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Multicriteria optimization: Scalarization tecniques

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Introduction

Why speaking about multicriteria optimization?

The answer is easy: even if we are economists and we know for sure what an optimization problem is, we usually do not realize that we optimize in each moment of our lives. Think about when you have to pick a red or a blue dress to pair with a specific pair of shoes, or when you have to choose between a sportive-aggressive car or a toasty-family driven car; it is always a matter of optimization. As we can see from real-life examples, sometimes we have multiple alternatives and multiple criteria on the basis of which we can decide which is the best solution: in the case of a car, we may look at the price, the consumption of oil and the powerfulness in deciding among an Audi, a BMW and a Volkswagen - this is exactly a multicriteria or multiobjective optimization problem. Actually, when we solve a problem of this kind, the main point is to collect - or create - a *set* of solutions where the decision-maker - you in the case of the car, the government in the case of taxation and so on - can find the suitable alternative(s). In fact, it is not necessarily true that the solution is one and only one; genuinely, for decision-makers it is better to have a range of alternatives rather than only one restrictive solution.

It is exactly here that issues starting arise. First of all, the creation of this set of optimal alternatives is not so easy as it seems to be; in some situations, the problem may display features that are not good for finding this set, such as non-convexities or disconnections. In addition to this, the multicriteria optimization problem may not be so easy to be solved and may presents some properties that are not very good for the analysis. For the latter reasons, we usually look at a "modified" version of the problem rather to the original one. The ways of "modifying" a multiobjective optimization problem can be divided into scalarized and non-scalarized methods: to the first group belongs the well-known weighted sum method, while a second-group member is the lexicographic method. Since both groups are made by tons of techniques, we concentrate only on the former group and on some particular methods.

Therefore, we briefly introduce what a multicriteria optimization problem is, also thanks to many examples, and we establish the concept of cone and of ordering through a cone, since this is useful for the following sections. Then we give a definition of efficient and nondominated points, by looking at their properties and features. The core of the work is in the second part. We start by introducing the broad-used weighted sum method, the most famous and adopted scalarization technique to solve a multiobjective optimization problem. Then we introduce the ε-constraint and the hybrid method, tight connected to the previous method. As we will highlight, the weighted sum method has many positive features - among which, the simplicity - but it has many weaknesses; for this reason, we study the Pascoletti-Serafini method that can be seen as a generalization of all the previous methods and that it does not only point out the pitfalls of the already presented methods, but it also overcomes them with its generality. Last but not least, we give a brief sketch of the new scalarization technique proposed by Burachik *et al.*

Chapter 1

Multicriteria Optimization

Life is generally made by decision making, choices and compromises, and usually we look for the best outcome among these factors, that it to say the optimal alternative that maximizes all of them. The problem here is the conflict (even only partial) between many objectives and goals. We are not only speaking about planning and pricing production systems, but also designing bridges, spacecraft or managing pollution problem, and even simpler problem like choosing a cellular phone, a car, a dress. In this framework, the traditional methods of single objective optimization are not enough and we need new methods for nonlinear multiobjective optimization.

Problems with multiple objectives and criteria are known as *multiplecriteria optimization* or *multiplecriteria decision-making problems*. According to MacCrimmon (1973), depending on the properties of the feasible solutions, we distinguish between *multiattribute decision analysis* and *multiobjective optimization*. In multiattribute decision analysis, the set of feasible alternative is discrete, finite and known. Possible examples are the selection of the locations of power plants and dumping sites or the purchasing of cars. In multiobjective optimization problems, the feasible alternatives are not known in advance, there exists an infinite number of them and they are represented by decision variables restricted by constraint functions. Our discussion will concentrate on the latter.

1.1 Notations, optimality, orders and cones

The multicriteria optimization is born from the necessity of finding an "optimal" solution suggested by criteria, i.e. the solution that satisfies all decision-makers under the considered constraints, when it is not possible to attain the "best" alternative. Consider some examples to clarify the idea.

Example 1.1 *A firm would like to maximize profits by fixing a level of production compatible with its budget constraints. At the same time, it is interested in maximizing investments in publicity and physical capital. Those objectives are not comparable and in conflict, since a decision to lower the investment in publicity could permit to build more plants (more physical capital) and/or buy more raw materials (more production and more profits).*

Example 1.2 *Suppose you are a businessman and you want to buy a computer; you can choose among an iMac, a MacBook Pro, an HP Envy TouchSmart and a HP PC All-in-One G1. A decision is taken by considering price, processor and portability. You prefer a cheap powerful notebook since you have to travel a lot for work. In this case you have four possible alternatives and three criteria. The problem is that the most powerful computer with the best performance is also the most expensive and it is a desktop computer, while if you choose to take the cheapest one, you will have to sacrifice performance and again portability. Of course, if you base your decision only on one criterium, e.g. the processor, you will end up with a simple decision problem and subsequent simple solution, i.e. an iMac.*

Example 1.3 *Consider the following maximization problem over the non-negative real line*

$$\text{“}\max_{x\in X}\text{”}(f_1(x), f_2(x)) \tag{1.1}$$

The criteria or objective functions are

$$f_1(x) = -\sqrt{x+2} \text{ and } f_2(x) = 8x - x^2 + 16. \tag{1.2}$$

and plotted in Figure 1.1. For each function individually the corresponding maximizers for $f_1(x)$ and $f_2(x)$ are $x_1 = 0$ and $x_2 = 4$, respectively.

Take Example 1.2, where we consider as a key criteria for the choice only the price and the portability. The set $\mathcal{X} = \{$iMac, MacBook Pro, HP Envy

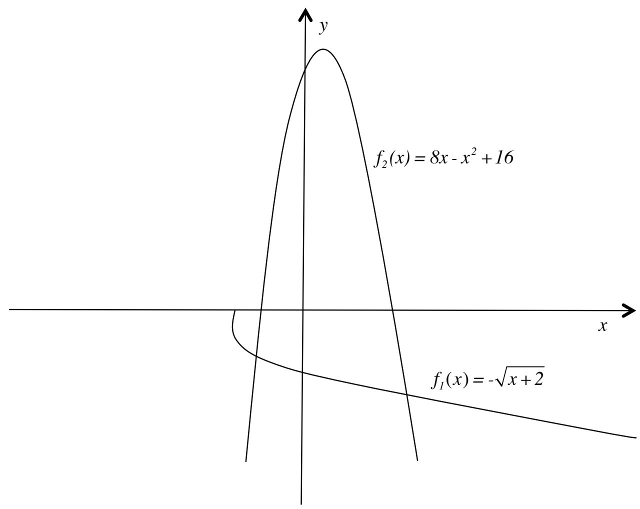

Figure 1.1: Objective functions of Example 1.3

TouchSmart, HP PC All-in-One G1} is the set of alternatives of the decision problem, or *feasible set*, and it is a subset of the *decision space*.

Denote price with f_1 and portability with f_2, the mapping $f_i : \mathcal{X} \to \mathbb{R}$ are criteria or objective functions and the optimization problem can be written as follows:

$$\text{``}\max_{x \in \mathcal{X}}\text{''}(f_1(x), f_2(x)) \tag{1.3}$$

The image of $\mathcal{X}$ under $f = (f_1, f_2)$ is denoted by $\mathcal{Y} := f(\mathcal{X}) := \{y \in \mathbb{R}^2 : y = f(x) \text{ for some } x \in \mathcal{X}\}$and it is called image of the feasible set, or the feasible set in the criterion space. The space from which the criterion values are taken is called *criterion space*.

To make it even clearer, take Example 1.3. Here the feasible set is

$$\mathcal{X} = \{x \in \mathbb{R} : x \geq 0\} \tag{1.4}$$

and the objective functions are

$$f_1(x) = -\sqrt{x+2} \text{ and } f_2(x) = 8x - x^2 + 16. \tag{1.5}$$

The decision space is $\mathbb{R}$ since $\mathcal{X} \in \mathbb{R}$, while the criterion space is $\mathbb{R}^2$ because $f(\mathcal{X}) \subset \mathbb{R}^2$. To find the image space of the feasible set in criterion space we substitute y_1 for $f_1(x)$ and y_2 for $f_2(x)$ to obtain $x = (y_1)^2 - 2$.

Therefore, we get $y_1 = -\sqrt{x+2}$ by solving for x and we obtain $y_2 = 12(y_1)^2 - y_1^2 - 4$ by substituting the definition of y_1.

Computing the maximum of y_2 as a function of y_1, the efficient solutions $x \in [0, 2]$ are equivalent to values of $y_1 = f_1(x) \in [\sqrt{2}, \sqrt{6}]$ and $y_2 = f_2(x) \in [16, 32]$. The points of $y_2(y_1)$ with $\sqrt{2} \leq y_1 \leq \sqrt{6}$ and $16 \leq y_2 \leq 32$ are called *nondominated points* and they represent the image of the set of efficient points.

Through those examples, we see that we often have many efficient solutions of a multicriteria optimization problem and a final choice has to be made among different efficient outcomes. The first problem that must be solved is the definition what we mean by saying minimize or maximize many objective functions. Although finding an efficient solution is one of the most common form of optimization, this point is crucial and for this reason until now we use the notation "max". Consider the following example.

Example 1.4 *Suppose we want to maximize both x_1 and x_2 of a circle with unitary ray*

$$\begin{array}{ll} \max & (x_1, x_2) \\ \text{subject to} & x_1^2 + x_2^2 \leq 1 \end{array} \tag{1.6}$$

Obviously, if we maximize for x_1 ($x_1 = 1$), we cannot maximize for x_2 and viceversa.

Example 1.5 *Suppose we want to minimize both x_1 and x_2 of a circle with unitary ray*

$$\begin{array}{ll} \min & (x_1, x_2) \\ \text{subject to} & x_1^2 + x_2^2 \geq 4 \end{array} \tag{1.7}$$

Obviously, if we minimize for x_1 ($x_1 = 4$), we cannot minimize for x_2 and viceversa.

Rational agents try to find the "best" solution over the set of feasible alternatives; but what do we mean for "best"?

In economics, the concept of "best" arises both in microeconomics, where decisions are taken by consumers and firms, and macroeconomics, where we have to optimize many criteria such as welfare of society and revenues of the government. An example is taxation: the government has to extract a certain amount of money to finance itself, but at the same time it must not disincentive people from working.

One of the first that dealt with those kind of tradeoffs was Francis Y. Edgeworth. For the first time, he defined the concept of optimum for multicriteria economic decision making, considering a multiutility problem where the consumer has two criteria, P and Π (Edgeworth, 1881):

> It is required to find a point (xy) such that, *in whatever direction* we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases.

Historically, probably the most popular solution concept is that of a contemporary of Edgeworth, the Pareto optimality. Indeed, using the words by Pareto (Pareto, 1906):

> Diremo che i componenti di una collettività godono, in una certa posizione, del *massimo di ofelimità*, quando è impossibile allontanarsi pochissimo da quella posizione giovando, o nuocendo, a tutti i componenti la collettività; ogni piccolissimo spostamento da quella posizione avendo necessariamente per effetto di giovare a parte dei componenti la collettività e nuocere ad altri [1].

Loosely speaking, a feasible solution is Pareto optimal if there is no other solution that is strictly better in one objective without being worse in another. However, the first sentence of the definition of maximum ophelimity at equilibrium contains an ambiguous passage, leading to a very carelessly definition:

> In such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases.

It may be thought that this is the result of a bad translation from Italian, but even adopting another translation such as “so as to benefit, or harm,

[1] In the English translation (Pareto, 1971):

> We will say that the members of a collectivity enjoy *maximum ophelimity* in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.

all the members of the community", the problem still remains and it is in the "or harm" part of the sentence. Similar wording can be found also in the Appendix of the *Manuel*. The second part of the definition excludes the possibility that all might be harmed, instead. Among the various reviews written about the *Manuel* by Pareto, the critical one written by Knut Wicksell (1913) for *Zeitschrift für Volkswirtschaft, Sozialpolitik und Verwaltung* is crucial for understanding this point.

He focused principally on the mathematical appendix and he concluded that "many truths are not new [...] and that most of the really new regrettably is not true". In particular, for what concerned the ambiguous sentence, Wicksell stated:

> The last adjunct sounds strange, for if the attained utility benefits cannot be decreased either, then one could just as well speak of a minimum d'ophélimité.

The problem is that Pareto did not have in hand the mathematical concept of *quasi-concavity*, needed for the general proof. He simply relied on *concavity* of the utility (ophelimity) function. According to Bergson (1948), it was left to Barone (1908) to give a correct definition of equilibrium concept, introduced by Pareto. Barone wrote:

> It must be impossible by any allocation of resources to enhance the welfare of one household without reducing that of another.

As we can see, this is the definition usually used; consider again Example 1.4: each point on the arch AB with A = (0, 1) and B = (1, 0) is an optimal solution in the sense of Pareto because we cannot increases x_1 without making worse off x_2 and viceversa.

While in the economic theory, multiobjective optimization problems have arisen first as maximization problems, in management science and in optimization theory, more attention have been devoted to minimization problems. In what follows we will refer to the latter ones; recalling that $\max f(x) = -\min -f(x)$ the presented results can be easily translated even for maximization problems.

Going back to the definition of minimization in a multiobjective framework, it is also possible that we are interested in assigning a priority among objectives, that is to say having a ranking among the objectives; e.g. we assign the first priority to the objective f_1, second priority to f_2 and so on, and an optimal solution of the problem is obtained by minimizing firstly the function f_1 and subsequently f_2 on the optimal solutions set of f_1, and so on i.e.:

> You solve the problem $P_1 : \min f_1(x), x \in S$ and let S_1 the set of optimal solutions of P_1.
>
> You solve the problem $P_2 : \min f_2(x), x \in S_1$ and let S_2 the set of optimal solutions of P_2.
>
> You solve the problem $P_p : \min f_m(x), x \in S_{p-1}$ and let S_p the set of optimal solutions of P_p.
>
> Every $x \in S_p$ is a solution of the problem $\min(f_1(x), \ldots, f_p(x))x \in S$.

If we do not want to exclude any objective, we can judge each single objectives through weights, i.e. you define $p_1 > 0$, $p_2 > 0$, $\ldots$, $p_p > 0$ and consider the problem $\min(p_1 f_1(x) + p_2 f_2(x) + \ldots + p_p f_p(x)), x \in S$. With $p_1 < p_2 < \ldots < p_p$ we give "more importance" to the first objective than the second one, the second one with respect to the third one, and so on.

Finally, we may think that for each value of the objective functions, we get a certain level of "cost" C; in this way, we obtain a function $C(f_1(x), \ldots, f_p(x))$ and we may be interested in minimizing C [2] (if the cost function is linear, then we end up with the previous case).

In order to define the meaning of "min" ("max"), we need to define how objective function vectors $(f_1(x), \ldots, f_p(x))$ have to be compared for different alternatives $x \in \mathcal{X}$: what do I have to consider to choose between an iMac and a PC: price, processor and/or portability?

In fact, from an analytical point of view, there is no doubt about the meaning of $\min f(x), x \in S$, $(\max f(x), x \in S)$ while doubts arise about the meaning of $\min(f_1(x), f_2(x)), x \in S$ $(\max(f_1(x), f_2(x)), x \in S)$.

In the first case $f(x)$ is a number and we have to determine the minimum (maximum) element for all possibile values taken by the function on the domain S. In $\mathbb{R}$ we have a complete order, i.e. a relation that, for every $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$; the minimum (maximum) element of a subset A of $\mathbb{R}$ is the number M such that $a \leq M, \forall a \in A$.

[2] In the case of maximization problems, we refer to a certain level of satisfaction or "utility"; we obtain a function $U(f_1(x), \ldots, f_p(x))$ and we may be interested in maximizing U.

In the second case $(f_1(x), f_2(x))$ is a couple of numbers. Here arises the question about how couple of numbers can be ordered and we can determine the minimum (maximum) element in a subset of $\mathbb{R}^p, p \geq 2$ where there is no canonical order as on $\mathbb{R}$. Definitions of order have to be introduced for this purpose.

Let $\mathcal{S}$ be any set. A *binary relation* on $\mathcal{S}$ is a subset of $\mathcal{R}$ of $\mathcal{S} \times \mathcal{S}$.

Definition 1.6 *A binary relation $\mathcal{R}$ on $\mathcal{S}$ has the following properties:*

- reflexive*: if $(s, s) \in \mathcal{R}$ for all $s \in \mathcal{S}$,*
- irreflexive*: if $(s, s) \notin \mathcal{R}$ for all $s \in \mathcal{S}$,*
- symmetric*: if $(s^1, s^2) \in \mathcal{R} \Longrightarrow (s^2, s^1) \in \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$,*
- asymmetric*: if $(s^1, s^2) \in \mathcal{R} \Longrightarrow (s^2, s^1) \notin \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$,*
- antisymmetric*: if $(s^1, s^2) \in \mathcal{R}$ and $(s^2, s^1) \in \mathcal{R} \Longrightarrow s^1 = s^2$ for all $s^1, s^2 \in \mathcal{S}$,*
- transitive*: if $(s^1, s^2) \in \mathcal{R}$ and $(s^2, s^3) \in \mathcal{R} \Longrightarrow (s^1, s^3) \in \mathcal{R}$ for all $s^1, s^2, s^3 \in \mathcal{S}$,*
- negatively transitive*: if $(s^1, s^2) \notin \mathcal{R}$ and $(s^2, s^3) \notin \mathcal{R} \Longrightarrow (s^1, s^3) \notin \mathcal{R}$ for all $s^1, s^2, s^3 \in \mathcal{S}$,*
- connected or complete*: if $(s^1, s^2) \in \mathcal{R}$ or $(s^2, s^1) \in \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$ with $s^1 \neq s^2$.*

Definition 1.7 *A binary relation $\mathcal{R}$ on $\mathcal{S}$ is*

- *an* equivalence relation *if it is reflexive, symmetric, and transitive,*
- *a* preorder (quasi-order) *if it is reflexive and transitive.*

For $(s^1, s^2) \in \mathcal{R}$ we can also write $s^1 \mathcal{R} s^2$. We write $s^1 \preceq s^2$ for $(s^1, s^2) \in \mathcal{R}$ and $s^1 \npreceq s^2$ for $(s^1, s^2) \notin \mathcal{R}$.

Given any preorder $\preceq$, we can define other two relations:

$$s^1 \prec s^2 :\Longleftrightarrow s^1 \preceq s^2 \text{ and } s^1 \npreceq s^2, \tag{1.8}$$

$$s^1 \sim s^2 :\Longleftrightarrow s^1 \preceq s^2 \text{ and } s^2 \preceq s^1. \tag{1.9}$$

where $\prec$ is the strict preference relation and $\sim$ is the indifference relation.

Definition 1.8 *A binary relation $\preceq$ on $\mathcal{S}$ is*

- *a* total preorder *if it is reflexive, transitive, and connected,*
- *a* total order *if it is an antisymmetric total preorder,*
- *a* strict weak order *if it is asymmetric and negatively transitive.*

Remark 1.9 *For some authors, reflexive and connected is meant to be complete.*

Example 1.10 *In $\mathbb{R}$ the relation: $a\mathcal{R}b \Leftrightarrow b - a \geq 0$ (i.e. the order $a \geq 0$) is a total order. The relation $a\mathcal{R}_1 b \Leftrightarrow b - a > 0$ is only transitive.*

Example 1.11 (Lexicographic order) *Define on $\mathbb{R}^2$ the following relation*

$$(x_1, y_1)\mathcal{R}(x_2, y_2) \textit{ if } x_1 < x_2 \textit{ or } x_1 = x_2,\ y_1 \leq y_2.$$

This is a total order and the relation can be rewritten as follow

$$(x_1, y_1) < (x_2, y_2) \textit{ if } x_1 < x_2 \textit{ or } x_1 = x_2,\ y_1 < y_2,$$

$$(x_1, y_1) = (x_2, y_2) \textit{ if } x_1 = x_2\ y_1 = y_2.$$

From a geometric point of view, fixing a point (a, b), the couple (x, y) such that $(a, b) < (x, y)$ are represented by the neither close nor open half-plane $\Gamma_+ = \{(x, y) : x > a \text{ or } x - a \text{ and } y \geq b\}$ (Figure 1.2).

The most important classes of relations in multicriteria optimization are partial orders and strict partial orders.

Definition 1.12 *A binary relation $\preceq$ is said to be*

- *a* partial order *if it is reflexive, transitive, and antisymmetric,*
- *a* strict partial order *if it is antisymmetric and transitive.*

Example 1.13 *In the set of positive integers consider the relation: $a\mathcal{R}b$ if a divides b that is if b is a multiple of a, i.e.*

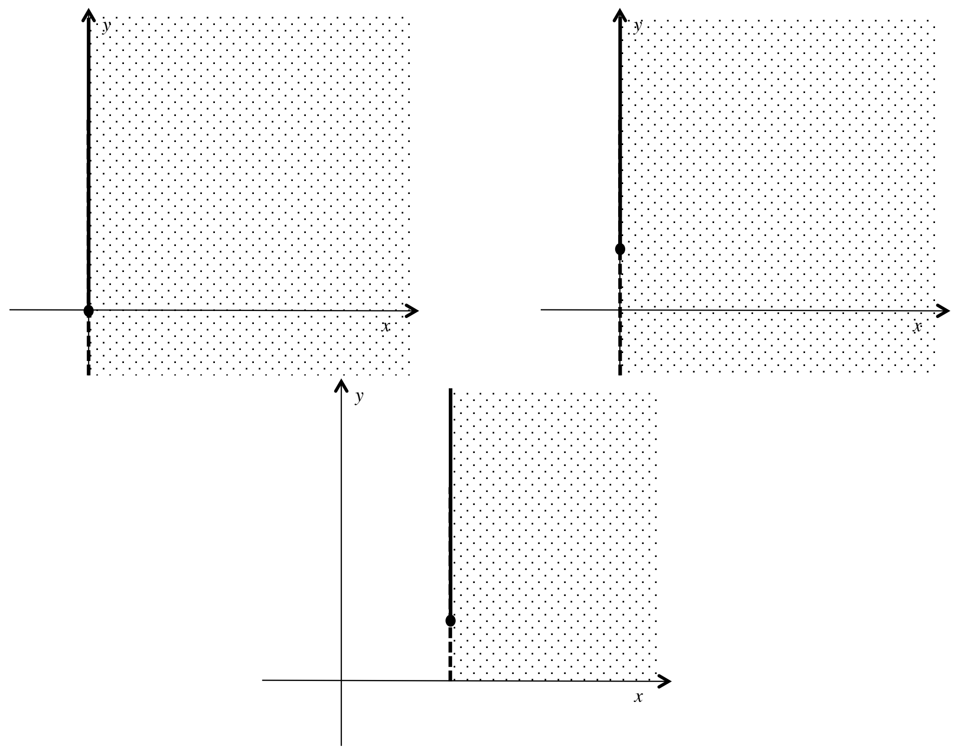

Figure 1.2: Examples of lexicographic orders

$$a\mathcal{R}b \Leftrightarrow b = ka, k \geq 1.$$

For example $2\mathcal{R}6$ *since* $6 = 3 \cdot 2$. *We can easily verifies that this a partial and not total order. In fact,*

1. *it is reflexive* $a\mathcal{R}a$ *since* $a = 1 \cdot a$,

2. *it is antisymmetric, i.e. if* $a\mathcal{R}b$ *and* $b\mathcal{R}a$, *then* $b = ka$ *and* $a = hb$, *from which we get* $b = (kh)b$, *and this is true if* $kh = 1$ *that, in the set of positive integers, implies* $k = h = 1$; *it follows that* $a = b$,

3. *it is transitive: if* $a\mathcal{R}b$ *and* $b\mathcal{R}c$, *that is if* $b = ka$ *and* $c = hb$, *then* $c = (kh)a$ *from which it follows that* $a\mathcal{R}c$,

4. *it is* not *connected because, for example, 3 and 7 are not "comparable", that is it is false either the relation* $3\mathcal{R}7$, *as 7 is not multiple of 3, or the relation* $7\mathcal{R}3$.

As a consequence, this relation is a partial order.

Example 1.14 *In the set of rectangles on the plane, consider the relation:*

$R_1 \mathcal{R} R_2$ if the area of R_1 is lower or equal than the area of rectangle R_2.

This relation is not antisymmetric as two rectangle having the same area are not necessarily equal. This implies that it is not a partial order and neither a total order.

Example 1.15 *On $\mathbb{R}^2$ define the relation*

$$(x_1, y_1) \leq (x_2, y_2) \text{ if } x_1 \leq x_2,\ y_1 \leq y_2 \text{ and } (x_1, y_1) < (x_2, y_2)$$

when at least one of the two previous inequalities is strictly verified.

It is easy to verify that this is a partial and not total order because, for example, couples like (2, 3) and (5, 1) cannot be compared with respect to the given relation.

Geometrically, all elements (x, y) such that $(a, b) \leq (x, y)$ with (a, b) fixed, belong to the cone of vertex (a, b) obtained by translating in (a, b) the cone $\mathbb{R}^2_+$.

Example 1.16 *Consider all the possible investments that can be done on two types of shares A and B. Be x the quantity invested on A and y the quantity invested on B for the generic investment I. As a convention, if $x > 0, y > 0$ we have a purchase of shares, while $x < 0, y < 0$ means a sell of shares.*

If $I_1 = (x_1, y_1)$ and $I_2 = (x_2, y_2)$ are two investments, express a "preference" through as follows

$$I_1 \mathcal{R} I_2 \text{ if } y_1 \leq y_2, \text{ and } x_1 + y_1 \leq x_2 + y_2.$$

The relation $y_1 \leq y_2$ expresses the idea that the quantity of shares of type B in the first investment must not exceed the quantity of shares of type B in the second one; the relation $x_1 + y_1 \leq x_2 + y_2$ expresses the fact that the total amount invested in I_1 must not exceed the amount invested on I_2.

If $I_0 = (a, b)$ is a generic investment, the set of preferred investments with respect of I_0 is given by the cone $\mathcal{C} = \{(x, y) : x + y \geq a + b, y \geq b\}$. This a partial order, but not a total one.

As shown in the examples, a partial order in $\mathbb{R}^p$ can be represented thorough a cone.

Definition 1.17 *A subset $\mathcal{C} \subseteq \mathbb{R}^p$ is called* cone*, if $\alpha d \in \mathcal{C}$ for all $d \in \mathcal{C}$ and for all $\alpha \in \mathbb{R}, \alpha > 0$.*

Example 1.18 *An example of cone is given by the nonnegative orthant and it is the cone of all those vectors which have nonnegative entries (Figure 1.3)*

$$\mathcal{C} = \{x \in \mathbb{R}^n : x \geq 0\}.$$

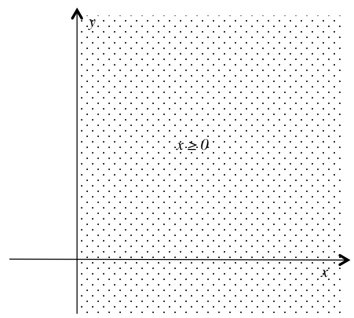

Figure 1.3: $\mathcal{C} = \{x \in \mathbb{R}^n : x \geq 0\}$

Generally, if $\mathcal{C} = \mathbb{R}^n_+$, we get an order "component by component" on the n-dimensional space. This order corresponds to the concept of optimal solution introduced by Pareto; in fact, $\mathbb{R}^n$ is also called Pareto cone (Figure 1.4).

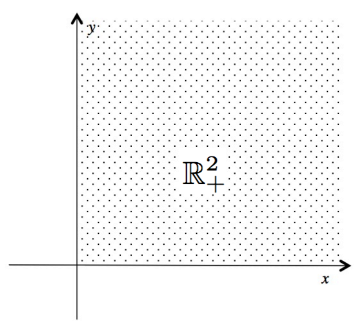

Figure 1.4: $\mathcal{C} = \mathbb{R}^2_+$

Another example is given by the so called ice cream cone *or* Lorentz cone *or* second-order cone*. It is the cone $\mathcal{C}$ defined over $\mathbb{R}^n$ such that*

$$\mathcal{C} = \{x \in \mathbb{R}^n : \sqrt{x_1^2 + \ldots + x_{n-1}^2} \leq x_n\}$$

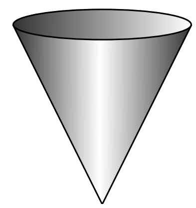

Figure 1.5: The Lorentz Cone in $\mathbb{R}^3$

Example 1.19 *By referring to our previous examples, we can see how orders can be represented through cones.*

In $\mathbb{R}^2$, if we choose the following cone

$$\mathcal{C} = \mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\},$$

we get the order of Example 1.15, plotted in Figure 1.6.

Instead, if we take the cone $\mathcal{C} = \{(x, y) : y \geq 0, x + y \geq 0\}$, we get the order of Example 1.16. It is represented in Figure 1.7.

By taking $\mathcal{C} = \{(x, y) : x > 0\} \cup \{(x, y) : x = 0, y \geq 0\}$, we obtain the lexicographic order of Example 1.11 (Figure 1.8).

Let $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2 \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}$. The multiplication of a set with a scalar is

$$\alpha\mathcal{S}_1 := \{\alpha \mathrm{s} : \mathrm{s} \in \mathcal{S}\}, \tag{1.10}$$

with $-\mathcal{S} = \{ -s : s \in \mathcal{S} \}$. The algebraic sum of $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by

$$\mathcal{S}_1 + \mathcal{S}_2 := \{s^1 + s^2 : s^1 \in \mathcal{S}_1, s^2 \in \mathcal{S}_2\}. \tag{1.11}$$

Definition 1.20 *A cone $\mathcal{C}$ in $\mathbb{R}^p$ is*

- nontrivial *or* proper *if $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq \mathbb{R}^n$,*
- convex *if $\alpha d^1 + (1 - \alpha)d^2 \in \mathcal{C}$ for all $d^1, d^2 \in \mathcal{C}$ and for all $0 < \alpha < 1$,*
- pointed *if for $d \in \mathcal{C}, d \neq 0, -d \notin \mathcal{C}$, i.e., $\mathcal{C} \cap (-\mathcal{C}) \subseteq \{0\}$.*
- acute *if there exists an open half space $H_\alpha = \{x \in \mathbb{R}^p : \langle x \ a\rangle > 0\}$ such that* $\mathrm{cl}C \subset H_\alpha \cup \{0\}$.

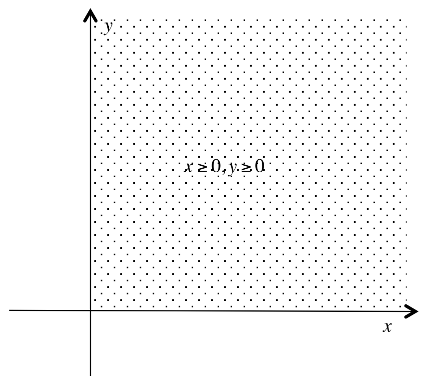

Figure 1.6: Order of Example 1.15

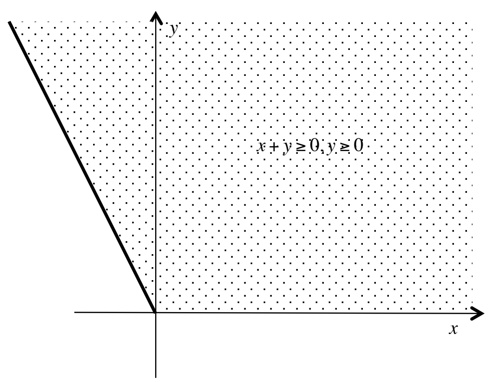

Figure 1.7: Order of Example 1.16

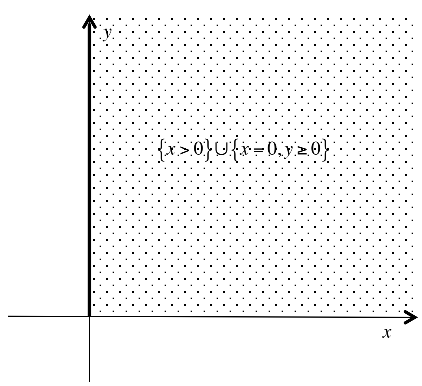

Figure 1.8: Order of Example 1.11

Remark 1.21 *If $\mathcal{C}$ is acute, then it is pointed.*
Suppose not. Then there exists $d \in \mathcal{C} : -d \in \mathcal{C}$. Since $\mathcal{C}$ is acute, then there exists α such that $\alpha x > 0, \forall x \in \mathcal{C}$; but if $\alpha d > 0$, then $\alpha(-d) < 0$ and this cannot be true.
The viceversa is not necessarily true, i.e. $\mathcal{C}$ is pointed then it is also acute, because, by considering the lexicographic order given by

$$\mathcal{C} = \{(x, y) :\in \mathbb{R}^2 : x > 0\} \bigcup \{x = 0, y > 0\},$$

it is easy to see that there does not exist $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\alpha x > 0, \forall x \in \mathcal{C}$.

Proposition 1.22 *Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a cone. Then $\mathcal{C}$ is convex if and only if $\forall d_1, d_2 \in \mathcal{C}$, $d_1 + d_2 \in \mathcal{C}$.*

Proof. Assume that $\mathcal{C}$ is a convex cone.

If $d_1, d_2 \in \mathcal{C}$, then $\forall \alpha \in [0, 1]$, $\alpha d_1 + (1 - \alpha)d_2 \in \mathcal{C}$ given the convexity of $\mathcal{C}$. Take $\alpha = \frac{1}{2}$, then $\frac{1}{2}d_1 + \frac{1}{2}d_2 \in \mathcal{C} \Rightarrow \frac{1}{2}(d_1 + d_2) \in \mathcal{C}$. If $z = \frac{1}{2}(d_1 + d_2) \in \mathcal{C}$, then $kz \in \mathcal{C}, \forall k > 0$. Take $k = 2 \Rightarrow d_1 + d_2 \in \mathcal{C}$.

Now assume that $\mathcal{C}$ is a cone. Take $d_1 \in \mathcal{C} \Rightarrow \alpha d_1 \in \mathcal{C}$ and $d_2 \in \mathcal{C} \Rightarrow (1 - \alpha)d_2 \in \mathcal{C}$, then $\alpha d_1 + (1 - \alpha)d_2 \in \mathcal{C}$. The graphical representation of the proof is given in Figure 1.9 ■

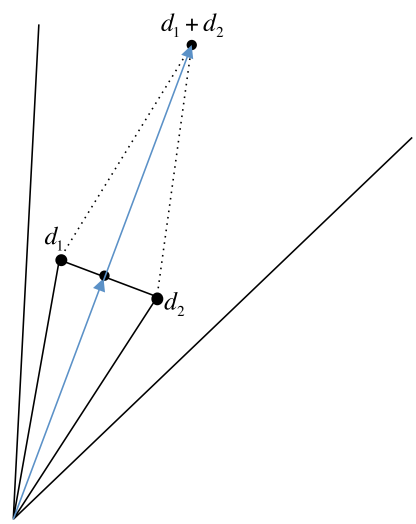

Figure 1.9: Proof of Proposition 1.22

Given an order relation $\mathcal{R}$ on $\mathbb{R}^p$, we can define a set

$$\mathcal{C}_{\mathcal{R}} := \{y^2 - y^1 : y^1 \mathcal{R} y^2\}, \tag{1.12}$$

that is to say the set of nonnegative elements of $\mathbb{R}^p$ according to $\mathcal{R}$. Shown below there are some relationships between the properties of $\mathcal{C}_{\mathcal{R}}$ and $\mathcal{R}$.

Proposition 1.23 *Let $\mathcal{R}$ be compatibile with scalar multiplication, i.e., for all $(y^1, y^2) \in \mathcal{R}$ and all $\alpha \in \mathbb{R}_>$ it holds that $(\alpha y^1, \alpha y^2) \in \mathcal{R}$. Then $\mathcal{C}_{\mathcal{R}}$ defined in* (1.12) *is a cone.*

In order to say that $\mathcal{C}_{\mathcal{R}}(y), y \in \mathbb{R}^p$ does not depend on y, we must be able to say that $\mathcal{R}$ is compatible with addition, i.e. $\mathcal{R}$ is said to be compatible with addition if $(y^1 + z, y^2 + z) \in \mathcal{R}$ for all $z \in \mathbb{R}^p$ and all $(y^1, y^2) \in \mathcal{R}$. The independence from y follows from Lemma 1.37.

Lemma 1.24 *If $\mathcal{R}$ is compatibile with addition and $d \in \mathcal{C}_{\mathcal{R}}$ then $0\mathcal{R}d$.*

Compatibility with addition of relations is useful for further results.

Theorem 1.25 *Let $\mathcal{R}$ be a binary relation on $\mathbb{R}^p$ which is compatible with scalar multiplication and addition. Then the following statements hold.*

1. *$0 \in \mathcal{C}_{\mathcal{R}}$ if and only if $\mathcal{R}$ is reflexive.*
2. *$\mathcal{C}_{\mathcal{R}}$ is pointed if and only if $\mathcal{R}$ is antisymmetric.*
3. *$\mathcal{C}_{\mathcal{R}}$ is convex if and only if $\mathcal{R}$ is transitive.*

Until now we have defined a cone $\mathcal{C}_{\mathcal{R}}$ given a relation $\mathcal{R}$, but we can use a cone to define an order relation. Let $\mathcal{C}$ be a cone and define $\mathcal{R}_{\mathcal{C}}$ by

$$y^1 \mathcal{R} y^2 \iff y^2 - y^1 \in \mathcal{C}. \tag{1.13}$$

Proposition 1.26 *Let $\mathcal{C}$ be a cone. Then $\mathcal{R}_{\mathcal{C}}$ define in* (1.13) *is compatible with scalar multiplication and addition in $\mathbb{R}^p$.*

Theorem 1.27 *Let $\mathcal{C}$ be a cone and $\mathcal{R}_{\mathcal{C}}$ be as defined in* (1.13)*. Then the following statements hold.*

1. *$\mathcal{R}_{\mathcal{C}}$ is reflexive if and only if $0 \in \mathcal{C}$.*
2. *$\mathcal{R}_{\mathcal{C}}$ is antisymmetric if and only if $\mathcal{C}$ is pointed.*
3. *$\mathcal{R}_{\mathcal{C}}$ is transitive if and only if $\mathcal{C}$ is convex.*

1.2 Efficiency and nondominance

Consider the multicriteria optimization problem of the kind:

$$\min_{x\in\mathcal{X}}(f_1(x),\dots,f_p(x)) \tag{1.14}$$

where the image of the feasible set $\mathcal{X}$ under the objective mapping f is $\mathcal{Y} := f(\mathcal{X})$. In order to present the concepts of efficient solution and of nondominated point, throughout this dissertation, we will refer to the following notations:

1. $f(x) \leqq f(\hat{x})$ means that $f_i(x) \leq f_i(\hat{x})$ for every i;
2. $f(x) \leq f(\hat{x})$ means that $f_i(x) \leq f_i(\hat{x})$ for every i and $f(x) \neq f(\hat{x})$;
3. $f(x) < f(\hat{x})$ means that $f_i(x) < f_i(\hat{x})$ for every i.

Definition 1.28 *A feasible solution $\hat{x} \in \mathcal{X}$ is called* efficient *or* Pareto optimal*, if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient, $f(\hat{x})$ is a* nondominated point*. If $x^1, x^2 \in \mathcal{X}$ and $f(x^1) \leq f(x^2)$ we say x^1* dominates *x^2 and $f(x^1)$* dominates *$f(x^2)$. The sets of all efficient solutions $\hat{x} \in \mathcal{X}$ is denoted $\mathcal{X}_E$ and called the* efficient set*. The set of all nondominated points $\hat{y} = f(\hat{x}) \in \mathcal{Y}$, where $\hat{x} \in \mathcal{X}_E$, is denoted $\mathcal{Y}_N$ and called the* nondominated set*.*

There are other equivalent definitions of efficiency. Indeed, $\hat{x}$ is efficient if

1. there is no $x \in \mathcal{X}$ such that $f_k(x) \leq f_k(\hat{x})$ for $k = 1,\dots,p$ and $f_i(x) < f_i(\hat{x})$ for some $i \in \{1,\dots,k\}$;
2. there is no $x \in X$ such that $f(x) - f(\hat{x}) \in -\mathbb{R}^p_{\geqq} \setminus \{0\}$;
3. $f(x) - f(\hat{x}) \in \mathbb{R}^p \setminus -\{\mathbb{R}^p_{\geqq} \setminus \{0\}\}$ for all $x \in \mathcal{X}$;
4. $f(\mathcal{X}) \cap (f(\hat{x}) - \mathbb{R}^p_{\geqq}) = \{f(\hat{x})\}$;
5. there is no $f(x) \in f(\mathcal{X}) \setminus \{f(\hat{x})\}$ with $f(x) \in f(\hat{x}) - \mathbb{R}^p_{\geqq}$;
6. $f(x) \leqq f(\hat{x})$ for some $x \in \mathcal{X}$ implies $f(x) = f(\hat{x})$.

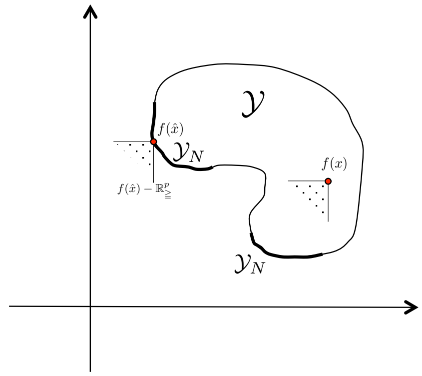

Figure 1.10: Representation of Definition 1., 4., and 5.

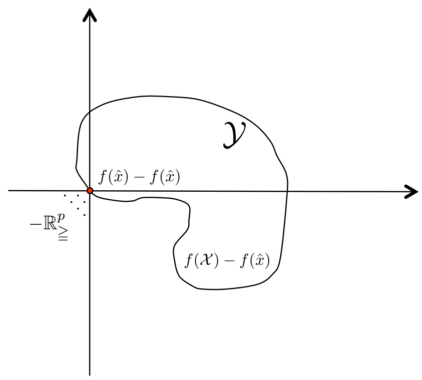

Figure 1.11: Representation of Definition 2. and 3.

Definition 1.28 is equivalent to definitions 1., 4., and 5.: take $f(\hat{x})$ and check for images of feasible solutions to the right and above of that point.

Equivalent definitions as 2. and 3. through $f(x) - f(\hat{x})$ translated the set $\mathcal{Y} = f(\mathcal{X})$ so that the origin coincides with $f(\hat{x})$, and the intersection of the translated set $\mathcal{Y}$ with the negative orthant is checked. This intersection contains only $f(\hat{x})$ if $\hat{x}$ is efficient.

Nondominated points are defined by the componentwise order on $\mathbb{R}^p$, but when we use the weak and strict componentwise order, we get definitions of strictly and weakly nondominated points.

Definition 1.29 *A feasible solution $\hat{x} \in \mathcal{X}$ is called* weakly efficient *or* weakly Pareto optimal *if there is no other $x \in \mathcal{X}$ such that $f(x) < f(\hat{x})$,* *i.e.* $f_k(x) < f_k(\hat{x})$ *for all* $k = 1, \ldots, p$. *The point* $\hat{y} = f(\hat{x})$ *is called* weakly nondominated.

A feasible solution $\hat{x} \in \mathcal{X}$ *is called* strictly efficient *or* strictly Pareto optimal *if there is no other* $x \in \mathcal{X}, x \neq \hat{x}$ *such that* $f(x) \leqq f(\hat{x})$. *The weakly (strictly) efficient and nondominated sets are* $\mathcal{X}_{wE}$ $(\mathcal{X}_{sE})$ *and* $\mathcal{Y}_{wN}$, *respectively.*

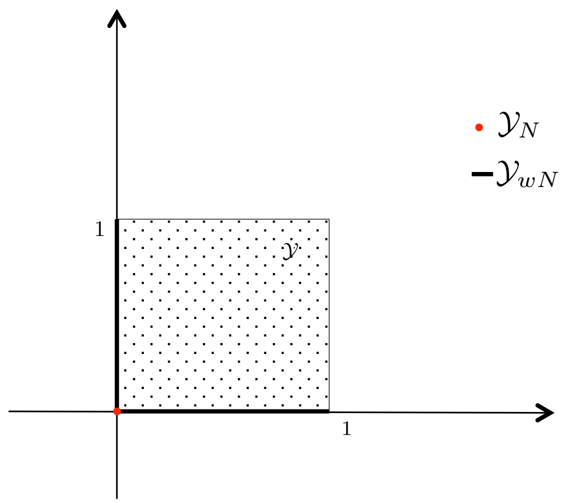

Figure 1.12: Nondominated points and weakly nondominated points

From the definitions

$$\mathcal{Y}_N \subset \mathcal{Y}_{wN} \tag{1.15}$$

and

$$\mathcal{X}_{sE} \subset \mathcal{X}_E \subset \mathcal{Y}_{wE}. \tag{1.16}$$

It is important to have in mind that what will follow depends on Definition 1.28, i.e. that we are able to find the minimum on the image space on the basis of an order defined by a Pareto cone. In fact, by changing the choice of the order, we would have another definition. Just as an example, consider lexicographic optimization problems; those problems arise when there is a conflict among different objects but those objects are hierarchically ordered. A problem of this kind can be stated in the following way:

$$\underset{x \in \mathcal{X}}{\operatorname{lexmin}} (f_1(x), \ldots, f_p(x)).$$

A feasible solution $\hat{x} \in \mathcal{X}$ is *lexicographically optimal* or a *lexicographic solution* if there is no $x \in \mathcal{X}$ such that $f(x) <_{lex} f(\hat{x})$. If this is true for all $x \in \mathcal{X}$ then $\hat{x} \in \mathcal{X}_E$. As we can see, by changing the definition of order we get a different concept of optimal and efficient solution.

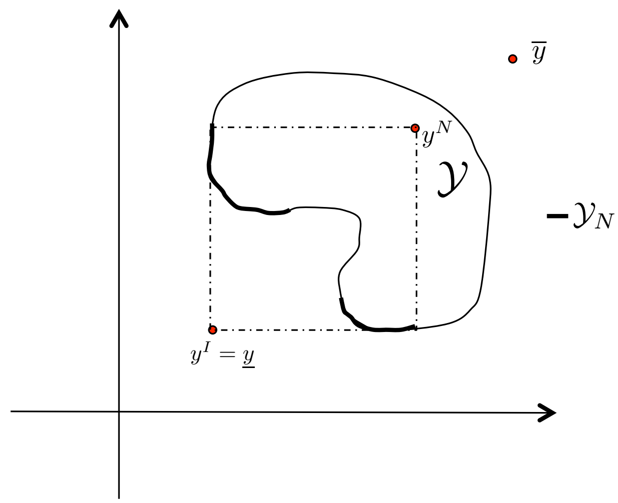

Figure 1.13: Nadir and ideal points

Another example is given by the definition of ideal and nadir points. Assume that $\mathcal{X}_E$ and $\mathcal{Y}_N$ are nonempty; we look for real numbers $\underline{y}_k$, $\bar{y}_k$, $k = 1, \dots, p$ with $\underline{y}_k \leq y_i \leq \bar{y}_k$ for all $y \in \mathcal{Y}_N$.

A possibility is to choose

$$\underline{y}_k := \min_{y \in \mathcal{Y}} y_i, \tag{1.17}$$

$$\bar{y}_k := \max_{y \in \mathcal{Y}} y_i. \tag{1.18}$$

While for the lower bound (1.17) there is no problem since there is always an efficient point $y \in \mathcal{Y}_N$ with $y_k = \underline{y}_k$, the upper bound (1.18) has a tendency to be far away from actual nondominated points. For this reason, the upper bound is usually defined as the maximum over nondominated points.

Definition 1.30 *1. The point $y^I = (y_1^I, \dots, y_p^I)$ given by*

$$y_k^I := \min_{x \in \mathcal{X}} f_k(x) = \min_{y \in \mathcal{Y}} y_k \tag{1.19}$$

is called the ideal point *of the multicriteria optimization problem.*

2. The point $y^N = (y_1^N, \dots, y_p^N)$ *given by*

$$y_k^N := \max_{x \in \mathcal{X}_E} f_k(x) = \max_{y \in \mathcal{Y}_N} y_k \tag{1.20}$$

is called the nadir point *of the multicriteria optimization problem.*

1.2.1 On the existence of nondominated points

Going back to Definition 1.28, the first issue that has to be analyzed is the existence and the properties of the efficient set and the nondominated set. The problem of existence of efficient points can be solved by simply looking for conditions that guarantee the existence of nondominated elements of the image space, ordered by the Pareto cone. Since $\mathbb{R}_{\geqq}$ is a convex and pointed cone we are going to state the definition of nondominated points and to give sufficient conditions for their existence in a more general context, namely by referring to a set $\mathcal{Y} \subset \mathbb{R}^p$ ordered by a convex pointed cone [3].

Definition 1.31 *Consider a set* $\mathcal{Y} \subset \mathbb{R}^p$ *ordered by a convex pointed cone* $\mathcal{C}$*;* $\hat{y} \in \mathcal{Y}$ *is said be nondominated or minimal point if there is no* $y \in \mathcal{Y}$ *such that* $y \in \hat{y} - \mathcal{C}$*. The set of all minimal points is denoted by* $\mathcal{Y}_N$*.*

Let us first consider the following properties of nondominated sets.

Proposition 1.32 *Let* $\mathcal{Y} \subset \mathbb{R}^p$ *ordered by a convex pointed cone* $\mathcal{C}$*.* $\mathcal{Y}_N = (\mathcal{Y} + \mathcal{C})_N$*.*

When $\mathcal{C} = \mathbb{R}^p_{\geqq}$, Proposition 1.32 allows us to say that nondominated points are located in "lower left part" of $\mathcal{Y}$, i.e. "adding" $\mathbb{R}^p_{\geqq}$ to $\mathcal{Y}$ does not change the nondominated set (Figure 1.10). Sometimes the new set obtained with the sum has more and better properties than the initial set.

A second intuitive result is that nondominated points must belong to the boundary of $\mathcal{Y}$.

Proposition 1.33 $\mathcal{Y}_N \subset cl(\mathcal{Y})$*.*

[3] Regarding the definition of nondominated and efficient points, we do not have a common reference framework; each author has worked in its own environment, by using different notations and wordings for the same concept, creating space for misunderstandings and confusion. A possible summary of different notations for the same concepts is given by Erghott, p. 60.

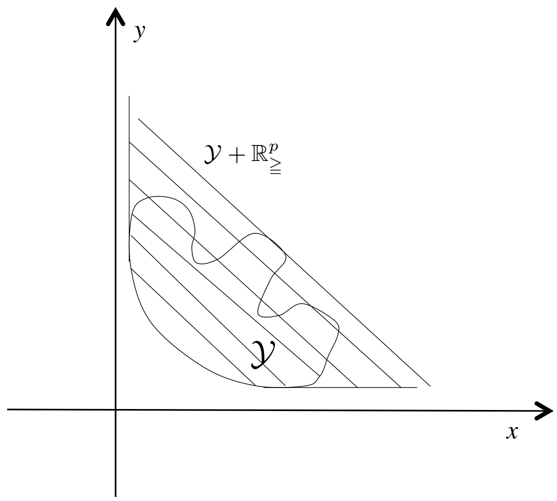

Figure 1.14: Nondominated points of $\mathcal{Y}$ and $\mathcal{Y}_N + \mathbb{R}^p_{\geqq}$ are the same

From Propositions 1.32 and 1.33 we get conditions for the emptiness of $\mathcal{Y}_N$.

Corollary 1.34 *If $\mathcal{Y}$ is open or if $\mathcal{Y} + \mathbb{R}^p_{\geqq}$ is open $\mathcal{Y}_N = \emptyset$.*

The following propositions are connected to the properties of nondominated set with respect to the algebraic sum of two sets and of a set multiplied by a positive scalar.

Proposition 1.35 $(\mathcal{Y}_1 + \mathcal{Y}_2)_N \subset (\mathcal{Y}_1)_N + (\mathcal{Y}_2)_N$.

Proposition 1.36 $(\alpha\mathcal{Y})_N = \alpha(\mathcal{Y}_N)$, *for* $\alpha \in \mathbb{R}, \alpha > 0$.

The following Lemma provides some sufficient conditions for the existence of minimal points of a set which is ordered by a convex pointed cone.

Lemma 1.37 *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^p$ and let $\mathcal{C}$ be a convex and pointed cone of $\mathbb{R}^p$.*

1. *If there exists an $y_0 \in \mathcal{Y}$ such that the set $(y_0 - cl\mathcal{C}) \cap \mathcal{Y}$ is compact, then $\mathcal{Y}_N \neq 0$.*

2. *If there exists a $y_0 \in \mathcal{Y} + cl\mathcal{C}$ such that the set $(y_0 - cl\mathcal{C}) \cap (\mathcal{Y} + cl\mathcal{C})$ is compact, then $\mathcal{Y}_N \neq 0$.*

The compactness of $(y_0 - cl\mathcal{C}) \cap \mathcal{Y}$ and $(y_0 - cl\mathcal{C}) \cap (\mathcal{Y} + cl\mathcal{C})$ is sufficient to guarantee the existence of efficient points in $\mathcal{Y}$. In order to define necessary and/or sufficient conditions to undertake such compactness, it is useful to introduce the concept of compact, closed and bounded set with respect to a given cone.

Definition 1.38 *Let $\mathcal{C}$ be a cone on $\mathbb{R}^p$ and $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$. $\mathcal{Y}$ is* $\mathcal{C}$-compact *if $(y - cl\mathcal{C}) \cap \mathcal{Y}$ is compact $\forall y \in \mathcal{Y}$.*

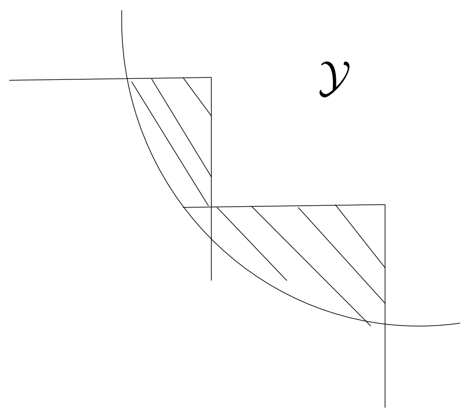

Figure 1.15: Geometric representation of $\mathcal{Y}$ compactness with respect to $\mathcal{C}$

Definition 1.39 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. $\mathcal{Y}$ is* $\mathcal{C}$-closed *if $\mathcal{Y} + cl\mathcal{C}$ is closed.*

Definition 1.40 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. $\mathcal{Y}$ is* $\mathcal{C}$-bounded *if $\mathcal{Y}^+ \cap -cl\mathcal{C} = \{0\}$, where $\mathcal{Y}^+$ is the recession cone defined in the following way:*

$$\mathcal{Y}^+ = \{x \in \mathbb{R}^p : \exists \alpha_k \longrightarrow 0, \alpha_k > 0, \{y^k\} \subset \mathcal{Y} : \alpha_k y^k \longrightarrow y\}.$$

Definition 1.41 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. $\mathcal{Y}$ is* L-bounded *if $(y - cl\mathcal{C}) \cap \mathcal{Y}$ is bounded $\forall y \in \mathcal{Y}$.*

A compact set is also $\mathcal{C}$-compact with respect to any $\mathcal{C}$. For what concerns the recession cone $\mathcal{Y}^+$, $\mathcal{Y}^+ = \{0\}$ if and only if $\mathcal{Y}$ is a bounded set.

The following examples show that the closedness of $\mathcal{Y}$ does not imply closedness of $\mathcal{Y} + cl\mathcal{C}$ and viceversa.

Example 1.42 *Let $\mathcal{Y} = \{(x, y) : xy = -1, x > 0\}$ and $\mathcal{C} = \mathbb{R}^2_+$. It is easy to recognize that $\mathcal{Y}$ is closed, but $\mathcal{Y} + cl\mathcal{C} = \{(x, y) : x > 0\}$ is not closed.*

Example 1.43 *Let $\mathcal{Y} = \{(x, y) : x^2 + y^2 < 1\} \cup \{(x, y) : x^2 + y^2 = 1, x \leq 0, y \leq 0\}$ and $\mathcal{C} = \mathbb{R}^2_+$. $\mathcal{Y}$ is not closed, while $\mathcal{Y} + cl\mathcal{C}$ is closed.*

A sufficient condition such that the closedness of $\mathcal{Y}$ implies the closedness of $\mathcal{Y} - cl\mathcal{C}$ is provided by the following theorem.

Theorem 1.44 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. If $\mathcal{Y}$ is closed and $\mathcal{C}$-*bounded*, then $\mathcal{Y}$ is $\mathcal{C}$-*closed*.*

Corollary 1.45 *A compact set $\mathcal{Y}$ is $\mathcal{C}$-*closed *with respect to any cone $\mathcal{C}$.*

Let's now exploit the relationship between L-bounded and $\mathcal{C}$-bounded sets.

Theorem 1.46 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. If $\mathcal{Y}$ is a $\mathcal{C}$-bounded set, then $\mathcal{Y}$ is L-bounded.*

Theorem 1.46 shows that the class of $\mathcal{C}$-bounded sets is contained in that of L-bounded set, while Example 1.47 clarifies that this inclusion is proper.

Example 1.47 *Let $\mathcal{Y} = \{(x, y) : xy = -1, x < 0\}$ and $\mathcal{C} = \mathbb{R}^2_+$. It is easy to verify that $\mathcal{Y}$ is L-bounded, while it is not $\mathcal{C}$-bounded because $(-1, 0), (0, 1) \in \mathcal{Y}^+ \cap -cl\mathcal{C} \neq \{0\}$.*

The L-bounded condition is weaker than the $\mathcal{C}$-bounded one since the latter implies the former but the viceversa does not hold; however, there are classes of sets with respect to which the L-bounded is equivalent to the $\mathcal{C}$-bounded condition. The following result holds.

Theorem 1.48 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by a convex pointed cone $\mathcal{C}$. If $\mathcal{Y}$ is convex, then $\mathcal{Y}$ is L-bounded if and only if $\mathcal{Y}$ is $\mathcal{C}$-bounded.*

A class of sets that are L-bounded and $\mathcal{C}$-bounded at the same time is given by lower bounded sets with respect to the cone $\mathcal{C}$.

Definition 1.49 *$\mathcal{Y}$ is bounded from below with respect to the cone $\mathcal{C}$ if $\exists a \in \mathbb{R}^p_{\geq}$ such that $\mathcal{Y} \subset a + cl\mathcal{C}$.*

When $\mathcal{C} = \mathbb{R}^p_{\geqq}$ (Pareto cone), Definition 1.49 is equivalent to ask $y_i \leq M, i = 1, \ldots, s, \forall y = (y_1, \ldots, y_s) \in \mathcal{Y}$.

Theorem 1.50 *Let $\mathcal{Y}$ be a non empty set on $\mathbb{R}^p$, ordered by an acute cone $\mathcal{C}$. If $\mathcal{Y}$ is bounded from below with respect to the cone $\mathcal{C}$, then it is $\mathcal{C}$-bounded and L-bounded.*

Theorem 1.50 establishes that a set bounded from above is both L-bounded and $\mathcal{C}$-bounded with respect to an acute cone $\mathcal{C}$, and, thus, gives us a sufficient condition for L-boundedness. The previous theorem is false if we consider a pointed cone instead of an acute cone. For example, consider the cone $\mathcal{C} = \{(x, y) : y > 0\} \bigcup \{(x, y) : y = 0, x \geq 0\}$ and the set $\mathcal{Y} = \{(x, y) : y = 0\}$, we have that $\mathcal{Y} \subset 0 - cl\mathcal{C}$ and $(1, 0) \in \mathcal{Y}^+ \cap cl\mathcal{C}$.

Theorem 1.51 *Let $\mathcal{Y}$ be a closed set ordered by a closed pointed cone $\mathcal{C}$. Then $\mathcal{Y}$ is L-bounded if and only if $\mathcal{Y}$ is $\mathcal{C}$-bounded.*

Corollary 1.52 *Let $\mathcal{Y}$ be a closed set ordered by a closed pointed cone $\mathcal{C}$. If $\mathcal{Y}$ is $\mathcal{C}$-bounded then $\mathcal{Y}$ is $\mathcal{C}$-compact.*

For closed sets, the $\mathcal{C}$-compactness is not equivalent to the $\mathcal{C}$-boundedness, differently for what happens for L-boundedness; the following theorem gives a sufficient condition for such equivalence.

Theorem 1.53 *Let $\mathcal{Y}$ be a closed and convex set and let $\mathcal{C}$ be a convex pointed cone. Then $\mathcal{Y}$ is $\mathcal{C}$-bounded if and only if $\mathcal{Y}$ is $\mathcal{C}$-compact.*

In Example 1.43, the set $\mathcal{Y}$ is $\mathcal{C}$-closed and $\mathcal{C}$-bounded but it is not $\mathcal{C}$-compact; Theorem 1.53, together with Theorem 1.44, provides a class of sets for which the hypothesis of $\mathcal{C}$-closedness and $\mathcal{C}$-boundedness are equivalent to the hypothesis of $\mathcal{C}$-compactness.

Theorem 1.54 *Let $\mathcal{C}$ be a closed, convex and pointed cone and $\mathcal{Y} \subset \mathbb{R}^p$ be a closed and convex set. Then $\mathcal{Y}$ is $\mathcal{C}$-closed and $\mathcal{C}$-bounded if and only if $\mathcal{Y}$ is $\mathcal{C}$-compact.*

All results about $\mathcal{C}$-compactness of $\mathcal{Y}$ and $\mathcal{Y} + cl\mathcal{C}$, through Lemma 1.37, allow to state many conditions for the existence of efficient points.

Theorem 1.55 *Let $\mathcal{Y} \subset \mathbb{R}^p$ be a nonempty set ordered by an acute convex cone $\mathcal{C} \subset \mathbb{R}^p$. The set $\mathcal{Y}_N$ of nondominated points of $\mathcal{Y}$ with respect to the cone $\mathcal{C}$ is nonempty when one of the following conditions is satisfied:*

1. *$\mathcal{Y}$ is $\mathcal{C}$-compact;*
2. *$\mathcal{Y}$ is $\mathcal{C}$-closed and $\mathcal{C}$-bounded;*
3. *$\mathcal{Y}$ is $\mathcal{C}$-closed and bounded from below;*
4. *$\mathcal{Y}$ is closed and L-bounded;*
5. *$\mathcal{Y}$ is closed and $\mathcal{C}$-bounded;*
6. *$\mathcal{Y}$ is closed and bounded from below.*

As we have already observed, L-boundedness condition is more general than $\mathcal{C}$-boundedness; in particular, the $\mathcal{C}$-closedness together with L-boundedness do not imply that a $\mathcal{Y} + cl\mathcal{C}$ is $\mathcal{C}$-compact. However, the existence of efficient points is ensured by the following theorem for a $\mathcal{C}$-closed and L-bounded set.

Theorem 1.56 *Let $\mathcal{C} \subset \mathbb{R}^p$ be a convex, closed and pointed cone and $\mathcal{Y} \subset \mathbb{R}^p$ be a nonempty set. If $\mathcal{Y}$ is $\mathcal{C}$-closed and L-bounded then $\mathcal{Y}_N \neq \emptyset$.*

Remark 1.57 *Until now, we worked on the image space and we looked at properties that this space must satisfy in order to get some important result. The issue is that, at the beginning of the problem, we have not the image space, but the objective functions. For this reason, we may want that those functions satisfy some important properties. Conditions that are easily satisfied are continuity of functions and compactness of the initial space.*

1.3 Proper efficiency and proper dominance

The definition of efficiency given until now allows for improvements of one criterion only through the decline of at least another criterion, i.e. it is not possible to improve one criterion by maintaining all others unchanged. These trade-offs can be measured through the increase f_i in terms of decrease in f_j, where f_i and f_j are two different objective functions, and connected to this concept it is possibile to introduce another definition of efficient solution with bounded trade-offs - the *properly efficient solutions*.

We have many definitions of properly efficient solution and many relationships occurs among them; we present each of them, by looking at how they are related.

Geoffrion's definition of proper efficiency The first definition of proper efficiency that we introduce is by Geoffrion.

Definition 1.58 (Geoffrion (1968)) *A feasible solution $\hat{x} \in \mathcal{X}$ is called* properly efficient*, if it is efficient and if there is a real number $M > 0$ such that for all i and $x \in \mathcal{X}$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index j such that $f_j(\hat{x}) < f_j(x)$ such that*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

The corresponding point $\hat{y} = f(\hat{x})$ is called properly nondominated*.*

According to the definition given by Geoffrion, points are properly efficient solutions when they have bounded trade-offs between the objectives. Properly efficient solutions can be obtained by minimizing a weighted sum of the objective functions with all positive weights.
The last point that must be highlighted when we speak about properly efficient points is that all the conditions given until now for the existence of nondominated and efficient points (in the general and weak meaning) do not guarantee the existence of properly nondominated points.

Borwein & Benson's definition of proper efficiency Another definition of proper efficiency was given by Borwein and Benson. Before stating the definition itself, we need for this purpose the definitions of a tangent cone and a conical hull.

Definition 1.59 *Let $\mathcal{Y} \subset \mathbb{R}^p$ and $y \in \mathcal{Y}$.*

1. *The* tangent cone *of $\mathcal{Y}$ at $y \in \mathcal{Y}$ is*

   $$T_{\mathcal{Y}}(y) := \{d \in \mathbb{R}^p : \exists \{t_k\} \subset \mathbb{R}, \{y^k\} \subset \mathcal{Y} \; s.t. \; y^k \longrightarrow y, t_k(y^k - y) \longrightarrow d\}.$$

2. *The* conical hull *of $\mathcal{Y}$ is*

   $$\text{cone}(\mathcal{Y}) = \{\alpha y : \alpha \geq 0, y \in \mathcal{Y}\} = \bigcup_{\alpha \geq 0} \alpha \mathcal{Y}.$$

Now we are able to give the definition of proper efficient points for Borwein and Benson.

Definition 1.60 *1. (Borwein (1977)) A solution $\hat{x} \in \mathcal{X}$ is called* properly efficient *(in Borwein's sense) if*

$$T_{\mathcal{Y}+\mathbb{R}^p_{\geqq}}(f(\hat{x})) \cap (-\mathbb{R}^p_{\geq}) = \{0\}.$$

2. (Benson (1979)) A solution $\hat{x} \in \mathcal{X}$ is called properly efficient *if*

$$\mathrm{cl}(\mathrm{cone}(\mathcal{Y} + \mathbb{R}^p_{\geqq} - f(\hat{x}))) \cap (-\mathbb{R}^p_{\geq}) = \{0\}.$$

From the definition of conical hull and tangent cone, it immediately follows that

$$T_{\mathcal{Y}+\mathbb{R}^p_{\geqq}}(f(\hat{x})) \subset \mathrm{cl}(\mathrm{cone}(\mathcal{Y} + \mathbb{R}^p_{\geqq} - f(\hat{x})))$$

meaning that Benson's definition is stricter than Borwein's one. The relationship between these two definitions is given by the following theorem.

Theorem 1.61 *1. If $\hat{x}$ is properly efficient in Benson's sense, it is also properly efficient in Borwein's sense.*

2. If $\mathcal{X}$ is convex and $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex then both definitions coincide.

Differently from the Geoffrion's definition, the latter definitions of proper efficiency do not require $\hat{x}$ to be efficient; the following proposition avoids any doubts about efficiency of proper efficient solutions in the sense of Borwein or Benson.

Proposition 1.62 *If $\hat{x}$ is properly efficient in Borwein's sense, then $\hat{x}$ is efficient.*

Another difference between these two definition and the Geoffrion's one of proper efficiency is given by the explicit use of the componentwise order, $\mathbb{R}^p_{\geqq}$: in fact, while Geoffrion used directly $\mathbb{R}^p_{\geqq}$, in Berson's and Borwein's cases we can substitute $\mathbb{R}^p_{\geq}$ with whatever arbitrary closed convex cone. When $\mathcal{C} = \mathbb{R}^p_{\geqq}$, Geoffrion's definition and Benson's definition of proper efficiency coincide and we can see the latter as a generalization of the former.

Theorem 1.63 (Benson (1979)) *Feasible solution $\hat{x} \in \mathcal{X}$ is properly efficient in Geoffrion's sense if and only if it is properly efficient in Benson's sense.*

Kuhn & Tucker's definition of proper efficiency The last definition of proper efficient solutions is suitable especially for the following type of problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \leqq 0, \end{array}$$

where both the objective functions, $f_i, i = 1, \ldots, p$ and the constraint functions $g_j, j = 1, \ldots, m$ are continuously differentiable. In this case, the definition of proper efficiency that must be used is the one provided by Kuhn and Tucker.

Definition 1.64 (Kuhn and Tucker (1951)) *A feasible solution $\hat{x} \in \mathcal{X}$ is called* properly efficient *(in Kuhn and Tucker's sense) if it is efficient and if there is no $d \in \mathbb{R}^n$ satisfying*

1. $\langle \nabla f_k(\hat{x}), d \rangle \leq 0, \forall k = 1, \ldots, p,$
2. $\langle \nabla f_i(\hat{x}), d \rangle < 0,$ for some $i \in \{1, \ldots, p\},$
3. $\langle \nabla g_j(\hat{x}), d \rangle \leq 0, \forall j \in \mathcal{J}(\hat{x}) = \{j = 1, \ldots, m : g_j(\hat{x}) = 0\},$

where $\mathcal{J}(\hat{x})$ represents the set of active indices.

This definition means that if a vector d satisfying 1., 2. and 3., exists, then moving from $\hat{x}$ in direction d causes that no objective function increases, one strictly decreases and the feasible set is not left. As in the case of Geoffrion's definition, the componentwise order is assumed and so it is not possible to use this definition for orders coming from closed and convex cone.

For the equivalence between the Kuhn and Tucker's definition and the Geoffrion's one, we need the subsequent definition.

Definition 1.65 *A differentiable multiobjective optimization problem satisfies the KT constraint qualification at $\hat{x} \in \mathcal{X}$ if for any $d \in \mathbb{R}^n$ with $\langle \nabla g_j(\hat{x}), h \rangle \leq 0$ for all $j \in \mathcal{J}(\hat{x})$ there is a real number $\bar{t} > 0$, a function $\theta : [0, \bar{t}] \longrightarrow \mathbb{R}^n$, and $\alpha > 0$ such that $\theta(0) = \hat{x}, g(\theta(t)) \leq 0$ for all $t \in [0, \bar{t}]$ and $\theta'(0) = \alpha d$.*

This definition gives us a constraint qualification for the equivalence between these definitions and it means that every feasible direction d can be written as the gradient of a feasible curve starting in $\hat{x}$.

Theorem 1.66 (Geoffrion (1968)) *If a differentiable multiobjective optimization problem satisfies the KT constraint qualification at $\hat{x}$ and $\hat{x}$ is properly efficient in the Geoffrion'sense, then it is properly efficient in the Kuhn and Tucker's sense.*

The reverse of the latter theorem holds without the constraint qualification.

Theorem 1.67 *Let $f_k, g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ be convex, continuously differentiable functions and suppose $\hat{x}$ is a properly efficient in Kuhn and Tucker's sense. Then $\hat{x}$ is properly efficient in the Geoffrion'sense.*

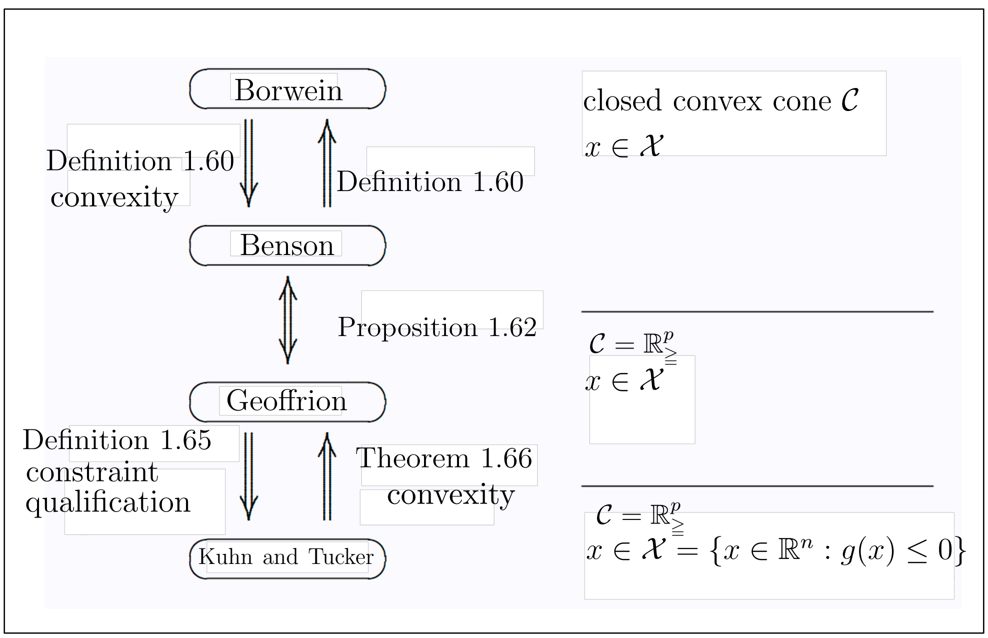

Figure 1.16: Link among different proper efficient definitions

1.4 Efficient frontier

After seeing the existence of nondominated points and efficient solutions and the different concepts of efficiency, we are going to use scalarization to study a very important topological property of efficient and nondominated set - connectedness. In fact, if $\mathcal{Y}_N$ or $\mathcal{X}_E$ is connect, then the whole set can be explored starting from a single nondominated/efficient point using local search. In addition to this, connectedness will make easier to select a final compromise solution from the set of efficient solutions $\mathcal{X}_E$.

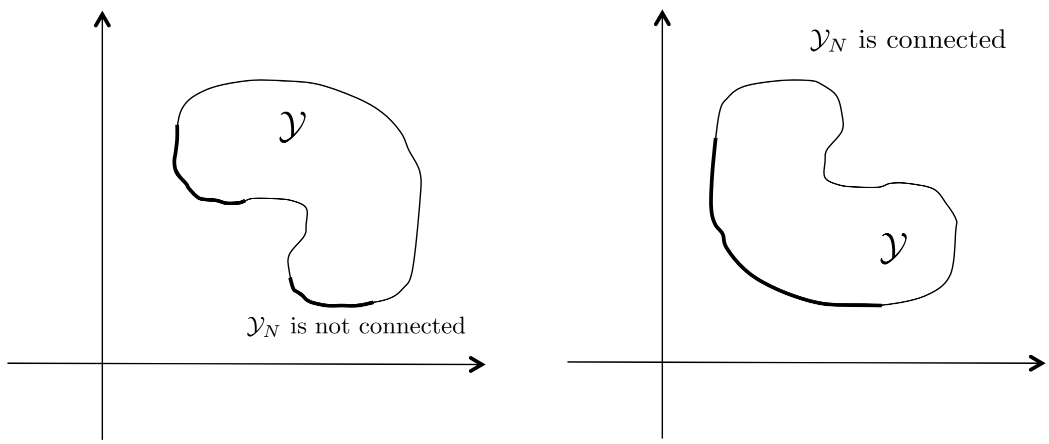

Figure 1.17: Example of connectedness and non-connectedness of $\mathcal{Y}_N$

Definition 1.68 *A set* $\mathcal{S} \in \mathbb{R}^p$ *is called* not connected *if it can be written as* $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$*, with* $\mathcal{S}_1, \mathcal{S}_2 \neq \emptyset$*,* $\mathrm{cl}\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_1 \cap \mathrm{cl}\mathcal{S}_2 = \emptyset$*. Equivalently,* $\mathcal{S}$ *is* not connected *if there exist open sets* $\mathcal{O}_1, \mathcal{O}_2$ *such that* $\mathcal{S} \subset \mathcal{O}_1 \cup \mathcal{O}_2, \mathcal{S} \cap \mathcal{O}_1 \neq \emptyset, \mathcal{S} \cap \mathcal{O}_2 \neq \emptyset, \mathcal{S} \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$*. Otherwise,* $\mathcal{S}$ *is* connected.

Proposition 1.69 *If* $\mathcal{Y}$ *is compact and convex then* $\mathcal{S}(\mathcal{Y})$ *is connected.*

Theorem 1.70 (Naccache (1978)) *If* $\mathcal{Y}$ *is closed, convex and* $\mathbb{R}^p_{\geqq}$*-compact then* $\mathcal{Y}_N$ *is connected.*

With Theorem 1.70 we have a criterion for connectedness on the objective space; for the decision space, by assuming that f is convex, we are able to say that also $\mathcal{X}_{wE}$ is connected.

Theorem 1.71 *Let* $\mathcal{X}$ *be a compact, convex set and assume that* $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$*,* $k = 1, \dots, p$ *are convex. Then* $\mathcal{X}_{wE}$ *is connected.*

Generalazing for $\mathcal{X}_E$.

Theorem 1.72 *Let $\mathcal{X} \in \mathbb{R}^n$ be a convex and compact set. Assume that all objective functions f_k are convex.Then $\mathcal{X}_E$ is connected.*

As a consequence of Theorems 1.71 and 1.72, the results on $\mathcal{X}$ derived on $\mathcal{Y}$ are given by the following Corollary.

Corollary 1.73 *If $\mathcal{X}$ is a convex, compact set and $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$, $k = 1, \ldots, p$ are convex functions then $\mathcal{Y}_{wN}$, $\mathcal{Y}_N$ and $\mathcal{Y}_{pE}$ are connected.*

Chapter 2

Scalarization Techniques

Generally, multiobjective optimization problems are solved by scalarization and scalarization means the replacement of a vector optimization problem with a scalar optimization problem, that is to say it means converting the problem into a single or a family of single objective optimization problems with real-valued objective function, called *scalarizing function*, depending, if possible, on some parameters. The crucial point is that the optimal solutions of a multicriteria optimization problem can be characterized as solutions of certain single objective optimization problems.

Sawaragi *et al.* (1985) define three requirement that scalarizing functions should respect:

1. it can cover any Pareto optimal solution,
2. every solution is Pareto optimal,
3. its solution is satisfying if the aspiration levels used are feasible (in the case in which the scalarizing functions is based on aspiration levels).

However, there is no scalarizing function that satisfy all three requirements.

There is a wide range of methods for solving a multiobjective problem. It is not possible to create a hierarchy among them since, depending on the features of the problem, a specific method would be better than the others. In addition to this, there are the preferences of the decision-maker that may lead to a choice based not on the deep knowledge of solution methods, but simply on the fact that he/she likes more a method rather than another one.

Through these kind of methods, we are able to generate many solutions for the multicriteria optimization problem, but the main focus is to generate the entire frontier of efficient solutions, so that the decision-maker has a wide range of values among which he can choose. There are many scalarization techniques that can be used for solving a multiobjective problem and in this work we simply concentrate on the weighted sum method, the ε-constraint method, the Serafini-Pascoletti method and the weighted-constraint method presented by Burachik.

In the following, we highlight strengths and weaknesses of each method and we make a brief comparison among them.

2.1 The weighted sum method

The final goal is to define a specific Pareto optima of the problem; however, this aim may require to generate all the frontier (if it is possibile) or a subset of it. To solve the multicriteria optimization problem, i.e. to find its efficient solutions, we can solve single objective problem of the type

$$\min_{x\in\mathcal{X}} \sum_{k=1}^{p} \lambda_k f_k(x), \tag{2.1}$$

where λ denotes the scalar product in $\mathbb{R}^p$. This optimization problem is called *weighted sum scalarization* of the multicriteria optimization problem.

Let $\mathcal{Y} \subset \mathbb{R}^p$. For a fixed $\lambda \in \mathbb{R}^p_{\geq}$ the set of optimal points of $\mathcal{Y}$ with respect to λ is given by

$$\mathcal{S}(\lambda,\mathcal{Y}) := \left\{ \hat{y} \in \mathcal{Y} : \langle \lambda, \hat{y} \rangle = \min_{y\in\mathcal{Y}} \langle \lambda, y \rangle \right\}. \tag{2.2}$$

Due to the definition of nondominated points, we have to consider only nonnegative weighting $\lambda \in \mathbb{R}^p_{\geq}$ and it is essential to make a distinction among optimal points with nonnegative and positive weights. For this purpose, define

$$\mathcal{S}(\mathcal{Y}) := \bigcup_{\lambda\in\mathbb{R}^p_{>}} \mathcal{S}(\lambda,\mathcal{Y}) = \bigcup_{\{\lambda>0:\sum_{k=1}^{p}\lambda_k=1\}} \mathcal{S}(\lambda,\mathcal{Y}) \tag{2.3}$$

$$\text{and } \mathcal{S}_0(\mathcal{Y}) := \bigcup_{\lambda\in\mathbb{R}^p_{\geq}} \mathcal{S}(\lambda,\mathcal{Y}) = \bigcup_{\{\lambda\geq 0:\sum_{k=1}^{p}\lambda_k=1\}} \mathcal{S}(\lambda,\mathcal{Y}). \tag{2.4}$$

As a simple notation, we have

$$\Lambda := \left\{ \lambda \in \mathbb{R}^p_{\geqq} : \sum_{k=1}^{p} \lambda_k = 1 \right\}$$

$$\Lambda^0 := ri\Lambda \left\{ \lambda \in \mathbb{R}^p_{>} : \sum_{k=1}^{p} \lambda_k = 1 \right\}.$$

We exclude the case in which $\lambda = 0$. Finally, following from the definition

$$\mathcal{S}(\mathcal{Y}) \subset \mathcal{S}_0(\mathcal{Y}). \tag{2.5}$$

For many of the following results, we need some convexity assumption; the problem is that convexity assumption on $\mathcal{Y}$ would be too restrictive. Having in mind that we are looking for nondominated points that are located in the "south-west" part of $\mathcal{Y}$, we define $\mathbb{R}^p_{\geqq}$-convex.

Definition 2.1 *A set $\mathcal{Y} \in \mathbb{R}^p_{\geqq}$ is called $\mathbb{R}^p_{\geqq}$-convex, if $\mathcal{Y} + \mathbb{R}^p_{\geqq}$ is convex.*

Every convex set $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex. A fundamental result from convex sets is that nonintersecting convex sets can be separated by a hyperplane.

Theorem 2.2 *Let $\mathcal{Y}_1, \mathcal{Y}_2 \subset \mathbb{R}^p$ be nonempty convex sets. There exists some $y^* \in \mathbb{R}^p$ such that*

$$\inf_{y \in \mathcal{Y}_1} \langle y, y^* \rangle \geq \sup_{y \in \mathcal{Y}_2} \langle y, y^* \rangle \tag{2.6}$$

$$\text{and} \sup_{y \in \mathcal{Y}_1} \langle y, y^* \rangle > \inf_{y \in \mathcal{Y}_2} \langle y, y^* \rangle \tag{2.7}$$

if and only if $\mathrm{ri}(\mathcal{Y}_1) \cap \mathrm{ri}(\mathcal{Y}_2) \neq \emptyset$. In this case $\mathcal{Y}_1, \mathcal{Y}_2$ are properly separated by a hyperplane with normal y^*.

Recall that $\mathrm{ri}(\mathcal{Y}_i)$ is the interior in the space of appropriate dimension $\dim(\mathcal{Y}_i) \leq p$.

Theorem 2.3 *Let $\mathcal{Y} \subset \mathbb{R}^p$ be a nonempty, closed, convex set and let $y^0 \in \mathbb{R}^p \setminus \mathcal{Y}$. Then there exists a $y^* \in \mathbb{R}^p \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that*

$$\langle y^*, y^0 \rangle < \alpha < \langle y^*, y \rangle$$

for all $y \in \mathcal{Y}$.

2.1.1 Efficiency and weak efficiency

Optimal solutions of the weighted sum problem with positive (nonnegative) weights are always (weakly) efficient and under convexity assumptions all (weakly) efficient solutions are optimal solutions of scalarized problems with positive (nonnegative) weights.

Theorem 2.4 *For any set $\mathcal{Y} \subset \mathbb{R}^p$ we have $\mathcal{S}(\mathcal{Y}) \subset S_0(\mathcal{Y}) \subset \mathcal{Y}_{wN}$.*

For $\mathbb{R}^p_{\geqq}$-convex sets it is also true the converse.

Theorem 2.5 *If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex, then $\mathcal{S}(\mathcal{Y}) \subset S_0(\mathcal{Y}) = \mathcal{Y}_{wN}$.*

Theorem 2.6 allows us to relate $\mathcal{S}(\mathcal{Y})$ and $S_0(\mathcal{Y})$ to $\mathcal{Y}_N$.

Theorem 2.6 *Let $\mathcal{Y} \subset \mathbb{R}^p$. Then $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N$.*

Corollary 2.7 *If $\mathcal{Y}$ is an $\mathbb{R}^p_{\geqq}$-convex set, then $\mathcal{Y}_N \subset S_0(\mathcal{Y}) = \mathcal{Y}_{wN}$.*

An extension of Theorem 2.6 is given by the following proposition.

Proposition 2.8 *If $\hat{y}$ is the unique element of $S(\lambda, \mathcal{Y})$ for some $\lambda \in \mathbb{R}^p_{\geq}$ then $\hat{y} \in \mathcal{Y}_N$.*

Now turn all those results in terms of decision space, that is to say referring to (weakly) efficient solutions of multicriteria optimization problems.

Proposition 2.9 *Suppose that $\hat{x}$ is an optimal solution of the weighted sum optimization problem with $\lambda \in \mathbb{R}^p_{\geq}$. Then the following statements hold.*

1. *If $\lambda \in \mathbb{R}^p_{\geq}$ then $\hat{x} \in \mathcal{X}_{wE}$.*
2. *If $\lambda \in \mathbb{R}^p_{>}$ then $\hat{x} \in \mathcal{X}_E$.*
3. *If If $\lambda \in \mathbb{R}^p_{\geq}$ and $\hat{x}$ is a unique solution of the problem, then $\hat{x} \in \mathcal{X}_{sE}$.*

Proposition 2.10 *Let $\mathcal{X}$ be a convex set, and let f_k be convex functions, $k = 1, \dots, p$. If $\hat{x} \in \mathcal{X}_E$ there is some $\lambda \in \mathbb{R}^p_{\geq}$ such that $\hat{x}$ is an optimal solution of the weighted sum optimization problem.*

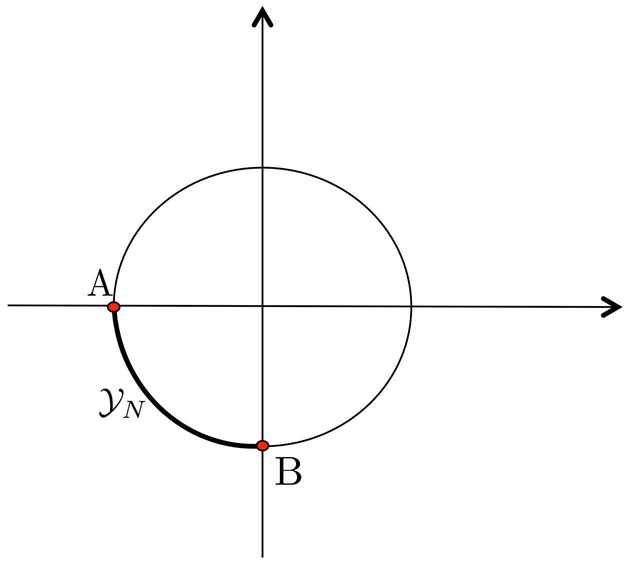

Figure 2.1: Case in which $\mathcal{Y}_N = \bigcup_{\lambda>0} \min \sum_{i=1}^{p} \lambda_i f(x)$ is not true

Now the first question that arises is the following: are we able to say that $\mathcal{Y}_N = \bigcup_{\lambda>0} \min \sum_{i=1}^{p} \lambda_i f(x)$?
The answer is negative. Consider the circle with center in the origin and unitary ray, and let $f(x) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$: as it is shown in Figure 2.1, the black bold line indicates the set $\mathcal{Y}_N$ and it is immediately clear that A and B are no possible solution of the weighted sum problem for $\lambda > 0$, i.e. $\{(0, -1); (-1, 0) \notin \bigcup_{\lambda>0} \min \sum_{i=1}^{p} \lambda_i f(x)\ \}$.

The second question is directly connected to the first one: is it true that $\mathcal{Y}_N = \bigcup_{\lambda\geq 0} \min \sum_{i=1}^{p} \lambda_i f(x)$?
Unfortunately, we cannot answer in a general way because if we consider again Figure 2.1, in that case the answer to the second question is positive, i.e. $\mathcal{Y}_N = \bigcup_{\lambda\geq 0} \min \sum_{i=1}^{p} \lambda_i f(x)$, but now turn to Figure 2.2: in this case, all the points on AB are solutions of the weighted sum problem, but only A is an efficient solution belonging to $\mathcal{Y}_N$, while all the others are weakly efficient solution, i.e. they belong to $\mathcal{Y}_{wN}$.

2.1.2 Proper efficiency

In the previous section, we show that optimal solution of the weighted sum problem are (weakly) efficient solutions with positive (nonnegative) weights

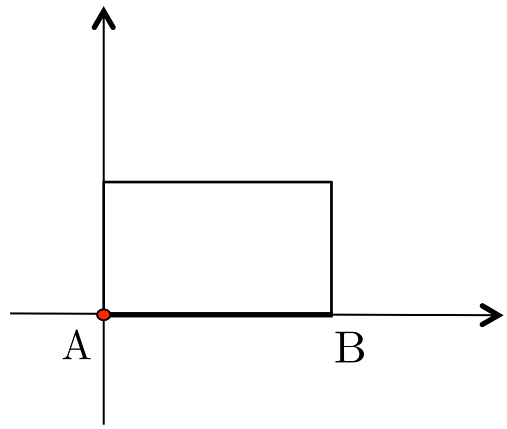

Figure 2.2: Case in which $\mathcal{Y}_N = \bigcup_{\lambda \geq 0} \min \sum_{i=1}^{p} \lambda_i f(x)$ is not true

and that also the viceversa holds true under convexity assumptions. Since we have spoken about also proper efficient and nondominated points, we need to analyze the relationship between solutions of the weighted sum problem and proper nondominated points.

Denote with $\mathcal{Y}_{pN}$ the set of proper efficient points in the Geoffrion's sense and $\mathcal{X}_{pE}$ the set of proper efficient solutions of the multiobjective problem in the Geoffrion's sense.

Theorem 2.11 (Geoffrion (1968)) *Let $\lambda_k > 0, k = 1, \ldots, p$ with $\sum_{k=1}^{p} \lambda_k = 1$ be positive weights. If $\hat{x}$ is an optimal solution of the weighted sum problem then $\hat{x}$ is a proper efficient solution of the multiobjective problem.*

In other words, Theorem 2.11 tells us that an optimal solution of the weighted sum problem is a proper efficient solution of the multiobjective optimization problem if $\lambda > 0$.

Corollary 2.12 *Let $\mathcal{Y} \subset \mathbb{R}^p$. Then $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_{pN}$.*

Theorem 2.13 *If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex then $\mathcal{Y}_{pN} \subset \mathcal{S}(\mathcal{Y})$.*

Theorem 2.14 (Geoffrion (1968)) *Let $\mathcal{X} \in \mathbb{R}^n$ be convex and assume $f_k : \mathcal{X} \longrightarrow \mathbb{R}$ are convex for $k = 1, \ldots, p$. Then $\hat{x} \in \mathcal{X}$ is properly efficient if and only if $\hat{x}$ is an optimal solution of the weighted sum problem, with strictly positive weights $\lambda_k, k = 1, \ldots, p$.*

Thanks to those results about proper nondominance and efficiency, we can state the following relationships:

$$\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_{pN} \subset \mathcal{Y}_N \text{ and } \mathcal{S}_0(\mathcal{Y}) \subset \mathcal{Y}_{wN}$$

holds for general sets, while for $\mathbb{R}^p_{\geqq}$-convex sets

$$\mathcal{S}(\mathcal{Y}) = \mathcal{Y}_{pN} \subset \mathcal{Y}_N \subset \mathcal{Y}_{wN} = \mathcal{S}(\mathcal{Y}).$$

The gap between $\mathcal{Y}_{wN}$ and $\mathcal{Y}_N$ may be quite large also in the case of convex sets, while this cannot be true for the gap between $\mathcal{Y}_{pN}$ and $\mathcal{Y}_N$.

Theorem 2.15 (Hartley (1978)) *If $\mathcal{Y} \neq \emptyset$ is $\mathbb{R}^p_{\geqq}$-closed and $\mathbb{R}^p_{\geqq}$-convex, the following inclusions hold*

$$\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N \subset \text{ cl}\mathcal{S}(\mathcal{Y}) = \text{ cl}\mathcal{Y}_{pN}.$$

Even if Theorem 2.15 tells us that $\mathcal{Y}_N \subset \text{ cl}\mathcal{Y}_{pN}$, the inclusion cl $\mathcal{Y}_{pN} \subset \mathcal{Y}_N$ is not necessarily always true.

2.1.3 Optimality condition for weak efficiency

At this point, let us introduce some necessary and sufficient conditions for weak efficiency of solutions of a multicriteria optimization problem.

Recall the Karush-Kuhn-Tucker necessary and sufficient condition of optimality for single objective problem.

Theorem 2.16 *Let $f, g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously differentiable functions and consider the single objective optimization problem*

$$\min\{f(x) : g_j(x) \leqq 0, j = 1, \dots, m\}. \tag{2.8}$$

Denote $\mathcal{X} := \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = \{1, \dots, m\}\}$.

- *If a qualification constraint condition holds and if $\hat{x} \in \mathcal{X}$ is a (locally) optimal solution of* (2.8) *there is some $\hat{\mu} \in \mathbb{R}^m_{\geqq}$ such that*

$$\nabla f(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j \nabla g_j(\hat{x}) = 0, \tag{2.9}$$

$$\sum_{j=1}^{m} \hat{\mu} g_j(\hat{x}) = 0. \tag{2.10}$$

- *If f, g_j are convex and there are $\hat{x} \in \mathcal{X}$ and $\hat{\mu} \in \mathbb{R}^m_{\geqq}$ such that* (2.9) *and* (2.10) *hold then $\hat{x}$ is a locally, thus globally, optimal solution of* (2.8).

Conditions for weak efficiency is given by Theorem 2.17.

Theorem 2.17 *Suppose that the KT constraint qualification (Definition 1.65 is satisfied at $\hat{x} \in \mathcal{X}$. If $\hat{x}$ is weakly efficient there exist $\hat{\lambda} \in \mathbb{R}^p_{\geq}$ and $\hat{\mu} \in \mathbb{R}^m_{\geqq}$ such that*

$$\sum_{k=1}^{p} \hat{\lambda}_k \nabla f_k(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j \nabla g_j(\hat{x}) = 0 \tag{2.11}$$

$$\sum_{j=1}^{m} \hat{\mu}_j g_j(\hat{x}) = 0 \tag{2.12}$$

$$\hat{\lambda} \geq 0 \tag{2.13}$$

$$\hat{\mu} \geqq 0 \tag{2.14}$$

For convex functions, we have a sufficient condition for weakly efficient solutions.

Corollary 2.18 *Under the assumption of Theorem 2.17 and the additional assumption that all functions f_k and g_j are convex (2.11) - (2.14) with $\hat{\lambda} \geq 0$ and $\hat{\mu} \geqq 0$ in Theorem 2.17 are sufficient for $\hat{x}$ to be weakly efficient.*

2.1.4 Optimality condition for proper efficiency

For what concerns proper efficient solutions, consider first the definition of Kuhn and Tucker based on the following inequalities

$$\langle \nabla f_k(\hat{x}, d) \rangle \leq 0 \;\; \forall\, k = 1, \ldots, p \tag{2.15}$$

$$\langle \nabla f_i(\hat{x}, d) \rangle < 0 \;\; \text{for some } i \in \{1, \ldots, p\} \tag{2.16}$$

$$\langle \nabla g_j(\hat{x}, d) \rangle \leq 0 \;\; \forall\, j \in \mathcal{J}(\hat{x}) = \{j = 1, \ldots, m : g_j(\hat{x}) = 0\}. \tag{2.17}$$

Theorem 2.19 *If $\hat{x}$ is properly efficient in KT sense there exist $\hat{\lambda} \in \mathbb{R}^p$ and $\hat{\mu} \in \mathbb{R}^m$ such that*

$$\sum_{k=1}^{p} \hat{\lambda}_k \nabla f_k(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j \nabla g_j(\hat{x}) = 0 \tag{2.18}$$

$$\sum_{j=1}^{m} \hat{\mu}_j g_j(\hat{x}) = 0 \tag{2.19}$$

$$\hat{\lambda} \geq 0 \tag{2.20}$$

$$\hat{\mu} \geqq 0 \tag{2.21}$$

Theorem 2.19 gives us necessary conditions for Kuhn-Tucker proper efficiency. Knowing that Geoffrion's proper efficiency implies Kunh and Tucker's proper efficiency under the constraint qualification (Theorem 1.66), we get the following corollary.

Corollary 2.20 *If $\hat{x}$ is properly efficient in Geoffrion's sense and the KT constraint qualification is satisfied at $\hat{x}$, then (2.18) - (2.21).*

To obtain the relationship among different proper efficiency definitions we use the Karush-Kuhn-Tucker sufficient optimality conditions seen in Theorem 2.16 and apply them to the weighted sum problem, getting the following theorem.

Theorem 2.21 *Assume that $f_k, g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex, continuously differentiable functions. Suppose that there are $\hat{x} \in \mathcal{X}, \hat{\lambda} \in \mathbb{R}^p$ and $\hat{\mu} \in \mathbb{R}^p$ satisfying (2.18) - (2.21). Then $\hat{x}$ is properly efficient in the sense of Geoffrion.*

Two corollaries follow.

Corollary 2.22 *Let $f_k, g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ be convex, continuously differentiable functions and suppose that $\hat{x}$ is properly efficient in Kuhn and Tucker's sense. Then $\hat{x}$ is properly efficient in Geoffrion's sense.*

Corollary 2.22 tells us that proper efficiency in Kuhn-Tucker's sense implies proper efficiency in Geoffrion's sense for convex problems.

Corollary 2.23 *If, in addition to assumptions of Theorem 2.21, the KT constraint qualification is satisfied at $\hat{x}$, (2.18) - (2.21) are sufficient for $\hat{x}$ to be properly efficient in KT's sense.*

Instead, Corollary 2.23 gives us sufficient conditions for proper efficiency in Kuhn and Tucker's sense, and it follows from Theorems 2.21 and 1.66.

Until now we have shown necessary and sufficient conditions for weakly and strictly efficient solutions, but actually they also include the conditions for efficient solutions. In fact, since $\mathcal{X}_E \subset \mathcal{X}_{wE}$, the necessary condition of Theorem 2.17 holds for efficient solutions, too and as $\mathcal{S}(\mathcal{Y}) = \mathcal{X}_{pE} \subset \mathcal{X}_E$ for convex problems, the sufficient condition of Theorem 2.21 are sufficient for $\hat{x}$ to be efficient, too.

2.2 The ε-constraint method

The ε-constraint method is the most known method to solve multi-criteria problems, especially in engineering design, and it consists in minimizing only one original objective and let the others being the constraints. The scalarized problem can be stated in the following way

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X}} & f_j(x) \\ \text{subject to} & f_k(x) \leq \varepsilon_k, k = 1, \ldots, p, k \neq j, \end{array} \tag{2.22}$$

where $\varepsilon \in \mathbb{R}^p$.

Proposition 2.24 *Let $\hat{x}$ be an optimal solution of* (2.22) *for some j. Then $\hat{x}$ is weakly efficient.*

This Proposition 2.52 becomes stronger if we require the optimal solution to be unique.

Proposition 2.25 *Let $\hat{x}$ be a unique optimal solution of* (2.22) *for some j. Then $\hat{x} \in \mathcal{X}_{sE}$ (and so $\hat{x} \in \mathcal{X}_E$).*

Theorem 2.26 *The feasible solution $\hat{x} \in \mathcal{X}$ is efficient if and only if there exists an $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution of* (2.22) *for all $j = 1, \ldots, p$.*

Theorem 2.26 tells us that under appropriate choices of ε all efficient solutions can be found. However, these ε_j values are equal to the actual values of the efficient solution one would like to find. We have a confirmation of efficiency and not a discovery of efficient solutions.

Denote by

$$\mathcal{E}_j := \{\varepsilon \in \mathbb{R}^p : \{x \in \mathcal{X} : f_k(x) \leq \varepsilon_k, k \neq j\} \neq \emptyset\}$$

the sets of right hand sides for which (2.22) is feasible and by

$$\mathcal{X}_j(\varepsilon) := \{x \in \mathcal{X} : \text{ is an optimal solution of } (2.22)\}$$

for $\varepsilon \in \mathcal{E}_j$ the set of optimal solution of (2.22). From Theorem 2.26 and Proposition 2.52 we have that for each $\varepsilon \in \cap_{j=1}^{p} \mathcal{E}_j$

$$\bigcap_{j=1}^{p} \mathcal{X}_j(\varepsilon) \subset \mathcal{X}_E \subset \mathcal{X}_j(\varepsilon) \subset \mathcal{X}_{wE} \tag{2.23}$$

for all $j = 1, \ldots, p$.

Theorem 2.27 (Chankong and Haimes (1983)) 1. *Suppose $\hat{x}$ is an optimal solution of $\min_{x \in \mathcal{X}} \sum_{k=1}^{p} \lambda_k f_k(x)$. If $\lambda > 0$ there exists $\hat{\varepsilon}$ such that $\hat{x}$ is an optimal solution of* (2.22), *too.*

2. *Suppose $\mathcal{X}$ is a convex set and $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions. If $\hat{x}$ is an optimal solution of* (2.22) *for some j, there exists $\hat{\lambda} \in \mathbb{R}^p_{\geq}$ such that $\hat{x}$ is optimal for $\min_{x \in \mathcal{X}} \sum_{k=1}^{p} \hat{\lambda}_k f_k(x)$.*

This theorem gives us the link between the weighted sum method and the ε-constraint method.

2.3 The hybrid method

The hybrid method can be obtained by combining the weighted sum method with the ε-constraint method. The problem becomes a scalarized one to be solved with a weighted sum objective and constraints on all objectives. Let x^0 be an arbitrary feasible point for a multicriteria optimization problem and consider the following problem:

$$
\begin{aligned}
&\min \sum_{k=1}^{p} \lambda_k f_k(x) \\
&\text{subject to } f_k(x) \leq f_k(x^0) \;\; k = 1, \ldots, p \\
&x \in \mathcal{X}
\end{aligned} \tag{2.24}
$$

where $\lambda \in \mathbb{R}^p_{\geq}$.

Theorem 2.28 (Guaddat et *al.* (1985)) *Let $\lambda \in \mathbb{R}^p_{>}$. A feasible solution $x^0 \in \mathcal{X}$ is an optimal solution of the problem (2.24) if and only if $x^0 \in \mathcal{X}_E$.*

2.4 The Pascoletti-Serafini method

As we sketch previously, all methods presented until now - weighted sum, ε-constraint, hybrid - have some weaknesses. For instance, consider the most used method - the weighted sum method; it is not appropriate for non-convex problems. Look at the following example.

Example 2.29 *Let $\mathcal{X} = \{x \in \mathbb{R}^2_{\geqq} : x_1^2 + x_2^2 \geq 1\}$ and $f(x) = x$. The set of efficient solutions is given by $\mathcal{X}_E = \{x \in \mathcal{X} : x_1^2 + x_2^2 = 1\}$; with the exception of the points $\hat{x}^1 = (1, 0)$ and $\hat{x}^2 = (1, 0)$, none of the points $x \in \mathcal{X}_E$ can be obtained as an optimal solution of (2.1), by a suitable choice of $\lambda \geq 0$.*

In order to understand better those drawbacks, let us now concentrate on the Pascoletti-Serafini method from which we can study all other methods as specification of this more general scalarization technique.

Pascoletti and Serafini started from the question about effectiveness of the transformation of a vector problem into a scalar one, since the latter provides more effective ways of computation of optima than vector problems do. In particular, they stated that two are the requirements that a scalarization should respect to be effective:

- the possibility of finding all vector optima among the scalar optima obtained by varying the parameters,
- the well-behavior dependence of optima on the parameters.

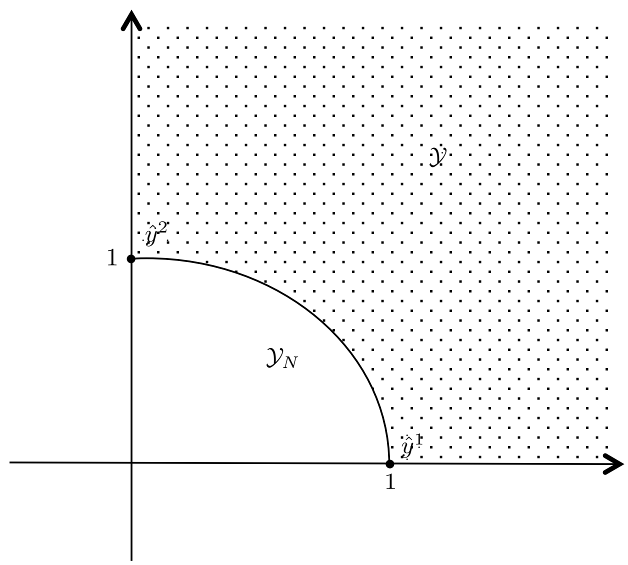

Figure 2.3: Failure of weighted sum method in presence of non-convexities

For a decision maker, it is important to have in hand all possible alternatives of the problem, but the quality of information depends on the approximation obtained by varying the parameters. As a consequence, a good choice of parameters is needed. Pascoletti and Serafini provided a particular scalarization technique effective in full generality and it is possibile to see other scalarization techniques as a modification of this method, i.e. the Pascoletti-Serafini method.

What we look for is a solution for the following multiobjective or vector optimization problem

$$\begin{aligned} &\min && f(x) = (f_1(x), \ldots, f_n(x)) \\ &\text{subject to} && x \in \mathcal{X} \subset \mathbb{R}^n. \end{aligned}$$

In the following, consider a set $\mathcal{X}$, a real linear space $\mathcal{Y}$, a function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and a closed pointed convex cone $\mathcal{K} = \mathbb{R}^p_{\geqq}$[1].

[1]In the original paper by Pascoletti and Serafini, this method is presented in a more general framework; the ordering is made with respect to any cone, $\mathcal{K}$, while we used as reference cone the Pareto cone, $\mathcal{K} = \mathbb{R}^p_{\geqq}$. This choice is made for the sake of simplicity and uniformity with previous sections.

The set of all efficient points is $\mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$, while the set of all weakly efficient points is given by $\mathcal{X}_{wE}(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$. As a consequence, we have $\mathcal{Y}_N(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) = \{f(x) \in \mathbb{R}^p : x \in \mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq})\}$, while $\mathcal{Y}_{wN}(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) = \{f(x) \in \mathbb{R}^p : x \in \mathcal{X}_{wE}(f(\mathcal{X}), \mathbb{R}^p_{\geqq})\}$.

The scalarized problem can be stated as follow:

$$\begin{aligned} \min \quad & \xi \\ \text{subject to} \quad & p + \xi q - f(x) \in \mathbb{R}^p_{\geqq}, \\ & \xi \in \mathbb{R}, \\ & x \in \mathcal{X}. \end{aligned}$$

As we said, this scalar problem is solved by moving the cone $-\mathbb{R}^p_{\geqq}$ along the line $p + \xi q, \xi \in \mathbb{R}$, starting from the point p in direction $-q$ until the set $(p + \xi q - \mathbb{R}^p_{\geqq}) \cap f(\mathcal{X}) = \emptyset$. The smallest value of $\hat{\xi}$ for which $(p + \hat{\xi} q - \mathbb{R}^p_{\geqq}) \cap f(\mathcal{X}) \neq \emptyset$ is the minimal value of the scalar problem.

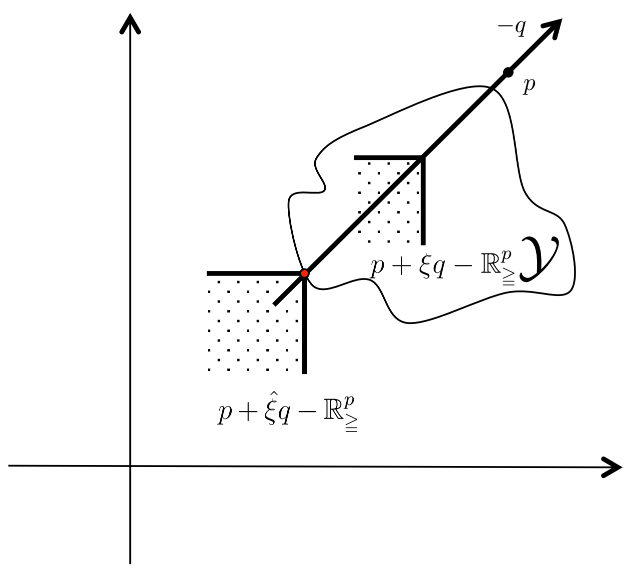

Figure 2.4: Way of solving the scalar problem in Pascoletti-Serafini

The problem $P(p, q)$ has all features a scalarization method for determining minimal solutions of a multi-objective optimization problem should have: if the couple $(\hat{\xi}, \hat{x})$ is a minimal solution of the scalar problem, then the point

$\hat{x}$ is, at least, a weakly efficient solution of the vector optimization problem. Thus, by varying the two parameters (p, q), it would be possibile to obtain all efficient points of the original problem as solution of $P(p, q)$.

The vector and scalar problems are equivalent if

- for any efficient solution x_0, there exists some pair (p, q) such that the scalarized problem has a solution (ξ, x) with $x = x_0$,
- for any pair (p, q), the problem $P(p, q)$ has solutions (ξ, x), with x as an efficient solution.

The following theorems show how the vector and the scalar problems are equivalent. We consider the scalar optimization problem $P(p, q)$ and the multiobjective optimization problem.

Theorem 2.30 *Let $\hat{x} \in \mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$, then $(0, \hat{x})$ is a minimal solution of $P(p, q)$ with the parameter $p = f(\hat{x})$ and $q \in \mathbb{R}^p_{\geqq} \backslash \{0\}$. Let $\hat{x} \in \mathcal{X}_{wE}(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$, then $(0, \hat{x})$ is a minimal solution of $P(p, q)$ with the parameter $p = f(\hat{x})$ and $q \in int(\mathbb{R}^p_{\geqq})$.*

Theorem 2.31 *For any solution $(\hat{\xi}, \hat{x})$ of $P(p, q)$, $\hat{x} \in \mathcal{X}_{wE}(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$ and $p + \hat{\xi} q - f(\hat{x}) \in \partial \mathbb{R}^p_{\geqq}$ with $\partial \mathbb{R}^p_{\geqq}$ the boundary of the cone $\mathbb{R}^p_{\geqq}$.*

Theorem 2.32 *Let $\hat{x}$ be a locally weakly efficient solution of the multiobjective problem, then $(0, \hat{x})$ is a local minimal solution of $P(p, q)$ for parameter $p = f(\hat{x})$ and for arbitrary $q \in int(\mathbb{R}^p_{\geqq})$. Let $\hat{x}$ be a locally efficient solution of the multiobjective problem, then $(0, \hat{x})$ is a local minimal solution of $P(p, q)$ for parameter $p = f(\hat{x})$ and for arbitrary $q \in \mathbb{R}^p_{\geqq} \setminus \{0\}$.*

Theorem 2.33 *Let $(\hat{\xi}, \hat{x})$ be a local minimal solution of $P(p, q)$, then $\hat{x}$ is a locally weakly efficient solution of the multi-objective optimization problem and $p + \hat{\xi} q - f(\hat{x}) \in \partial \mathbb{R}^p_{\geqq}$.*

Theorem 2.31 does not always ensure that we obtain a $\hat{x}$ weakly efficient point for arbitrary (p, q); hence, it is possible that the problem $P(p, q)$ has no minimal solution, as shown in Figure 2.5

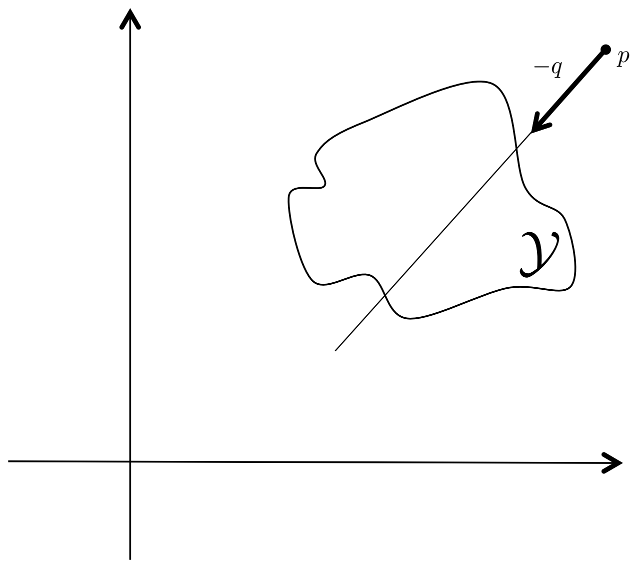

Figure 2.5: Case of no minimal solution of scalar problem

Theorem 2.34 *Let the set $f(\mathcal{X})$ be $\mathbb{R}^p_{\geqq}$-closed and $\mathbb{R}^p_{\geqq}$-convex. Assume $\mathcal{Y}_N(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) \neq \emptyset$. Then*

$$\left\{(p, q) \in \mathbb{R}^p \times int(\mathbb{R}^p_{\geqq}) : \sum(p, q) \neq \emptyset\right\} = \mathbb{R}^p \times int(\mathbb{R}^p_{\geqq}),$$

i.e. for any choice of parameters (p, q), $P(p, q)$ has feasible points.
Besides all parameters (p, q), there exists a minimal solution of the scalar problem $P(p, q)$.

Corollary 2.35 *Let the set $f(\mathcal{X})$ be $\mathbb{R}^p_{\geqq}$-closed and $\mathbb{R}^p_{\geqq}$-convex. If there is a parameter $(p, q) \in \mathbb{R}^p \times int(\mathbb{R}^p_{\geqq})$ such that $P(p, q)$ has no minimal solution then $\mathcal{Y}_N(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) = \emptyset$.*

The meaning of Theorem 2.34 is that if for some parameter $p \in \mathbb{R}^p$ and $q \in int(\mathbb{R}^p_{\geqq})$ $P(p, q)$ has no minimal solution, then under some further assumptions we can said that the related multiobjective problem has no efficient solution at all. Instead, if the scalar problem for arbitrary parameters satisfies Corollary 2.35, then we either obtain a weakly efficient solution or we get information about the fact that there are no efficient points of the problem.

According to Theorem 2.30, it is not always true that a point $\hat{x}$ of $P(p,q)$ is efficient; it may be weakly efficient and, thus, there exists another point x' which dominates $\hat{x}$. However, x' corresponds to some other solution of $P(p,q)$ with same values of p and q. Hence, the scalar problem is a perfect representation of vector problem.

Theorem 2.36 *If the point $(\hat{\xi},\hat{x})$ is a minimal solution of $P(p,q)$ with $\hat{k} = p + \hat{\xi}q - f(\hat{x})$ and if there is a point $y' = f(x') \in f(\mathcal{X})$ dominating the point $f(\hat{x})$ w. r. t. the cone $\mathbb{R}^p_{\geqq}$, then the point $(\hat{\xi},x')$ is also a minimal solution $P(p,q)$ and there exists a $k' \in \partial\mathbb{R}^p_{\geqq}, k \neq 0$, with $p + \hat{\xi}q - f(x') = \hat{k} + k'$.*

Corollary 2.37 *If the point $(\hat{\xi},\hat{x})$ is an image-unique minimal solution of the scalar problem $P(p,q)$ w. r. t. f, i. e. there is no other minimal solution (ξ',x') with $f(x') = f(\hat{x})$, then $\hat{x}$ is an efficient solution of the multiobjective optimization problem.*

A further characterization of solutions of the scalarized problem is needed in order to test efficiency; in fact, it is possible that points that are weakly efficient, are not efficient. However, this possibility is atypical, while it is more probable that solutions of $P(p,q)$ give efficient solutions. A sufficient assumption to get efficient solution is to require the strict optimality from solutions of $P(p,q)$.

Theorem 2.38 *The point $\hat{x}$ is efficient for the multiobjective optimization problemif and only if*

1. *there is some $\hat{\xi} \in \mathbb{R}$ so that $(\hat{\xi},\hat{x})$ is a minimal solution of $P(p,q)$ for some parameters $p \in \mathbb{R}$ and $q \in int(\mathbb{R}^p_{\geqq})$,*

2. *for $k := p + \hat{\xi}q - f(\hat{x})$ it is:*

$$((p + \hat{\xi}q) - \partial\mathbb{R}^p_{\geqq}) \cap (f(\hat{x}) - \partial\mathbb{R}^p_{\geqq}) \cap f(\mathcal{X}) = \{f(\hat{x})\}.$$

Theorem 2.38 means that if $(\hat{\xi},\hat{x})$ is a minimal solution of the scalar problem $P(p,q)$, with $q \in int(\mathbb{R}^p_{\geqq})$, then $\hat{x}$ is a weakly efficient solution and for checking if $\hat{x}$ is also efficient it is sufficient to test the points $((p + \hat{\xi}q) - \partial\mathbb{R}^p_{\geqq}) \cap (f(\hat{x}) - \partial\mathbb{R}^p_{\geqq})$ of the set $f(\mathcal{X})$.

The aim of this method is to get all efficient solutions of the multiobjective optimization problem throughout the scalar problem with different parameters. On the basis of Theorem 2.30, we can get all efficient solutions given $q \in \mathbb{R}^p_{\geqq} \setminus \{0\}$ and varying p. However, it is possible to further restrict the set of all values of p and still obtaining all efficient solutions. Before considering briefly the bicriteria case, consider the following theorem.

Theorem 2.39 *Let $\hat{x}$ be an efficient solution for the multiobjective optimization problem and define a hyperplane*

$$H = \{y \in \mathbb{R}^p : b^T y = \beta\}$$

with $b \in \mathbb{R}^p \setminus \{0\}$ and $\beta \in \mathbb{R}$. Let $q \in \mathbb{R}^n_{\geqq}$ with $b^T q \neq 0$ be arbitrarily given. Then there is a parameter $p \in H$ and some $\hat{\xi} \in \mathbb{R}$ so that $(\hat{\xi}, \hat{x})$ is a minimal solution of $P(p, q)$. This holds for instance for

$$\hat{\xi} = \frac{b^T f(\hat{x}) - \beta}{b^T q}$$

and

$$p = f(\hat{x}) - \hat{\xi} q.$$

Now consider the bicriteria case; assume $\hat{x}^i, i = 1, 2$ are the minimal solutions of $\min\limits_{x \in \mathcal{X}} f_i(x), i = 1, 2$, and choose a parameter $q \in \mathbb{R}^p_{\geqq}$ and a hyperplane $H = \{y \in \mathbb{R}^p : b^T y = \beta\}$ with $b \in \mathbb{R}^p$, $\beta \in \mathbb{R}$ and $b^T q \neq 0$. Then set

$$\hat{\xi}^i = \frac{b^T f(\hat{x}^i) - \beta}{b^T q}$$

and

$$\hat{p}^i = f(\hat{x}^i) - \hat{\xi}^i q.$$

Now we have just to consider parameters $p \in H^p$ with

$$H^p := \{y \in H : y = \lambda \hat{p}^1 + (1 - \lambda)\hat{p}^2, \lambda \in [0, 1]\}$$

the line segment on the hyperplane H between points $\hat{p}^1$ and $\hat{p}^2$.

Theorem 2.40 *We consider the multiobjective optimization problem with $p = 2$ and $\mathcal{K} = \mathbb{R}^2_+$. For any $\hat{x} \in \mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^2_+)$ there exists a parameter $p \in H^p \subset H$ and a scalar $\hat{\xi} \in \mathbb{R}$ with $(\hat{\xi}, \hat{x})$ a minimal solution of $P(p, q)$.*

In the general case, things are a little bit tricky. Even if the ordering cone $\mathbb{R}^p_{\geqq}$ is finitely generated by three vectors, we cannot generalized the results in the bicriteria case for determining the set H^p. However, a weaker restriction of the set H for the parameter p is obtained by projecting the image set $f(\mathcal{X})$ in direction q onto the set H, determining in this way the set

$$\tilde{H} := \{y \in H : y + \xi q = f(x), \xi \in \mathbb{R}, x \in \mathcal{X}\} \subset H. \tag{2.25}$$

Since $\tilde{H} \subset H$ has generally an irregular boundary, making itself unsuitable for systematic procedure, we embed the set $\tilde{H}$ in a $(p-1)$-dimensional cuboid $H^0 \subset \mathbb{R}^p$, chosen as minimal as possible. To get H^0, first determine $p-1$ vectors $v^1, \ldots, v^{p-1}$, which span the hyperplane H with $\tilde{H} \subset H$ and which are orthogonal and normalized by one, that is to say

$$v^{iT} v^j = \begin{cases} 0, \text{for } i \neq j, & i, j \in \{1, \ldots, p-1\}, \\ 1, \text{for } i = j, & i, j \in \{1, \ldots, p-1\}. \end{cases} \tag{2.26}$$

These vectors form an orthonormal basis of the smallest subspace of $\mathbb{R}^p$ containing H. We have that $v^i \in H, i = 1, \ldots, p-1$, i.e.

$$b^T v^i = \beta, i = 1, \ldots, p-1. \tag{2.27}$$

This leads to the representation

$$H = \{y \in \mathbb{R}^p : y = \sum_{i=1}^{p-1} s_i v^i, s \in \mathbb{R}^{p-1}\} \tag{2.28}$$

of the hyperplane H. Then, we can find the cuboid by solving the $2(p-1)$ scalar-values optimization problems

$$\begin{aligned} \min \quad & s_j \\ \text{subject to} \quad & \sum_{i=1}^{p-1} s_i v^i + \xi q = f(x), \\ & \xi \in \mathbb{R}, \\ & x \in \mathcal{X}, \\ & s \in \mathbb{R}^{p-1}. \end{aligned} \tag{2.29}$$

for $j \in \{1, \ldots, p-1\}$ with minimal solution $(\xi^{\min,j}, x^{\min,j}, s^{\min,j})$ and minimal value $s_j^{\min,j}$ and

$$
\begin{aligned}
\min \quad & -s_j \\
\text{subject to} \quad & \sum_{i=1}^{p-1} s_i v^i + \xi q = f(x), \\
& \xi \in \mathbb{R}, \\
& x \in \mathcal{X}, \\
& s \in \mathbb{R}^{p-1}.
\end{aligned}
\tag{2.30}
$$

for $j \in \{1, \ldots, p-1\}$ with minimal solution $(\xi^{\max,j}, x^{\max,j}, s^{\max,j})$ and minimal value $s_j^{\max,j}$. We can now define H^0 as follow

$$
H^0 := \{y \in \mathbb{R}^p : y = \sum_{i=1}^{p-1} s_i v^i, s_i \in [s_i^{\min,i}, s_i^{\max,i}], i = 1, \ldots, p-1\}
$$

with $\tilde{H} \subset H^0$, leading us to the suitable restriction of H for which we were looking for.

Lemma 2.41 *Let $\hat{x}$ be an efficient solution of the multiobjective optimization problem. Let $q \in \mathbb{R}^p_{\geqq} \setminus \{0\}$. Then there is a parameter $\hat{p} \in H^0$ and some $\hat{\xi} \in \mathbb{R}$ so that $(\hat{\xi}, \hat{x})$ is a minimal solution of $P(\hat{p}, q)$.*

In this way, we restrict the parameter set in the case of more than two objectives and arbitrary ordering cones $\mathbb{R}^p_{\geqq}$.

2.5 Modified Pascoletti-Serafini scalarization

A modified version of the Pascoletti-Serafini problem is given by $\bar{P}(p, q)$

$$
\begin{aligned}
\min \quad & \xi \\
\text{subject to} \quad & p + \xi q - f(x) = 0, \\
& \xi \in \mathbb{R}, \\
& x \in \mathcal{X},
\end{aligned}
$$

where $p + \xi q - f(x) \in \mathbb{R}^p_{\geqq}$ is replaced with $p + \xi q - f(x) = 0$.

Theorem 2.42 *Let a hyperplane* $H = \{y \in \mathbb{R}^p : b^T y = \beta\}$ *with* $b \in \mathbb{R}^p \setminus \{0\}$ *and* $\beta \in \mathbb{R}$ *be given. Let* $(\hat{\xi}, \hat{x})$ *be a minimal solution of the scalar optimization problem* $P(p, q)$ *for the parameters* $p \in \mathbb{R}^p$ *and* $q \in \mathbb{R}^p$ *with* $b^T q \neq 0$. *Hence there is a* $\hat{k} \in \mathbb{R}^p_{\geqq}$ *with*

$$p + \hat{\xi} q - f(\hat{x}) = \hat{k}.$$

Then there is a parameter $p' \in H$ *and some* $\xi' \in \mathbb{R}$ *so that* $(\xi', \hat{x})$ *is a minimal solution of* $P(p', q)$ *with*

$$p' + \xi' q - f(\hat{x}) = 0.$$

Theorem 2.42 gives us the link between $P(p, q)$ and $\bar{P}(p, q)$, i.e. for a minimal solution $(\hat{\xi}, \hat{x})$ of the scalar optimization problem $P(p, q)$ with

$$p + \hat{\xi} q - f(\hat{x}) = \hat{k}, \hat{k} \neq 0,$$

there exists a parameter $p' \in H$ and some $\xi' \in \mathbb{R}$ so that the couple $(\xi', \hat{x})$ is a minimal solution of $P(p', q)$ with

$$p' + \xi' q - f(\hat{x}) = 0$$

and hence $(\xi', \hat{x})$ is also a minimal solution of $\bar{P}(p', q)$.

It is important to point out that minimal solution of the modified scalar problem not necessarily are weakly efficient points of the multiobjective optimization problem. However, thanks to Theorem 2.42, it is still possible to find all efficient points only by varying the parameter p on a hyperplane.

Theorem 2.43 *Let a hyperplane* $H = \{y \in \mathbb{R}^p : b^T y = \beta\}$ *with* $b \in \mathbb{R}^p$, $\beta \in \mathbb{R}$ *be given. Let* $\hat{x} \in \mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$ *and* $q \in \mathbb{R}^p_{\geqq} \setminus \{0\}$ *with* $b^T q \neq 0$. *Then there is a parameter* $p \in H$ *and some* $\hat{\xi} \in \mathbb{R}$ *so that* $(\hat{\xi}, \hat{x})$ *is a minimal solution of the scalar optimization problem* $\bar{P}(p, q)$.

2.6 Relationship between scalarizations

As we have seen, the Pascoletti-Serafini method has many interesting properties and, as we said previously, other scalarization techniques can be seen as a special case or as a modification of this method. We concentrate on the ε-constraint method and on the weighted sum method.

Since Lagrange multipliers will play a role in the following sections, it is necessary to consider them and then extend Theorem 2.42. Consider the following scalar optimization problem

$$\begin{aligned} \min \quad & F(x) \\ \text{subject to} \quad & G(x) \in \mathcal{C}, \\ & H(x) = 0, \\ & x \in S, \end{aligned}$$

where $\mathcal{C} \subset \mathbb{R}^m$ is a closed convex cone, $\hat{S} \subset \mathbb{R}^n$ is an open subset, $S \subset \hat{S}$ is a closed convex set and functions $F : \hat{S} \longrightarrow \mathbb{R}, G : \hat{S} \longrightarrow \mathbb{R}^p, H : \hat{S} \longrightarrow \mathbb{R}^q$ are continuously differentiable. Then the related Langrange function is given by $\mathcal{L} : \mathbb{R}^n \times C^* \times \mathbb{R}^q \longrightarrow \mathbb{R}$,

$$\mathcal{L}(x, \mu, \phi) := F(x) - \mu^T G(x) - \phi^T H(x).$$

If x is feasible and there exists $(\mu, \phi) \in C^* \times \mathbb{R}^q$ with

$$\nabla_x \mathcal{L}(x, \mu, \phi)^T (s - x) \geq 0, \forall s \in S$$

and

$$\mu^T G(x) = 0$$

then μ and ϕ are called Lagrange multipliers to the point x.

Given the following assumption,

Assumption 2.44 *Let $\mathcal{K}$ be a closed pointed convex cone in $\mathbb{R}^p$ and $\mathcal{C}$ a closed convex cone in $\mathbb{R}^m$. Let $\hat{S}$ be a nonempty open subset of $\mathbb{R}^n$ and assume $S \subset \hat{S}$ to be closed and convex. Let the functions $f : \hat{S} \longrightarrow \mathbb{R}^m, g : \hat{S} \longrightarrow \mathbb{R}^p, h : \hat{S} \longrightarrow \mathbb{R}^q$ be continuously differentiable on $\hat{S}$.*

we restate Theorem 2.42 as follow.

Lemma 2.45 *We consider the scalar optimization problem $P(p, q)$ under the Assumption 2.44. Let $(\hat{\xi}, \hat{x})$ be a minimal solution and assume there exist Lagrange multipliers $(\mu, \upsilon, \phi) \in \mathcal{K}^* \times \mathcal{C}^* \times \mathbb{R}^q$ to the point $(\hat{\xi}, \hat{x})$. According to Theorem 2.27 there exists a parameter $p' \in H$ and some $\xi' \in \mathbb{R}$ so that $(\xi', \hat{x})$ is a minimal solution of $P(p', q)$ and $p' + \xi' q = f(\hat{x})$.*

Then (μ, υ, ϕ) are Lagrange multipliers to the point $(\xi', \hat{x})$ for the problem $P(p', q)$, too.

2.6.1 Relationship with ε-constraint method

Recall the ε-constraint problem for an arbitrary $j \in \{1, \ldots, n\}$ and parameters $\varepsilon \in \mathbb{R}$, $k \in \{1, \ldots, p\} \setminus \{k\}$ as follow

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X}} & f_j(x) \\ \text{subject to} & f_k(x) \leq \varepsilon_k, k \in \{1, \ldots, p\} \setminus \{k\}. \end{array}$$

The ε-constraint method can be seen as a special case of the Pascoletti-Serafini one if you consider the ordering cone $\mathcal{K} = \mathbb{R}^p_{\geqq}$; another way of seeing the connection between these two methods is by considering the Lagrange multipliers.

Theorem 2.46 *Let Assumption 2.44 hold and let $\mathcal{K} = \mathbb{R}^p_{\geqq}$, $\mathcal{C} = \mathbb{R}^m_+$, $\hat{S} = S = \mathbb{R}^n$. A point $\hat{x}$ is a minimal solution of the ε-constraint problem with Lagrange multipliers $\hat{\mu}_k \in \mathbb{R}_+$ for $k \in \{1, \ldots, p\} \setminus \{k\}$, $\hat{v} \in \mathbb{R}^m_+$, and $\hat{\phi} \in \mathbb{R}^q$, if and only if $(f_k(\hat{x}, \hat{x}))$ is a minimal solution of $P(p, q)$ with Lagrange multipliers $(\hat{\mu}, \hat{v}, \hat{\phi})$ with $\hat{\mu}_k = 1$, and*

$$p_k = \varepsilon_k, \forall k \in \{1, \ldots, p\} \setminus \{k\}, p_k = 0 \text{ and} q = e_k, \tag{2.31}$$

with e_k the k^{th} unit vector in $\mathbb{R}^p$.

The ε-constraint method is a restriction of the Pascoletti-Serafini scalarization because parameter p is chosen from the hyperplane $H = \{y \in \mathbb{R}^p : y_j = 0\}$, while q is maintained constant and equal to e_k. From Theorem 2.30, any efficient solution $\hat{x}$ of the multiobjective optimization problem can be obtained by solving the ε-constraint problem for the parameters $\varepsilon_k = f_k(\hat{x}), k \in \{1, \ldots, p\} \setminus \{k\}$. The difference with respect to the Pascoletti-Serafini method is that not all efficient solution can be determined by solving the ε-constraint problem because $q \in \partial \mathcal{K} = \partial \mathbb{R}^p_{\geqq}$ in the ε-constraint method. However, from Theorem 2.30, any efficient solution of the ε-constraint problem is, at least, a weakly efficient solution of the multiobjective optimization problem.

Corollary 2.47 *If $\hat{x}$ is a minimal solution of the ε-constraint problem, then $\hat{x} \in \mathcal{X}_{wE}(f(\mathcal{X}), \mathbb{R}^p_{\geqq})$.*

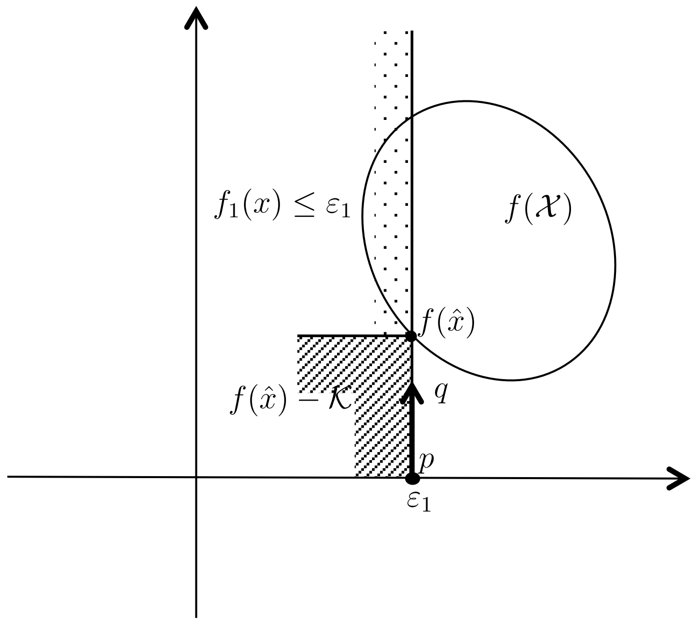

Figure 2.6: Relationship between Pascoletti-Serafini and ε-constraint method

According to Corollary 2.47, for $\mathcal{Y}_N(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) \neq \emptyset$ and for closed and convex $f(\mathcal{X})$, there exists a minimal solution of $P(p, q)$ for any choice of $(p, q) \in \mathbb{R}^p \times int(\mathbb{R}^p_{\geqq})$. The main problem with ε-constraint method is exactly here: it may happen that the ε-constraint problem is solved for a wide range of parameters without getting any solution or by obtaining the only information that $\mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) = \emptyset$. It is possible to restrict the set of parameters to be considered in order to find an arbitrary efficient point by using Theorem 2.40.

Corollary 2.48 *Let $p = 2$, $\mathcal{K} = \mathbb{R}^2_+$, and $\hat{x} \in \mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^2_+)$. Let $\hat{x}^1$ be a minimal solution of $min_{x \in \mathcal{X}} f_1(x)$ and $\hat{x}^2$ be a minimal solution of $min_{x \in \mathcal{X}} f_2(x)$. Then there exists a parameter $\varepsilon \in \{y \in \mathbb{R} : f_1(\hat{x}^1) \leq y \leq f_1(\hat{x}^2)\}$ such that $\hat{x}$ is a minimal solution of the second ε-constraint problem.*

The same for the first ε-constraint problem.

2.6.2 Relationship with weighted sum method

We turn now to analyze the most used scalarization method; as we said before, the problem to be solved is

$$\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} w_i f_i(x) = w^T f(x) \\ \text{subject to} & x \in \mathcal{X}, \end{array}$$

with $w \geq 0$. The connection with the Pascoletti-Serafini method is given in the following theorem.

Theorem 2.49 *A point $\hat{x}$ is a minimal solution of the weighted sum problem for the parameter $w \geq 0$ if and only if there is some $\hat{\xi}$ so that $(\hat{\xi}, \hat{x})$ is a minimal solution of $P(p, q)$ with $p \in \mathbb{R}^p$ arbitrarily chosen and $q \in int(\mathbb{R}^p_{\geqq})$.*

By varying the weights w, we get a variation in the ordering cone $\mathbb{R}^p_{\geqq}$; the latter cone is a closed convex polyhedral cone, but it is not pointed. For this reason, Theorem 2.30 cannot be used for the weighted sum method and it is also the reason why the multiobjective problem cannot be solved, i.e. it is not possibile to find all efficient points, by means of this method in general and in the case of non-convex set. However, it is possible to find all efficient points of the problem in the convex set case. From Theorem 2.30, it is possibile to get the following result.

Corollary 2.50 *Let $\hat{x}$ be a minimal point of the weighted sum problem with parameter $w \geq 0$, then $\hat{x}$ is weakly efficient for multiobjective optimization problem.*

The weighted sum method has the same pitfall with respect to the Pascoletti-Serafini method as the ε-constraint method, that is not for any choice of the parameters $w \geq 0$ there exists a minimal solution, even in the case in which $\mathcal{X}_E(f(\mathcal{X}), \mathbb{R}^p_{\geqq}) \neq \emptyset$.

2.7 The weighted-constraint method

As already said, the main goal of multiobjective optimization problem is to find the best "compromise" among different criteria, i.e. to find all Pareto optimum solutions that composes the Pareto frontier. In general, this frontier is an infinite set and it can be created by solving the scalar problem over the whole set of the parameters of the scalarization. However, the main issues are: 1) it may be necessary to solve the scalar problem many times in order to get an approximation of the frontier; 2) this procedure is quite costly; 3) the Pareto frontier and/or the feasible set may be disconnected, things that further complicate the computation.

Burachik, Kaya and Rizvi (2013) proposed a new approach that can be used not only in presence of disconnected Pareto frontier, but also in presence of disconnected feasible set - the so called k^{th}-objective weighted-constraint method. The name comes from the fact that this technique is based on the minimization of the weighted k^{th}-objective function, for each fixed k, given that all other functions are taken as constraints.

First of all, defined the set of non-negative weights

$$\Lambda := \{\lambda \in \mathbb{R}^p_{\geq} : \sum_{i=1}^{p} \lambda_i = 1\},$$

and the set of positive weights

$$\Lambda_0 := \{\lambda \in \mathbb{R}^p_{>} : \sum_{i=1}^{p} \lambda_i = 1\}.$$

Then we are interested in the following problem

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X}} & \lambda_k f_k(x) \\ \text{subject to} & \lambda_i f_i(x) \leq \lambda_k f_k(x), \\ & i = 1, \ldots, p, i \neq k. \end{array}$$

This is the k^{th}-objective weighted-constraint problem; the feasible set (for fixed k and λ) is given by

$$\mathcal{X}^k_\lambda := \{x \in \mathcal{X} : \lambda_i f_i(x) \leq \lambda_k f_k(x), \forall i \neq k\},$$

while the solution set is

$$S^k_\lambda := \{x \in \mathcal{X} : x \text{ solves the problem}\}.$$

For each fixed $\lambda \in \Lambda_0$, we have

$$\mathcal{X} := \bigcup_{k=1}^{p} \mathcal{X}_{\lambda}^{k}.$$

Define also the following set

$$\Lambda(x) := \{\lambda \in \Lambda_0 : x \in S_{\lambda}^{k}, \forall k = 1, \dots, p\},$$

this set may be empty for some $x \in \mathcal{X}$.

Theorem 2.51 *$\hat{x} \in \mathcal{X}$ is a weak efficient solution of the multiobjective optimization problem, if and only if there exists some $\lambda \in \Lambda_0$ such that $\hat{x}$ solves the k^{th}-objective weighted-constraint problem for all $k \in \{1, \dots, p\}$.*

It is important to point out that the "only if" part holds for efficient points that, by definition, are weakly efficient; but if a point solves the k^{th}-objective weighted-constraint problem for all k, then it is not necessarily an efficient point, unless all objective functions are strictly convex.

Proposition 2.52 *Suppose that $\mathcal{X}$ is convex and $f_i = 1, \dots, p$, are strictly convex functions defined on $\mathcal{X}$. If $\hat{x} \in \mathcal{X}$ is a weak efficient solution of the multiobjective optimization problem, then $\hat{x}$ is an efficient solution of the latter problem.*

Proposition 2.53 *If $\Lambda(\hat{x}) \neq \emptyset$, then $\Lambda(\hat{x})$ is a singleton. Consequently, $x \in \mathcal{X}_{wE}$ if and only if $\hat{x} \in \bigcap_{k=1}^{p} S_{\hat{\lambda}}^{k}$ where $\hat{\lambda}_i = (1/f_i(\hat{x}))/(\sum_{j=1}^{p} 1/f_j(\hat{x})$.*

Proposition 2.53 tells us that if there exists $\lambda \in \Lambda_0$ such that a point solves the k^{th}-objective weighted-constraint problem for all k, then this λ is unique.

Theorem 2.51 implies that, for all $\lambda' \in \Lambda_0$, we have

$$\bigcap_{k=1}^{p} S_{\lambda'}^{k} \subseteq \mathcal{X}_{wE} \subset \bigcup_{\lambda \in \Lambda_0} \left[\bigcap_{k=1}^{p} S_{\lambda}^{k} \right].$$

In the case in which, for some $\lambda' \in \Lambda_0$, $\bigcap_{k=1}^{p} S_{\lambda'}^{k} \neq \emptyset$, then the left hand side of the previous statement gives a way of computing weak efficient points. Instead, if $\bigcap_{k=1}^{p} S_{\lambda'}^{k} = \emptyset$, Proposition 2.54 and Corollary 2.56 provide tools for generating new weak efficient points.

Proposition 2.54 *Assume* $\exists \lambda \in \Lambda_0$ *such that* $S_\lambda^j \neq, \forall j = 1, \ldots, p$. *Suppose that, for some* $k \in \{1, \ldots, p\}, \exists \hat{x}_k \in S_\lambda^k$ *such that* $\forall r \neq k, \exists \hat{x}_r \in S_\lambda^r$, *which satisfies*

$$f_r(\hat{x}_r) \geq f_r(\hat{x}_k).$$

Then $\hat{x}_k \in \mathcal{X}_{wE}$.

Corollary 2.55 *Let* $\lambda \in \Lambda_0$. *Suppose that* $(\hat{x}_1, \ldots, \hat{x}_l) \in S_\lambda^1 \times \ldots \times S_\lambda^l$ *and that* $\forall r, k \in \{1, \ldots, p\}$,

$$f_r(\hat{x}_r) \geq f_r(\hat{x}_k).$$

Then $\hat{x}_k \in \mathcal{X}_{wE}$, *for every* $k = 1, \ldots, p$.

2.7.1 Relationship with the weighted sum method

The relationship between the k^{th}-objective weighted-constraint approach and the weighted sum method is given by the fact that the Pareto frontier created by means of the latter can be reproduced by the former, but the converse is not true, i.e. the k^{th}-objective weighted-constraint method provides us with some Pareto points that are not achievable with weighted sum problem.

Corollary 2.56 *If there exists* $\lambda \in \Lambda$ *such that* $\hat{x}$ *solves the weighted sum problem, then there exists* $\alpha \in \Lambda_0$ *such that* $\hat{x}$ *solves the* k^{th}*-objective weighted-constraint problem for all* k.

The following example by Burachik *et al.* shows that the converse of Corollary 2.56 is not true.

Example 2.57 *Consider the problem*

$$\min_{x \in \mathcal{X}} (x_1, x_2)$$

with

$$\mathcal{X} = \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 - 1 \leq 0, 0.3 - x_1^2 - x_2^2 \leq 0\}.$$

Consider as weights $\lambda_1 = 0.48$ *and* $\lambda_2 = 0.52$. *For* $k = 1, 2$, *the weighted sum problems can be written as* $\min_{x \in \mathcal{X}} \lambda_1 x_1$ *s.t.* $\lambda_2 x_2 - \lambda_1 x_1 \leq 0$, *and* $\min_{x \in \mathcal{X}} \lambda_2 x_2$ *s.t.* $\lambda_1 x_1 - \lambda_2 x_2 \leq 0$. *Here,* $\hat{x} = (0.4, 0.37)$ *is located on the concave part of the front and this means that* $\hat{x}$ *cannot be obtained as a solution of the weighted sum problem for* $\lambda \in \Lambda$.

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