From a car-following model with reaction time to a macroscopic convection-diffusion traffic flow model

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Microscopic model

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Pursuit law

The microscopic model comes from the generalized Newell (1961) one

\[ \dot{x}_i(t + \tau) = W(\Delta x_i(t)), \quad (i, t) \in \mathbb{Z} \times (0, +\infty) \]

with \( \tau \) the reaction time (if positive), \( \Delta x_i(t) = x_{i+1}(t) - x_i(t) \) the spacing and \( W(\cdot) \) the equilibrium (or optimal) speed function

Writing \( \tau \) in rhs and applying a linear approximation for small \( \tau \) we get

\[ \dot{x}_i(t) = W(\Delta x_i(t) - \tau[\dot{x}_{i+1}(t) - \dot{x}_i(t)]) \]

The model is then obtained by substituting the speeds \( \dot{x} \) in rhs by the optimal speed \( W(\Delta x) \):

\[ \dot{x}_i(t) = W\left(\Delta x_i(t) - \tau[W(\Delta x_{i+1}(t)) - W(\Delta x_i(t))]\right) \quad (1) \]
Pursuit law

Microscopic model:

\[ \dot{x}_i(t) = W\left( \Delta x_i(t) - \tau [W(\Delta x_{i+1}(t)) - W(\Delta x_i(t))] \right), \quad (i, t) \in \mathbb{Z} \times (0, +\infty) \]

- Speed model with two predecessors in interaction
- Collision-free (by construction):
  \[ \Delta x_i \geq \ell \quad \forall i, t \]
  if \( W(\cdot) \geq 0 \) and \( W(s) = 0 \) for all \( s < \ell \)
- Same linear stability condition for homogeneous solutions as original Newell model (or OVM by Bando (1998)):
  \[ |\tau| W' < 1/2 \]
  (Homogenization for small \( \tau \), stop-and-go for high \( \tau \))
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Rewriting the microscopic model

Same methodology as in [Aw et al. (2002)] considering the density at vehicle $i$ and time $t$, $\rho_i(t)$, as the inverse of the spacing

$$\rho_i(t) := \frac{1}{\Delta x_i(t)}. \tag{2}$$

The microscopic model becomes

$$\dot{x}_i = W \left( \frac{1}{\rho_i(t)} - \tau \left[ W \left( \frac{1}{\rho_{i+1}(t)} \right) - W \left( \frac{1}{\rho_i(t)} \right) \right] \right) =: \tilde{V}(\rho_{i+1}, \rho_i), \tag{3}$$

Then,

$$\partial_t \frac{1}{\rho_i(t)} = \partial_t \Delta x_i(t) = \left( \tilde{V}(\rho_{i+2}, \rho_{i+1}) - \tilde{V}(\rho_{i+1}, \rho_i) \right). \tag{4}$$

(semi–discretized version of hyperbolic partial differential equation in the space of vehicle indices)
Derivation

$y \in \mathbb{R}$ such that $y_i = i\Delta y$ with $\Delta y$ proportional to $\ell$

Piecewise constant density $\rho(t, y)$ such that $\frac{1}{\rho_i(t)} = \frac{1}{\Delta y} \int_{y_{i-\Delta y/2}}^{y_{i+\Delta y/2}} \frac{1}{\rho(t, z)} dz$.

Rescaling of time $t \to t\Delta y$ and reaction time $\tau \to \tau\Delta y$ to obtain

$$\partial_t \frac{1}{\rho_i(t)} - \frac{1}{\Delta y} \left( V\left( \frac{\rho_{i+1}}{1 - \rho_{i+1}\tau \frac{Z_{i+1}}{\Delta y}} \right) - V\left( \frac{\rho_i}{1 - \tau \rho_i \frac{Z_i}{\Delta y}} \right) \right) = 0,$$

(5)

where $Z_i := V(\rho_{i+1}) - V(\rho_i)$ and $V(x) = W\left( \frac{1}{x} \right)$ for $x > 0$ (non-increasing).

(5) is an upwind discretization in the rescaled time and in the limit $\Delta y$ (i.e. $i \to \infty$ and $\ell \to 0$) of the macroscopic equation

$$\partial_t \frac{1}{\rho} - \partial_y V\left( \frac{\rho}{1 - \tau \rho \partial_y V(\rho)} \right) = 0.$$ (6)
Macroscopic model in Eulerian coordinates

Coordinate transformation \((t, y) \rightarrow (t, x)\) where \(y = \int_{-\infty}^{x} \rho(t, x) dx\).

In the Eulerian coordinates \((t, x)\), the macroscopic model Eq. (6) reads

\[
\partial_t \rho + \partial_x \left( \rho V \left( \frac{\rho}{1 - \tau \partial_x V(\rho)} \right) \right) = 0. \tag{7}
\]

Extension of the LWR model with FD \(\rho \mapsto V(\rho/(1 - \tau \partial_x V(\rho)))\).

Taylor expansion in terms of \(\tau\) for equation (7) yields

\[
\partial_t \rho + \partial_x (\rho V(\rho)) = -\tau \partial_x \left( (\rho V'(\rho))^2 \partial_x \rho \right) \tag{8}
\]

Note that for constant densities \(\rho\) (or \(\tau = 0\)) the additional term vanishes and we recover the classical LWR.
Fundamental diagram

Figure: Illustration for the FD obtained in the macroscopic model with constant inhomogeneity $\tau \partial_x V(\rho) = \pm 0.3$. Triangular FD $V : \rho \mapsto \max\{\min\{2, 1/\rho - 1\}\}$. Bounded FD as in [Colombo (2003), Goatin (2006), Colombo et al. (2010)]
Macroscopic model

(Eulerian coordinates)

\[ \partial_t \rho + \partial_x (\rho V(\rho)) = -\tau \partial_x \left((\rho V'(\rho))^2 \partial_x \rho \right) \]  

\( (9) \)

- \( \tau < 0 \): Convection-diffusion equation (LWR with diffusion – cf. Burger equations) with variable diffusion coefficient (cf. Fick equations)
- \( \tau = 0 \): LWR model
- \( \tau > 0 \): Negative diffusion??
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Discrete macroscopic models

\[ \rho_i(t + \delta t) = \rho_i(t) + \frac{\delta t}{\delta x} (f_{i-1}(t) - f_i(t)) \]  \hspace{1em} (10)

- Godunov/Euler scheme

\[ f_i = G(\rho_i, \rho_{i+1}) + \frac{\tau}{\delta x} (\rho_i V' (\rho_i))^2 (\rho_{i+1} - \rho_i) \]  \hspace{1em} (D1)

- Simple Godunov scheme

\[ f_i = G \left( \frac{\rho_i}{1 - \frac{\tau}{\delta x} (V(\rho_{i+1}) - V(\rho_i))}, \frac{\rho_{i+1}}{1 - \frac{\tau}{\delta x} (V(\rho_{i+2}) - V(\rho_{i+1}))} \right) \]  \hspace{1em} (D2)

- Double Godunov scheme

\[ f_i = G(\rho_i, \rho_{i+1}) + \frac{\tau}{\delta x} \rho_i V'(\rho_i) [G(\rho_{i+1}, \rho_{i+2}) - G(\rho_i, \rho_{i+1})] \]  \hspace{1em} (D3)

with \( G(x, y) = \min\{\Delta(x), \Sigma(y)\} \) the Godunov scheme, \( \Delta(\cdot) \) and \( \Sigma(\cdot) \) are the demand and supply functions.
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- Numerical schemes
- Stability analysis
- Simulation results
Stability analysis for the continuous model

Stability analysis of the homogeneous solution where $\rho(x, t) = \rho_E$ for all $x, t$ ($\rho_E$ being the mean density)

Perturbation to homogeneous solution

$$\varepsilon(x, t) = \rho(x, t) - \rho_E$$

Linearisation:

$$\varepsilon_t = F(\rho_E + \varepsilon, \varepsilon_x, \varepsilon_{xx}) \approx \alpha \varepsilon + \beta \varepsilon_x + \gamma \varepsilon_{xx}$$

with $F(\rho, \rho_x, \rho_{xx}) = -\partial_x(\rho V(\rho)) - \tau \partial_x((\rho V'(\rho))^2 \partial_x), \alpha = \frac{\partial F}{\partial \rho}(\rho_E, \rho_E, \rho_E) = 0, \\beta = \frac{\partial F}{\partial \rho_x} = -V(\rho_E) - \rho_E V'(\rho_E)$, and $\gamma = \frac{\partial F}{\partial \rho_{xx}} = -\tau(\rho_E V'(\rho_E))^2$

Linear system: $\varepsilon = z e^{\lambda t - il}, \varepsilon_t = \lambda \varepsilon, \varepsilon_x = -il \varepsilon, \varepsilon_{xx} = -l^2 \varepsilon$

$$\lambda = \tau(l \rho_E V'(\rho_E))^2 + il(V(\rho_E) + \rho_E V'(\rho_E)) \quad \text{— Stable if} \, \Re(\lambda) < 0 \, \forall l > 0$$

$\rightarrow$ Homogeneous solution linearly stable if $\tau < 0$ (positive diffusion)
Stability analysis for the discrete schemes

Perturbation to homogeneous solution

\[ \varepsilon_i(t) = \rho_i(t) - \rho_E \]

Linearisation of the perturbed system:

\[ \varepsilon_i(t + \delta t) = \rho_i(t + \delta t) - \rho_E = F(\rho_i(t), \rho_{i+1}(t), \rho_{i+2}(t), \rho_{i-1}(t)) - \rho_E \]

\[ \approx \alpha \varepsilon_i(t) + \beta \varepsilon_{i+1}(t) + \gamma \varepsilon_{i+2}(t) + \xi \varepsilon_{i-1}(t) \]

with

\[ \alpha = \frac{\partial F}{\partial \rho_i}(\rho_E, \rho_E, \rho_E, \rho_E) \]
\[ \beta = \frac{\partial F}{\partial \rho_{i+1}}(\rho_E, \rho_E, \rho_E, \rho_E) \]
\[ \gamma = \frac{\partial F}{\partial \rho_{i+2}}(\rho_E, \rho_E, \rho_E, \rho_E) \]
\[ \xi = \frac{\partial F}{\partial \rho_{i-1}}(\rho_E, \rho_E, \rho_E, \rho_E) \]
General conditions for stability of the discrete schemes

$N$ cells with periodic boundary conditions — The linear dynamics are

$$\vec{e}'(t + \delta t) = M \vec{e}'(t) \text{ with } \vec{e}' = (e_1, \ldots, e_N) \text{ and } M \text{ a sparse matrix with } (\xi, \alpha, \beta, \gamma) \text{ on the diagonal}$$

If $M = PDP^{-1}$ with $D$ a diagonal matrix, then $\vec{e}'(t) = PD^{t/\delta t}P^{-1} \vec{e}'(0) \to \vec{0}$ if all the coefficients of $D$ are less than one excepted one.

$M$ is circulant therefore the eigenvectors of $M$ are $z(l^0, l^1, \ldots, l^{m-1})$ with

$$l = \exp \left( i \frac{2\pi l}{N} \right) \text{ and } z \in \mathbb{Z}, \text{ and the eigenvectors are } \lambda_l = \alpha + \beta l + \gamma l^2 + \xi l^{-1}$$

The system is linearly stable if $|\lambda_l| < 1$ for all $l = 1, \ldots, N - 1$ with

$$\lambda_l^2 = \alpha^2 + \beta^2 + \gamma^2 + \xi^2 - 2\alpha\gamma - 2\beta\xi + 2f(c_l), \quad c_l = \cos(2\pi l/N) \text{ and } f(x) = (\alpha\beta + \alpha\xi + \beta\xi - 3\gamma\xi)x + 2(\alpha\gamma + \beta\xi)x^2 + 4\gamma\xi x^3$$
Stability analysis for the scheme (D1)

Affine speed function \( V(\rho) = \frac{1}{T} (1/\rho - \ell) \), with \( T > 0 \) the time gap between the vehicles and \( \ell > 0 \) their size — Godunov scheme is \( G(x, y) = \frac{1}{T} (1 - y \ell) \)

The scheme (D1) is

\[
F_1(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t}{\delta x T} \left( \ell (\rho_{i+1} - \rho_i) + \frac{\tau}{\delta x T} \left( \frac{\rho_i - \rho_{i-1}}{\rho_{i-1}^2} - \frac{\rho_{i+1} - \rho_i}{\rho_i^2} \right) \right)
\]

and (with \( A = \frac{\delta t \ell}{\delta x T} \) and \( B = \frac{\delta t \tau}{(\delta x T \rho E)^2} \))

\[
\alpha = 1 - A + 2B \quad \beta = A - B \\
\gamma = 0 \quad \xi = -B
\]
Signs of the partial derivative $\alpha \beta \gamma$

\[ A = \frac{\delta t \ell}{\delta x T} \quad \text{and} \quad B = \frac{\delta t \tau}{(T \delta x \rho_E)^2} \]

- $\alpha = 1 - A + 2B$ is positive if
  \[ \delta t < \frac{\delta x T}{\ell} \left( 1 - \frac{2\tau}{T \ell \delta x \rho_E^2} \right)^{-1} \]  
  \[ (P_\alpha) \]

  if $\tau < \frac{1}{2} T \ell \delta x \rho_E^2$, or for all $\delta t \geq 0$ if $\tau \geq \frac{1}{2} T \ell \delta x \rho_E^2$

  Moreover $1 - \alpha \geq 0$ iff $\tau < \frac{1}{2} T \ell \delta x \rho_E^2$

- $\beta = A - B$ is positive iff
  \[ \tau < T \ell \delta x \rho_E^2 \]  
  \[ (P_\beta) \]

- The sign of $\xi = -B$ is the one of $-\tau$
Case $\tau < 0$

If $\tau < 0$ and $(P_\alpha)$ holds, $f(x) = \alpha(1 - \alpha)x + 2\beta\xi x^2$ is convex and is maximal on $[-1, 1]$ for $x = -1$ or $x = 1$

Therefore the model is stable if $f(-1) < f(1)$; this is simply

$$-\alpha(1 - \alpha) < \alpha(1 - \alpha)$$

that is always true since $\alpha > 0$ if $(P_\alpha)$ holds and $1 - \alpha > 0$ on $\tau < 0$

Therefore the system is stable for all $\tau < 0$
Case $\tau > 0$

Several cases have to be distinguished:

- $0 < \tau < \frac{1}{2} T \ell \delta x \rho_E^2 \quad \alpha, 1 - \alpha, \beta > 0, \xi < 0$
  
  $f(x) = \alpha(1 - \alpha)x + 2\beta\xi x^2$ is concave and maximal for $x_0 = -\frac{\alpha(1-\alpha)}{4\beta\xi} > 0$;
  
  The model is stable if $x_0 > 1$, this is $\delta t < \frac{\delta x T}{\ell} \left(1 - \frac{2\tau}{T \ell \delta x \rho_E^2}\right)$

- $\frac{1}{2} T \ell \delta x \rho_E^2 < \tau < T \ell \delta x \rho_E^2 \quad \alpha, \beta > 0, 1 - \alpha, \xi < 0$
  
  We have $f(-1) > f(1)$ therefore the model is unstable; $f$ maximal for $x_0 < -1$ (shortest wave) if $\delta t < \frac{\delta x T}{2\ell} \left(\frac{2\tau}{T \ell \delta x \rho_E^2} - 1\right) \left(\frac{2\tau}{T \ell \delta x \rho_E^2}\right)^{-2}$

- $\tau > T \ell \delta x \rho_E^2 \quad \alpha > 0, 1 - \alpha, \beta, \xi < 0$
  
  Unstable for all $\delta t$ with shortest wavelength since $f$ convex and $f(-1) > f(1)$
Scheme (D1) — Summary

\[ \delta t < \frac{T \delta x}{\ell} - \frac{2 \tau}{T \delta x \rho_E^2} \]

- **Stable**
  \[ \delta t < \frac{T \delta x}{\ell} - \frac{2 \tau}{T \delta x \rho_E^2} \]

- **Unstable**
  \[ \delta t < \frac{\tau - \frac{1}{2} T \ell \rho_E^2}{\left(\frac{2 \tau}{T \delta x \rho_E^2}\right)^2} \]

\[ \ell \delta x \rho_E^2 \]

\[ 0 \quad \frac{1}{2} T \ell \delta x \rho_E^2 \quad T \ell \delta x \rho_E^2 \]

\[ \tau \]

→ The same conditions as the continuous macroscopic model for:

\[ \delta x \rightarrow 0 \quad \text{(and } \delta t \rightarrow 0 \text{ such that } \delta t / \delta x \rightarrow 0) \]
Stability analysis for the schemes (D2) and (D3)

Affine speed function $V(\rho) = \frac{1}{T}(1/\rho - \ell)$, with $T > 0$ the time gap between the vehicles and $\ell > 0$ their size — Godunov scheme is $G(x, y) = \frac{1}{T}(1 - y\ell)$

The schemes (D2) and (D3) are

$$F_2(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t \ell}{\delta x T} \left( \frac{\rho_{i+1}}{1 - \tau \rho_{i+2} - \tau \rho_{i+2}} - 1 - \frac{\rho_i}{\tau \rho_{i+1} - 1 - \rho_i} \right)$$

$$F_3(\rho_i, \rho_{i+1}, \rho_{i+2}, \rho_{i-1}) = \rho_i + \frac{\delta t \ell}{\delta x T} \left( \rho_{i+1} - \rho_i + \frac{\tau}{\delta x T} \left( \frac{\rho_{i+1} - \rho_{i+2}}{\rho_i} - \frac{\rho_i - \rho_{i+1}}{\rho_{i-1}} \right) \right)$$

By construction, both gives (with $A = \frac{\delta t \ell}{\delta x T}$ and $B = \frac{\tau}{T \delta x \rho E}$)

$$\alpha = 1 - A(1 + B) \quad \beta = A(1 + 2B)$$

$$\gamma = -AB \quad \xi = 0$$


**Signs of the partial derivative $\alpha \beta \gamma$**

$$A = \frac{\delta t \ell}{\delta x T} \quad \text{and} \quad B = \frac{\tau}{\delta x T \rho_E}$$

- $\alpha = 1 - A(1 + B)$ is positive if

  $$\delta t < \frac{\delta x T}{\ell} \left( 1 + \frac{\tau}{T \delta x \rho_E} \right)^{-1}$$

  if $\tau > -T \delta x \rho_E$, or for all $\delta t \geq 0$ if $\tau \leq -T \delta x \rho_E$

- $\beta = A(1 + 2B)$ is positive iff

  $$\tau > -\frac{1}{2} T \delta x \rho_E$$

  Moreover $1 - \beta > 0$ if $(P_\alpha)$ holds

- The sign of $\gamma = -AB$ is the one of $-\tau$
Case $\tau < 0$

If $\tau < 0$ and $(P_\alpha)$ holds, $f(x) = \beta(1 - \beta)x + 2\alpha \gamma x^2$ is convex is maximal on $[-1, 1]$ for $x = -1$ or $x = 1$.

Therefore the model is stable if $f(-1) < f(1)$; this is

$$\tau > -\frac{1}{2} T \delta x \rho E$$

and

$$\delta t < \frac{\delta x T}{\ell} \left( 1 + \frac{2\tau}{T \delta x \rho E} \right)^{-1}$$

The condition on $\delta t$ is weaker than $(P_\alpha)$.

If $\tau \leq -\frac{1}{2} T \delta x \rho E$ then the system is unstable at the shortest wave-length frequency $\cos^{-1}(-1)$.

A sufficiently condition for that the finite system produces the frequency $\cos^{-1}(-1)$ is simply $N \geq 2$. 
Case $\tau > 0$

$(P_\alpha)$ holds then $f(x) = \beta(1 - \beta)x + 2\alpha\gamma x^2$ is concave and is maximum at
arg sup$_{x} f(x) = x_0 = -\frac{\beta(1-\beta)}{4\alpha\gamma} > 0$

We know that $\lambda_0^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta + f(1) = 1 \ (\text{case } l = 0)$

Therefore the model is stable if $x_0 > 1$; this is

$$\tau < \frac{1}{2} T \delta x \rho_E \quad \text{and} \quad \delta t < \frac{\delta x T}{\ell} \left(1 - \frac{2\tau}{T \delta x \rho_E}\right)$$

The condition on $\delta t$ is stronger than $(P_\alpha)$

If $\tau \geq \frac{1}{2} T \delta x \rho_E$ then the system is unstable at the frequency $\cos^{-1}(x_0)$ that is reachable in the finite system if $N > 2\pi/\cos^{-1}(x_0)$

We have $x_0 \to \frac{1}{2} + \frac{T \delta x \rho_E}{4\tau}$ going from 1 to 1/2 according to $\tau$ (long-waves)
Schemes (D2) and (D3) — Summary

<table>
<thead>
<tr>
<th>Unstable</th>
<th>Unstable</th>
<th>Stable</th>
<th>Stable</th>
<th>Unstable</th>
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<tbody>
<tr>
<td>for all ( \delta t &gt; 0 )</td>
<td>Shortest wavelength</td>
<td>( \delta t &lt; \frac{T \delta x}{\ell + \frac{T \tau}{\delta x \rho E}} )</td>
<td>( \delta t &lt; \frac{T \delta x}{\ell + \frac{T \tau}{\delta x \rho E}} - \frac{2\tau}{\ell \rho E} )</td>
<td>Wavelength from ( N/2 ) to ( N/6 )</td>
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\[ -T \delta x \rho_E \quad -\frac{1}{2} T \delta x \rho_E \quad 0 \quad \frac{1}{2} T \delta x \rho_E \]

→ The same conditions as the microscopic model for:

\[ \delta t \to 0 \quad \text{and} \quad \delta x = 1/\rho_E = \text{mean spacing} \]
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Simulation

Models
Microscopic: Euler explicit scheme
Macroscopic: Godunov/Godonov scheme (D3)

Setting
\[ \rho_E = 2, \ \delta x = 1/\rho_E, \ \delta t = 1e-2, \ V : \rho \mapsto \max\{\min\{2, 1/\rho - 1\}\} \]

Environment
Ring (periodic conditions)
Initial condition: jam, random, perturbed
Trajectories

Jam initial configuration

Time

Space
Perturbed initial configuration

Trajectories
Fundamental diagram

Microscopic model

Macroscopic model

V(·)
Bounds

Density

Flow

Time