LOCALIZATION IN RINGS OF FINITE UNIFORM DIMENSIONS

by

Maung He Tin

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ABSTRACT

A definition of torsion theory τ on the category R-mod of left R-modules is given, using an equivalence relation on the class of all injective left R-modules. For a torsion theory τ, the category R-mod/τ of τ-torsion-free and τ-injective modules is shown to be abelian. A necessary and sufficient condition is given under which a ring A has finite left uniform dimension and zero left singular ideal.

Let I denote an ideal of a ring R, and σ the torsion radical cogenerated by the R-injective envelope E(R/I) of R/I. For a left R-module M, we denote $M_\sigma = Q_\sigma(M)$, where Q_σ is the quotient functor corresponding to the torsion radical σ. The focal point of our dissertation is the following:

THEOREM (Beachy). Let I be an ideal of R and let σ be the torsion radical cogenerated by E(R/I).

Then the following conditions are equivalent.

(1) $(R/I)_\sigma$ is a finite direct sum of simple objects in the category of σ-torsion-free and σ-injective modules.

(2) The ring R/I has finite left uniform dimension and zero left singular ideal.

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INTRODUCTION

Let R be a ring with unit element. We define a preordering on the class $\underline{E}$ of injective left R-modules by setting $E_2 \leq E_1$ if and only if E_1 can be embedded in a direct product of copies of E_2. Then we define an equivalence relation as follows: E_1 and E_2 are equivalent if and only if $E_1 \leq E_2$ and $E_2 \leq E_1$. An equivalence class of $\underline{E}$ is called a <u>torsion</u> <u>theory</u> on R-mod, the category of left R-modules. This torsion theory turns out to be hereditary. Let R-tors denote the space of all torsion theories on R-mod.

In Chapter 1, we discuss when a torsion theory $\tau \in$ R-tors forms a "torsion theory" as defined in Faith [1973]. The material in Chapter 1 as in Chapter 2 is taken mainly from Golan [1975]. We prove that the category R-mod/τ of all τ-torsion-free and τ-injective modules is abelian, following Beachy [1974].

Chapter 2 concerns some special torsion theories such as prime torsion theories and semiprime torsion theories. Conditions are given under which a ring is left semiartinian. We discuss some applications to HNP-rings.

In Chapter 3, we discuss the following theorem:

THEOREM. The following conditions for a ring A are equivalent.

(1) A has finite left uniform dimension and zero

left singular ideal.

(2) A is left quasi-order of a semisimple ring.

Another equivalent condition for a ring A to have finite left uniform dimension and zero left singular ideal is given in the following theorem, which is due to Beachy [1976]. Let I denote a two-sided ideal of a ring R, and σ the torsion radical cogenerated by the R-injective envelope E(R/I) of R/I. For a left R-module M, we denote $M_\sigma = Q_\sigma(M)$, where Q_σ is the quotient functor corresponding to the torsion radical σ.

THEOREM (Beachy). Let I be an ideal of R and let σ be the torsion radical defined by E(R/I).

Then the following conditions are equivalent.

(1) $(R/I)_\sigma$ is a finite direct sum of simple objects in R-mod/σ

(2) The ring R/I has finite left uniform dimension and zero left singular ideal.

This theorem has the following corollary, which is due to Lambek and Michler.

COROLLARY. The following conditions are equivalent:

(1) $I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple artinian.

(2) σ is perfect, I_σ is an ideal of R_σ, and the ring R/I has finite left uniform dimension and zero left singular ideal.

CHAPTER 1

BACKGROUND RESULTS IN TORSION THEORY

1.1 Preliminaries

All rings of this discussion will be associative with unit element and all modules unital. Our first interest is the term "torsion" for arbitary rings and we shall define a "torsion theory" which turns out to be hereditary.

We have met the term "torsion" when we study the following familiar theorems of classical algebra:

THEOREM (A). A finitely generated module over a principal ideal domain is free if and only if it is torsion-free.

THEOREM (B). A finitely generated module over a principal ideal domain is bounded if and only if it is a torsion module.

Theorems (A) and (B) are corollaries of the following:

THEOREM (C). Every finitely generated module over a principal ideal domain is a direct sum of cyclic submodules.

In the following discussion, a few results from elementary localization theory will be proved and some will be illustrated by examples. Generally, definitions and results from text books such as Cohn [1971], Faith [1973], Golan [1975], Stenström [1975] will be freely used. Propositions whose proofs are omitted are indicated by letters A, B, C etc.

Let us first recall the following definition:

Definition: A reflexive and transitive relation on a set S is called a preordering on S.

We shall consider the class $\underline{E}$ of all injective left R-modules. We may preorder the class $\underline{E}$ by setting $E_2 \leq E_1$ if and only if E_1 can be embedded in a direct product of copies of E_2. We say that two injective left R-modules E_1 and E_2 are equivalent if and only if $E_1 \leq E_2$ and $E_2 \leq E_1$. The preordering $\leq$ in the class $\underline{E}$ will then be characterized by using the sets

$S_i = \{ M \in \text{R-mod} \mid \text{Hom}_R(M, E_i) = 0 \}$ where R-mod denotes the category of left R-modules.

We shall prove in PROPOSITION (1.1.1.) that $E_1 \leq E_2$ if and only if $S_1 \subseteq S_2$.

We call an equivalence class of $\underline{E}$ a torsion theory on R-mod.

We will denote the collection of all such torsion theories by R-tors.

If $\tau \in$ R-tors we say that a left R-module M is τ-torsion-free if and only if $E \leq E(M)$ for some E in τ, where E(M) denotes the injective hull of M. Sometimes we shall just say M is torsion-free if it is clear from the context which torsion theory is meant. The class of all torsion-free left R-modules will be denoted by $\underline{\underline{F}}$ (or by $\underline{\underline{F}}_\tau$ if we have to indicate the torsion theory concerned).

We say that a left R-module M is τ-torsion if and only if $\mathrm{Hom}_R(M, E) = 0$ for some E in τ. The class of all torsion left R-modules will be denoted by $\underline{\underline{T}}$ (or by $\underline{\underline{T}}_\tau$ according to the convention mentioned above). Sometimes the torsion theory will be denoted by $(\underline{\underline{T}}, \underline{\underline{F}})$, $\underline{\underline{T}}$ by Ann $\underline{\underline{F}}$ and $\underline{\underline{F}}$ by Ann $\underline{\underline{T}}$.

Let us denote, for any class B of R-modules,

$$B^R = \{ Y \in R\text{-mod} \mid \mathrm{Hom}_R(X, E(Y)) = 0 \text{ for all } X \in B \}$$

and for any class C of R-modules,

$$C^L = \{ X \in R\text{-mod} \mid \mathrm{Hom}_R(X, E(Y)) = 0 \text{ for all } Y \in C \}$$

Also we shall use the following notation:

$$B^{\perp} = \{ Y \in R\text{-mod} \mid \mathrm{Hom}_R(X, Y) = 0 \text{ for all } X \in B \}$$

and $${}^{\perp}C = \{ X \in R\text{-mod} \mid \mathrm{Hom}_R(X, Y) = 0 \text{ for all } Y \in C \}.$$

We shall prove in PROPOSITION (1.1.3) that for a torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$, $\underline{\underline{T}} = \underline{\underline{F}}^L$ and $\underline{\underline{F}} = \underline{\underline{T}}^R$. This will establish the fact that the ordered pair $(\underline{\underline{T}}, \underline{\underline{F}})$ in the category R-mod forms a "torsion theory" as defined in Faith [1973]. From Proposition 16.8B of Faith, we get the following:

PROPOSITION (1.1.A). If $\tau \in$ R-tors and M $\in$ R-mod, then

(1) There exists a radical T on R-mod such that $N \subseteq M$ implies that $T(M) \cap N = T(N)$

(2) $\underline{\underline{T}} = \{ M \mid T(M) = M \}$ and $\underline{\underline{F}} = \{ M \mid T(M) = 0 \}$

T(M) in the above proposition is called the τ-torsion submodule of M.

If we are dealing with more than one torsion theory we will use a suffix to indicate the torsion theory concerned as: $T_\tau(-)$. Sometimes we denote $\mathrm{rad}_\tau(-) = T_\tau(-)$ and it is called the τ-torsion-functor or a torsion radical.

Given any class of left R-modules, we can obtain a torsion class containing it and its associated torsion-free class.

For any class $\underline{A}$ of left R-modules, let us consider the complete set $\underline{A}_o$ of representatives of isomorphism classes of cyclic submodules of members of $\underline{A}$ and form the injective left R-module $E_o = \prod \{E(M) \mid M \in \underline{A}_o\}$. We denote the equivalence class of E_o by $\chi(\underline{A})$ and call it the torsion theory cogenerated by $\underline{A}$. Then we have $\underline{A} \subseteq \underline{\underline{F}}_{\chi(\underline{A})}$. If $\underline{A}$ contains a single member M, we shall write $\chi(M)$ for $\chi(\{M\})$. Then $\chi(M)$ is precisely the equivalence class of E(M). We shall denote the torsion theory cogenerated by $\underline{A}^R$ by $\xi(\underline{A})$ and call it the torsion theory generated by $\underline{A}$. Then we have $\underline{A} \subseteq \underline{\underline{T}}$ where $\underline{\underline{T}}$ is the $\xi(\underline{A})$-torsion class.

If ξ is the class of all injective cogenerators of R-mod, then the ξ-torsion class is just $\{0\}$ and if χ is the class containing only the 0 module then the χ-torsion class is R-mod, where $\xi = (0, \text{R-mod})$ and $\chi = (\text{R-mod}, 0)$.

Let $\mathcal{A}$ be an abelian category. A semi-Serre class or subcategory of $\mathcal{A}$ is a non-empty full subcategory $\mathcal{S}$ such that for every exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, we have B in $\mathcal{S}$ whenever A and C are in $\mathcal{S}$.

$\mathcal{S}$ is a Serre class or subcategory of $\mathcal{A}$ if it is true that B is in $\mathcal{S}$ if and only if A and C are in $\mathcal{S}$.

Let $T: \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S}$ by the canonical functor and

$S: \mathcal{A}/\mathcal{S} \to \mathcal{A}$ the section functor (if it exists). $\mathcal{S}$ is called a <u>localizing subcategory</u> of $\mathcal{A}$ whenever the section functor $S: \mathcal{A}/\mathcal{S} \to \mathcal{A}$ exists. Then, the composite functor $ST: \mathcal{A} \to \mathcal{A}$ is called the <u>localizing functor</u>.

PROPOSITION (1.1.1). For any two injective modules E_i, $i = 1, 2$, let us denote

$$S_i = \{M \in R\text{-mod} \mid \mathrm{Hom}_R(M, E_i) = 0\},\ i = 1, 2.$$

Then $E_1 \leq E_2$ if and only if $S_1 \subseteq S_2$.

<u>Proof</u>. Suppose $S_1 \subseteq S_2$ and let the cardinality of the set $\mathrm{Hom}_R(E_2, E_1)$ be denoted by d. Form the product E of d copies of E_1. If we take an element f in $\mathrm{Hom}_R(E_2, E_1)$ there exists a canonical R-homomorphism $\hat{f}$ of E_2 into E given by

$$\hat{f} : x \longmapsto \langle xf \rangle$$

where f ranges over all $\mathrm{Hom}_R(E_2, E_1)$ as shown in the diagram:

$$\begin{array}{ccc} & & E_2 \\ & \hat{f} \swarrow & \downarrow f \\ E & \longrightarrow & E_1 \end{array}$$

Let $K = \ker(\hat{f})$. Since any R-homomorphism $g: K \to E_1$ can be extended to E_2 by the injectivity of E_1, $\mathrm{Hom}_R(K, E_1) = 0$. So we find that K belongs to S_1 and hence it belongs to S_2 by assumption; hence $\mathrm{Hom}_R(K, E_2) = 0$. Now K is a submodule of E_2 and so $K = 0$. This means that $\hat{f} : E_2 \to E$ is an embedding and so we find that $E_1 \leq E_2$.

Conversely, suppose that $E_1 \leq E_2$. Let us take an element N in the set S_1. Then $\mathrm{Hom}_R(N, E_1) = 0$. Thus by the assumption E_2 can be embedded in a direct product E' of copies of E_1. Let $g: E_2 \longrightarrow E'$ be the embedding and $p: E' \longrightarrow E_1$ be the projection. Then the composition map

$$N \xrightarrow{h} E_2 \xrightarrow{g} E' \xrightarrow{p} E_1$$

is necessarily the zero map. Hence $gh = 0$ and $h = 0$, so that $\mathrm{Hom}_R(N, E_2) = 0$. Then N is in the set S_2. Hence we have proved the proposition.

From the above proposition we may deduce the following:

COROLLARY (1.1.2). Injective modules E_1, E_2 in R-mod are equivalent if and only if $S_1 = S_2$.

Let us consider some simple consequences of these definitions. Let us denote the injective hull of M by E(M).

If M is τ-torsion-free, then $E_i \leq E(M)$ for some $E_i \in \tau$ and hence $E_j \leq E(M)$ for all E_j in τ.

If M is τ-torsion, then $\mathrm{Hom}_R(M, E_i) = 0$ for some E_i in τ. Here also we may prove that $\mathrm{Hom}_R(M, E_j) = 0$ for all E_j in τ.

PROPOSITION (1.1.3). If $\tau = (\underline{\underline{T}}, \underline{\underline{F}})$, then $\underline{\underline{T}} = \underline{\underline{F}}^L$ and $\underline{\underline{F}} = \underline{\underline{T}}^R$.

<u>Proof</u>: Suppose M is in $\underline{\underline{T}}$ and N is in $\underline{\underline{F}}$. Since N is τ-torsion-free, $E \leq E(N)$ for some E in τ. Suppose, if possible, $\mathrm{Hom}_R(M, E(N)) \neq 0$. Since E(N) is embeddable in a direct product of copies of E we would then have a nonzero R-homomorphism $M \longrightarrow E(N) \longrightarrow \prod E \longrightarrow E$. But M is in $\underline{\underline{T}}$ and so $\mathrm{Hom}_R(M, E) = 0$ for

all E in τ, which is a contradiction. So $\mathrm{Hom}_R(M, E(N)) = 0$, i.e. M is in $\underline{\underline{F}}^L$.

Conversely, suppose M is in $\underline{\underline{F}}^L$. Then in particular $\mathrm{Hom}_R(M, E) = 0$ for all E in τ, since $\mathrm{Hom}_R(M, E(N)) = 0$ for all N in $\underline{\underline{F}}$. So M is in $\underline{\underline{T}}$. Thus $\underline{\underline{T}} = \underline{\underline{F}}^L$.

To prove that $\underline{\underline{F}} = \underline{\underline{T}}^R$, take N in $\underline{\underline{F}}$. Then $\mathrm{Hom}_R(M, E(N)) = 0$ for all M in $\underline{\underline{T}}$ from the first part. So $\underline{\underline{F}} \subseteq \underline{\underline{T}}^R$.

Conversely, assume that N is in $\underline{\underline{T}}^R$, i.e. $\mathrm{Hom}_R(M, E(N)) = 0$ for all M in $\underline{\underline{T}}$. Let E be in τ and let X be a set whose cardinality equals the cardinality of $\mathrm{Hom}_R(E(N), E)$. Then we have a canonical R-homomorphism $g\colon E(N) \longrightarrow E^X$ given by $x \longmapsto \langle x f \rangle$ where the f ranges over all $\mathrm{Hom}_R(E(N), E)$. Let $K = \ker(g)$. Then as in the proof of PROPOSITION (1.1.1) we find that $\mathrm{Hom}_R(K, E) = 0$ so that K is in $\underline{\underline{T}}$. By assumption $\mathrm{Hom}_R(K, E(N)) = 0$. Since K is a submodule of E(N), we have $K = 0$ and thus we get the embedding of E(N) into E^X. So N is in $\underline{\underline{F}}$ and we have established the equality of the two sets. This proves the proposition.

This proposition establishes the fact that the ordered pair $(\underline{\underline{T}}, \underline{\underline{F}})$ in R-mod forms a "torsion theory" as defined in Faith. The notion of a torsion theory due to Dickson may be found, for example, in Stenström. From Proposition 16.8B of Faith, we get the following proposition which shows the relation between $B^{\perp}$ and B^R, and the relation between ${}^{\perp}C$ and C^L.

PROPOSITION (1.1.B). An ordered pair $(\underline{\underline{T}}, \underline{\underline{F}})$ of full subcategories of R-mod which is a torsion theory satisfies the following equivalent conditions:

(1) $\underline{\underline{T}}$ is a localizing subcategory and $\underline{\underline{F}} = \underline{\underline{T}}^{\perp}$

(2) $\underline{\underline{F}}$ is a semi-Serre class, containing the product and injective hulls of any family of objects of $\underline{\underline{F}}$ and $\underline{\underline{T}} = {}^{\perp}\underline{\underline{F}}$.

We will need the following later on.

PROPOSITION(1.1.C). If $\tau \in$ R-tors and $M \in$ R-mod, then

(1) $T_{\tau}(M) = M$ if and only if M is τ-torsion

(2) $T_{\tau}(M) = 0$ if and only if M is τ-torsion-free.

(3) If $f \in \mathrm{Hom}_R(M, N)$ then $T_{\tau}(M)f \subseteq T_{\tau}(N)$;

(4) $T_{\tau}(M / T_{\tau}(M)) = 0$

(5) If N is a submodule of M then $T_{\tau}(M) \cap N = T_{\tau}(N)$.

1.2 Absolute closure

If $\tau \in$ R-tors, then a submodule N of a left R-module M is said to be τ-dense in M if and only if M/N is τ-torsion.

The set of all τ-dense left ideals of the ring R will be denoted by $\mathcal{L}_{\tau}$.

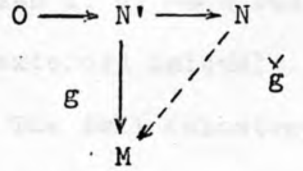
A submodule N of a left R-module M is said to be τ-**closed** in M if and only if M/N is τ-torsion-free.

A unique minimal element of the family of all τ-**closed** submodules of M containing N is called the τ-**closure** of N in M.

We call the τ-**closure** of M in inj hull M the τ-injective hull of M and denote it by $E_{\tau}(M)$. We denote

$E_\tau(M / T_\tau(M))$ by $Q_\tau(M)$.

A module M in R-mod is called τ-<u>injective</u> if and only if every diagram of the form



where N/N' is τ-torsion, can be completed commutatively.

If a left R-module M is τ-torsion-free and τ-injective we say that M is <u>absolutely τ-closed</u>

Sometimes τ-injective modules are called <u>divisible</u>, absolutely τ-closed modules are called <u>closed</u>.

Let T_σ be the torsion radical corresponding to a torsion theory $\sigma \in$ R-tors. Setting $C_\sigma(N)$ to be the inverse image in M of $\mathrm{rad}_\sigma(M/N)$, we find that N is σ-dense if and only if $C_\sigma(N) = M$ and N is σ-closed if and only if $C_\sigma(N) = N$. Thus a torsion radical T_σ defines a closure operation on submodules $N \subseteq M$. The closure of N can be described as

$$C_\sigma(N) = \{ m \in M \mid Im \subseteq N \text{ for some } I \in \mathcal{L}_\sigma \}.$$

If $W \in$ R-mod, then for any left R-module M, let $\mathrm{rad}_W(M)$ be the intersection of all kernels of homomorphisms from M into W. Then $C_\sigma(N)$ is $\{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, W)$ such that $f(N) = 0\}$. $E_\sigma(M)$ is the σ-closure of M in its injective hull E(M).

A module M is σ-injective if each homomorphism f: N'→M such that N' is a σ-dense submodule of N can be extended to N. This reduces to the usual definition of injectivity if T_σ is the identity functor. Now we find that M is σ-injective if and only if M is σ-closed in E(M), i.e. if and only if $M = E_\sigma(M)$.

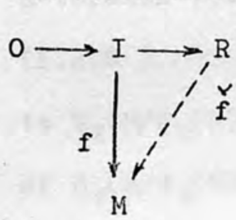
A module M is absolutely σ-closed if and only if each homomorphism $f: N' \to M$ such that N' is a σ-dense submodule of N can be extended uniquely to N.

The full subcategory of R-mod determined by all absolutely σ-closed modules will be denoted by R-mod/σ. We will need the following propositions later on.

PROPOSITION (1.2.A). Let $\tau \in$ R-tors. For a left R-module M the following conditions are equivalent:

(1) M is τ-injective

(2) Each R-homomorphism $f: I \to M$ such that $I \in \mathcal{L}_\tau$ can be extended to R in the following diagram



(3) M is τ-closed in E(M)

(4) $\mathrm{Ext}^1_R(N, M) = 0$ for every τ-torsion left R-module N

(5) M is a direct summand of a τ-closed submodule of any left R-module containing it.

PROPOSITION (1.2.B). Let $\tau \in$ R-tors. For a left R-module M the following conditions are equivalent:

(1) M is absolutely τ-closed

(2) M is τ-torsion-free and is a τ-closed submodule of every τ-torsion-free left R-module containing it

(3) Any diagram of the form

$$\begin{array}{ccl} 0 \longrightarrow N' & \longrightarrow & N \qquad (N/N' \in \underline{\underline{T}}_{\tau}) \\ f \big\downarrow & \swarrow \check{f} & \\ M & & \end{array}$$

can be completed commutatively in a unique manner.

1.3 Localization

Let $\tau \in$ R-tors. We can define a functor Q_τ on R-mod as follows:

(1) If M is a left R-module, set $Q_\tau(M) = E_\tau(M/T_\tau(M))$

(2) If $N \in$ R-mod/τ and $f: M \longrightarrow N$, f factors through $M/T_\tau(M)$ since $(T_\tau(M))f \subseteq T_\tau(N) = 0$. Now N is τ-torsion-free and τ-injective. So by PROPOSITION (1.2.B), there is a unique extension of f to $E_\tau(M/T_\tau(M))$ since $M/T_\tau(M)$ is a τ-dense submodule of $E_\tau(M/T_\tau(M))$. This shows how to define the functor Q_τ on homomorphisms (and also shows that Q_τ is a left adjoint of the inclusion functor R-mod/$\tau \xrightarrow{U_\tau}$ R-mod.) If $f: M \longrightarrow N$ is an R-homomorphism, then $T_\tau(M) f \subseteq T_\tau(N)$ by PROPOSITION (1.1.C) and so f induces an R-homomorphism $\bar{f} : M/T_\tau(M) \longrightarrow N/T_\tau(N)$. By τ-injectivity there exists an R-homomorphism g making the diagram

$$\begin{array}{ccc} 0 \longrightarrow M/T_\tau(M) & \longrightarrow & E_\tau(M/T_\tau(M)) \\ \bar{f} \big\downarrow & & g \big\downarrow \\ 0 \longrightarrow N/T_\tau(N) & \longrightarrow & E_\tau(N/T_\tau(N)) \end{array}$$

commute and g is unique by PROPOSITION (1.2.B).

Then set $U_\tau Q_\tau(f) = g$. We call $U_\tau Q_\tau$ the <u>localizing functor</u>.

We now have the following:

PROPOSITION (1.3.A). For $\tau \in$ R-tors, the functor $U_\tau Q_\tau$ is left exact.

PROPOSITION (1.3.1). R-mod/τ is an abelian category.

<u>Proof</u>. If f is in Hom_R (M, N) and M, N are in R-mod/τ then ker(f) is τ-closed in M, because N is τ-torsion-free. Since any submodule of a τ-torsion-free module is τ-torsion-free, and a τ-closed submodule of a τ-injective module is τ-injective, ker(f) is in R-mod/τ.

The fact that a τ-injective submodule of a τ-torsion-free module is τ-closed implies that N/Mf is τ-torsion-free for a monomorphism f, since Mf is τ-injective. So M is the kernel of the natural homomorphism $N \longrightarrow E_\tau(N/Mf)$.

This shows that R-mod/τ has kernels, and that every monomorphism is a kernel.

Now if $f : M \rightarrow N$ is a homomorphism in R-mod/τ then Q_τ(coker (f)) serves as the cokernel of f in R-mod/τ.

To show that every epimorphism is a cokernel, let f be an epimorphism in R-mod/τ. Then Q_τ(coker (f)) = 0 and so Mf must be τ-dense in N, which shows that N is the cokernel of the inclusion ker(f) $\longrightarrow$ M. Also R-mod/τ has finite direct sums and zero. Thus R-mod/τ is an abelian category.

PROPOSITION (1.3.2). Q_τ is an exact functor.

Proof. Since Q_τ is a left adjoint of U_τ, it is right exact. It is also left exact, since Q_τ preserves monomorphisms. So Q_τ is exact.

The functor Q_τ is called the quotient functor.

1.4 Special torsion theories

For a ring R we select a complete set of representatives of the isomorphism classes of simple left R-modules and denote it by R-simp. If $\tau \in$ R-tors we say that τ is semisimple if and only if $\tau = \xi(\underline{A})$ for some subset $\underline{A}$ of R-simp. Or equivalently a (hereditary) torsion class is semisimple if it is generated by a class of simple modules. (In Stenström, it is referred to as simple.) If, in particular, $\underline{A} = \{M\}$, we say that τ is simple following Golan.

A left R-module M is said to be semiartinian if and only if M is ξ(R-simp)-torsion, i.e., if and only if every nonzero homomorphic image of M has a nonzero socle. A ring R is said to be left semiartinian if and only if R is semiartinian as a left module over itself. This is equivalent to the fact that $\xi(\text{R-simp}) = \chi$.

R is left semiartinian if and only if every nontrivial $\tau \in$ R-tors is semisimple. This is equivalent to the fact that the lattice R-tors is boolean.

Let $\tau \in$ R-tors. For any left R-module M, the R-homomorphism $\bar{\tau}_M : M \to Q_\tau(M)$ can be factored as

$$M \xrightarrow{g_M} R_\tau \otimes_R M \xrightarrow{\check{\tau}_M} Q_\tau(M)$$

where $g_M : m \longmapsto 1 \otimes m$ and $\check{\tau}_M : \sum(s_i \otimes m_i) \longmapsto \sum s_i (m_i \overline{\tau}_M)$. If $\check{\tau}_M$ is an isomorphism in the above factorization, we say that τ is a <u>perfect</u> torsion theory.

τ is a perfect torsion theory if and only if every left R_τ-module is τ-torsion-free as a left R-module.

1.5 Examples

All examples have been concentrated in the last sections of the chapters. Some examples are given to illustrate the concepts introduced in the text and some are given to introduce new concepts to be used later on. Proofs given in these examples are usually relatively more elementary than those given elsewhere.

<u>Example</u> (1.5.1). Let R be a left Ore ring and let T(M) be the set of all torsion elements of M.

Then (1) T(M) is a submodule of M

(2) T(M) is a torsion module and $M/_{T(M)}$ is a torsion-free module

(3) T is a radical on R-mod

(4) If $L \subseteq M$, then $T(L) = T(M) \cap L$.

(5) Let $B = \{ M \mid T(M) = M \}$ and

$C = \{ M \mid T(M) = 0 \}$.

Then (B, C) is a torsion theory on R-mod.

Example (1.5.2). Let R be a commutative PID.

Then a finitely generated module M over R is torsion-free if and only if it is free; and M is a torsion module if and only if it is bounded.

Example (1.5.3). Let k be a commutative field and $k[x_1, x_2, \ldots]$ be a polynomial ring over k in a countably infinite set of indeterminates. Let A be the factor ring produced from it by dividing out the ideal generated by all expressions $x_i x_j$, $i \neq j$ and $x_i^2 - x_i$. By denoting the image of x_i in A by $\overline{x}_i$, let I be the ideal of A generated by $\overline{x}_i$, $i = 1, 2, \ldots$

Then: (i) $\underline{D} = \{A, I\}$ is a left Gabriel topology on A

(ii) $\underline{D} = \mathcal{L}_\tau$ for some $\tau = (\underline{\underline{T}}, \underline{\underline{F}})$ in A-tors such that $\underline{\underline{T}}$ is a TTF-class.†

(iii) $\underline{D} = \{{}_AH \subseteq A \mid I(A/H) = 0\} = \{{}_AH \subseteq A \mid I \subseteq H\}$

(iv) $\underline{S} = \{M \in A\text{-mod} \mid IM = 0\}$ is a prelocalizing subcategory and is precisely the set $\{M \in A\text{-mod} \mid \mathrm{Hom}_A(I, M) = 0\}$

(v) $Q_\tau(M) \cong \mathrm{Hom}_A(I, M/IM)$.

Proof (i). We can easily verify the following properties:

T1. A belongs to $\underline{D}$, and every left ideal containing a member of $\underline{D}$ belongs to $\underline{D}$.

T2. The intersection of any two members of $\underline{D}$ belongs to $\underline{D}$. Since A is a member of $\underline{D}$, $\underline{D}$ is a filter on A.

Now we shall verify the following condition:

T3. If $\mathfrak{J}$ belongs to $\underline{D}$ and r belongs to A, then $\mathfrak{J}r^{-1}$ belongs to $\underline{D}$, where $\mathfrak{J}r^{-1} = \{x \in A \mid xr \in \mathfrak{J}\}$.

† A TTF-class is a class of modules which is a torsion class in one torsion theory and a torsion-free class in another.

In the literature Ir^{-1} is sometimes denoted by $ann_A\ (r + I)/I$ or $^{\perp}(r/I)$ or $ann_A\ (r/I)$ or $(I : r)$.

For instance, take Ir^{-1} which is clearly a subset of I. If r is not in I, then Ir^{-1} is a left ideal similar to I and is a maximal left ideal. So $Ir^{-1} = I$. If r is in I, then clearly $Ir^{-1} = A$. Hence D satisfies T3.

It remains to show that D satisfies the following:

T4. If J belongs to D and if Id^{-1} belongs to D for all d in J, then I belongs to D.

Here also we have to check only two cases. If we take A in D, we have already shown above that $Id^{-1} = I$ for all d in A and I is in D. Now $Ad^{-1} = A$ and so D satisfies T4 as well. Hence we have proved that D is a (left) Gabriel topology.

Alternative terms for a Gabriel topology is a topologizing family or idempotent filter or idempotent topologizing filter (IT-filter). Formally, a nonempty set D of left ideals of A is called an <u>IT-filter</u> provided D satisfies T3 and T4.

Then it may be proved that D satisfies T_2 and assuming A is in D it satisfies T_1 also.

<u>Proof</u> (ii). For the given Gabriel topology D we have a torsion theory $\tau = (T, F)$. We find that $D = \mathcal{L}_\tau$ the set of all τ-dense left ideals of A. Since the intersection of all τ-dense left ideals of A belongs to D, T is a TTF-class. Sometimes such a torsion theory τ is called jansian.

<u>Proof</u> (iii). Let us consider the set $S = \{ M \in A\text{-mod} \mid IM = 0\}$. Then it is easy to see that S is closed under taking submodules, homomorphic images, direct sums, and extensions. So S is the

class of all τ'-torsion modules for some τ' in A-tors. Then D may be described as stated. This proves (iii).

Proof (iv). If M is not τ-torsion, then there exists a nonzero element m in M such that Im $\neq 0$ and so m defines a nonzero A-homomorphism $I \longrightarrow M$ given by $a \longmapsto am$. So $\{M \in A\text{-mod} \mid Hom_A\ (I, M) = 0\}$ is a subset of the τ-torsion class.

Conversely, let M be a τ-torsion module. Then we find that $Hom_A\ (I, A/I) = 0$ since I is idempotent and so A/I is τ-torsion. If g belongs to $Hom_A\ (I, M)$, then from the exact sequence $0 \longrightarrow ker(g) \longrightarrow I \longrightarrow M$ we get I/ker(g) is τ-torsion. Now let us consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(g) & \longrightarrow & ker(g) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I/ker(g) & \longrightarrow & A/ker(g) & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

From the exactness of the last line we find that A/ker(g) is τ-torsion, i.e. $I \subseteq ker(g)$. This means that $g = 0$, so that S is precisely the set

$$\{M \in A\text{-mod} \mid Hom_A\ (I, M) = 0\}.$$

Now the subclass S of A-mod defines a subcategory S where S-homomorphisms are the same as in A-mod and they are composed as in A-mod. Then S is a full subcategory of A-mod which is closed under taking submodules, homomorphic images and direct sums, i.e. S is a prelocalizing subcategory. Since it is also stable for extensions, it is a localizing subcategory. Indeed S is an exact subcategory of A-mod and is a Grothendieck category.

Proof (v). To prove this we need the fact that $E_\tau(\overline{M})$ is τ-injective, where we have denoted $M/T_\tau(M)$ by $\overline{M}$. In our example

$T_\tau(M) = IM$. Now let $f : I \longrightarrow \overline{M}$ be an A-homomorphism. By the τ-injectivity of $E_\tau(\overline{M})$ there exists a unique A-homomorphism g making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & A \\ & & \downarrow f & & \downarrow g \\ & & \overline{M} & \longrightarrow & E_\tau(\overline{M}) \end{array}$$

commute. Define a map $\varphi : \mathrm{Hom}_A(I, \overline{M}) \longrightarrow E_\tau(\overline{M})$ by the rule $\varphi : f \longmapsto 1g$. By the uniqueness of g, φ is a well-defined A-homomorphism. Moreover, φ is monic. Let x be an element of $E_\tau(\overline{M})$. Then we can describe g as an A-homomorphism defined by the rule $r \longmapsto rx$. Let $h: E_\tau(\overline{M}) \longrightarrow E_\tau(\overline{M})/\overline{M}$ be the canonical A-epimorphism. Then $E_\tau(\overline{M})/\overline{M}$ is τ-torsion and so A g h is also τ-torsion. Therefore ker (g h) belongs to $\underline{D}$ and so it contains I. Thus f may be taken as the restriction of g to I and $\mathrm{im}(f) \subseteq \overline{M}$. Since $1g = \varphi(f)$ we see that φ is epic and so is an isomorphism.

CHAPTER 2

SOME RESULTS ON HNP-RINGS

2.1 Preliminaries

In this chapter we look at some applications of results from torsion theory to the study of HNP-rings. Conditions are given under which an HNP-ring R modulo its Jacobson radical J(R) is isomorphic to the endomorphism ring of a left vector space V of finite dimension over a field D.

We have cited some results concerning the conditions under which $R_\tau / J(R_\tau)$ is simple artinian as a prelude to the study of localization of a ring R at an ideal in Chapter 3. The proofs of these results may be found in Golan's Localization of Non-commutative Rings. First let us record the terminology.

We can partially order the set R-tors by setting $\tau' \leq \tau$ if and only if $E' \leq E$ for some E in τ and E' in τ'. Then $\tau' \leq \tau$ if and only if $\underline{\underline{T}}_{\tau'} \subseteq \underline{\underline{T}}_{\tau}$ or equivalently $\underline{\underline{F}}_{\tau} \subseteq \underline{\underline{F}}_{\tau'}$.

If $\tau \leq \tau'$ we say that τ' is a generalization of τ and that τ is a specialization of τ'. We will denote the set of all generalizations of τ by gen(τ).

A torsion theory is said to be proper if and only if $\tau \neq$ (R-mod, 0) $= \chi$. We shall denote the set of all proper torsion theories on R-mod by R-prop.

If U is a subset of R-tors and if $E_\tau \in \tau$ for every $\tau \in U$, let $E = \prod\{E_\tau \mid \tau \in U\}$. We call the equivalence class of

E the meet of U and denote it by $\wedge U$. Likewise, let $V = \bigcap\{\text{gen}(\tau) \mid \tau \in U\}$. Pick $E_{\tau'} \in \tau'$ for all $\tau' \in V$ and denote the set $\prod\{E_{\tau'} \mid \tau' \in V\}$ by E. We call the equivalence class of E the join of U and denote it by $\vee U$.

Then R-tors forms a complete lattice under $\wedge$ and $\vee$. We note that $\chi = \chi(0) = \vee(\text{R-tors}) = (\text{R-mod}, 0)$ and $\xi = \xi(0) = \wedge(\text{R-tors}) = (0, \text{R-mod})$.

A torsion theory $\tau \in$ R-tors is said to be faithful if and only if $R \in \underline{\underline{F}}_{\tau}$. R-prop has a unique maximal faithful torsion theory, namely $\chi(R)$. This is called the Lambek torsion theory.

We say that τ is stable if and only if $\underline{\underline{T}}_{\tau}$ is closed under taking injective hulls.

A ring R is left local if and only if the lattice R-tors has a unique atom, or equivalently all simple left R-modules are isomorphic.

Let $\tau \in$ R-tors. We say that τ is noetherian if and only if, for every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R with $\bigcup I_j \in \mathcal{L}_{\tau}$, there exists a positive integer k for which $I_k \in \mathcal{L}_{\tau}$.

Then τ is noetherian if and only if R-mod/τ is closed under taking direct sums or equivalently $U_{\tau}Q_{\tau}(\ -\ )$ commutes with direct sums.

A left module M is said to be τ-finitely generated if and only if M has a finitely generated τ-dense submodule.

A torsion theory τ is said to be of finite type if and only if every $I \in \mathcal{L}_{\tau}$ is τ-finitely generated.

A torsion theory τ is of finite type if and only if

$T_\tau(-)$ commutes with direct limits.

If the localizing functor $U_\tau Q_\tau$ is exact, we say that $\tau \in$ R-tors is an exact torsion theory.

If R is left hereditary, then every $\tau \in$ R-tors is exact.

If R is a left noetherian hereditary ring, then every $\tau \in$ R-tors is perfect.

τ is perfect if and only if τ is exact and noetherian or equivalently τ is exact and of finite type.

We denote the set of perfect torsion theories $\tau \in$ R-tors by R-perf.

2.2 Prime torsion theories

A nonzero left R-module M is said to be τ-cocritical if and only if M is τ-torsion-free and every nonzero submodule of M is τ-dense in M.

We say that a nonzero left R-module M is cocritical if and only if it is $\chi(M)$-cocritical.

We say that a left ideal I of R is τ-critical if and only if the cyclic left R-module R/I is τ-cocritical. A left ideal I is called critical if and only if it is $\chi(R/I)$-critical.

A ring R is left seminoetherian if and only if every $\tau \in$ R-prop has a τ-cocritical left R-module.

If a ring is left semiartinian, then it is left seminoetherian.

We first note a few properties of a τ-cocritical left R-module.

PROPOSITION (2.2.1). Let $\tau \in$ R-tors and let M be a τ-cocritical left R-module.

Then (1) Every nonzero submodule of M is τ-cocritical.

(2) If N is τ-torsion-free, then any nonzero R-homomorphism $M \to N$ is a monomorphism.

(3) If $\tau = \chi(N)$, then N has a nonzero τ-cocritical submodule.

(4) M is uniform.

<u>Proof</u> (1). Let N be a nonzero submodule of a τ-cocritical module M. So N is τ-dense in M and M is τ-torsion-free. So, in particular, N is also τ-torsion-free. If N' is a nonzero submodule of N, then N/N' is a submodule of M/N'. So N/N' is τ-torsion, since N' is τ-dense in M. Also N, as a submodule of τ-torsion-free module M, is τ-torsion-free. So by definition, N is τ-cocritical.

(2). If N is τ-torsion-free and $f:M \to N$ is a nonzero R-homomorphism, then M/ker(f) is isomorphic to a submodule of N and so is τ-torsion-free. Since M is τ-cocritical, any nonzero submodule of M is τ-dense in M.

Suppose, if possible, ker(f) $\neq$ 0. Then M/ker(f) is both τ-torsion and τ-torsion-free. Since f is a nonzero homomorphism ker(f) = 0. Hence f is a monomorphism.

(3). Let $\tau = \chi(N)$. Then N is τ-torsion-free and so is its injective hull E(N). So there exists a nonzero R-homomorphism $f:M \to E(N)$ which must be a monomorphism. We note that $Mf \cap N \neq 0$ and it is a nonzero submodule of Mf which is τ-cocritical. So by (1), it is τ-cocritical.

(4). Suppose, if possible, we have nonzero submodules N and N' of M with $N \cap N' = 0$. Then N is τ-torsion-free. On the other hand N is a submodule of M/N' which is τ-torsion, so that N itself is τ-torsion. So we have a contradiction. Hence M is uniform. This proves the proposition.

We also note that M is τ-cocritical if and only if Rm is τ-cocritical for every $0 \neq m \in M$.

We have a characterization of a left local ring in the following proposition. The condition for a ring R to be left local is the "trivial" torsion theory $\xi = (0, R\text{-mod})$ must be prime as defined below:

We say that a torsion theory τ is <u>prime</u> if and only if $\tau = \chi(M)$ for some cocritical left R-module M. The set of all prime members of R-tors will be called the <u>left</u> <u>spectrum</u> of R and will be denoted by R-sp.

PROPOSITION (2.2.2). A ring R is left local if and only if $\xi = (0, R\text{-mod})$ is prime.

<u>Proof</u>. Suppose R is left local. Then R-simp $= \{M\}$ for some left R-module M. Suppose, if possible, ξ is not equal to $\chi(M)$, the torsion theory cogenerated by M. So M is $\chi(M)$-torsion-free. Let $\xi(M)$ be the torsion theory generated by M. Then $\xi(M) \leq \chi(M)$, so that $M \in \underline{\underline{T}}_{\xi(M)} \subseteq \underline{\underline{T}}_{\chi(M)}$ which implies that M is $\chi(M)$-torsion; a contradiction. So $\xi = \chi(M)$ which is prime, since M is a simple left R-module.

Conversely, assume that ξ is prime. Then $\xi = \chi(M)$

for some cocritical left R-module M. So M is a ξ-cocritical module; hence M is ξ-torsion-free and for any submodule $N \subseteq M$ with $N \neq 0$, M/N is ξ-torsion, i.e. M/N = 0. So M is a simple left R-module.

Suppose M' is a simple left R-module not isomorphic to M. For the torsion theory $\xi(M')$ generated by M', M' is $\xi(M')$-torsion. So M cannot be $\xi(M')$-torsion. Thus M is $\xi(M')$-torsion-free. The fact that $\chi(M) = \vee\{\tau \in R\text{-tors} \mid M \in \underline{\underline{F}}_{\tau}\}$ implies that $\xi(M') \leq \chi(M)$. Therefore $\xi(M') \leq \chi(M) = \xi$, which is a contradiction. So all simple left R-modules are isomorphic, i.e. R is left local. This proves the proposition.

If $\tau' \leq \tau$ are prime torsion theories, then there exists a τ-critical left ideal I of R and a τ'-critical left ideal I' of R with $I \subseteq I'$. If the converse holds we say that R is <u>left balanced</u>. That is to say, R is left balanced if and only if, for critical left ideals $I \subseteq I'$ of R, we have $\chi(R/I') \leq \chi(R/I)$.

A ring for which every left ideal is two-sided is called a <u>left duo ring</u>.

Then a left duo ring is left balanced.

We define the <u>assassinator</u> ass(M) of a left R-module M by

$$\text{ass}(M) = \{\tau \in R\text{-sp} \mid \text{there exists a } \tau\text{-cocritical submodule of } M\}.$$

We say that M is a <u>D-module</u> if and only if $\text{ass}(M/N) \neq \emptyset$ for every proper submodule N of M. A ring R is called a <u>left D-ring</u> if and only if every left R-module is a

D-module.

We note that for a left noetherian ring R, the space of prime torsion theories R-sp satisfies the descending chain condition.

If M is a nonzero noetherian left R-module, then ass(M) is finite.

We call a left R-module M <u>coprimary</u> if and only if ass(M) consists of precisely one member.

The maximal elements of the space of all proper torsion theories R-prop are called <u>coatoms</u> of R-tors.

The atoms of R-tors are precisely the simple torsion theories.

For τ in R-tors, denote $\bar{\tau} = \chi(R/T_{\tau}(R))$, called the <u>saturation</u> of τ. We say that τ is <u>saturated</u> if $\bar{\tau} = \tau$.

Now let us recall a few basic facts about rings of fractions.

Let A be a ring and let S be a subset of A. Then S is called a <u>multiplicatively closed subset</u> of A if and only if (i) 1 belongs to S and (ii) if s and t belong to S, then st belongs to S.

We say that S is <u>left permutable</u> if it satisfies:

(S1). If s belongs to S and a belongs to A, there exists s' in S and a' in A such that a's = s'a.

We say that S is <u>left simplifiable</u> if it satisfies:

(S2). For any pair of elements a, b of A and an element s in S such that as = bs, there exists an element s' in S such that s'a = s'b.

Sometimes S is called a <u>left Ore set</u> if S is a multiplicatively closed subset of A and S satisfies (S1).

We say that S is left calculable if S is multiplicative, left permutable and left simplifiable.

Let S be a multiplicatively closed subset of a ring A. We define a left ring of fractions of A with respect to S as a ring $[S^{-1}]A$ together with a ring homomorphism $h: A \to [S^{-1}]A$ satisfying:

(F1). h(s) is invertible for every s in S

(F2). Every element in $[S^{-1}]A$ has the form $h(s)^{-1}\, h(a)$ with s in S and a in A.

(F3). h(a) = 0 if and only if sa = 0 for some s in S.

We say that an element a in A is left regular if the left annihilator l(a) = 0 and it is called right regular if the right annihilator r(a) = 0. It is called regular if it is both left and right regular.

Then an element a in A is regular if and only if for any x in A the relation "xa = 0 or ax = 0" implies that x = 0.

We say that A admits cancellation by S, if for any pair of elements a, a' in A and any element s in S, the relation "as = a's or sa = sa'" implies that a = a'.

We say that S is left reversible if it satisfies:

(S3). If as = 0 with s in S, then s'a = 0 for some s' in S.

We call S a left denominator set when it is multiplicatively closed and satisfies (S1) and (S3).

We say that A satisfies the left Ore condition if and only if for a and r in A with r regular, there exists a' and r' in A with r' regular such that r'a = a'r.

Let S be the set of all regular elements of A. The ring of fractions $[S^{-1}]A$ is called the classical left ring of quotients of A. We will denote it by $Q^{1}_{cl}(A)$ or simply by Q_{cl}.

A ring T is a left order in a ring Q if T is a subring of Q such that given any x in Q there exist elements a, b in T such that a is a unit of Q and $x = a^{-1}b$.

We say that a ring Q is a (classical) left quotient ring of a subring A if every regular element x in A has an inverse in Q and

$$Q = \{ b^{-1} a \mid a, b \text{ in } A, b \text{ regular} \} .$$

Then A is a left order in Q. The right analogues of these definitions are defined in an obvious way. We note the following:

PROPOSITION (2.2.A.). A ring A has a left quotient ring if and only if A satisfies the left Ore condition.

PROPOSITION (2.2.B.). Let S be the set of all regular elements of the ring A and let

$$\underline{L} = \{ {}_{A}I \subseteq A \mid I \cap S \neq \emptyset \}.$$

Then $\underline{L}$ is a left Gabriel topology if and only if A is a left Ore ring. In this case $\underline{L}$ defines the classical torsion theory denoted by $\mu(S)$.

PROPOSITION (2.2.C.). The following conditions are equivalent for a multiplicatively closed subset S of A:

(1) S is a left Ore set

(2) For every left R-module M, $T_{\mu(S)}$ is the set $\{m \in M \mid (0:m) \cap S \neq \emptyset\}$

(3) For every s in S, As is $\mu(S)$-dense left ideal.

We have defined the localization functor for τ in R-tors in Chapter 1, denoted by $U_\tau Q_\tau$, where U_τ is the inclusion functor from R-mod/τ to R-mod, sometimes called the section functor and Q_τ is the canonical functor, sometimes called the quotient functor. For brevity let L_τ denote the functor $U_\tau Q_\tau$. Let R_τ be the endomorphism ring of the left R-module $L_\tau(R)$. Then R_τ is canonically a left R-module which is isomorphic to $L_\tau(R)$. The ring R_τ is called the <u>localization</u> of the ring R at τ.

We will state a result from Golan (Prop. 17.19) concerning localization with respect to a perfect torsion theory. We shall study a generalization of this proposition in Chapter 3.

PROPOSITION (2.2.D). For $\tau \in$ R-perf, the following conditions are equivalent:

(1) τ is a coatom of R-tors

(2) (i) R_τ is left local

(ii) $J(R_\tau)$ is right T-nilpotent; and

(iii) $R_\tau/J(R_\tau)$ is simple artinian

(3) (i) τ is saturated; and

(ii) every nonzero injective left R_τ-module is faithful.

PROPOSITION (2.2.3). Let I be a two sided ideal of a ring R that is critical as a left ideal of R, then I is completely prime.

Proof. Let a, b be elements of R such that ab is in I but a is not in I. Set H = Ra + I. Since I is a proper subset of H, R/H is $\chi(R/I)$-torsion. So there exists no nonzero R-homomorphism $R/H \to R/I$. In particular, the map defined by $r+H \mapsto rb+I$ must be the zero map, which implies that b belongs to I. This proves that I is completely prime.

PROPOSITION (2.2.4). The following conditions are equivalent for a two sided ideal I of a ring R:

(1) R/I is a left Ore domain

(2) I is a critical left ideal of R.

Proof: (1) $\Rightarrow$ (2). To prove (2), it suffices to show that for every left ideal H of R properly containing I, we have $H \in \mathcal{L}_{\chi(R/I)}$. Here also we only need to consider H of the form Ra + I where $a \in R \setminus I$.

Therefore let H = Ra + I for a fixed element a in $R \setminus I$. Suppose, if possible, we have a nonzero R-homomorphism $f: R/H \to E(R/I)$. Then there exists $b \in R \setminus H$ such that $(b + H)f = c + I \in R/I$. Therefore, $(H:b) \subseteq (I:c)$ and $(I:c) = I$ since R/I is an integral domain. But for the left Ore domain R/I, there exist a' in $R \setminus H$ and b' in R such that $a'b - b'a \in I$ and so $(H:b) \not\subseteq I$, which is a contradiction. Thus there is no nonzero R-homomorphisms $R/H \to E(R/I)$ and so $H \in \mathcal{L}_{\chi(R/I)}$.

The part of the proof (2) $\Rightarrow$ (1) will not be needed in the sequel and so will be omitted.

PROPOSITION (2.2.5). Let $\tau \in$ R-sp and M a τ-cocritical left R-module. Then $\tau = \chi(M)$.

Proof. Since τ is prime, $\tau = \chi(M')$ for some cocritical left R-module M'. Since M' is uniform, E(M') is indecomposable. Also we note that E(M') is a member of τ.

Suppose M is τ-cocritical. Then there exists an R-homomorphism $M \rightarrow E(M')$ that can be extended by injectivity to an R-homomorphism $E(M) \rightarrow E(M')$. By the indecomposability of E(M') this is an isomorphism, and so E(M) belongs to τ as well. Thus $\tau = \chi(M)$. This proves the proposition.

For the converse of the above proposition, see Golan, Prop. 19.2, pp. 197-198. We note that τ is prime if and only if it is the torsion theory cogenerated by R/I where I is a critical left ideal of R. We will need the following result due to Webber as quoted in Faith [1973], p.378.

PROPOSITION (2.2.E). Let R be a left and right hereditary noetherian prime ring and I be a nonzero ideal. Then the factor ring R/I is artinian.

2.3 Semiprime torsion theories

With each $\tau \in$ R-tors, we associate the subset pgen(τ) consisting of all prime generalizations of τ. Let $\tau \in$ R-prop. We define the root of τ by the meet $\wedge$ pgen(τ), and we denote

$$\sqrt{\tau} = \bigwedge \text{pgen}(\tau).$$

In general $\tau \leq \sqrt{\tau}$. If $\tau = \sqrt{\tau}$, we say that τ is <u>semiprime</u>.

We note that $\sqrt{\tau \wedge \tau'} = \sqrt{\tau} \wedge \sqrt{\tau'}$. Hence we may deduce that if τ, τ' are semiprime, so is $\tau \wedge \tau'$.

PROPOSITION (2.3.1). If every $\tau \in$ R-prop is semiprime, then the following conditions are equivalent for $\tau \in$ R-prop:

(1) τ is a maximal element of R-prop

(2) τ is a maximal element of R-sp

<u>Proof</u> (1) $\Rightarrow$ (2): If τ is a maximal element of R-prop, then the set of generalizations of τ is just $\{\tau, \chi\}$ and so pgen$(\tau) \subseteq \{\tau\}$. Since τ is semiprime, $\tau = \bigwedge$ pgen(τ) and so pgen(τ) is nonempty. Therefore τ is prime and so (2) follows.

(2) $\Rightarrow$ (1): If $\tau' \in$ R-prop exists such that $\tau < \tau'$, then by the maximality of τ we have pgen $(\tau') = \emptyset$, contradicting the fact that $\tau' = \bigwedge$pgen(τ'). This proves the proposition.

PROPOSITION (2.3.2). A ring R is a left D-ring if and only if R is a left D-module.

<u>Proof</u>. If R is a left D-ring, then R is a left D-module by definition.

Conversely, let R be a left D-module and let M be a nonzero left R-module. If $0 \neq m \in M$, then Rm is isomorphic to a homomorphic image of R and so ass(Rm) $\neq \emptyset$. Thus, ass(M) $\neq \emptyset$. This proves the result.

We note the following characterization of semiartinian rings from Golan, pp.207-208.

PROPOSITION (2.3.A). The following conditions are equivalent for a ring R:

(1) R is left semiartinian

(2) (i) Every $\tau \in$ R-sp has a simple τ-cocritical module;

(ii) $\mathcal{X}(E)$ is prime for every indecomposable injective left R-module E; and

(iii) Every nonzero left R-module contains a nonzero uniform submodule.

Now we prove the following:

PROPOSITION (2.3.3). The following conditions are equivalent for a ring R:

(1) R is left semiartinian

(2) (i) R is a left D-ring; and

(ii) Every $\tau \in$ R-sp has a simple τ-cocritical module.

<u>Proof</u> (2)$\Rightarrow$(1): Let M be a nonzero left R-module. Then by (2i) ass(M) $\neq \emptyset$ and so M has a cocritical submodule N. By (2ii), N has a simple τ-cocritical module N'. By PROPOSITION (2.2.1) there exists a submodule of N isomorphic to N'. Therefore, M has a simple submodule, proving (1).

(1)$\Rightarrow$(2): Clearly left semiartinian rings are left D-rings. We use PROPOSITION (2.3.A) to prove (2ii). This proves the proposition.

PROPOSITION (2.3.4). The following conditions are equivalent for a ring R:

(1) R is left artinian

(2) (i) R is left noetherian; and

(ii) Every $\tau \in$ R-sp has a simple τ-cocritical module.

Proof: (1) $\Rightarrow$ (2) follows from the above proposition and the fact that every left artinian ring is left noetherian.

(2) $\Rightarrow$ (1) follows from the above proposition. This proves the proposition.

We note that there exists a torsion theory τ_D in R-tors such that the set of all D-modules is precisely the τ_D-torsion class. If R is a left D-ring, then $\tau_D = \chi$. Also if $\tau \in$ R-tors and $\tau \vee \tau_D = \chi$, then τ is semiprime. Hence we may deduce the following result from Golan, p.233.

PROPOSITION (2.3.B). If R is a left D-ring, then every τ in R-prop is semiprime.

PROPOSITION (2.3.5). If I is a two-sided ideal of a left D-ring R, then R/I is a left D-ring.

Proof. Let $g : R \to R/I$ be the canonical ring surjection. If N is a nonzero left R/I-module, then ${}_RN$ is a nonzero left R-module and so has a cocritical submodule N'. Since IN = 0, we have IN' = 0 and so N' is also a left R/I-submodule of N and N' is $\chi(N')$-cocritical as a left R/I-submodule. Thus $\chi(N') \in$ ass(N). Thus ass(N) $\neq \emptyset$ and R/I is a left D-ring. This proves the proposition.

PROPOSITION (2.3.6). Let R be a left semiartinian ring. Then (1) a nonzero left R-module M is coprimary if and only if (2) any two simple submodules of M are isomorphic.

Proof: (1) $\Rightarrow$(2): If M' is a simple submodule of M, then M' is cocritical and so $\tau' = \chi(M')$ is prime. Thus τ' is in ass(M). If M" is another simple submodule of M, then by (1) we have $\chi(M") = \tau'$. Therefore, there exists a nonzero R-homomorphism M" $\rightarrow$E(M') which is an isomorphism, since M' and M" are simple.

(2) $\Rightarrow$(1): Since R is left semiartinian, M has at least one simple submodule. Moreover, for every simple submodule M' of M, $\chi(M') \in$ ass(M). Thus ass(M) $\neq \emptyset$. If $\tau \in$ ass(M), then there exists a τ-cocritical submodule N of M. Since $0 \neq$ N, the module N has a simple submodule N' which is also τ-cocritical by PROPOSITION (2.2.1). For a prime torsion theory τ, a cocritical left R-module M is τ-cocritical if and only if $\tau = \chi(M)$. Thus every τ in ass(M) is of the form $\chi(M')$ for some simple submodule M' of M. Thus (2) implies (1). This proves the proposition.

We note the following result whose proof may be found in Golan, pp.239-240.

PROPOSITION (2.3.C). Every nonzero uniform noetherian left R-module is coprimary.

PROPOSITION (2.3.7). Every nonzero noetherian left R-module is a D-module.

<u>Proof</u>: If M is noetherian, then so is M/N for every proper submodule N of M. So it is sufficient to prove that for every nonzero noetherian left R-module M, ass(M) $\neq \emptyset$. By PROPOSITION (2.3.C), it suffices to show that every nonzero noetherian left R-module contains a nonzero uniform submodule.

Suppose, if possible, M is not uniform; then there exist nonzero submodules N_1 and N_1' of M such that $N_1 \cap N_1' = 0$. If N_1' is not uniform, then there exist nonzero submodules N_2 and N_2' of N_1' with $N_2 \cap N_2' = 0$. Continuing this process, if we do not obtain a nonzero uniform submodule of M in a finite number of steps, then we obtain an infinite ascending chain $N_1 \subset N_1 \oplus N_2 \subset \ldots$, contradicting the fact that M is noetherian. Hence M must be uniform. This proves the proposition.

We note the following result, whose proof requires a series of propositions:

PROPOSITION (2.3.D). The following conditions on a ring R are equivalent:

(1) R is left seminoetherian and left balanced.

(2) (i) R is left balanced

(ii) Every $\tau \in$ R-prop is semiprime; and

(iii) R-sp satisfies the descending chain condition.

2.4 HNP-rings

We shall now apply the results mentioned in the previous sections to left and right hereditary, left and right noetherian prime rings, which will be abreviated as HNP-ring from now on.

We recall that the ring R is prime if there are no nonzero two-sided ideals A and B such that AB = 0, and A is semiprime if it has no nonzero nilpotent ideals. A two-sided ideal A of R is called semiprime if R/A is a semiprime ring. A prime ideal in R is a two-sided ideal $A \neq R$ with the property that if B and C are two-sided ideals such that $BC \subseteq A$, then either B or C is contained in A. In other words, A is a prime ideal in R if and only if R/A is a prime ring. The intersection of all prime ideals of R is called the prime radical of R, and is denoted by N(R). Then N(R) is the smallest ideal A of R such that R/A is a semiprime ring, and R is a semiprime ring if and only if N(R) = 0. In general the Jacobson radical J(R) strictly contains N(R). We call a ring R semilocal if and only if R/J(R) is semisimple. A ring R is local if and only if R/J(R) is a field.

LEMMA (2.4.1). Let R be a HNP-ring such that $J(R) \neq 0$.

Then R is semilocal.

Proof: Since R is prime and $J(R) \neq 0$, we find that the ring A = R/J(R) is artinian by PROPOSITION (2.2.E). Then by Wedderburn-Artin theorem, the fact that A is semiprime and artinian implies that A is semisimple. Hence R is semilocal.

This proves the lemma.

Now we wish to consider HNP-rings with nonzero Jacobson radical. Let R be an HNP-ring with $J(R) \neq 0$. Let A denote the ring $R/J(R)$. Then by PROPOSITION (2.2.E), A is artinian; see Robson [1972], where we can also find conditions under which HNP-ring has nonzero Jacobson radical. If in addition A is a prime ring, then we may deduce from Wedderburn-Artin theorem that A is isomorphic to the matrix ring D_n, for some field D and a non-negative integer n. If, however, A is a domain, then A is a left Ore domain, and hence A is a uniform left A-module.

Now we prove the main theorem of this chapter.

THEOREM (2.4.2). Let R be a HNP-ring such that $J(R) \neq 0$ and let $R/J(R)$ be denoted by A. Suppose A is a uniform left A-module. **Then A is a field.**

<u>Proof</u>. By PROPOSITION (2.2.E), A is artinian. Thus A is left semiartinian; this remark may be proved by using PROPOSITIONS (2.3.3) and (2.3.4).

Thus A is a nonzero uniform noetherian left A-module. Hence it is coprimary by PROPOSITION (2.3.C). For a left semiartinian ring A, the fact that A is a nonzero coprimary left A-module implies that any two simple submodules of A are isomorphic by PROPOSITION (2.3.6).

Since A is semiprime and artinian, it is semisimple.

So it is a direct sum of simple modules, which are mutually isomorphic. Thus A is a simple ring. Then by Wedderburn's theorem, A is isomorphic to the endomorphism ring of a left vector space V over a field D. Being uniform, A must be 1-dimensional, i.e. A is a field. This proves the theorem .

2.5 Examples

We now introduce the following examples, bearing in mind the remarks made in Section 1.5.

Example (2.5.1). Let Z be the ring of integers and $p_1, \ldots, p_m$ a set of non-associated primes. Let S be the set of all integers not divisible by any p_i. Form the ring $R = [S^{-1}]Z$.

Then the coatoms of R-tors are the maximal elements of R-sp.

Proof. Since every nonzero noetherian left R-module is a D-module, R is a D-module; hence it is a left D-ring by PROPOSITION (2.3.2). By PROPOSITION (2.3.B), every proper torsion theory is semiprime. By definition, the coatoms are the maximal elements of R-prop. Thus by PROPOSITION (2.3.1) the coatoms are the maximal elements of R-sp. This proves the assertion.

We record the following result due to J. Kuzamanovich as quoted in Golan, p.315.

Example (2.5.2). Let R be a HNP-ring. Define the subfunctor T of the identity functor R-mod $\longrightarrow$ R-mod by

$T(M) = \{ m \in M \mid Im = 0$ for some invertible two-sided ideal I of R $\}$ and

$T(f)$ = restriction of f to T (domain f).

Then T is a torsion radical, defining a torsion theory $\tau \in$ R-tors. Moreover, $\mathcal{L}_\tau$ is the set

$\{ {}_R I \subseteq R \mid$ I is a large left ideal of R containing an invertible two-sided ideal of R $\}$.

<u>Example</u> (2.5.3). Let R be an arbitary ring and let A be a multiplicatively closed subset of R.

Then the set

$\{ M \in$ R-mod $\mid$ for each $m \in M$ there exists an a in A with $am = 0 \}$

is a torsion class $\underline{\underline{T}}_\tau$ for some τ in R-tors. We denote this torsion theory by $\mu(A)$.

<u>Example</u> (2.5.4). If R is commutative and P is a prime ideal of R, then $\mu(R \setminus P) = \chi(R/P)$.

<u>Example</u> (2.5.5). Let R be a left noetherian ring and let P be a prime ideal of R.

Then $C(P) = \{ a \in R \mid ab \in P$ implies that $b \in P \}$ is a multiplicatively closed set and the torsion theory $\mu(C(P))$ coincides with the torsion theory $\chi(R/P)$ cogenerated by R/P.

<u>Example</u> (2.5.6). Let R be a left and right discrete

valuation ring . Then R is a duo ring which is also a left and right D-ring and R-sp satisfies DCC.

<u>Example</u> (2.5.7). Let k be a commutative field and R be the free algebra $k\langle x_1, x_2, \ldots, x_n\rangle$. Let I be the ideal generated by $x_1, x_2, \ldots, x_{n-1}$. Then I is critical as a left ideal of R.

CHAPTER 3

LOCALIZATION AT AN IDEAL

3.1 Preliminaries

In this chapter we present the result of Beachy concerning generalization of localization in noetherian rings at a semiprime ideal to localization in a class of more general rings at an ideal. His result may be found in the paper, "On localization at an ideal" (Preprint).

We will study the extension of the following well known result.

PROPOSITION (3.1.A). The following properties of the ring A are equivalent:

(1) A is a left order in a semisimple ring.

(2) A has left finite rank, satisfies ACC on left annihilators and is a semiprime ring.

(3) A left ideal of A is essential if and only if it contains a regular element.

In Chapter 2, we have studied the process of forming fractions which is due to Ore; see Cohn [1971], Faith [1973] and Stenström [1975]. Another method of forming fractions depends on the notion of injective hulls; it is usually expressed within the framework of torsion theories; see Popescu [1973] and Stenström [1975]. This method may be called the <u>injective</u>

method following Cohn, in contrast to another way of generalizing Ore's method, called the inversive method due to Cohn.

We recall that a ring R may be regarded as a category with a single object R, the morphisms being the elements of R. We denote this category by the same symbol R. The opposite or dual category is a ring, denoted by R^o. A ring homomorphism $R \to R'$ defines a (covariant) functor, and conversely. The category R is then a preadditive category, i.e. $\mathrm{Hom}_R(a, b)$ is always an additive group. We define ModR as the category of all additive functors from R to Ab and Mod R^o as the category of all additive functors from R^o to Ab. Then an object of Mod R is a left module over R and an object of Mod R^o is a right module over R. Sometimes we denote Mod R by R-mod and Mod R^o by Mod-R.

Let $\tau \in$ R-tors. We have defined a τ-localization functor $U_\tau Q_\tau$ in Chapter 1. Let us denote it by L_τ for brevity. We recall that

$$L_\tau(M) = E_\tau(M/T_\tau(M)).$$

We have a natural transformation u: $\mathrm{IdModR} \to L_\tau$ and an R-homomorphism $u_X: X \to L_\tau(X)$ which is just the composition of the canonical projection $X \to X/T_\tau(X)$ and the canonical embedding $X/T_\tau(X) \to L_\tau(X)$. Thus $\ker(u_X) = T_\tau(X)$ and so u_X is a monomorphism if and only if X is τ-torsion-free, and it is a zero map if and only if X is τ-torsion. The fact that L_τ is idempotent follows from the definition.

Now let $g: M \to N$ be an **R-monomorphism**. Then g induces an R-homomorphism $\bar{g}$

$$\bar{g}: M/T_\tau(M) \to N/T_\tau(N)$$

which is also a monomorphism by the closure of $\underline{\underline{T}}_\tau$ under taking submodules and extensions. Since $M/T_\tau(M)$ is large in $L_\tau(M)$, this implies that $L_\tau(g)\colon L_\tau(M) \to L_\tau(N)$ is also a monomorphism. It follows that L_τ is left exact.

PROPOSITION (3.1.1). Let $\tau \in$ R-tors. Then the following conditions are equivalent for an R-monomorphism $g\colon M \to N$:

(1) Mg is τ-dense in N

(2) $L_\tau(g)$ is an isomorphism.

<u>Proof</u>. We have an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow N/Mg \longrightarrow 0.$$

By the left exactness of L_τ, we have an exact sequence

$$0 \longrightarrow L_\tau(M) \xrightarrow{L_\tau(g)} L_\tau(N) \longrightarrow L_\tau(\widetilde{N}),$$

where we denote $\widetilde{N} = N/Mg$ for short.

(1) $\Rightarrow$ (2): By (1), $\widetilde{N}$ is τ-torsion. Since $\widetilde{N}$ is τ-injective we find that $L_\tau(\widetilde{N}) = 0$. Thus $L_\tau(g)$ is an isomorphism.

(2) $\Rightarrow$ (1): By the definition of L_τ we have the following commutative diagram:

$$\begin{array}{ccccc}
N & \longrightarrow & N/T_\tau(N) & \xrightarrow{h_1} & L_\tau(N) \\
\downarrow f & & \downarrow \bar{f} & & \downarrow L_\tau(f) \\
\widetilde{N} & \longrightarrow & \widetilde{N}/T_\tau(\widetilde{N}) & \xrightarrow{h_2} & L_\tau(\widetilde{N})
\end{array}$$

where h_1 and h_2 are monomorphisms. By assumption (2), $L_\tau(f) = 0$ and so $\bar{f} = 0$. Since f is an epimorphism, so is $\bar{f}$ and so $\widetilde{N} = T_\tau(\widetilde{N})$. Thus $\widetilde{N}$ is τ-torsion and hence Mg is

τ-dense in N. This proves the proposition.

We note the following proposition which characterizes $L_\tau(M)$ up to isomorphism.

PROPOSITION (3.1.2). For $\tau \in$ R-tors and for an R-homomorphism $g: M \longrightarrow N$, the following conditions are equivalent:

(1) There exists a unique R-homomorphism $\check{u}_M: N \longrightarrow L_\tau(M)$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ g\ } & N \\ {\scriptstyle u_M}\downarrow & \swarrow {\scriptstyle \check{u}_M} & \\ L_\tau(M) & & \end{array}$$

commute, where $\check{u}_M$ is in fact an isomorphism.

(2) (i) ker (g) and coker(g) are τ-torsion

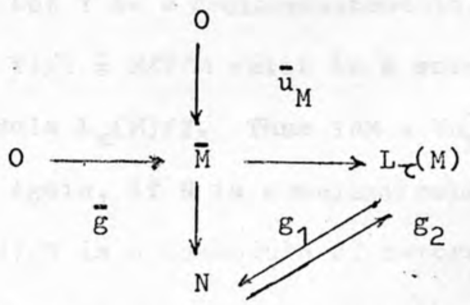
(ii) N and E(N)/N are τ-torsion-free.

<u>Proof</u>. Let $\bar{M} = M/T_\tau(M)$.

(1) $\Rightarrow$ (2): By (1) we can deduce that $\ker(g) = \ker(u_M)$ and $\mathrm{coker}\,(g) \cong E_\tau(\bar{M})/\bar{M}$ so that (i) follows. Now $N \cong L_\tau(M)$ which is τ-injective. So by PROPOSITION (1.2.A), N is τ-closed in E(N), i.e. E(N)/N is τ-torsion-free. Clearly N is τ-torsion-free.

(2) $\Rightarrow$ (1): Since ker(g) is τ-torsion, we have $\ker(g) \subseteq T_\tau(M) = \ker(u_M)$. Suppose, if possible, $\ker(g) \neq \ker(u_M)$. Then there exists a non-zero R-homomorphism $T_\tau(M) \longrightarrow N$, contradicting the fact that N is τ-torsion-free. Thus $\ker(g) = \ker(u_M)$. Therefore g induces monomorphism $\bar{g}: \bar{M} \rightarrow N$ and u_M induces monomorphism $\bar{u}_M : \bar{M} \rightarrow L_\tau(M)$. By (2 ii), E(N)/N is τ-torsion-free and so N is τ-closed in E(N). By

PROPOSITION (1.2.A), N is τ-injective. Also we know that $L_\tau(M)$ is τ-injective. Therefore there exist R-homomorphisms g_1 and g_2 making the diagram



commute. Then g_1 and g_2 are unique by PROPOSITION (1.2.B). So g_1g_2 and g_2g_1 are identity maps. Thus $g_2 = \check{u}_M$ is an isomorphism.

The following proposition investigates the relation between the sets of τ-closed submodules of M and $L_\tau(M)$.

PROPOSITION (3.1.3). Let $\tau \in$ R-tors. Then there is a bijective correspondence between the sets of τ-closed submodules of M and $L_\tau(M)$.

Proof. If N is a τ-closed submodule of M and if $\gamma : M \twoheadrightarrow M/N$ is the canonical surjection, then $T_\tau(M)\gamma \subseteq M/N$ which is τ-torsion-free and so N contains $T_\tau(M)$. Thus, $N/T_\tau(M)$ is τ-closed in $M/T_\tau(M)$ if and only if N is τ-closed in M, and so without loss of generality we can take $T_\tau(M) = 0$. Thus $u_M : M \rightarrow L_\tau(M)$ is a monomorphism. Identifying M with its image under u_M, we find that for every submodule Y of $L_\tau(M)$, we have $Y u_M^{-1} = Y \cap M$.

If N is a τ-closed submodule of M, then by the left exactness of L_τ, we have an exact sequence

$$0 \longrightarrow L_{\tau}(N) \longrightarrow L_{\tau}(M) \longrightarrow L_{\tau}(M/N)$$

and so $L_{\tau}(M)/L_{\tau}(N)$ is isomorphic to a submodule of $L_{\tau}(M/N)$ which is τ-torsion-free. Thus $L_{\tau}(N)$ is τ-closed in $L_{\tau}(M)$.

Let Y be a τ-closed submodule of $L_{\tau}(M)$. Then we find that $(M + Y)/Y \cong M/Y \cap M$ which is a submodule of the τ-torsion-free left R-module $L_{\tau}(M)/Y$. Thus $Y \cap M = Yu_M^{-1}$ is τ-closed in M.

Again, if N is a τ-closed submodule of M, then $(M \cap L_{\tau}(N))/N$ is a submodule of τ-torsion-free module M/N and so it is τ-torsion-free. On the other hand, it is also a submodule of $L_{\tau}(N)/N$ and so is τ-torsion. Thus, we find that $N = M \cap L_{\tau}(N)$. So the map

$$N \longmapsto L_{\tau}(N) \longmapsto M \cap L_{\tau}(N)$$

is the identity.

Finally, let Y be a τ-closed submodule of $L_{\tau}(M)$, so that $L_{\tau}(M)/Y$ is τ-torsion-free. Also we have the isomorphism $Y/Y \cap M \cong (Y \cap M)/M$ which is a submodule of the τ-torsion module $L_{\tau}(M)/M$. Let $f : Y \cap M \rightarrow Y$ be the inclusion map. Then we have a map $\check{u}_{Y\cap M}$ making the following diagram

$$\begin{array}{ccc} Y \cap M & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle u_{Y\cap M}} \downarrow & \swarrow {\scriptstyle \check{u}_{Y\cap M}} & \\ L_{\tau}(Y \cap M) & & \end{array}$$

commute . Here $\check{u}_{Y\cap M}$ is also inclusion map.

By the exactness of the sequence

$$0 \longrightarrow L_{\tau}(M)/Y \longrightarrow E(M)/Y \longrightarrow E(M)/L_{\tau}(M) \longrightarrow 0$$

where $L_{\tau}(M)/Y$ is τ-torsion-free and $E(M)/L_{\tau}(M)$ is τ-torsion-free by the absolute τ-closure of $L_{\tau}(M)$, we find that $E(M)/Y$ is also τ-torsion-free. Thus $E(Y)/Y$ as a submodule of $E(M)/Y$ is τ-torsion-free. Now by PROPOSITION (3.1.2), the R-homomorphism $\check{u}_{Y\cap M}$

is an isomorphism so that $Y = L_\tau(Y \cap M)$. Therefore, the map

$$Y \longrightarrow Y \cap M \longrightarrow L_\tau(Y \cap M)$$

is the identity. So the correspondence is bijective. This proves the proposition.

We note a related result from Beachy [1974].

PROPOSITION (3.1.B). For a left R-module M, and a torsion radical σ, there is a one-to-one correspondence between the subobjects of $Q_\sigma(M)$ in R-mod/σ and σ-closed submodules of M.

PROPOSITION (3.1.4). Let $\tau \in$ R-tors and let R_τ be the endomorphism ring of the left R-module $L_\tau(R)$. Then R_τ is canonically a left R-module which is isomorphic to $L_\tau(R)$.

<u>Proof</u>. If x belongs to $L_\tau(R)$, then x induces an R-homomorphism $\rho_x : R \to L_\tau(R)$ defined by $r \mapsto rx$. Since $L_\tau(R)$ is τ-torsion-free, $T_\tau(R) \subseteq \ker(\rho_x)$ and so ρ_x induces an R-homomorphism $\bar{\rho}_x : R/T_\tau(R) \to L_\tau(R)$. By PROPOSITION (1.2.B), $\bar{\rho}_x$ can be extended uniquely to an R-homomorphism $\beta_x : L_\tau(R) \to L_\tau(R)$.

Let r belong to R and $\bar{r} = r + T_\tau(R)$. Then we can define a left R-module structure on R_τ by setting $r\alpha = \beta_{\bar{r}}\alpha$ for every r in R and α in R_τ.

Now let $\theta : L_\tau(R) \to R_\tau$ be the function defined by $x \longmapsto \beta_x$. If $x, y \in L_\tau(R)$, then $\beta_x + \beta_y$ and β_{x+y} both extend $\bar{\rho}_x + \bar{\rho}_y$ and so, by PROPOSITION (1.2.B), they are equal. Similarly $r\beta_x = \beta_{rx}$. Thus θ is an R-homomorphism. Since $\alpha = \beta_{\bar{1}\alpha}$, θ is an epimorphism. Also since $x \neq 0$ implies that $\bar{1}\beta_x = x \neq 0$, θ is a monomorphism. Thus θ is an isomorphism. This proves the proposition.

Then θ and u_R give us an R-homomorphism $\hat{\tau} : R \rightarrow R_\tau$. The ring R_τ is the localization of the ring R at τ.

3.2 Localization at an ideal

Let I be a two sided ideal of a ring A. We denote by F(A/I) the left Gabriel topology corresponding to the torsion theory cogenerated by the injective hull E(A/I), i.e. the Gabriel topology formed by the left ideals B such that Hom (A/B, E(A/I)) = 0.

When I = (0), F(A) is the family D of dense left ideals. The ring A_D is called the maximal (or complete) left ring of quotients of A, and it will be denoted by $Q^l_{max}(A)$ or simply Q_{max}. Since A is D-torsion-free we have $Q_{max} = \varinjlim Hom(B, A)$, $B \in D$ and A is a subring of Q_{max}. We call D the dense topology.

Let E denote the set of all essential left ideals of A. Then E is a topology and is sometimes called the Goldie prelocalizing system. With E we associate the weakest Gabriel topology, denoted by J(E) stronger than E. J(E) is called the Goldie topology of A and is denoted by G. We always have $D \subseteq E$.

We call a module M non-singular if and only if its singular submodule Z(M) is zero. Here Z is the left exact preradical corresponding to the topology E.

We note that a module is Goldie torsion-free if and only if it is non-singular. When A is left non-singular, the Goldie topology G and the dense topology D coincide; and

$Q_{max} = A_{\underline{G}} \cong E(A)$.

Let $\underline{F}(M)$ denotes the filter of open submodules of M in the $\underline{F}$-topology, i.e. the submodules L of M for which M/L is an $\underline{F}$-torsion module.

An $\underline{F}$-injective envelope of M is an essential monomorphism $M \longrightarrow E$ such that E is $\underline{F}$-injective and M is in $\underline{F}(E)$. We will denote the $\underline{F}$-injective envelope of M by $E_{\underline{F}}(M)$. We note the following Lemma VI, 3.8, Stenström [1975].

LEMMA (3.2.A). If L and M are modules, then $Hom(L, E(M)) = 0$ if and only if $Hom(C, M) = 0$ for every cyclic submodule C of L.

PROPOSITION (3.2.1). Let I be a two-sided ideal of A and $\sigma = (\underline{\underline{T}}, \underline{\underline{F}})$ be the torsion theory cogenerated by E(A/I). Let $T_{\sigma}(M) = \sum \{Am \mid m \text{ belongs to } M \text{ and } Am \text{ belongs to } \underline{\underline{T}}\}$ and $\sigma(M) = \{m \in M \mid f(m) = 0 \text{ for all } f \text{ in } Hom_A(M, E(A/I))\}$.

Then $T_{\sigma}(M) = \sigma(M)$.

Proof. (We note here that $\sigma(-)$ may be looked upon as the torsion radical $T_{\sigma}(-)$ to simplify notation.) Now $\sigma = (\underline{\underline{T}}, \underline{\underline{F}}) = \chi(E(A/I))$ and $\underline{\underline{F}}$ is the smallest σ-torsion-free class containing E(A/I) and $T_{\sigma}(M)$ is a unique submodule of M such that $T_{\sigma}(M) \in \underline{\underline{T}}$ and $M/T_{\sigma}(M)$ is in $\underline{\underline{F}}$.

Since E(A/I) is in $\underline{\underline{F}}$, $Hom_A(M, E(A/I)) = 0$ for all M in $\underline{\underline{T}}$. In particular, $Hom_A(Am, E(A/I)) = 0$ for all m in M, so that $T_{\sigma}(M) \subseteq \sigma(M)$. For the reverse inclusion, we use LEMMA (3.2.A). This proves the proposition.

We note the following proposition; see Stenström (IX, 2.3).

PROPOSITION (3.2.B). If C is an injective module cogenerating the torsion theory associated to $\underline{F}$, then

$$E_{\underline{F}}(M) = \{ x \in E(M) \mid f(x) = 0 \text{ for all } f: E(M) \to C \text{ with } f(M) = 0 \}.$$

Using the above proposition $E_{\underline{F}}(M/\sigma(M))$ may be described as:

$$\{ x \in E(M/\sigma(M)) \mid f(x) = 0 \text{ for all } f \in \mathrm{Hom}_A(E(M)/\sigma(M)), E(A/I))$$
$$\text{such that } f(M/\sigma(M)) = 0 \}.$$

We note that $M_{\underline{F}} \cong E_{\underline{F}}(M/\sigma(M))$. For brevity we will denote $M_{\underline{F}}$ by M_σ and $E_{\underline{F}}(M/\sigma(M))$ by $Q_\sigma(M)$. We call Q_σ the <u>quotient functor</u> corresponding to the torsion radical σ.

The quotient category Mod A/σ of Mod A defined by σ is the full subcategory of A-modules M for which $M_\sigma = M$; Mod A/σ is also a full subcategory of Mod A_σ. The torsion radical σ is called <u>perfect</u> if Mod A/σ coincides with Mod A_σ.

We have noted that the functor q : Mod A $\to$ Mod A_σ is left exact so that the exact sequence

$$0 \to I \to A \to A/I \to 0$$

of A-modules gives rise to the exact sequence

$$0 \to I_\sigma \to A_\sigma \to (A/I)_\sigma$$

of A -modules. Hence we may identify A_σ/I_σ with the corresponding submodule of $(A/I)_\sigma$.

A left linear topology on A is sometimes called a left prelocalizing system and a left Gabriel topology is called a left localizing system.

Let P be a prelocalizing system A and s(P) the full subcategory of Mod A consisting of all the objects M such that for any x in M, we deduce that ann(x) is a left ideal in P. Then s(P) is a prelocalizing subcategory, i.e. a full subcategory of Mod A which is closed under taking submodules, quotient modules and direct sums.

A localizing system is called improper if the corresponding localizing subcategory is Mod A. Otherwise it is called proper.

Let P be a proper localizing system. Then define P^2 as follows : P^2 is formed by all left ideals B of A for which there exists a left ideal B' in P such that for any x in B' we can deduce that (B:x) is in P.

Then for the Goldie prelocalizing system $\underline{E}$, we always have $\underline{D} \subseteq \underline{E}$ and $\underline{E}^2$ is a localizing system. Let $\mathcal{E}$ be the localizing subcategory of Mod A corresponding to $\underline{E}$ and for any module M in Mod A there is a largest subobject in $\mathcal{E}$ denoted by $M_{\mathcal{E}}$. We note that $M_{\mathcal{E}} = Z(M)$ the singular submodule of M. For the nonzero A-module A/I such that Z(A/I)=0, we find that $\underline{E} \subseteq F(A/I)$ = the left Gabriel topology for the torsion theory cogenerated by E(A/I). We also note that the transfinite process which leads from Z to the Goldie torsion radical gets stationary: $G = Z_2$. And so does the system $\underline{E}$, namely, $\underline{E}^2 = \underline{E}^3 = \ldots$. $\underline{E}^2$ is the closure $\tilde{\underline{E}}$ of $\underline{E}$. The ring $A_{\tilde{\underline{E}}} = G(A)$ is a regular ring called the left regular ring of quotients of A.

A basis for the topology $\underline{F}$ is a subset $\underline{B}$ of $\underline{F}$ such that every left ideal in $\underline{F}$ contains some member of $\underline{B}$. For a Gabriel topology $\underline{F}$ on A, the corresponding torsion class consists of all members which are discrete in their $\underline{F}$-topology, or equivalently, for which all elements are annihilated by left ideals in $\underline{F}$. These modules will be called $\underline{F}$-torsion modules.

An A-module M is $\underline{F}$-injective if the canonical homomorphisms

$$\mathrm{Hom}_A(A, M) \longrightarrow \mathrm{Hom}_A(I, M)$$

are epimorphisms for all I in $\underline{F}$; and M is called $\underline{F}$-closed if they are isomorphisms.

Then M is $\underline{F}$-closed if and only if M is $\underline{F}$-injective and $\underline{F}$-torsion-free. We note the following:

PROPOSITION (3.2.C). The full subcategory of Mod $A_{\underline{F}}$ consisting of modules of the form $M_{\underline{F}}$ is equivalent to the full subcategory of Mod A consisting of $\underline{F}$-closed modules.

We let Mod (A, $\underline{F}$) denote the full subcategory of Mod A consisting of $\underline{F}$-closed modules and call it the quotient category of Mod A with respect to $\underline{F}$. If σ is the torsion radical corresponding to $\underline{F}$, we sometimes denote Mod(A, $\underline{F}$) by Mod A/σ.

Let i be the inclusion functor on Mod(A, $\underline{F}$) into Mod A and a : $M \longmapsto M_{\underline{F}}$ the left adjoint of i. Then ia = LL. Let j be a functor on Mod(A, $\underline{F}$) which considers each $\underline{F}$-closed modules as an $A_{\underline{F}}$-module. Then ja = q. Here both i and q are left exact.

We call a Grothendieck category $\underline{\underline{G}}$ a <u>spectral</u> <u>category</u> if every short exact sequence in $\underline{\underline{G}}$ splits. In other words, a spectral category is a Grothendieck category in which every object is injective (and also projective). We note the following; see Stenström [1975], p.215 and Beachy [1974], p.412.

PROPOSITION (3.2.D). An $\underline{F}$-closed module M is an injective object in Mod(A, $\underline{F}$) if and only if M is injective in Mod A.

PROPOSITION (3.2.E). If $\underline{G}$ denotes the Goldie topology on A, then every $\underline{G}$-injective module is injective and the functor q : Mod A $\longrightarrow$ Mod $A_{\underline{G}}$ is exact.

Moreover, Mod(A, $\underline{G}$) is a spectral category.

Using the terminology and notation of Popescu [1973] we have the following:

PROPOSITION (3.2.F). Let $\underline{E}$ be the prelocalizing system of left large ideals of A and $\tilde{\underline{E}} = \underline{E}^2$ the closure of $\underline{E}$. Then:

(1) Any $\tilde{\underline{E}}$-closed module is injective

(2) The category Mod A/$\tilde{\underline{E}}$ is a spectral category and the functor S : Mod A/$\tilde{\underline{E}}$ $\longrightarrow$ Mod A is exact

(3) The ring G(A) = $A_{\tilde{\underline{E}}}$ is a regular ring.

PROPOSITION (3.2.G). A Gabriel topology $\underline{F}$ is perfect if and only if the functor q : Mod A ⟶ Mod $A_{\underline{F}}$ is exact and $\underline{F}$ has a basis consisting of finitely generated left ideals.

A module M is called <u>finite dimensional</u> or of <u>finite uniform dimension</u> if it does not contain any infinite family of nonzero submodules M_j such that their sum $\sum_i M_i$ is direct.

Then the ring A has finite uniform dimension, with dim(R) = n, if and only if there is an essential direct sum $U_1 \oplus \dots \oplus U_n$ of uniform left ideals.

LEMMA (3.2.H). Any A-module M of finite dimension contains a finitely generated essential submodule.

PROPOSITION (3.2.2). Let A be a ring which is left non-singular and is left finite dimensional. Then the Goldie topology $\underline{G}$ on A is perfect.

<u>Proof.</u> We have observed that for a left non-singular A, the Goldie topology and dense topology coincide, i.e. $\underline{G} = \underline{D}$. For the Goldie topology $\underline{G}$, the functor q : Mod A ⟶ Mod $A_{\underline{G}}$ is exact. Since A is left finite dimensional, the Goldie topology $\underline{G}$ has a basis consisting of finitely generated left ideals by LEMMA (3.2.H). Hence the Goldie topology $\underline{G}$ on A is perfect by PROPOSITION (3.2.G). This proves the proposition.

Let f : A ⟶ B be a ring homomorphism. Then we say that f is an epimorphism if for any ring C and homomorphisms g, h : B $\longrightarrow$ C, gf = hf implies that g = h. If f : A $\longrightarrow$ B is an epimorphism of rings which makes B into a flat left A-module, then we say that B is a perfect right localization of A, or B is called a flat epimorphic right ring of quotients of A and f is a left flat epimorphism.

We define monomorphism of rings dually. We say that a ring homomorphism f : A $\longrightarrow$ B is a bimorphism if it is both an epimorphism and a monomorphism. For a left flat epimorphism of rings f : A $\longrightarrow$ B, B is called a left flat epimorphic extension of A, if f is a left flat bimorphism of rings or equivalently if ker(f) = 0. Then we also say that f defines a left quasi-order of B. Sometimes we say that a subring A of a ring B is a left quasi-order of B if the inclusion A $\longrightarrow$ B is a left flat bimorphism of rings. We now have the following result:

THEOREM (3.2.3). Let A be a ring. Then the following conditions on A are equivalent:

(1) A is left non-singular and is left finite dimensional.

(2) A is a left quasi-order of a semi-simple ring.

Proof. (1) $\Rightarrow$ (2). We have noted that $\underline{G}$ is perfect, so that the inclusion A $\longrightarrow$ Q_{max} is a left flat epimorphism of rings. We have also noted that Mod(A, $\underline{G}$) is a spectral category. Now the fact that Mod $A_{\underline{G}}$ is equivalent to Mod (A,$\underline{G}$) implies that Mod $A_{\underline{G}}$ is a spectral category; hence $A_{\underline{G}}$ or Q_{max} is semisimple.

To prove (2) $\Rightarrow$ (1), assume that A is a left quasi-order in a semisimple ring B. So B has a left finite dimension. Let $H = \{H_i\}$ be a set of left ideals of A such that the sum $\sum_i H_i$ is direct. Then the sum $\sum_i BH_i \cong B \otimes_A (\sum_i H_i)$ is also direct in B. Hence the set H is finite. So A has a left finite dimension.

We note that for any left ideal K of B one has $K = B(K \cap A)$. If L is essential left ideal of A, then BL is an essential left ideal of B, hence BL = B. Suppose that, for an essential left ideal L of A, x is an element of A such that Lx = 0. Then Bx = BLx = 0 so that x = 0. Hence Z(A) = 0. Since B is the injective hull of A, we find that $B = Q_{max}$ and also $B = Q_{tot}$, where $Q_{tot} = Q^l_{tot}$ is the maximal flat epimorphic left ring of quotients, which is unique up to isomorphism. This proves the theorem.

For the existence of Q_{tot}, see Stenström, (XI, 4.1). In the above theorem, since Q_{max} is flat as a left A-module, we find that $B = Q_{max} = Q^r_{max}(A)$.

We also note that for the dense topology $\underline{D}$, $A_{\underline{D}} = Q_{max}$ is semisimple artinian if and only if A is left finite dimensional and left non-singular. Beachy has extended this result to the localization at any ideal I, thereby extending the result of Lambek and Michler concerning the conditions under which $I_{\mathcal{T}} = J(A_{\sigma})$ and $A_{\sigma}/J(A_{\sigma})$ is semisimple artinian, where I is a semiprime ideal and A is left noetherian ring. The assumption that I is semiprime and A is left noetherian is thereby replaced by the assumption that A/I is left finite dimensional and is left non-singular for any two-sided ideal I of A. THEOREM (3.2.4) and its corollaries (3.2.I), (3.2.J) and (3.2.K) may be found in Beachy's forthcoming paper "On localization at an ideal" (preprint).

THEOREM (3.2.4). Let I be an ideal of R, and let σ be the torsion radical defined by E(R/I).

Then the following conditions are equivalent.

(1) $(R/I)_\sigma$ is a finite direct sum of simple objects in R-mod/σ

(2) The ring R/I has finite left uniform dimension and zero left singular ideal.

<u>Proof</u>. We note that both R/I and E(R/I) are σ-torsion-free. So we find that $(R/I)_\sigma = E_\sigma(R/I)$. Thus R/I is an essential submodule of $E_\sigma(R/I)$ and $(R/I)_\sigma$ is a σ-injective submodule of E(R/I).

Let A = R/I, B = $(R/I)_\sigma$ for brevity.

(1) $\Rightarrow$ (2): By (1) B is a finite direct sum of simple objects in R-mod/σ. Let J = $\{A_i\}$ be a set of left ideals of A such that the sum $\sum_i A_i$ is direct. Then the sum

$$\sum_i BA_i \cong B \otimes_A (\sum_i A_i)$$

is also direct in B. Since B has finite uniform dimension, the set J is finite. So A has finite uniform dimension.

To prove that sing(A) = 0 we proceed as follows: If K is a left ideal of R such that K/I is essential in R/I, then $(K/I)_\sigma$ is essential in $(R/I)_\sigma$. So $(K/I)_\sigma$ contains every simple subobjects of $(R/I)_\sigma$, i.e. $(K/I)_\sigma = (R/I)_\sigma$.

Since $Q_\sigma(-) = (-)_\sigma$ is an exact functor, we have an exact sequence in R-mod/σ:

$$0 \longrightarrow (K/I)_{\sigma} \longrightarrow (R/I)_{\sigma} \longrightarrow (R/K)_{\sigma} \longrightarrow 0.$$

Thus we find that

$$(R/K)_{\sigma} = E_{\sigma}((R/K)/\sigma(R/K)) = 0;$$

whence $R/K = \sigma(R/K)$. Since R/I is σ-torsion, for each nonzero $\bar{r}$ in R/I, $K\bar{r} \neq 0$. We note that

$$\text{sing}(R/I) = \{m \in R/I \mid \text{ann}_R(m) \text{ is an essential left ideal of } R\}$$

where $\text{ann}_R(m) = \{b \in R \mid bm = 0\}$.

Since $Am \neq 0$ for nonzero m in R/I we find that $\text{ann}_R(m) \cap A = 0$. Thus $\text{sing}(R/I) = 0$. This proves (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (1): If M is an R/I-module and $f \in \text{Hom}_R(M, E(R/I))$, then since $\text{Im}(f)$ is an R/I-module, it is contained in $E_{R/I}(R/I)$, which by assumption has zero singular submodule. Thus if $N \subseteq M$ is an essential submodule, then $\text{Hom}_R(M/N, E(R/I)) = 0$. Since R/I has finite uniform dimension, it must contain an essential direct sum $\oplus_{i=1}^{n} U_i$ of uniform submodules. Now letting $K/I = \oplus_{i=1}^{n} U_i$, it follows that $\text{Hom}_R(R/K, E(R/I)) = 0$ and $\text{Hom}_R(U_i/U_i', E(R/I)) = 0$ for any nonzero submodule $U_i' \subseteq U_i$, since every nonzero submodule of U_i is essential. This shows that $(R/I)_{\sigma} = (K/I)_{\sigma} = \oplus_{i=1}^{n} (U_i)_{\sigma}$, where $(U_i)_{\sigma}$ is simple in R-mod/σ since $(U_i')_{\sigma} = (U_i)_{\sigma}$ for any nonzero submodule $U_i' \subseteq U_i$. This proves the theorem.

This proposition has the following corollaries:

COROLLARY (3.2.I). If the conditions of the theorem are satisfied, then σ is perfect if and only if R_{σ}/I_{σ} is a direct sum of simple R_{σ}-modules and contains an isomorphic copy of each simple left R_σ-module.

COROLLARY (3.2.J). If the conditions of the theorem are satisfied and σ is perfect, then the localization M_σ of any R/I-module M is a direct sum of simple R_σ-modules.

COROLLARY (3.2.K). The following conditions are equivalent.

(1) $I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple artinian

(2) σ is perfect, I_σ is an ideal of R_σ, and the ring R/I has finite left uniform dimension and zero left singular ideal.

We have described an application of a hereditary torsion theory to rings of finite uniform dimensions and zero singular ideals. For the generalization of THEOREM C in section 1.1 to the finitely generated modules over a semifir and torsion modules over firs and semifirs, the reader is referred to Cohn [1971].

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