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RUSSELL'S HYPERSURFACE FROM A GEOMETRIC POINT OF VIEW

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Abstract

The famous Russell hypersurface is a smooth complex affine threefold which is diffeomorphic to a euclidean space but not algebraically isomorphic to the three dimensional affine space. This fact was first established by Makar-Limanov, using algebraic minded techniques. In this article, we give an elementary argument which adds a greater insight to the geometry behind the original proof and which also may be applicable in other situations.

1. Introduction

Russell's hypersurface

$$X := \{(x, y, z, t) \in \mathbb{C}^4 \mid x + x^2 y + z^3 + t^2 = 0\} \hookrightarrow \mathbb{C}^4,$$

is one of the most prominent examples of an exotic variety, i.e. a variety which is diffeomorphic to an affine space ([1], [6, Lemma 5.1]), but not isomorphic to it. The latter is an immediate consequence of Theorem 1 below, which states that there are not sufficiently many actions of the additive group $\mathbb{G}_a$ on X, and the aim of this paper is to give an elementary argument for this theorem. It includes some important elements of the original proof, but gives a greater geometrical insight to the situation.

The study of exotic varieties goes back to a paper of Ramanujam [16], where a nontrivial example of a topologically contractible smooth affine algebraic surface S over $\mathbb{C}$ is constructed. Ramanujam observed that $S \times \mathbb{C}$ is diffeomorphic to $\mathbb{C}^3$, and asked whether this product is also isomorphic to $\mathbb{C}^3$. This was later proven not to be the case, and thus the algebraic structure on $\mathbb{C}^3$ coming from $S \times \mathbb{C}^2$ is exotic [17]. Later on, many other exotic structures on $\mathbb{C}^3$ have been constructed, see e.g. the introduction of [18] for a list. Note also that there are no exotic structures on affine space in dimension ≤ 2 [16].

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The motivation for studying Russell's hypersurface originally came from the linearization conjecture for $\mathbb{C}^3$, which claims that each $\mathbb{G}_m$-action on $\mathbb{C}^3$ is linearizable. In the proof of this result by Koras and Russell, they described a list of smooth affine threefolds diffeomorphic to $\mathbb{C}^3$ which contains all the potential counterexamples to the conjecture, and thus it was reduced to determining whether all of these so called Koras–Russell threefolds are exotic [13]. Kaliman and Makar-Limanov established exoticity for some of them [10], and Russell's hypersurface is the "most simple" among the remaining ones. The difficulty with Russell's hypersurface was that all the usual algebraic and geometric invariants failed to distinguish it from $\mathbb{C}^3$. Makar-Limanov finally established exoticity of Russell's hypersurface (Theorem 1), and later on Kaliman and Makar-Limanov were able to prove exoticity of the remaining Koras–Russell threefolds [8, 9] as well, elaborating on Makar-Limanov's methods. This confirmed the linearization conjecture [7].

From now on, we will focus on Makar-Limanov's result, stated in the following theorem.

Theorem 1 (Makar-Limanov, [14])**.** *The projection* $\mathrm{pr}_1\colon X \to \mathbb{C}$, $(x, y, z, t) \mapsto x$ *is invariant with respect to any* $\mathbb{G}_a$*-action on* X.

Some years after Makar-Limanov proved Theorem 1, Kaliman proved, using non-elementary birational geometry, that morphisms $\mathbb{C}^3 \to \mathbb{C}$ with generic fiber $\mathbb{C}^2$ cannot have any other fibers [5]. Since all the fibers of $\mathrm{pr}_1\colon X \to \mathbb{C}$ are $\mathbb{C}^2$ except the zero fiber $\mathrm{pr}_1^{-1}(0)$, it follows also from Kaliman's result that $X \ncong \mathbb{C}^3$. In 2005, Makar-Limanov gave another proof of the exoticity of Russell's hypersurface [15]; yet another proof was given by Derksen [3], and Crachiola also proved the exoticity in the positive characteristic case [2]. The original proof of Theorem 1 used algebraic techniques, while we rather focus on a geometric approach using fibrations and quotient maps.

An outline of our proof. In order to prove Theorem 1, we make use of an isomorphism $X \cong U \subset M$ with an open subset U of a blowup $\pi\colon M \to \mathbb{C}^3$, such that $D := M \setminus U$ is the strict transform of $\{0\} \times \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ and

$$X \cong U \subset M \to \mathbb{C}^3$$

is the map $(x, y, z, t) \mapsto (x, z, t)$. That is, X is isomorphic to an affine modification U of $\mathbb{C}^3$. The key-result is then that $\mathcal{O}(M) \subset \mathcal{O}(X)$ is invariant for any $\mathbb{G}_a$-action on X. Since $\mathcal{O}(M) \cong \mathcal{O}(\mathbb{C}^3)$, this allows us to conclude that for any given $\mathbb{G}_a$-action on X, there is an induced $\mathbb{G}_a$-action on $\mathbb{C}^3$ which makes $\pi|_X\colon X \to \mathbb{C}^3$ equivariant. Then $\pi(U)$ is obviously invariant, and it follows that its interior $\mathbb{C}^* \times \mathbb{C}^2$ is invariant as well. Theorem 1 is obtained from this by observing that any $\mathbb{G}_a$-action on $\mathbb{C}^* \times \mathbb{C}^2$ leaves the first coordinate invariant: a nontrivial $\mathbb{G}_a$-orbit is isomorphic to $\mathbb{C}$, but there are no non-constant morphisms from $\mathbb{C}$ to $\mathbb{C}^*$.

2. Russell's hypersurface in a blowup of $\mathbb{C}^3$

We recall the realization of Russell's hypersurface as an affine modification of $\mathbb{C}^3$, see also [11, Example 1.5]. Let $N \hookrightarrow \mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[z, t])$ denote the affine cuspidal cubic curve given by

$$N := \{(z, t) \in \mathbb{C}^2 \mid z^3 + t^2 = 0\},$$

and let $I := (g, h) \subset \mathbb{C}[x, z, t]$ denote the ideal which is generated by the two relatively prime polynomials $g(x, z, t) = x^2$ and $h(x, z, t) = x + z^3 + t^2$. The zero set of I is $\{0\} \times N$, and the blowup

$$M := Bl_I(\mathbb{C}^3) \cong \{((x, z, t), [u : v]) \in \mathbb{C}^3 \times \mathbb{P}^1 \mid h(x, z, t)u + g(x, z, t)v = 0\}$$

of $\mathbb{C}^3$ along I is a hypersurface in $\mathbb{C}^3 \times \mathbb{P}^1$ with singular locus of codimension two: $\operatorname{Sing}(M) = \{0\} \times N \times \{[0 : 1]\}$. In particular, M is a normal variety.

REMARK 2.1. Russell's hypersurface is isomorphic to the open subset U of M given by $u \neq 0$, via the embedding $X \hookrightarrow M$, $(x, y, z, t) \mapsto ((x, z, t), [1 : y])$.

We denote the complement of U in M by D. Note that D is then given by $u = 0$, and that the image of U under the blowup morphism is $\pi(U) = \mathbb{C}^* \times \mathbb{C}^2 \cup (\{0\} \times N)$.

3. Additive group actions on Russell's hypersurface

In order to see that $\mathcal{O}(M) \subset A := \mathcal{O}(X)$ is invariant for every $\mathbb{G}_a$-action on X, we show the equivalent fact that $\mathcal{O}(M) \subset A$ is stable under every locally nilpotent derivation $\partial \colon A \to A$. This obviously holds for the trivial $\mathbb{G}_a$-action on X, so we may assume that $\partial \neq 0$. The first step is to characterize $\mathcal{O}(M)$ in terms of a filtration on A.

REMARK 3.1. With the filtration

$$A_{\leq n} := \mathcal{O}_{nD}(M) = \{ f \in \mathbb{C}(M)^* \mid \operatorname{div}(f) \geq -nD\} \cup \{0\}$$

we have $A_{\leq 0} = \mathcal{O}(M) = \pi^*(\mathcal{O}(\mathbb{C}^3))$, so $\pi^*(\mathcal{O}(\mathbb{C}^3))$ is stable with respect to a locally nilpotent derivation $\partial \colon A \to A$ if and only if $\partial(A_{\leq 0}) \subseteq A_{\leq 0}$.

In order to understand the above filtration, we treat A as a subset of $\mathbb{C}(x, z, t)$ and note that multiplicities along D are simply multiplicities along $\{0\} \times \mathbb{C}^2$; so x, y, z and t have multiplicities 1, -2, 0 and 0, respectively. Now every element $f \in A \setminus \{0\}$ can be written in the form

$$f = \sum_{i=0}^{k} y^i p_i(x, z, t),$$

where each p_i is at most linear in x for $i \geq 1$ (and $p_k \neq 0$). Thus

$$A = \bigoplus_{k=-\infty}^{\infty} A_k$$

is a direct sum of free $\mathbb{C}[z, t]$-modules A_k of rank 1 defined as

$$A_k := \begin{cases} \mathbb{C}[z, t]x^{|k|}, & \text{if} \quad k \leq 0, \\ \mathbb{C}[z, t]y^l, & \text{if} \quad k = 2l > 0, \\ \mathbb{C}[z, t]xy^l, & \text{if} \quad k = 2l - 1 > 0, \end{cases} \tag{1}$$

and one can check that

$$A_{\leq n} = \bigoplus_{k \leq n} A_k.$$

We thus obtain an explicit description of the associated graded algebra

$$B := \mathrm{Gr}(A) = \bigoplus_{n \in \mathbb{Z}} B_n \quad \text{with} \quad B_n := A_{\leq n}/A_{\leq n-1}.$$

It is generated by the elements $\mathrm{gr}(x) \in B_{-1}$, $\mathrm{gr}(y) \in B_2$, $\mathrm{gr}(z)$, $\mathrm{gr}(t) \in B_0$ and

$$W := \mathrm{Spec}(B) \cong \{(x, y, z, t) \in \mathbb{C}^4 \mid x^2y + z^3 + t^2 = 0\}.$$

In particular $B_0 = \mathbb{C}[\mathrm{gr}(z), \mathrm{gr}(t)] \simeq \mathbb{C}[z, t]$.

REMARK 3.2. This grading was also used by M. Zaidenberg, see [18, Lemma 7.4].

Let $l = l(\partial) \in \mathbb{Z}$ be minimal with the property that $\partial(A_{\leq n}) \subset A_{\leq n+l}$ for all $n \in \mathbb{Z}$; the existence of such an l follows from the fact that both $A_{\leq 0}$ and B are finitely generated graded algebras, and $l \neq -\infty$ since we consider a nontrivial $\mathbb{G}_a$-action. It follows that $\partial\colon A \to A$ induces a nontrivial homogeneous locally nilpotent derivation on B of degree l; we will denote it by δ. With this notation it is enough to show that $l \leq 0$ in order to obtain $\partial(A_{\geq 0}) \subseteq A_{\geq 0}$. In fact, more is true:

Proposition 3.3. *With B as above, any nontrivial locally nilpotent homogeneous derivation $\delta\colon B \to B$ has degree $l < 0$.*

Before going into the proof, let us start with a discussion of the geometry of $W \hookrightarrow \mathbb{C}^4$ and prove Lemma 3.6 below. As a hypersurface in $\mathbb{C}^4$, W is a normal variety since its singular set $\mathrm{Sing}(W) = \{0\} \times \mathbb{C} \times \{0\} \times \{0\}$ has codimension two. It admits two different group actions: the $\mathbb{G}_a$-action $(\tau, w) \mapsto \tau \cdot w$ corresponding to the locally nilpotent derivation $\delta\colon B \to B$; and the $\mathbb{G}_m$-action corresponding to the grading of B. The latter is given by

$$\mathbb{G}_m \times W \to W, \quad (\lambda, (x, y, z, t)) \mapsto (\lambda^{-1}x, \lambda^2 y, z, t),$$

and since $B_0 = \mathbb{C}[z, t]$, the $\mathbb{G}_m$-quotient morphism is given by

$$p\colon W \to \mathbb{C}^2 \cong \mathrm{Spec}(\mathbb{C}[z, t]), \quad (x, y, z, t) \mapsto (z, t).$$

It is trivial above $\mathbb{C}^2 \setminus N$: the map

$$(\mathbb{C}^2 \setminus N) \times \mathbb{G}_m \xrightarrow{\sim} p^{-1}(\mathbb{C}^2 \setminus N), \quad ((z, t), \lambda) \mapsto (\lambda^{-1}, -(z^3 + t^2)\lambda^2, z, t),$$

is a $\mathbb{G}_m$-equivariant isomorphism with inverse

$$p^{-1}(\mathbb{C}^2 \setminus N) \xrightarrow{\sim} (\mathbb{C}^2 \setminus N) \times \mathbb{G}_m, \quad (x, y, z, t) \mapsto ((z, t), x^{-1}).$$

As for N, we have $p^{-1}(N) = F_- \cup F_+$, where F_- and F_+ are the subsets of $p^{-1}(N)$ given by $y = 0$ and $x = 0$ respectively.

REMARK 3.4. The set F_- consists exactly of the points $w \in W$ for which $\lim_{\lambda\to\infty} \lambda w$ exists, and F_+ consists exactly of the points $w \in W$ for which $\lim_{\lambda\to 0} \lambda w$ exists.

REMARK 3.5. The above trivialization extends to a trivialization $\mathbb{C}^2 \times \mathbb{G}_m \xrightarrow{\sim} W \setminus F_+$, but for $W \setminus F_-$ there is no such trivialization since the $\mathbb{G}_m$-isotropy group of a point in $F_+ \setminus F_-$ has order 2.

Now let us turn to the $\mathbb{G}_a$-action $\mathbb{G}_a \times W \to W$, $(\tau, w) \mapsto \tau \cdot w$, corresponding to $\delta\colon B \to B$. Since δ is homogeneous of degree l, it is normalized by the $\mathbb{G}_m$-action, i.e. for $w \in W$, $\tau \in \mathbb{G}_a$ and $\lambda \in \mathbb{G}_m$, we have

$$(\lambda^{-l}\tau) \cdot (\lambda w) = \lambda(\tau \cdot w).$$

In particular this implies that λO is a $\mathbb{G}_a$-orbit for any $\mathbb{G}_a$-orbit O.

Lemma 3.6. *Let $\delta\colon B \to B$ be a nontrivial locally nilpotent derivation, homogeneous of degree l. Then either*
(1) *$l < 0$ and F_+ is invariant, or*
(2) *$l > 0$ and F_- is invariant.*

Proof. Since the locally nilpotent derivation $\delta\colon B \to B$ is homogeneous, its kernel

$$B^\delta := \{ f \in B \mid \delta(f) = 0\} = \{ f \in B \mid f(\tau \cdot w) = f(w), \ \forall \tau \in \mathbb{G}_a\}$$

is a graded subalgebra, i.e.:

$$B^\delta = \bigoplus_{n\in\mathbb{Z}} B^\delta_n.$$

Given $f \in B_k \setminus \{0\}$, we have $\delta^\nu f \in B^\delta_{k+\nu l} \setminus \{0\}$ for a suitable $\nu \in \mathbb{N}$. It follows that
(1) if $l = 0$, we have $B^\delta_n \neq \{0\}$ for all $n \neq 0$,
(2) if $l > 0$ we have $B^\delta_n \neq \{0\}$ for some $n > 0$,
(3) if $l < 0$ we have $B^\delta_n \neq \{0\}$ for some $n < 0$.

First assume that $l > 0$, so that $B^\delta_n \neq 0$ for some n, and let $f \in B^\delta_n \setminus \{0\}$. Then f vanishes on F_- since $f(\lambda x) = \lambda^n f(x)$ and since $\lim_{\lambda\to\infty} \lambda x$ exists in W for $x \in F_-$. It follows that F_- is invariant since it is an irreducible component of the invariant set $V(f) \subset W$ of dimension two. If $l < 0$, it follows analogously that $F_+ \subset W$ is invariant. It remains to show that l cannot be zero.

If $l = 0$, both F_- and F_+ are invariant. So $p^{-1}(N)$ is invariant and $W \setminus p^{-1}(N)$ as well. Then for any nontrivial $\mathbb{G}_a$-orbit $O \subset W \setminus p^{-1}(N)$ the map $(z^3+t^2) \circ p|_O$ has no zeros, and thus must be constant, say with value $a \in \mathbb{C}^*$, since $O \cong \mathbb{C}$. However, any morphism $p|_O \colon O \to V(\mathbb{C}^2; z^3 + t^2 - a)$ from the complex line to the smooth affine elliptic curve $V(\mathbb{C}^2; z^3 + t^2 - a)$ is constant, so O is contained in a p-fiber. Since $p(O) \in \mathbb{C}^2 \setminus N$, this p-fiber is isomorphic to $\mathbb{G}_m$, as p is a $\mathbb{G}_m$-principal bundle over $\mathbb{C}^2 \setminus N$. This gives a contradiction since $\mathbb{C}$ cannot be embedded into $\mathbb{G}_m$. □

Proof of Proposition 3.3. By Lemma 3.6 it is enough to show show that F_-, given by $y = 0$, is not invariant. Suppose to the contrary that F_- is invariant; then its complement in W, given by $y \neq 0$, is invariant as well. Since there is no non-constant invertible function on $\mathbb{G}_a$-orbits, all $\mathbb{G}_a$-orbits in $W \setminus F_-$ are contained in level hypersurfaces of y. In particular the hypersurface $V \subset W$ which is given by $y = 1$ is invariant and we have $V \simeq \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + z^3 + t^2 = 0\}$. The restriction of the $\mathbb{G}_m$-quotient projection

$$\psi := p|_V \colon V \to \mathbb{C}^2, \quad (x, z, t) \mapsto (z, t)$$

is a two sheeted branched covering of $\mathbb{C}^2$ with branch locus $\psi^{-1}(N) \cong N$ and deck transformation

$$\sigma \colon V \to V, \quad (x, z, t) \mapsto (-x, z, t),$$

which is simply the action of $-1 \in \mathbb{G}_m$. In particular $\sigma(O)$ is a $\mathbb{G}_a$-orbit for any $\mathbb{G}_a$-orbit O. Assume for the moment that every nontrivial $\mathbb{G}_a$-orbit intersects $\psi^{-1}(N)$ exactly once. Since the hypersurface V is a normal surface, there is a quotient map

$$\chi \colon V \to V/\!/\mathbb{G}_a := \operatorname{Spec}(\mathcal{O}(V)^{\mathbb{G}_a}),$$

the generic fiber of which is a $\mathbb{G}_a$-orbit [4, Lemma 1.1]. Thus the restriction

$$\chi|_{\psi^{-1}(N)} \colon \psi^{-1}(N) \to V/\!/\mathbb{G}_a$$

is injective on a nonempty open subset of $\psi^{-1}(N)$. Hence, $V/\!/\mathbb{G}_a$ being a smooth curve, it follows from Zariski's main theorem that this restriction is an open embedding. However, this is a contradiction since the affine cuspidal cubic curve $\psi^{-1}(N)$ has a singular point.

Finally, any nontrivial $\mathbb{G}_a$-orbit O intersects $\psi^{-1}(N)$: otherwise $\psi|_O$ would be a non-constant morphism to an affine elliptic curve $z^3+t^2=a$ for some $a \in \mathbb{C}^*$, which is impossible. Note that a point in $O \cap \psi^{-1}(N)$ is a common point of the two $\mathbb{G}_a$-orbits $\sigma(O)$ and O, so $\sigma(O) = O$. Choose an equivariant isomorphism $\mathbb{C} \cong O$ such that $0 \in \mathbb{C}$ corresponds to a point in $\psi^{-1}(N)$. Then the involution $\sigma\colon O \to O$ corresponds to $\mathbb{C} \to \mathbb{C}$, $\zeta \mapsto -\zeta$, and as a consequence every nontrivial $\mathbb{G}_a$-orbit $O \hookrightarrow V$ intersects the branch locus $\psi^{-1}(N) = W^\sigma$ (the fixed point set of σ) in exactly one point. $\square$

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