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Wang's premium principle: overview and comparison with classical principles

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Wang's premium principle: overview and comparison with classical principles

Abstract. A premium principle is an economic assessment regulation used by the insurer in order to settle on the amount of net premium for each individual risk in his portfolio. In this research, we will practically examine the performance, by comparing with other principles, of Wang's (1996) proposed premium principle based on transforming the premium layer density. Theoretically, Wang's principle is the best premium principle among all existing premium calculation principles as it satisfies most of the properties of a premium principle.

Keywords: premium principle, insurer, insured, stochastic dominance, comonotonicity, non-life insurance
CERCS code: P160 Statistics, operation research, programming, actuarial mathematics.

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1 Introduction

Insurance is a practice of transforming the risk to the (re)insurer for a fixed premium. Premium calculation for a possible risk, in actuarial science, is considered as the crucial task. there are several premium principles which are widely used in insurance industries. Finding of a generally accepted premium calculation principle is the main concern of actuarial experts. But it is still far away from the best principle. In insurance industries, it is well known that policyholders are more risk averse than insurers, i.e the policyholders are willing to pay more premium for higher risk which is known as risk adjusted premium.

According to Meyers (1991), individual insurers are to price-makers, but not price-takers, as the insurer have to compete with their counterparts. Hence, in a competitive market, the insurance prices are to be measured by the combined efforts of all situations. According to S. Wang, V.R. Young and H. Panjer (1997) for a given market, the price of an insurance risk X depends only on its distribution. As insurance risks are becoming more incorporated, unified premium principles are becoming more demandable. Many researchers are giving efforts to put together different existing premium principles or trying to introduce new premium principle to meet the demand.

A number of researchers proposed several premium principles. Most of those are depending on first and second moments included, on some principle third moment are used as well. But there is a lot of inconsistency in moment based premium principles. Some researchers proposed premium principle which is based on utility theory, especially exponential utility (Freifelder 1979 and Gerber 1979) and Esscher principle (Buhlmann 1980) which are generally theoretical. Wang (1995, 1996) proposed premium principles by transforming survival function (de-cumulative distribution function) and proportional hazard rate. Wang suggested to calculate premium by transforming the layer premium density (1996).

There are mainly four sections, in section one we have discussed some basic properties which should be satisfied by a premium principle to be an ideal premium calculation principle. Also, in this section, we have discussed some existing premium principles.

Section two contains some preliminary ideas for tail based premium principle which includes comonotonicity, proportional hazard, layer net premium density, stochastic dominance and the idea of transforming the survival function.

In section three we have exposed Wang's premium principle by transferring the layer premium density, which satisfies most of the necessary conditions of a sound premium principle. Finally, in section four we give two examples where we compare the performance of Wang's premium calculation principle with other premium principles by calculating the errors. We estimate the error of the calculated premiums with randomly generated claims. In our first example we use Pareto distribution II, and in the second example, we use exponential distribution.

2 Different premium principles and properties of a premium principle

This chapter mainly gives some overview on some essential basics on insurance risk and how the insurer charged the premium for a risk. Firstly we define " risk" and "premium principle. Then we discuss some essential properties which should be satisfied by a premium calculation principle, to be an ideal principle. After that, we define some existing popular premium calculation principles. We also establish a simple relation among charged premium, premium in view of insurer and premium in view of insured.

We refer to Leaven & Goovaerts (2011) for the definition of risk , definition premium calculation principle, and properties of premium principles (see also textbook "Insurance Risk and Ruin" by Dickson)

Definition 2.1 (Risk) *Let us assume that* (Ω, F) *is a measurable space where* Ω *is the outcome space and* F *is a* σ*-algebra defined on it. A risk is a random variable defined on* (Ω, F)*; that is* $X : \omega \to \Re$ *is a risk if* $X^{-1}((-\infty, x]) \in F$ *for all* $x \in \Re$.

A risk represents the final net loss of a position or contingency currently held. If $X > 0$ we referred it as a loss, on the other hand if $X \leq 0$ we say it is a gain. We assume the set of all random variables on $(\Omega$,F$)$ by X.

Definition 2.2 (Premium calculation principle) *A premium calculation principle* P_X *is a function* X *to the real number space i.e.* $\Re$*. In other words* P_X *is a functional that assign a real number to any random variable on* (Ω, F).

2.1 Properties of premium principles

2.1.1 Law invariance

A premium calculation principle P_X is said to be probability law invariant if premiums for two risks are equal for probabilities of the risks, less than any real value, are equal. That is, P_X is a $\mathbf{P}$ - law invariant (where $\mathbf{P}$ represents probability measure on $(\Omega$,F$)$) if $P_X = P_Y$ when $\mathbf{P}(X \leq x) = \mathbf{P}(Y \leq x)$ for all real x.

2.1.2 Monotonicity

A premium principle is said to be monotonic if in any trajectory premium for less risky risk is less than premium which corresponds to more risky risk. In other words, P_X is monotonic if $P_X \leq P_Y$ when $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. And P_X is **P**-monotonic if $P_X \leq P_Y$ for $X \leq Y$ **P**-almost surely.

2.1.3 Sub-additivity

If X_1 and X_2 are two risks, then the premium for the combined risk is less than the sum of the individual risk's premium. That is the premium principle should hold the following inequality

$$P_{X_1+X_2} \leq P_{X_1} + P_{X_2}.$$

2.1.4 Additivity

If X_1 and X_2 are two independent risks, then the premium for the combined risk should be equal to the sum of their individual premiums. That is

$$P_{X_1+X_2} = P_{X_1} + P_{X_2}.$$

2.1.5 Scale invariance or positive homogeneity

If for all real $a > 0$, there exists a risk Y such that $Y = aX$ then premium for risk Y is equal to premium for risk X multiplied by the constant 'a'. That is according to scale invariance property

$$P_Y = aP_X.$$

2.1.6 Consistency or translation invariance

If for all real $c > 0$, there exists a risk Z such that $Z = X + c$, that is if the distribution of Z is the distribution of X shifted by c units then the premium for risk Z is equal to the premium for risk X increased by c units i.e

$$P_Z = P_X + c.$$

2.1.7 No ripoff

If there is a finite maximum claim amount for the risk, x_m, then according to this property the premium should be less than the maximum claim amount, i.e $P_X \leq x_m$. It is said that if this property is not satisfied, then there is no incentive for an individual to effect insurance.

2.1.8 Convexity

The premium principle P_X is said to be convex if it holds the following inequality

$$P_X[\alpha X + (1-\alpha)Y] \leq \alpha P_X + (1-\alpha)P_Y.$$

2.1.9 Preserving stop-loss order (SL)

P_X preserves stop-loss order if $P_X \leq P_Y$ when $E[(X-d)_+] \leq E[(Y-d)_+]$ for $d \in \Re$. Where the $+$ sign indicate that if the difference is positive then it will be counted otherwise it will be considered as zero.

2.2 Different types of premium principles

We refer the textbook "Insurance Risk and Ruin" by Dickson for the following premium principles, Pure premium principle , Expected value premium principle, Variance premium principle, Standard deviation premium principle, the principle of zero utility, Esscher premium principle. We take quantile principle from the textbook "Risk modelling in General Insurance" by Gray and Pitts. Also,we refer to Swiss premium principle and Dutch premium principle Laeven & Goovaert(2011) and for tail standard deviation premium principle Furman & Lands man(2006).

2.2.1 Pure premium principle

The pure premium is equal to the insurer's expected claims under the risk X, i.e

$$P_X = E[X].$$

The pure premium principle is not an attractive principle to the insurers. This principle does not contain any loading for profit it just covers the insurer's expected claim from the risk. We can say that the insurer who exercise this principle will not survive in this business in long run.

2.2.2 The expected value principle

For any $\theta > 0$, where θ is known as the relative security loading on the pure premium. So, the loading in the premium is $\theta E[X]$ and the expected value principle is pure premium plus the premium loading . Hence,

$$P_X = (1+\theta)E[X].$$

The expected value principle is the simplest one but its major negative side is that it assign the same premium for all risk with the same mean. In practice, we know, risks with the same mean but different variances should have different premiums.

2.2.3 The variance principle

The variance principle for premium calculation is as follows:

$$P_X = E[X] + \theta Var[X], \quad \text{where} \;\; \theta > 0.$$

E[X] and Var[X] represents the pure premium and variance of the loss distribution of X respectively. We may again refer θ as the relative security loading which is, in this case, proportional to the variance.

2.2.4 The standard deviation principle

For $\theta > 0$ (referred as the relative security loading), the standard deviation premium principle is:

$$P_X = E[X] + \theta\sqrt{(Var[X])}.$$

where E[X] and Var[X] represents the pure premium variance of the loss distribution of X respectively. That is, the pure premium is increased by a percentage of the standard deviation of the risk. Though this principle is same as variance principle, but they do not satisfy same properties always.

2.2.5 Combined variational principle

The combined variational principle is the combination of two principles, the variance principle and the standard deviation principle. This principle is also known as compromise principle. According to this principle, for $\alpha, \beta > 0$

$$P_X = E[X] + \alpha\sqrt{(Var[X])} + \beta Var[X].$$

2.2.6 The quantile premium principle

The quantile principle which is known as percentile principle, can derive by solving the following inequality

$$\mathbf{P}\{X \geq P_X\} \leq \varepsilon, \quad \varepsilon > 0.$$

Where ε is the maximum tolerance of ruin probability. In other words, one can find the premium which grantees that the probability that this premium is sufficient to cover the claims at least $1-\varepsilon$. That is, we have

$$P_X = min_{P_X}[\mathbf{P}\{X \leq P_X\} > 1-\varepsilon].$$

To use this principle we need to have a model for the distribution of X, or at least know the values of certain higher percentiles of the distribution.

Let the insurer wants to have probability of at least 0.90 to cover the risk X i.e $\mathbf{P}(X \leq P_X) = 0.90$, with $X \sim N(\mu, \sigma)$ and the 90^{th} percentile of the standard normal distribution is 1.28, so we need P_X to satisfy $\mathbf{P}(X \leq P_X) = 0.90$ and we know 90th percentile of the standard normal distribution is 1.28. We need P_X to satisfy

$$\frac{P_X - E[X]}{\sqrt{Var(X)}} = 1.28$$

which gives $P_X = E[X] + 1.28\sqrt{Var[X]}$. That is, in this case, the quantile premium principle is equivalent to standard deviation principle with relative safety loading $\alpha = 1.28$ fixed by the insurer.

2.2.7 Swiss principle

For a given non-negative and non-decreasing real valued function v and a given parameter $0 \leq q \leq 1$, the Swiss principle is the solution of the following equation:

$$E[v(X - qP_X)] = v((1-q)P_X).$$

We see that this principle includes both zero utility principle and Expected value principle as particular cases. As

$$v((1-q)P_X) = E[v(X - qP_X)]$$

which is the principle of zero utility and gives us (using Jensen's inequality)

$$\Rightarrow v((1-q)P_X) \leq v(E[X] - qP_X$$
$$\Rightarrow (1-q)P_X = E[X] - qP_X$$
$$\Rightarrow P_X = E[X].$$

2.2.8 Esscher principle

The Esscher premium principle named after the Swedish actuary F. Esscher, who proposed this principle as well, and the principle is as follows:

$$P_X(X) = \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]}, \quad \text{where} \quad \theta > 0.$$

That is, if we consider the cumulative generating function (cgf) of a random risk X as a function of relative loading factor as follows $f_X = \ln(E[e^{\theta X}])$ then Esscher premium principle represents the first derivative of the cgf with respective to the parameter θ.

2.2.9 Dutch principle

This principle was introduced by Van Heerwaarden and Kaas in 1992, they defined the principle in the following way

$$P_X = E[X] + \theta E[(X - \alpha E[X])_+], \quad \alpha \geq 1, \quad 0 < \theta \leq 1.$$

That is, according to the Dutch principle premium is equal to the pure premium plus relative safety loading factor times the expected value of positive difference of the risk over the certain percentage of the expected loss, where the value of the relative safety loading factor is any positive real number which is less than 1.

2.2.10 Tail standard deviation principle

Among all the premium principles the standard deviation principle becomes the most popular premium calculation principle as it is the simplest premium calculation principle. But, since this premium principle usages only first two moments (mean and variance) hence it overlooks the shape of the risk distribution. This is a great disadvantage of the standard deviation premium principle.

Furman and Landsman (2005) developed a tail standard deviation premium principle $P_{X_{TSD}}$ as alternative to the standard deviation principle. Their proposed tail base premium principle is as follows:

$$P_{X_{TSD}}(x_q) = E[X|X > x_q] + \theta\sqrt{Var(X|X > x_q)} \quad \text{for} \quad \theta \geq 0.$$

where, $E[X|X > x_q]$ and $Var(X|X > x_q)$ represents the tail conditional expectation and the tail conditional variance respectively.

It is clear that if we let $q \to 0$ then we see that the standard deviation principle is a particular case of tail standard deviation premium principle.

2.2.11 The principle of zero utility or exponential principle

Let us assume a person or an asset, belongs to someone, as 'insured' who is carrying a certain risk, even though he decided not to buy the insurance policy which he is offered by an insurance company or 'insurer'.

If the wealth of the insurer at the beginning of the period is W and the amount of loss occurred due to the insured risk on the mentioned period is X. Hence, the wealth of the insurer at the end of the period is $W + P_X - X$.

The loss X is a non-negative random variable as it is uncertain which has the distribution $\mathbf{P}[X \leq x]$. Consider that both insurer and insured have the inclination to obey the expected utility hypothesis.

Let, $u : \Re \to \Re$ be the insurer's utility function then the insurer's expected utility is $E[u(W + P_X - X)]$.

The equivalent utility principle or zero utility principle says that the following property hold for both insurer's and insured's point of view:

$$E[u(W+P_X-X)] = u(W).$$

Solution P_X of this equation is treat as premium of the respective risk. The expression shows that in general, premium depends on wealth W but if the utility function is an exponential function i.e for $\beta > 0, \quad u(x) = -e^{-\beta x}$ then according to exponential principle, the premium is defined as follows:

$$P_X = \frac{1}{\beta}\ln(E[e^{\beta X}]).$$

Which indicates that premium does not depend on insurer's(or insured's) wealth.

2.3 The relation between premium and expected loss

From Jensen's inequality, we know that if u is a concave function then $E[u(x)] \le u(E[x])$ and the reverse holds for convex function. Now, exponential function is a convex function so, if we apply the Jensen's inequality on $P_X = \frac{1}{\beta}\ln(E[e^{\beta X}])$ we get,

$$P_X \ge \frac{1}{\beta}\ln(e^{E[\beta X]})$$

$$\Rightarrow P_X \ge \frac{1}{\beta}\beta E[X]$$

$$i.e \quad P_X \ge E[X].$$

So, the premium should be always more than the expected loss. This property is also known as non-negative loading.

2.4 The optimal value of the premium

In the previous section, we see that premium is greater than the expected loss, the question is, does the premium has any upper and lower bound? In this section, we will examine that.

In view of Insured: Let u_1 and W_1 be the exponential utility and wealth of the insured respectively, an insurance treaty that for a given uncertain loss X leaves insured with final wealth $W_1 - P_X$, will be preferred to full self insurance, which leaves the insured with final wealth $W_1 - X$, if and only if $u_1(w_1 - P_X) \geq E[u_1(w_1 - X)]$. The equivalent utility premium, in view of insured (P_X^{ird}) is obtained by solving the following equation

$$u_1(w_1 - P_X) = E[u_1(w_1 - X)].$$

The insured will buy the policy if $\quad P_X \leq P_X^{ird}$.

In view of Insurer: The equivalent utility expression for insurer will be as follows expected utility if coverage is sold $\geq$ expected utility if no coverage punched that is

$$u_2(w_2 + P_X) = E[u_2(W_2 - X)].$$

Where u_2, W_2 indicates utility function and wealth of the insurer respectively. So, the final wealth of the insurer after receiving the premium is $W_2 + P_X$, and the final wealth after the incurred claim X is $W_2 - X$. So according to the insurance treaty, the insurer will prefer the insurance if and only if, $u_2(w_2 + P_X) \geq E[u_2(w_2 - X)]$.

If we denote the premium from the insurer's view by P_X^{ir}, the insurer will sell the policy if $P_X^{ir} \leq P_X$.

Hence, we conclude that the premium will satisfy

$$P_X^{ir} \leq P_X \leq P_X^{ird}.$$

In the following section we will discuss some preliminary ideas which are important for tail based premium principle.

3 Some important preliminary ideas for tail based premium principle

In this chapter, we will discuss comonotonicity, hazard rate and proportional hazard transform of a risk X, the net layer premium density, first stochastic dominance and second stochastic dominance and the idea of transforming the survival function. We refer Wang (1996) for these properties where details of all these properties are also available.

3.1 Comonotonicity

The idea of comonotonicity was first introduced by Yaari (1987). But, in actuary Wang first applied the concept of comonotonicity in 1996.

Definition 3.1 *Two risks X , $Y \in F$ are said to be comonotonic iff there exists a risk $Z \in F$ and two non-decreasing mappings f and g such that $X = f(Z)$ and $Y = g(Z)$.*

Wang & Dhaene (1997), Wang, Yong & Goovaerts (1997), examine the application of comonotonicity in actuarial science.

The insurance is a risk sharing an idea, either between the insurer and insured or between the first insurer and reinsurer guide to partial risk which are comonotonic. But both of the risk sharing partners have to bear more portion if the underlying total claim amount rise.

Also, premiums for two comonotonic risks should be additive, i.e. if X, $Y \in F$ are two comonotonic risks then $P_X(X+Y) = P_X(X) + P_X(Y)$. According to S. Wang layer premiums must satisfy additive condition, that is for any sequence of layers $0 < x_0 < x_1 \cdots < x_n < \cdots$

$$P_X = \sum_{i=1}^{\infty} P_X(I_{(X_{i-1}-X_i)}).$$

In sub-section 2.3 we will discuss layer premium density.

3.2 Proportional hazard transform and risk-adjusted premium principle

The insurance risk X which is a non-negative random variable and is defined by its cumulative distribution function and $F_X(x)$ and survival function $S_X(x)$.

Let we assume that the distribution of claim sizes $F_X(x)$ as a continuous function, then the hazard rate of the risk X is

$$H_X(x) = \frac{F_X'(x)}{(1 - F_X(x))} = -\frac{d}{dx} \ln S_X(x).$$

An unfavourable claim reported means that the occurred loss is larger than expected loss. To require a safety margin, we can reduce the hazard rate as follows:

$$H_Y(x) = \frac{1}{\alpha} H_X(x), \quad \alpha > 1, \quad x \geq 0.$$

For this relation we get another random variable Y which survival function satisfies $S_Y(x) = S_X(x)^{\frac{1}{\alpha}}$ and the mapping $P_\alpha : X \to Y$ is called proportional hazard transform.

Definition 3.2 *If X is continuous with density function $f_X(x)$, $x \in I$, then $Y = P_\alpha(X)$ is also continuous with density function*

$$f_Y(x) = [\frac{1}{\alpha} S_X(x)^{\frac{1}{\alpha}-1}] f_X(x), \quad x \in I.$$

The weight function $\frac{1}{\alpha} S_X(x)^{\frac{1}{\alpha}-1}$ increases with the loss size x, thus gives more weight to the unfavourable events.

Definition 3.3 *The risk adjusted premium for a risk X is given by*

$$P_X^\alpha = E[P_X^\alpha(x)] = \int_0^\infty S_X(x)^{\frac{1}{\alpha}} dx, \quad \alpha \geq 1.$$

Where α-defined as risk averse.

Since we know that

$$E[X] = \int_0^\infty S_X(x)dx.$$

So, for $\alpha = 1, \quad P_X^1 = E[X]$.

Wang (1995) showed that the risk-adjusted premium principle satisfies positive loading, no ripoff, scale invariant, transitivity, subadditivity properties and it also preserves stochastic dominance.

3.3 Layer net premium density

As we mentioned before that insurance risk X is a non-negative uncertain variable. We can define its distribution by two ways, either in terms of its cumulative distribution function $F_X(x)$, or its survival distribution function$S_X(x)$, as follows:

$$(i) F_X(x) = Pr(X \leq x)$$

and

$$(ii) S_X(x) = Pr(X > x)$$

The expected loss of a risk X can be determined by its survival function in the following way

$$E(X) = \int_0^\infty S_X(x)dx.$$

Since most insurance contracts contain provisions, for example, deductible and maximum breaking point or limit point it is helpful to utilise a general term of excess-of-loss layers.

Let us consider the following definition at first,

Definition 3.4 *A layer $(a, a+h)$ of a risk X is defined as the loss from an excess-of-loss cover*

$$I_{(a,a+h]}(X) = \begin{cases} 0 & if \quad 0 \leq X < a \\ (X-a) & if \quad a \leq X < a+h \\ h & if \quad a+h \leq X \end{cases}$$

Shaun Wang (1996) defined the expected loss or net premium for a layer $(a, a+h]$ using the survival function as follows:

$$E(I_{a,a+h}) = \int_a^{a+h} S_X(x)dx$$

where the survival function for a layer $I_{(a,a+h]}$ is

$$S_{I_{(a,a+h]}}(x) = \begin{cases} S_X(a+x) & \text{if} \quad x < h \\ 0 & \text{if} \quad x \geq h \end{cases}$$

It is clear that $S_X(x)$ plays a vital role as layer net premium density as $S_X(x)dx$ illustrates the net premium for an infinitesimal layer $(x, x+dx]$.

3.4 Stochastic dominance

Definition 3.5 ***First stochastic dominance (FSD)*** *If X_1 and X_2 are two positions of a risk X then X_1 is said to be precedes of X_2 iff the survival function of X_1 is everywhere lower than the survival function of X_2 i.e $X_1 \prec X_2$, iff*

$$S_{X_1}(t) \leq S_{X_2}(t), \quad \text{for all } \ t \geq 0.$$

If we consider two layers of equal width $(a, a+h]$ and $(b, b+h]$ for a risk X, and if $a < b$ then premium at $(b, b+h]$ should less than the premium at $(a, a+h]$.

Definition 3.6 ***Second Stochastic dominance(SSD)*** *According to (Kaas t al. 1994) $X_1 \prec X_2$ under second stochastic dominance if any one of the following two equivalent conditions holds:*

1. *There exists $U_i (i = 1, 2, \cdots, n)$ such that*

$$X_1 \prec U_1, \quad U_1 \prec U_2, \cdots, U_n \prec X_2$$

or

2.
$$\int_x^\infty S_{X_1}(t)dt \leq \int_x^\infty S_{X_2}(t)dt \quad for\ \ all\ \ x \geq 0.$$

According to Wang, if $(a, a+h_1]$ and $(b, b+h_2]$ are two layers of different width for a particular risk X with the same net average loss i.e $E[I_{(a,a+h_1]}] = E[I_{(b,b+h_1]}]$, then layer at $(b, b+h_2]$ is a higher risky, i.e $\quad I_{(a,a+h_1]} \prec I_{(b,b+h_1]}$.

And if we assume that $P_X(X) : X \to [0, \infty)$ preserve SSD, then

$$P_X(I_{(a,a+h_1]}) < P_X(I_{(b,b+h_2]})$$

$$\Rightarrow \frac{P_X(I_{(a,a+h_1]})}{E[I_{(a,a+h_1]}]} < \frac{P_X(I_{(b,b+h_2]})}{E[I_{(b,b+h_1]}]}.$$

That is, if a premium principle preserves SSD then for any risk, the higher layer should have the higher percentage of loading.

3.5 Transforming the survival function

Since, we see that the survival function $S_X(x)$ represents the layer net premium density and using the concept of risk adjusted premium principle, Wang defined, risk-adjusted layer premium density $S_Y(x)$ as follows:

$$S_Y(x) = g[S_X(x)].$$

where g is a non-decreasing mapping form unit interval to unit interval, i.e $g : [0,1] \to [0,1]$, with $g(0) = 0$ and $g(1) = 1$.

Then $S_Y(x) = g[S_X(x)]$ represents a new survival function.

4 Wang's premium calculation principle

In this chapter, we will describe the premium principle, based on transforming layer premium density which is proposed by Wang, also, we include a table which illustrates different existing premium principles satisfiability of different properties.

Based on the above observations (2.1-2.5), Wang (1996) proposed the following premium principle:

$$P_X(x) = \int_0^\infty g[S_X(x)]dx.$$

And this premium principle satisfies all the following conditions:

(i) $E[X] \leq P_X \leq \max(X)$

(ii) $P_X(aX+b) = aP_X(X)+b$

(iii) Layer additivity

(iv) Comonotonically additive

(v) Preserve stochastic dominance

Now, we will include a table which describes theoretically the performance of different premium principles depending on their satisfying different properties of a premium principle.

The following table illustrates the comparison among several existing premium calculation principles, in accordance to satisfying different properties of premium principle. (see Virginia, Roger(2007))

Name of Premium Principle	M	TI	A	SA	C	CA	NR	SL
Expected value principle	+	+	+	+	+	+	- (+ for $\theta = 0$)	+
Variance principle	-	+	-	-	+	-	-	-
Standard deviation principle	-	+	-	+	+	-	-	-
Exponential principle	+	+	-	-	+	-	+	+
Esscher principle	-	+	-	-	-	-	+	-
Swiss principle	+	-	-	-	-	-	+	+(if v is convex)
Dutch principle	+	-	-	+	+	-	+	+
Tail standard deviation	-	+	-	+	+	-	-	-
Wang principle	+	+	+	+	-	+	-	+

M=monotonocity, TI=translation invariant, A=additivity, SA=sub-additivity, C=convexity, CA=comonotonic additivity, NR=no ripoff, SL=stop-loss order.

In the next section, we will examine the performance of different premium calculation principles with two examples. At first, we will calculate premiums using our assigned distribution then we will generate random claims and after that, we will calculate the error of each premium principle. Here, we consider two portfolios, each of which contains 1000 policies and follow, in first example Pareto distribution II and in second example exponential distribution. We will estimate error e by using the following formula

$$e = \sqrt{\frac{\sum_{i=1}^{1000}(x_{1,i} - P_{X_1})^2 + \sum_{i=1}^{1000}(x_{2,i} - P_{X_2})^2}{1000}}.$$

In first example, we will repeat the process to see how will it affect, if we change the parameters of our distributions.

5 Comparison of different premium principles

5.1 Example 1: Using Pareto distribution

Let us consider two portfolios of risks, each of which has 1000 policyholders. Let for both portfolios for all risks with probability 0.9 there is no claim, with probability 0.1 there is a claim. Let the risks from the both portfolios follow Pareto II (American Pareto) distribution.

We choose American Pareto distribution, in our example, because it is most widely used in non-life insurance as a model for claim severity.

We know that if a random variable X follow American Pareto distribution, i.e $X \sim Pa(\alpha, \beta)$, where α is scale parameter and β is shape parameter, then cumulative distribution function, probability density function, survival function, mean, variance, p-th quantile function and moments of the distribution are as follows:

- Dstribution function $F_X(x) = 1 - \frac{\alpha^\beta}{(x+\alpha)^\beta}, \quad x \geq 0.$
- Density function $f_X(x) = \frac{\beta \alpha^\beta}{(x+\alpha)^{\beta+1}}, \quad x \geq 0.$
- Survival function $S_X(x) = \frac{\alpha^\beta}{(x+\alpha)^\beta}, \quad x \geq 0.$
- Expectation $E[X] = \frac{\alpha}{\beta-1}, \quad \beta > 1.$
- Variance $Var[X] = \frac{\alpha^2 \beta}{(\beta-1)^2(\beta-2)}, \quad \beta > 2.$
- Quantile function $F^{-1}(p) = \frac{\alpha}{(1-p)^{1/\beta}}, \quad 0 < p < 1.$
- Moments $EX^n = \frac{\alpha^n n!}{\prod_{i=1}^n (\beta - i)}, \quad \beta > n.$ But $\quad EX^n = \infty, \quad \beta \leq n.$

Let us consider $X_1 \sim Pa(800, 3)$ and $X_2 \sim Pa(1000, 5)$, whereX_1 and X_2 stands for risks from first portfolio and second portfolio respectively.

Since we know that mean of Pareto distribution is $EX = \frac{\alpha}{\beta-1}$. So, mean of first Pareto distribution is $EX_1 = \frac{800}{3-1} = 400$, therefore, mean loss of risks from first portfolio is $400 \cdot 0.1 = 40$.

Hence expected loss of the first portfolio is $ES_1 = 40 \cdot 1000 = 40,000$.

Similarly, mean of second Pareto distribution is $EX_2 = \frac{1000}{5-1} = 250$. Therefore overall expected loss for second portfolio is $ES_2 = (250 \cdot 0.1 \cdot 1000) = 25,000$.

That is, the overall expected loss is approximately 65,000. Let us assume that the overall risk premium (i.e. the risk premium + risk loading) is 75,000.

Also, we know that variance of Pareto distribution is $VarX = \frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$. So, variance of the first Pareto distribution is $VarX_1 = \frac{800^2 \cdot 3}{2^2 \cdot 1} = 480,000$. Similarly variance of the second distribution is $VarX_2 = \frac{1000^2 \cdot 5}{4^2 \cdot 3} = 104,166.7$.

We know that aggregate variance is $Var(S) = (EX)^2 \cdot VarN + EN \cdot VarX$. In our problem $N \sim Bin(1000, 0.1)$ so, $EN = 1000 \cdot 0.1 = 100$ and $VarN = 1000 \cdot 0.1 \cdot (1 - 0.9) = 90$.

Hence variance of the first portfolio is $VarS_1 = 400^2 \cdot 90 + 100 \cdot 480000 = 62,400,000$. And variance of the second portfolio is $VarS_2 = 250^2 \cdot 90 + 100 \cdot 104166.7 = 16,041,670$.

Now we will calculate premiums for each portfolio (and also for each policy) using different premium calculation principles.

5.1.1 Premiums using expected value principle

We know the expected value premium calculation principle is

$$P_X = (1+\theta)EX.$$

That is

$$75000 = (1+\theta) \cdot (ES_1 + ES_2)$$
$$\Rightarrow 75000 = (1+\theta) \cdot 65000$$
$$\Rightarrow \theta = \frac{2}{13}.$$

Therefore, total premium for first portfolio is

$$P_{S_1} = 40000 \cdot (1+\theta) = 46,153.85.$$

Hence premium for each policy of first portfolio is

$$P_{X_1} = \frac{46,153.85}{1000} = 46.15.$$

Similarly total premium for the second portfolio is

$$P_{S_2} = 25000(1+\theta) = 28,846.15.$$

And hence premium for each policy of the second portfolio is

$$P_{X_2} = \frac{28,846.15}{1000} = 28.85.$$

5.1.2 Premiums using variance principle

We know tat variance principle for premium calculation is

$$P_X = EX + \theta \cdot VarX.$$

So, in our problem

$$75000 = ES_1 + ES_2 + \theta \cdot (VarS_1 + VarS_2)$$

That is

$$75000 = 65000 + \theta(62400000 + 16041670)$$
$$\Rightarrow \theta = \frac{10000}{62400000 + 16041670} = 0.0001274833.$$

There for total premium of the first portfolio is

$$P_{S_1} = 40000 + \theta \cdot 62400000 = 47954.96.$$

And premium of each policy of the first portfolio is

$$P_{X_1} = \frac{47954.96}{1000} = 47.95.$$

Similarly total premium of the second portfolio is

$$P_{S_2} = 25000 + \theta 16041670 = 27045.04.$$

And premium of per policy of second portfolio is

$$P_{X_2} = \frac{27045.04}{1000} = 27.05.$$

5.1.3 Premiums using standard deviation principle

The formula for premium using standard deviation principle is

$$P_X = EX + \theta \cdot \sqrt{(VarX)}.$$

So, in our case

$$75000 = (ES_1 + ES_2) + \theta \cdot (\sqrt{VarS_1} + \sqrt{VarS_2})$$

That is

$$75000 = 65000 + \theta(\sqrt{62400000} + \sqrt{16041670})$$

$$\Rightarrow \theta = \frac{10000}{(\sqrt{62400000} + \sqrt{16041670})} = 0.8400134.$$

Therefore, total premium for first portfolio is

$$P_{S_1} = 40000 + \theta\sqrt{62400000} = 46635.57.$$

Hence premium for each policy of first portfolio is

$$P_{X_1} = \frac{46635.57}{1000} = 46.64.$$

Similarly total premium and premium per policy of second portfolio is

$$P_{S_2} = 25000 + \theta\sqrt{16041670} = 28364.43.$$

And premium per policy of second portfolio is

$$P_{X_2} = 28.36.$$

5.1.4 Premiums using quantile principle

We know p-th quantile for Pareto II distribution is

$$F^{-1}(p) = \frac{\alpha}{(1-p)^{1/\beta}}.$$

So, p-th quantile of our first and second portfolios are

$$F_1^{-1}(p) = \frac{800}{(1-p)^{1/3}}.$$

and

$$F_2^{-1}(p) = \frac{1000}{(1-p)^{1/5}}.$$

We know that quantile function does not depend on survival function. Hence

$$75000 = F_1^{-1}(p) + F_2^{-1}(p)$$

$$\Rightarrow 75000 = \frac{800}{(1-p)^{1/3}} + \frac{1000}{(1-p)^{1/5}}$$

and after simplifying we get $p = 0.9999977955145566079751046\,4$. Now using this value we find total premium for first portfolio is

$$P_{S_1} = \frac{800}{(1-p)^{1/3}} = 61468.73.$$

So premium per policy of the first portfolio is

$$P_{X_1} = \frac{61468.73}{1000} = 61.47.$$

Similarly, premium of each policy of the second portfolio is

$$P_{X_1} = \frac{13531.27}{1000} = 13.53.$$

5.1.5 Calculation of premiums using Wang's principle

For our convenience, let us consider $g(x) = x^c \quad 0 < c < 1$. We know that survival function of Pareto distribution is

$$S_X = \frac{\alpha^\beta}{(x+\alpha)^\beta}, \quad x \geq 0.$$

So survival functions of the first and second portfolio considering with the probability are

$$S_{S_1} = 0.1 \cdot 1000 \cdot S_{X_1} = 0.1 \cdot 1000 \cdot \frac{800^{3c}}{(x+800)^{3c}}.$$

and

$$S_{S_2} = 0.1 \cdot 1000 \cdot S_{X_2} = 0.1 \cdot 1000 \cdot \frac{1000^{5c}}{(x+1000)^{5c}}.$$

Now we know that premium using Wang's principle is

$$P_X = \int_0^\infty g(S_X(x))dx.$$

Therefore in our case

$$75 = 0.1 \cdot (\int_0^\infty (\frac{800}{x+800})^{3c} dx + \int_0^\infty (\frac{1000}{x+1000})^{5c} dx)$$

$$\Rightarrow 750 = 800^{3c}[\frac{(x+800)^{1-3c}}{1-3c}]_0^\infty + 1000^{5c}[\frac{(x+1000)^{1-5c}}{1-5c}]_0^\infty$$

So, $1-3c < 0$ or $c > \frac{1}{3}$ and $1-5c < 0$ i.e $c > \frac{1}{3}$.

$$\Rightarrow 750 = 800^{3c}(0 - \frac{800^{1-3c}}{1-3c}) + 1000^{5c}(0 - \frac{1000^{1-5c}}{1-5c})$$

$$\Rightarrow 750 = \frac{-800}{1-3c} - \frac{1000}{1-5c}$$

$$\Rightarrow c = \frac{26 \pm \sqrt{217}}{45}$$

$$\therefore c_1 = 0.250424, c_2 = 0.905132.$$

But $c > \frac{1}{3}$, hence $c = 0.905132$. Now using this value, if we integrate the transferred survival functions of both portfolios we get,

$$P_{X_1} = 0.1 \cdot (\int_0^\infty (\frac{800}{x+800})^{3c} dx$$

$$\Rightarrow P_{X_1} = 0.1 \cdot 466.3646 = 46.64.$$

i.e premium for each policy of the first portfolio is

$$P_{X_1} = \frac{46636.46}{1000} = 46.64.$$

In similar way we get premium for each policy of the second portfolio is

$$P_{X_2} = 28.36.$$

We know that moment generating function of Pareto distribution does not exist. So, we can not calculate premium using exponential principle.

The following table summarises the value of the relative risk loading factor, premium per policy of the first portfolio, premium per policy of the second portfolio and the estimated error for different premium principles.

Name of principle	Value of risk loading factor	P_{X_1}	P_{X_2}	e
Expected value	$\frac{2}{13}$	46.15	28.85	225.3914
Variance	0.0001274833	47.95	27.05	225.3843
Standard deviation	0.8400134	46.64	28.36	225.3863
Quantile p-th	0.99999779551455661	61.47	13.53	226.2509
Wang	c=0.905132	46.64	28.36	225.3863

We see that if our unfavourable risk follows Pareto distribution then variance principle performs the best as it gives the least error. On the other hand performance of the quantile principle is the worst because it gives the highest error.Though the rounded value of standard deviation principle and Wang's principle are same, practically they are not exactly same but their errors are exactly same.

Now, if we redefine our distributions as follows: $X_1 \sim Pa(1200, 15)$ and $X_2 \sim Pa(850, 8)$.

We know that mean of Pareto distribution is $EX = \frac{\alpha}{\beta - 1}$. So, mean of first Pareto distribution is $EX_1 = \frac{1200}{15-1} = \frac{600}{7}$, therefore, mean loss of risks from first portfolio is $\frac{600}{7} \cdot 0.1 = \frac{60}{7} = 8.571429$ Therefore expected loss of the first portfolio is

$$ES_1 = \frac{60}{7} \cdot 1000 = 8571.43.$$

Similarly, expected loss per policy of the second portfolio is $\frac{850}{8-1} \cdot 0.1 = 12.14$.

$$ES_2 = \frac{850}{8-1} \cdot 0.1 \cdot 1000 = 12142.86.$$

That is, the overall total expected loss is 20714.29. Let us assume that the overall risk premium (i.e. the risk premium + risk loading) is 25,000.

Also, the variance of the first distribution is

$$VarS_1 = (85.714)^2 \cdot 90 + 100 \cdot (\frac{1200^2 \cdot 15}{14^2 \cdot 13}) = 1508944.$$

And similarly, variance of the second portfolio is

$$VarS_1 = (\frac{850}{7})^2 \cdot 90 + 100 \cdot (\frac{850^2 \cdot 8}{7^2 \cdot 6}) = 3293027.$$

For Wang's principle we use $g(x) = x^c \quad 0 < c < 1$, and after simplification we get $c = 0.846049$.

The following table represents the corresponding relative risk loading factors, premiums and e-values for different premium principles.

Name of principle	Value of risk loading factor	P_{X_1}	P_{X_2}	e
Expected value	0.2068963	10.34	14.66	55.33749
Variance	0.0008924898	9.92	15.08	55.35431
Standard deviation	1.408355	10.30	14.70	55.33896
Wang	c=0.846049	10.27	14.73	55.33998

Here we see that by changing of parameters of Pareto distribution change the performance level. In this case expected value principle performs the best and variance principle performs the worst.

Now, we will calculate premiums and corresponding errors of different premium principles, considering that our random claims follow a light tail distribution, e.g exponential distribution.

5.2 Example 2: Comparison of different premium principles using exponential distribution

Let us again consider two portfolios of risks, each of which has 1000 policyholders. Let for both portfolios for all risks with probability 0.9 there is no claim, with probability 0.1 there is a claim. Let the risks from the both portfolios follow the exponential distribution.

This time, we choose exponential distribution because it is a light tailed distribution, and in our first example we used Pareto distribution II which is a heavy tail distribution.

We know that if a random variable X follow exponential distribution, i.e $X \sim Exp(\lambda)$, where $\lambda > 0$, then cumulative distribution function, probability density function, survival function, moment generating function, quantile generating function, mean, variance and moments and of the distribution are as follows:

- Distribution function $F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0$.

- Density function $f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$
- Survival function $S_X(x) = e^{-\lambda x}, \quad x \geq 0.$
- Moment generating function $M_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}, \quad \lambda > t.$
- Quantile function $F^{-1}(p) = \frac{-\ln(1-p)}{\lambda}, \quad 0 \leq p < 1.$
- Expectation $E[X] = \frac{1}{\lambda}, \quad \lambda > 0.$
- Variance $Var[X] = \frac{1}{\lambda^2}, \quad \lambda > 0.$
- Moments $EX^n = \frac{n!}{\lambda^n}, \quad \lambda > 0.$

Let us consider $X_1 \sim Exp(0.002)$ and $X_2 \sim Exp(0.005)$, whereX_1 and X_2 stands for risks from first portfolio and second portfolio respectively.

Since we know that mean of exponential distribution is $EX = \frac{1}{\lambda}$. So, mean of first exponential distribution is $EX_1 = \frac{1}{0.002} = 500$, therefore, mean loss of risks from first portfolio is $500 \cdot 0.1 = 50$

Hence expected loss of the first portfolio is $ES_1 = 50 \cdot 1000 = 50,000$.

Similarly, mean of second exponential distribution is $EX_2 = \frac{1}{0.005} = 200$. Therefore overall expected loss for second portfolio is $ES_2 = (200 \cdot 0.1 \cdot 1000) = 20,000$.

That is, the overall expected loss is approximately 70,000. Let us assume that the overall risk premium (i.e. the risk premium + risk loading) is 80,000.

Also, we know that variance of exponential distribution is $VarX = \frac{1}{\lambda^2}$. So, variance of the first exponential distribution is $VarX_1 = \frac{1}{0.002^2} = 250,000$. Similarly variance of the second distribution is $VarX_2 = \frac{1}{0.005^2} = 40,000$.

We know that aggregate variance is $Var(S) = (EX)^2 \cdot VarN + EN \cdot VarX$. In our problem $N \sim Bin(1000, 0.1)$ so, $EN = 1000 \cdot 0.1 = 100$ and $VarN = 1000 \cdot 0.1 \cdot (1 - 0.9) = 90$.

Hence variance of the first portfolio is $VarS_1 = 500^2 \cdot 90 + 100 \cdot 250,000 = 47,500,000$. And variance of the second portfolio is $VarS_2 = 200^2 \cdot 90 + 100 \cdot 40,000 = 7,600,000$.

Now we will calculate premiums for each portfolio (and also for each policy) using different premium calculation principles.

5.2.1 Premiums using expected value principle

We know the expected value premium calculation principle is

$$P_X = (1+\theta)EX.$$

That is

$$80000 = (1+\theta)\cdot(ES_1+ES_2)$$
$$\Rightarrow 80000 = (1+\theta)\cdot 70000$$
$$\Rightarrow \theta = \frac{1}{7}.$$

Therefore, total premium for first portfolio is

$$P_{S_1} = 50000\cdot(1+\theta) = 57,142.86.$$

Hence premium for each policy of first portfolio is

$$P_{X_1} = \frac{57,142.86}{1000} = 57.14.$$

Similarly total premium for the second portfolio is

$$P_{S_2} = 20000(1+\theta) = 22,857.14.$$

And hence premium for each policy of the second portfolio is

$$P_{X_2} = \frac{22857.14}{1000} = 22.86.$$

5.2.2 Premiums using variance principle

We know tat variance principle for premium calculation is

$$P_X = EX + \theta\cdot VarX.$$

So, in our problem

$$80000 = ES_1+ES_2+\theta\cdot(VarS_1+VarS_2)$$

That is

$$80000 = 70000 + \theta(47500000 + 7600000)$$

$$\Rightarrow \theta = \frac{20000}{47500000 + 7600000} = 0.0001814882.$$

Therefore total premium of the first portfolio is

$$P_{S_1} = 50000 + \theta \cdot 47500000 = 58620.69.$$

And premium of each policy of the first portfolio is

$$P_{X_1} = \frac{58620.69}{1000} = 58.62.$$

Similarly total premium of the second portfolio is

$$P_{S_2} = 20000 + \theta \cdot 7600000 = 21379.31.$$

And premium of per policy of second portfolio is

$$P_{X_2} = \frac{21379.31}{1000} = 21.38.$$

5.2.3 Premiums using standard deviation principle

The formula for premium using standard deviation principle is

$$P_X = EX + \theta \cdot \sqrt{(VarX)}.$$

So, in our case

$$80000 = (ES_1 + ES_2) + \theta \cdot (\sqrt{VarS_1} + \sqrt{VarS_2})$$

That is

$$80000 = 70000 + \theta(\sqrt{47500000} + \sqrt{7600000})$$

$$\Rightarrow \theta = \frac{10000}{(\sqrt{47500000} + \sqrt{7600000})} = 1.036395.$$

Therefore, total premium for first portfolio is

$$P_{S_1} = 50000 + \theta\sqrt{47500000} = 57142.86.$$

Hence premium for each policy of first portfolio is

$$P_{X_1} = \frac{57142.86}{1000} = 57.14.$$

Similarly total premium and premium per policy of second portfolio is

$$P_{S_2} = 20000 + \theta\sqrt{7600000} = 22857.14.$$

And premium per policy of second portfolio is

$$P_{X_2} = 22.86.$$

5.2.4 Premium using exponential principle

We know that formula for premium using exponential principle is

$$P_X = \frac{1}{\beta}\ln(E[e^{\beta X}]).$$

And we know

$$E[e^{\beta X}] = M_X(\beta) = \frac{\lambda}{\lambda - \beta}.$$

Therefore

$$80 = \frac{1}{\beta}(\frac{0.002}{0.002-\beta} + \frac{0.005}{0.005-\beta}).$$

Solving of this equation for β gives

$$\beta = -0.082602293187943872821.$$

Now using this value we get premium of each policy of first portfolio is

$$P_{X_1} = 45.34.$$

and premium of each policy of second portfolio is

$$P_{X_2} = 34.66.$$

5.2.5 Premium using quantile principle

We know that p-th quantile of exponential distribution is

$$F^{-1}(p) = \frac{-\ln(1-p)}{\lambda}, \quad 0 \le p < 1.$$

So, p-th quantile of our first and second distributions are

$$F_1^{-1}(p) = \frac{-\ln(1-p)}{0.002},$$

and

$$F_1^{-1}(p) = \frac{-\ln(1-p)}{0.005},$$

So,

$$80 = \frac{-\ln(1-p)}{0.002} + \frac{-\ln(1-p)}{0.005}$$

i.e

$$80 = 500 \cdot (-\ln(1-p)) + 200 \cdot (-\ln(1-p))$$

After simplifying we get

$$p = 0.107997.$$

Therefore, premium per policy of the first portfolio is

$$P_{X_1} = \frac{-\ln(1-p)}{0.002} = 57.14.$$

And premium for each policy of the second portfolio is

$$P_{X_2} = \frac{-\ln(1-p)}{0.005} = 22.86.$$

5.2.6 Calculation of premiums using Wang's principle

For our convenience, again let us consider $g(x) = x^c \quad 0 < c < 1$. We know that survival function of exponential distribution is

$$S_X = e^{-\lambda x}, \quad \text{x} \ge 0.$$

So survival functions of the first and second portfolio considering with the probability are

$$S_{S_1} = 0.1 \cdot 1000 \cdot S_{X_1} = 0.1 \cdot 1000 \cdot e^{-\lambda_1 x},$$

i.e

$$S_{S_1} = 0.1 \cdot 1000 \cdot e^{-0.002x},$$

and

$$S_{S_2} = 0.1 \cdot 1000 \cdot S_{X_2} = 0.1 \cdot 1000 \cdot e^{-\lambda_2 x},$$

i.e

$$S_{S_2} = 0.1 \cdot 1000 \cdot e^{-0.005x}.$$

Now we know that premium using Wang's principle is

$$P_X = \int_0^\infty S_X(x)dx.$$

Therefore in our case

$$80 = 0.1 \cdot (\int_0^\infty e^{-0.002\cdot \text{ cx}} dx + \int_0^\infty (e^{-0.005\cdot \text{cx}} dx)$$

$$\Rightarrow 800 = [\frac{e^{-0.002\cdot \text{c}x}}{-0.002c} + \frac{e^{-0.005\text{c}x}}{-0.005c}]_0^\infty$$

$$\Rightarrow 800 = \frac{1}{0.002c} + \frac{1}{0.005c}$$

$$\Rightarrow c = \frac{7}{8}.$$

Now using this value, if we integrate the transferred survival functions of both portfolios we get,

$$P_{S_1} = 0.1 \cdot 1000 \cdot (\int_0^\infty e^{-0.002cx} dx)$$

$$\Rightarrow P_{S_1} = 100 \cdot 571.4236 = 57142.86,$$

i.e premium for each policy of the first portfolio is

$$P_{X_1} = \frac{57142.86}{1000} = 57.14.$$

In similar way we get

$$P_{S_2} = 22857.14,$$

i.e premium for each policy of the second portfolio is

$$P_{X_1} = \frac{22857.14}{1000} = 22.86.$$

The following table summarises the value of the relative risk loading factor, premium per policy of the first portfolio, premium per policy of the second portfolio and the estimated error for different premium principles.

Name of principle	Value of risk loading factor	P_{X_1}	P_{X_2}	e
Expected value	$\frac{1}{7}$	57.14	22.86	254.4231
Variance	0.0001814882	58.62	21.38	254.3714
Standard deviation	1.036395	57.14	22.86	254.4231
Exponential	-0.08260229318794387	45.34	34.66	255.4484
Quantile p-th	p=0.107997	57.14	22.86	254.4229
Wang	c=$\frac{7}{8}$	57.14	22.86	254.4229

We see that expected value principle and standard deviation principle gives exactly same answer,this is obvious because we know the expectation and standard deviation of exponential distribution both are equal to $\frac{1}{\lambda}$. Also, see that premiums and errors of quantile principle and Wang's principle are exactly same this is because both of them are considering the tail distribution. Here we represent them as rounding to two decimal places. Moreover, we see that if our random claims follow exponential distribution the variance principle performs the best.

In our practical examples, we do not calculate premium using combined variational, tail standard deviation and Dutch principles as they have two parameters and need further research to use practically. We also do not use Swiss and Esscher principles to find premiums as they seem to us more theoretical and required more research to use them practically.

6 Conclusion

The crucial task to actuaries is to determine the risk loading for premiums, there is no such premium calculation principle which is accepted by all (re)insurer to determine the appropriate risk loading.

In our research, we calculate the risk loading and corresponding premiums using different premium calculation principles. We also examine the performance level of each premium principle.

Theoretically, we know that among all existing premium calculation principles, Wang's premium principle, by transforming the layer premium density, is the best as it satisfies most of the properties of a premium principle but in practice we see that:

- In the case of heavy tail distribution (e.g Pareto II), variance principle performs the best, whereas quantile principle performs the worst. On the other hand Wang principle performs same as standard deviation principle though the premium values are slightly different.
- In the case of light tail distribution (e.g Exponential), expected value principle and standard deviation principle performs exactly same. Again variance principle performs the best and the worst performance goes to exponential principle. Moreover, the performance of quantile principle and wang principle are exactly same.

We also see that changing the value of parameters of the distribution does not affect the performance level of the wang principle comparing with other principles. That is if we change the value of the parameters the wang premium calculation principle perform almost same as before.

Obviously, these small scenarios are not sufficient to say that Wang's premium principle does not perform well. But it gives a hint that the choice of premium principle might also depend on the nature of a particular risk portfolio. Also, the comparison by justification of premiums is just one possibility to measure the quality of a premium principle. The list of nice properties still makes the Wang's principle an appealing one and thus it certainly requires further attention.

References

[1] Buhlmann,H (1980) *An economic premium principle* ASTIN Bulletin, 11, 52-60.

[2] Denneberg, D (1990) *Premium calculation why standard deviation should be replaced by absolute deviation* ASTIN Bulletin, 20, 181-190.

[3] Denneberg,D.(1994), *Non-Additive Measure And Integral* Kluwer Academic Publishers, Boston.

[4] Fishburn, P. (1982), *The Foundation of Expected Utility* Reidel,Dordrecht,Holland.

[5] E.Furman and Z.Landsman (2006), *On Some Risk-Adjusted Tail-Based Premium Calculation Principles*, Journal of Actuarial Practice, 13

[6] Gerber, H. U (1979), *An Introduction to Mathematical Risk Theory Huebner Foundation Monograph*, Wharton School, University of Pennsylvania.

[7] Machina, M. (1982), *'Expected Utility' analysis without the independence axiom* Econometrica, 50(2), 217-323.

[8] Roger J.A. Laeven and Marc J. Goovaerts (2007), *Premium Calculation and Insurance Pricing*.

[9] Virginia R. Young, *Premium principles*, Encyclopedia of Actuarial Science, http://www.wiley.com/legacy/wileychi/eoas/pdfs/TAP027.

[10] Wang S. (1995), *Insurance Pricing And Increased Limits Ratemaking By Propotional Hazards Transforms* Insurance Mathematics and Economics, 17, 43-54.

[11] Wang S. (1996), *Premium Calculation by Transforming The Layer Premium Density* ASTIN Bulletin, 26, 71-92

[12] Wang S. Young R. and Panjer H.(1997), *Axiomatic Characterization of Insurance Prices* Insurance Mathematics and Economics, 21, 173-183.

Books

[13] David C. M. Dickson: *Insurance Risk and Ruin*.

[14] Pavel Cizek, Wolfgang Kari Hardle, Rafat Weron: *Statistical tools for Finance and Insurance*.

[15] Roger J. Gray and Susan M. Pitts: *Risk Modelling in general Insurance*.

[16] De Vylder, F. Etienne, Gorvaerts, Marc, Haezendonck,J: *Premium Calculation in Insurance.*

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