sense that there is not arbitrariness in the choice of the switch time (there is only one switch, and it happens when the $Y$ crosses the turnpike). Then, for this strategy we do not need the $\theta(t)$.

The plot in figure 3.15a shows the trend of the total time $T_{\text{total}}$ for any value of $u_{\text{max}}$ between 5.2 and 50. We can see a regular decreasing trend until a value
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\[ t_f + t_1^+ + s_1^+ = t_0^+ \]

Figure 3.13: Two examples of ordinary trajectories and the corresponding \( \theta(t) \)s in the pseudoperiodic range but for the case (2B) (trajectories cross the turnpike). Figures 3.13a and 3.13b for \( u_{\text{max}} = 8 \), figures 3.13c and 3.13d for \( u_{\text{max}} = 13 \). \( t_f \) is the first time when \( Y \) crosses \( C \) in the second quadrant leaving the remaining part of the spiral at negative values of the \( y \) coordinate.

of \( u_{\text{max}} \) of about 22, where a sort of “phase transition” happens. The value \( u_{\text{max}} \simeq 22 \) corresponds to the first trajectory that reaches the trace distance circle before touching the turnpike. In particular, for \( u_{\text{max}} \lesssim 22 \) the \( Y \) reaches the turnpike from above the FP (as in figure 3.14a). Until this value, the time taken by the concatenation \( Y * Z \) on the turnpike is gradually shorter. Increasing \( u_{\text{max}} \), the \( Y \) trajectory becomes wider and wider: until \( u_{\text{max}} \simeq 25 \) it still reaches the target before crossing the turnpike. For \( u_{\text{max}} \gtrsim 25 \) the \( Y \)
Figure 3.14: The four different strategies included in the time optimal synthesis. The switch-points are indicated by \textit{sw}. Note that the strategy (4) (fig. 3.14d) is of the same type of the strategy (2) except by the fact that in (4) the switch is randomly selected in the second quadrant without using the \textit{θ}-function.

Strategies:

(a) Strategy (1): \(Y \ast Z\).
(b) Strategy (2): \(Y \ast X \ast Z\).
(c) Strategy (3): \(Y \ast Z \ast Y \ast Z\).
(d) Strategy (4): \(Y \ast X \ast Z\) (no \(θ\)).

Trajectories cross the turnpike below the target, but from now on, since the trajectories are gradually wider, the time taken on the turnpike after the switch is gradually longer: this explains the increasing trend of \(T_{\text{total}}\) for \(u_{\text{max}} \gtrsim 25\).

**Strategy (2) \(Y \ast X \ast Z\):** The ordinary \(Y\) trajectory leaves the starting point. Then it crosses the singular (antiturnpike) set and switches at a suitable point in the second quadrant. After reaching the \(z\)-axis as an
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Figure 3.15: Plot of the total times as a function of $u_{\text{max}} \in [5.2, 50]$ for the strategies (1), in fig 3.15a and (2), fig 3.15b for $\epsilon = 0.01$. The horizontal dashed and solid lines are the pure relaxation time and the lower bound $T_{\text{opt}}^{LB}$ (3.8) respectively. In fig 3.15c the two previous plots are merged.
X trajectory, it relaxes towards the FP target. The plot of $T_{\text{total}}$ vs $u_{\text{max}}$ is shown in figure 3.15b. Differently from the strategy (1), now there is an arbitrariness in the selection of the first switch time ($sw$, in the figure) within the time interval $[s_1^+, t_1^+]$ given by the $\theta(t)$. The best switch time is found numerically: for any $u_{\text{max}}$ our algorithm makes 100 random samplings from the time interval $[s_1^+, t_1^+]$. Once the first switch time is chosen, the second switch is completely determined (it is the point where the X trajectory crosses the turnpike). Thus, each point shown in figure 3.15b is the minimum global time resulting from the comparison among 100 possible switching times. The randomness in the algorithm explains the spread of the points. From the data analysis a regular trend of the minimum times emerges. In particular, as shown in figure 3.15c, for $u_{\text{max}} \lesssim 25$ strategy (2) is better than strategy (1). As a matter of fact we will see that (2) is always the globally optimal strategy for $u_{\text{max}} \in [5.2, 25]$. Instead, for $u_{\text{max}} \gtrsim 25$ the strategies (1) and (2) are very similar: this is explained by noting that for increasing $u_{\text{max}}$ the switch-point $sw$ tends to be closer to the turnpike and then the two strategies become almost indistinguishable up to a reflection with respect the $z$-axis. This behaviour is shown in figure 3.16.

**Strategy (3) $Y \ast Z \ast Y \ast Z$: figure 3.14c** This strategy is deduced by similarity to the optimal strategy in the case of unconstrained control discussed in [8]. When the ordinary Y trajectory reaches the turnpike for the first time, it relaxes. Then, before the Z touches the set $C^{\text{‡‡}}$, we again turn the control on until the trajectory reaches the turnpike for the second time. Then it relaxes again towards the FP target. This corresponds to a concatenation $Y \ast Z \ast Y \ast Z$ (note that, by the symmetry with respect to the $z$-axis, this strategy is completely equivalent to e.g. $Y \ast Z \ast X \ast Z$). In this case the first switch is completely fixed by the first intersection of the Y trajectory with the turnpike, while the second one (the one after the first $Z$) is randomly selected. A plot of the total times is shown in figure 3.17 for values of $u_{\text{max}}$ smaller than 150. For control strengths $u_{\text{max}} \lesssim 25$ the curve $T_{\text{total}}(u_{\text{max}})$ is rapidly decreasing and follows a trend similar to that of strategy (1), then a sudden slope change occurs and the total time slowly asymptotically tends to the lower limit found in [8]. We note a similarity between strategy (3) and strategy (1) in the range of controls $u_{\text{max}} \in [22, 25]$. This is explained by the fact that until this value the second switch $sw2$ in figure 3.14c tends to happen very close to the first one $sw1$, in order to reduce the time taken by the concatenation within the turnpike. As in the previous discussion (when we analysed the similarity

\footnote{We have written an original code implemented in the software Mathematica.}

\footnote{In section 3.6.4 we motivate this prescription.}
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Figure 3.16: Fig. 3.16a shows the comparison between the strategies (1),(2),(3) in the range $u_{\text{max}} \in [22, 25]$ (see fig. 3.23) where they have a similar shape and then reach the FP in about the same total time. Fig. 3.16b is a zoom of the neighborhood of the point $sw$ (in fig 3.16a); trajectories (1), (2) and (3) start as $Y$ (solid black curve); in $sw(2)$ the strategy (2) prescribes a switch and the trajectory becomes a $X$ (black dashed curve). Instead the Ys (1) and (3) continue until the turnpike following the same (solid red) trajectory. In $sw(3)$ the strategy (3) has a switch and it continues on the turnpike until its second switch-point (indicated by $sw(3)\text{bis}$), following the dot dashed trajectory. On the other hand, from the point $sw(3)$, strategy (1) continues as $Y$ without switching (gray dotted line).

between (1) and (2)), for $u_{\text{max}} \gtrsim 25$ the strategies (1) and (3) are identical up to a reflection with respect the $z$-axis. A comparison between the strategies (1), (2) and (3) is shown in figure 3.16a.

Strategy (4) $Y \ast X \ast Z$: figure 3.14d The three previous strategies make sense for $u_{\text{max}} > 5.2$ (case 2B), since until such a control strength the ordinary $Y$ trajectory does not reach the turnpike and the $\theta$-function tells us that switches in the second quadrant are not convenient. On the other hand, switches in the first quadrant are permitted for any $u_{\text{max}} > 0$, but they are not convenient, as shown in section 3.6.1. Actually, the $\theta(t)$ gives us only a suggestion: nothing prevents us trying to make switches in the second quadrant for $u_{\text{max}} < 5.2$ too. The resulting simulation is shown in figure 3.18. For $u_{\text{max}} \gtrsim 0.5$ the free relaxation time is more convenient. A sort of “plateau” starts around $u_{\text{max}} \simeq 0.5$ and ends at $u_{\text{max}} \simeq 1$ after which the decreasing trend of the total times restarts. The times of the plateau are higher than the
relaxation time and then the pure relaxation is always more convenient up to \( u_{\text{max}} \approx 1 \). The appearance of the plateau at \( u_{\text{max}} \approx 0.5 \) has nothing to do with the change of the ordinary trajectories from aperiodic to pseudoperiodic (change that happen at \( u_{\text{max}} = 0.5 \) too). Instead, it is related to the tolerance \( \epsilon \) allowed in reaching the target, as deduced by the plot in figure 3.19 which shows, for \( \epsilon = 0.05 \), that there is still a plateau, but starting from \( u_{\text{max}} \approx 1.4 \). Strategy (4) is equivalent to strategy (2), in the sense that they are both of the type \( Y^*X^*Z \). Strategy (4) differs by (2) only by the fact the switch-points in (2) are selected using the \( \theta \)-function. We can see the strategy (4) as a natural “continuation” of the strategy (2) for \( u_{\text{max}} < 5.2 \): figure 3.20 combines the total time of these two strategies in the range \( u_{\text{max}} < 10 \).

### 3.6.4 Excluded strategies

We now discuss which strategies can be easily excluded to belong in the optimal synthesis. Referring to the strategy (2) (or (4)) we can ask if it is convenient, instead of relaxing once the \( X \) reaches the turnpike, to cross the turnpike and then to make a switch in the third quadrant, i.e. to make a concatenation of the form \( Y*X*Y*Z \) as shown in figure 3.21a. This strategy is quickly excluded by using the clock form in the third quadrant, where \( g < 0 \). If instead the switch happens in the second quadrant (figure 3.21b), the \( Y \) trajectory tends to gets away from the FP. A similar reasoning holds for the strategy (1) if a switch happens in the third quadrant.
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Figure 3.18: Plot of the total times for $u_{\text{max}} < 5.2$ for the strategy (4) and for $\epsilon = 0.01$: the relaxation time (horizontal dashed line) and the corresponding relaxation trajectory are convenient until a control strength $u_{\text{max}} \simeq 1$. In the range of control $[0.5, 1]$ there is a plateau of the total times of the trajectory (4).

Figure 3.19: Plot of the total times for the strategy (4), $u_{\text{max}} < 5.2$ but for a trace distance tolerance $\epsilon = 0.05$: as in the $\epsilon = 0.01$ case there is still a plateau, but starting from $u_{\text{max}} \simeq 1.4$.

quadrant (see figure).

In the construction of the strategy (3), we have imposed that the second switch on the turnpike (i.e. the switch $Z * Y$), must happen on the positive $z$-axis. In fact switches in $z < 0$ (i.e. inside $C$) are not convenient, as shown in
Remark 3.4 A complete study of the $\theta(t)$s requires a classification of each of its stationary points, local maxima and minima and so on, as discussed in section 2.3.5. These points are related to the points where an ordinary trajectory crosses the set $S$. Furthermore we have seen that for high $u_{\text{max}}$ an $Y$ crosses each quadrant several times before its final time $t_f$. Since for each crossed quadrant in general there is a time interval for a potential switch, in principle we should classify, for each $u_{\text{max}}$, all $t_i^+, t_i^{+}, s_i^+, s_i^+$: an incredible task! We have limited the classification only to the first three quadrants and only to the study of reasonable concatenations of extremal trajectories. A complete study needs numerical tools and algorithms out of our reach. On the other hand, the benefit derived from such a study is uncertain: an emerging of a plethora of exotic extremal trajectories is indeed possible, but maybe physically useless in the applications.

3.6.5 The globally optimal strategy

In order to establish which is the globally optimal strategy for a given $u_{\text{max}}$ we combine the plots of the total times relative to each strategy in figure 3.22. In particular, for $\epsilon = 0.01$ we can see that the pure relaxation time is the optimal one until $u_{\text{max}} \approx 1$ (see also the plot 3.18). Increasing the control strength the
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(a) Excluded strategy: $Y \ast X \ast Y \ast Z$

(b) Excluded strategy: $Y \ast X \ast Y$

(c) Excluded strategy: in strategy (3) the second switch is not convenient if it happens inside the ellipse $C$.

Figure 3.21: Three non optimal, excluded strategies.
strategies (2) and (4) (one switch in the second quadrant) are optimal until $u_{\text{max}} \simeq 22$. In the range $u_{\text{max}} \simeq [22, 25]$ (as shown in the figure 3.23, where we have zoomed the plot 3.22) strategies (1),(2) and (3) are equivalent. Then, increasing the control, the optimal strategy is always the strategy (3), and the total optimal time tends, for $u_{\text{max}} \to \infty$, to the lower bound $T_{\text{opt}}^{LB}$ (3.8).

If we instead tolerate to reach the target with a trace distance $\epsilon = 0.05$, the comparison of the optimal strategies is shown in figure 3.24.

In the tables 3.24 and 3.25 the globally optimal strategies and the corresponding control range of optimality are summarized, respectively for $\epsilon = 0.01$ and $\epsilon = 0.05$. Figure 3.25 shows the final optimal times and the optimal strategies for each range of the control strength.
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Figure 3.23: A zoom of the plot 3.22 in the range of control strengths $u_{\text{max}} \in [22, 25]$ where all the strategies (1), (2) and (3) are optimal.

| Strategies ($\epsilon = 0.01$) | $u_{\text{max}}$ range | | | | | |
|---------------------------------|-------------------------|---|---|---|---|
| relaxation                      | OPT                     | * | * | * | * |
| (1)                             | *                       | * | OPT | * |
| (2)                             | *                       | * | OPT | OPT | * |
| (3)                             | *                       | OPT | * | OPT |
| (4)                             | OPT                     | * | * | * | * |

\[(3.24)\]

| Strategies ($\epsilon = 0.05$) | $u_{\text{max}}$ range | | | | | |
|---------------------------------|-------------------------|---|---|---|---|
| relaxation                      | OPT                     | * | * | * | * |
| (1)                             | *                       | * | OPT | * |
| (2)                             | *                       | * | OPT | OPT | * |
| (3)                             | *                       | * | OPT | OPT |
| (4)                             | OPT                     | * | * | * | * |

\[(3.25)\]
Figure 3.24: Mix of the total times of each strategy versus the control strength $u_{\text{max}}$, for $\epsilon = 0.05$. 
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Figure 3.25: Plot of the optimal times for $\epsilon = 0.01$. For each range of the control strengths, the corresponding optimal strategies are indicated.
Chapter 4

Discussion

In this thesis we have given a detailed study of the (coherent, open loop) time-optimal control of a dissipative two level quantum system subject to Generalized Amplitude Damping in the Markovian approximation. In particular we have focused on the acceleration of the relaxation of the system, starting from a state “more pure” than the fixed point (FP) associated with the bare open quantum system evolution. Roughly speaking, we have discussed how to speed up the heating of the qubit.

In chapter 1 we reviewed the theory of open quantum systems. In chapter 2 we have described a classical variational approach to the time-optimal control problem, based on the Pontryagin maximum principle (PMP). This principle, when applied to an affine control system on a 2-D manifold leads to a set of powerful mathematical tools which allow to solve a large class of physical problems.

At the beginning of chapter 3 (the original part of this thesis) we showed that an affine control system has a natural application in modelling a dissipative quantum system, when its dynamics is described by a master equation in Lindblad form. In particular, we formally explained this relationship for our model system, where the role of the control is played by an interaction electromagnetic laser field in resonance with the two-level system and the Rabi frequencies play the role of control parameters. The symmetry of the quantum system allows to reduce the 3-D state space (the Bloch ball) to a 2-D manifold (the projection on a plane of the Bloch ball).

In section 3.4 we analytically studied the solutions of the master equation for the controlled dynamics with constant control parameters. These “ordinary” trajectories are locally time optimal and are the building blocks of the extremal trajectories that we search in order to solve the time optimal problem.
We noted that increasing the control at our disposal, the ordinary trajectories change their behaviour: they are aperiodic for $u_{\text{max}} < 0.5$, while they are pseudoperiodic for $u_{\text{max}} > 0.5$. Differently from the works of other authors, which treated similar control systems but only in the aperiodic case, we have studied both types of these trajectories. Then we analytically described the properties of the other set of locally time optimal trajectories, i.e. the turnpike singular curves and characterised the geometry of the Bloch ball (section 3.5) by a division in sets of points in which the optimal trajectories can have only certain behaviours. Within the sets called “ordinary” the extremal trajectories correspond to a concatenation of controls that switch between $u_{\text{max}}$ and $-u_{\text{max}}$, while within the “singular turnpike” sets, trajectories can have only a given control $-u_{\text{max}} < u < u_{\text{max}}$. The “globally” time optimal trajectory is a concatenation of trajectories which are “locally” time optimal. In our case the turnpike is the $z$-axis (where both the initial state and the FP are located) and the singular control is null: in other words, the singular trajectory is a relaxation trajectory. Notably, this trajectory is globally optimal for low control strengths. In other words, if the control laser at our disposal in not sufficiently powerful, it does not help in the acceleration of the free dynamics.

We proved this statement analytically, using only the clock form (a differential form introduced in section 2.3.3) and geometrical reasoning.

For higher control strengths a pure analytical reasoning is not enough and we had to resort to a numerical (original) algorithm in order to find the best concatenation of extremal trajectories. In addition we used the $\theta$-function introduced in section 2.3.5 and we studied in detail this function for each starting trajectory and for each $u_{\text{max}}$. This help in selecting the time intervals where a trajectory can switch. Among a plethora of possible syntheses of globally optimal trajectories we selected four which we consider the most reasonable. Among these, three are selected using the the $\theta$-function, but for $u_{\text{max}} < 5.2$ the strategy was guessed, using only the prescription deriving from the PMP. The analysis of the total time required by each strategy and for any given $u_{\text{max}}$ showed that until a critical control strength free relaxation is always the most convenient way to reach the FP. Whereas increasing the control the optimal trajectories change and the optimal total times are monotonically decreasing, until the lower bound found in [8]. The main results of our analytical and numerical investigation are summarised in figures 3.22 and 3.24 and in tables 3.24 and 3.25 respectively for a trace distance tolerance in reaching the FP given by $\epsilon = 0.01$ and $\epsilon = 0.05$.

Summarizing, we have extended the results in [8, 9, 10, 11, 12, 13], finding the globally optimal trajectories for all the possible control strengths. In par-
ticular we rediscover the result in [8] as the limit case for $u_{\text{max}} \to \infty$. Instead a direct comparison of our results with the other cited works is impossible, because the authors use different dissipation channels, for example an GAD channel plus a dephasing channel and different starting and target points.

Finally, we would like to stress that the mathematical methods used in this thesis have been developed only recently and their applications to physical systems are not trivial. For these reasons further studies are needed to improve the numerical techniques capable to find the globally optimal trajectories and to exclude the non optimal one. Furthermore, a theoretical study of the “abnormal extremal” (see [7]) is necessary.

Other research directions could be to apply these methods to different dissipation channels (depolarizing and phase damping [6]) as well as to the speeding up of the relaxation starting from a generic initial state, including the cooling case, i.e. when the initial state is less pure that the FP. An more involved problem is to understand how these methods can be modified in order to study more complex system, for example two or more qubits coupled to an environment.
Appendix A

Master equation of the model system

In this appendix we show a detailed derivation of the affine control system (3.4) written in Cartesian coordinates.

We consider a controlled dissipative two-level system, assuming the hypothesis stated in section 1.3. The Hamiltonian of the total system, in Schrödinger picture, is of the form $H = H_S(t) + H_E + H_I$, where $H_S(t)$ is the Hamiltonian of the two level system, $H_E$ is the free Hamiltonian of the environment, $H_I$ is the interaction Hamiltonian between the system and the environment. Let us describe in more details these Hamiltonians.

$H_S(t)$ is composed of two pieces: $H_S(t) = H_0 + H_c(t)$. The free drift Hamiltonian $H_0$ is:

$$H_0 = \frac{1}{2} \hbar \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} = \frac{1}{2} \hbar \sigma_z.$$

$H_c(t)$ is the control Hamiltonian. It describes a dipole interaction $-\vec{D} \cdot \vec{E}_c(t)$ between the dipole operator $\vec{D}$ of the two-level system and a classical coherent electro magnetic field $\vec{E}_c(t) = \vec{\epsilon}_c e^{i\omega t} + \vec{\epsilon}_c^* e^{-i\omega t}$ which plays the role of control. Let us describe the procedure to obtain the rotating wave approximation. For this purpose we first consider that the field $\vec{E}_c(t)$ oscillates with a generic frequency $\omega_c$. In a second step we assume the field in resonance with the energy difference between the ground and the excited states of the system $S$, i.e. $\omega_c = \omega$. This hypothesis allows the RWA.

The off-diagonal elements describing the dipole transition are $\vec{d} = \langle 1|\vec{D}|0 \rangle$ and the respective h.c. $\vec{d}^* = \langle 0|\vec{D}|1 \rangle$. The diagonal elements of $\vec{D}$ vanish for invariance under parity of the e.m. interaction [38]. Thus we obtain the
following form of the control Hamiltonian:

\[ H_c(t) = -\vec{D} \cdot \vec{E}_c(t) \]

\[ = - (\vec{d} \sigma_+ + \vec{d}^* \sigma_-) \cdot (\vec{e} e^{i\omega_c t} + \vec{e}^* e^{-i\omega_c t}) \]

\[ = - (\vec{d} \cdot \vec{e} e^{i\omega_c t} + \vec{d}^* \cdot \vec{e}^* e^{-i\omega_c t}) \sigma_+ - (\vec{d}^* \cdot \vec{e} e^{i\omega_c t} + \vec{d} \cdot \vec{e}^* e^{-i\omega_c t}) \sigma_- . \]  

(A.1)

Switching to the interaction picture \( H_c(t) \rightarrow H'_c(t) = e^{iH_0t} H_c(t) e^{-iH_0t} \), and in (A.1) we have four kinds of exponentials: \( e^{\pm(\omega \pm \omega_c) t} \). In the resonant case, \( \omega_c \sim \omega \), the terms with \( \omega_c + \omega \) oscillate very rapidly respect to the others and can be neglected for sufficiently long times. We are then left with the following Hamiltonian (in the Schrödinger picture and for \( \omega_c = \omega \)):

\[ H_c(t) = -\vec{d} \cdot \vec{e}^* e^{-i\omega t} \sigma_+ - \vec{d}^* \cdot \vec{e} e^{i\omega t} \sigma_- \]

\[ = - \frac{1}{2} u e^{-i\omega t} \sigma_+ - \frac{1}{2} u^* e^{i\omega t} \sigma_- \]

\[ = \left( -\frac{1}{2} u^* e^{i\omega t} \begin{array}{c} 0 \\ \frac{1}{2} u e^{-i\omega t} \end{array} \right) . \]

(A.2)

\( u \equiv 2\vec{d} \cdot \vec{e}^* = u_1 + iu_2 \) is proportional to the complex Rabi frequency.

\( H_I \) and \( H_E \) have a role in the construction of the dissipative part \( \mathcal{L}_D \) of the master equation. The bath can be seen as a free quantized radiation field \( H_E = \sum_k \sum_{\lambda=1,2} \hbar \omega_k b_k^\dagger(\vec{k}) b_{\lambda}(\vec{k}) \). It follows that (see [4]) the dissipation part of the master equation (3.1) is given by:

\[ \mathcal{L}_D(\rho) \equiv \sum_{\mu=1}^2 \left( L_\mu \rho L_\mu^\dagger - \frac{1}{2} \{ L_\mu^\dagger L_\mu, \rho \} \right) , \]

where:

\[ L_1 \equiv \sqrt{\gamma_{01}} \sigma_+ , \quad L_2 \equiv \sqrt{\gamma_{10}} \sigma_- . \]

(A.3)

Now we have all the elements necessary to explicitly write the master equation:

\[ \frac{\partial}{\partial t} \rho(t) = -i[H_0 + H_I(t), \rho(t)] + \mathcal{L}_D(\rho(t)) . \]

(A.4)

The unitary part (the commutator) is:

\[ -i[H_0 + H_I(t), \rho] = \]

\[ -i \begin{pmatrix} -\frac{1}{2} u e^{-i\omega t} \rho_{10} + \frac{1}{2} u^* e^{i\omega t} \rho_{10} & \omega \rho_{01} - \frac{1}{2} u e^{-i\omega t} \rho_{11} + \frac{1}{2} u^* e^{i\omega t} \rho_{10} \\ -\omega \rho_{10} - \frac{1}{2} u^* e^{i\omega t} \rho_{00} + \frac{1}{2} u e^{i\omega t} \rho_{11} & -\frac{1}{2} u^* e^{i\omega t} \rho_{01} + \frac{1}{2} u e^{-i\omega t} \rho_{10} \end{pmatrix} . \]

(A.5)
The dissipative part is:

\[ \mathcal{L}_D(\rho(t)) = \gamma_{10} \begin{pmatrix} -\rho_{00} & -\frac{1}{2}\rho_{01} \\ -\frac{1}{2}\rho_{10} & \rho_{00} \end{pmatrix} + \gamma_{01} \begin{pmatrix} -\rho_{11} & -\frac{1}{2}\rho_{01} \\ -\frac{1}{2}\rho_{10} & \rho_{11} \end{pmatrix}. \]  \tag{A.6}

We can rewrite the master equation as a system of four first order differential equations:

\[
\begin{align*}
\dot{\rho}_{00} &= \frac{1}{2}ue^{-i\omega t}\rho_{10} - \frac{1}{2}u^*e^{i\omega t}\rho_{01} - \gamma_{10}\rho_{00} + \gamma_{01}\rho_{11}, \\
\dot{\rho}_{01} &= -i\omega\rho_{01} + \frac{1}{2}ue^{i\omega t}\rho_{11} - \frac{1}{2}ue^{-i\omega t}\rho_{00} - \Gamma\rho_{01}, \\
\dot{\rho}_{10} &= i\omega\rho_{10} + \frac{1}{2}ue^{-i\omega t}\rho_{00} - \frac{1}{2}u^*e^{i\omega t}\rho_{11} - \Gamma\rho_{10}, \\
\dot{\rho}_{11} &= -i\frac{1}{2}ue^{-i\omega t}\rho_{10} + \frac{1}{2}u^*e^{i\omega t}\rho_{01} + \gamma_{10}\rho_{00} - \gamma_{01}\rho_{11}.
\end{align*}
\]

This system of equation can be then expressed in a suitable matrix form:

\[
\begin{pmatrix}
\dot{\rho}_{00} \\
\dot{\rho}_{01} \\
\dot{\rho}_{10} \\
\dot{\rho}_{11}
\end{pmatrix} =
\begin{pmatrix}
-\gamma_{10} & -i\frac{u^*}{2}e^{i\omega t} & i\frac{u}{2}e^{-i\omega t} & \gamma_{01} \\
-i\frac{u}{2}e^{i\omega t} & -i\omega - \Gamma & 0 & i\frac{u}{2}e^{-i\omega t} \\
i\frac{u}{2}e^{-i\omega t} & 0 & i\omega - \Gamma & -i\frac{u^*}{2}e^{i\omega t} \\
\gamma_{10} & i\frac{u}{2}e^{i\omega t} & -i\frac{u^*}{2}e^{-i\omega t} & -\gamma_{01}
\end{pmatrix}
\begin{pmatrix}
\rho_{00} \\
\rho_{01} \\
\rho_{10} \\
\rho_{11}
\end{pmatrix}.
\tag{A.7}
\]

Until now we have worked in the Schrödinger picture. In order to switch to the interaction picture, we make the following change of variable: let \( \rho_I \) the density operator in the interaction picture and \( U_0 \equiv \exp \{-iH_0t\} \). Then \( \rho_I = U_0^\dagger \rho U_0 \).

We find that in vector form, this transformation is equivalent to:

\[
\begin{pmatrix}
\rho_{00} \\
\rho_{01} \\
\rho_{10} \\
\rho_{11}
\end{pmatrix}_I =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i\omega t} & 0 & 0 \\
0 & 0 & e^{-i\omega t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\rho_{00} \\
\rho_{01} \\
\rho_{10} \\
\rho_{11}
\end{pmatrix}.
\]

Let us define as \( V \) the \( 4 \times 4 \) matrix in the last equation. Substituting \( \rho \) with \( V^{-1}\rho_I \) in the equation \( \text{(A.7)} \), that for simplicity we rewrite as \( \dot{\rho} = A\rho \), we have:

\[
\begin{pmatrix}
\dot{\rho}_{00} \\
\dot{\rho}_{01} \\
\dot{\rho}_{10} \\
\dot{\rho}_{11}
\end{pmatrix}_I =
\begin{pmatrix}
-\gamma_{10} & -i\frac{u^*}{2} & i\frac{u}{2} & \gamma_{01} \\
-i\frac{u}{2} & -\Gamma & 0 & i\frac{u}{2} \\
i\frac{u}{2} & 0 & -\Gamma & -i\frac{u^*}{2} \\
\gamma_{10} & i\frac{u}{2} & -i\frac{u^*}{2} & -\gamma_{01}
\end{pmatrix}
\begin{pmatrix}
\rho_{00} \\
\rho_{01} \\
\rho_{10} \\
\rho_{11}
\end{pmatrix}_I.
\tag{A.8}
\]

Now we use the following change of variables:

\[
x \equiv 2\text{ Re}(\rho_{01}); \quad y \equiv -2\text{ Im}(\rho_{01}); \quad z \equiv \rho_{00} - \rho_{11},
\]

with
Master equation of the model system

and we proceed as follows (dropping the index I):

\[
\dot{\rho}_{01} = -i u \rho_{00} - \Gamma \rho_{01} + i u \rho_{11} \Rightarrow \\
\dot{x} = 2 \text{Re}(\dot{\rho}_{01}) = 2 \text{Re}\{-i(u_1 + u_2)\rho_{00} - \\
\Gamma[\text{Re}(\rho_{01} + i \text{Im}(\rho_{01})) + i(u_1 + iu_2)\rho_{11}]\} \\
= 2 \text{Re}\{-i u_1 \rho_{00} + u_2 \rho_{00} - \Gamma \text{Re}(\rho_{01}) - i \Gamma \text{Im}(\rho_{01}) + i u_1 \rho_{11} - u_2 \rho_{11}\}. \tag{A.9}
\]

By the fact that \(\rho_{00}, \rho_{11} \in \mathbb{R}\), the last line simplifies as:

\[
2 \text{Re}\{u_2 \rho_{00} - \Gamma \text{Re}(\rho_{01}) - u_2 \rho_{11}\} = 2 u_2 (\rho_{00} - \rho_{11}) - 2 \Gamma \text{Re}(\rho_{01}) \\
= 2 u_2 z - \Gamma x \equiv \dot{x},
\]

A similar calculation hold for \(\dot{y}, \dot{z}\). Finally we obtain the control system (3.4):

\[
\begin{cases}
\dot{x} = \frac{1}{2} \gamma \frac{x}{z_f} + u_2 z \\
\dot{y} = \frac{1}{2} \gamma \frac{y}{z_f} + u_1 z \\
\dot{z} = -\gamma + \gamma \frac{z}{z_f} - u_1 y - u_2 x.
\end{cases} \tag{A.10}
\]
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BIBLIOGRAPHY


