On Coarse Geometry and Coarse Embeddability

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Tiivistelmä — Referat — Abstract

Coarse structures are an abstract construction describing the behavior of a space at a large distance. In this thesis, a variety of existing results on coarse structures are presented, with the main focus being coarse embeddability into Hilbert spaces. The end goal is to present a hierarchy of three coarse invariants, namely coarse embeddability into a Hilbert space, a property of metric spaces known as Property A, and a finite-valued asymptotic dimension.

After outlining the necessary prerequisites and notation, the first main part of the thesis is an introduction to the basics of coarse geometry. Coarse structures are defined, and it is shown how a metric induces a coarse structure. Coarse maps, equivalences and embeddings are defined, and some of their basic properties are presented. Alongside this, comparisons are made to both topology and uniform topology, and results related to metrizability of coarse spaces are outlined.

Once the basics of coarse structures have been presented, the focus shifts to coarse embeddability into Hilbert spaces, which has become a point of interest due to its applications to several unsolved conjectures. Two concepts are presented related to coarse embeddability into Hilbert spaces, the first one being Property A. It is shown that Property A implies coarse embeddability into a Hilbert space, and that it is a coarse invariant.

The second main concept related to coarse embeddability is asymptotic dimension. Asymptotic dimension is a coarse counterpart to the Lebesgue dimension of topological spaces. Various definitions of asymptotic dimension are given and shown equivalent. The coarse invariance of asymptotic dimension is shown, and the dimensions of several example spaces are derived. Finally, it is shown that a finite asymptotic dimension implies coarse embeddability into a Hilbert space, and in the case of spaces with bounded geometry it also implies Property A.

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Chapter 1

Introduction

This master's thesis introduces the reader to coarse structures, and presents several concepts related to coarse embeddability into Hilbert spaces. Coarse structures are a way of characterizing how a metric space behaves at large distances, similar to the tools provided by topology for analyzing behavior at small distances. As in many other theories of mathematics, property-preserving mappings and equivalence are defined among coarse spaces. The resulting theory can be used to categorize discrete spaces, with for example the space of integers sharing an equivalence class with the space of real numbers.

After presenting the basics of coarse structures, coarse embeddings become a major focus for the thesis. Coarse embeddings have gained a lot of recent attention, as several major conjectures have obtained partial solutions based on the existence of a coarse embedding into a Hilbert space.

The thesis presents two major coarse invariants related to the aforementioned coarse embeddability. The first is a property of metric spaces called Property A, which implies coarse embeddability into a Hilbert space. The second is a coarse counterpart for the Lebesgue covering dimension called the asymptotic dimension. For metric spaces, a finite asymptotic dimension implies coarse embeddability into a Hilbert space. Basic properties of both invariants are proven, along with the previously mentioned embeddability results.

The reader is assumed to be familiar with basic topology and functional analysis. Although it isn't used in this thesis, a degree of familiarity with category theory may enhance the reader's understanding of coarse structures.

Chapter 2

Prerequisites

This chapter is a short review of concepts the reader should be familiar with. The chapter also goes over the notation used in this thesis.

2.1 General terminology and notations

The standard symbols of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ for integers, rational numbers and real numbers are used. The set of natural numbers $\mathbb{N}$ is assumed to consist of all nonnegative integers. The set of all positive integers is denoted by $\mathbb{Z}_+$. Similarly, the set of all positive real numbers is denoted by $\mathbb{R}_+$. For the set of all nonnegative real numbers, the notation $\mathbb{R}_0$ is adopted.

For an applicable function f, the term *increasing* is used, if the condition $f(x) \leq f(y)$ holds whenever x is less than y. For the strict version, the term *strictly increasing* is used. The terms *decreasing* and *strictly decreasing* are defined correspondingly.

A metric space (X, d) or a topological space $(X, \mathcal{T})$ may be abbreviated as just X, if the metric or topology in question is clear from the context. Similar abbreviations may also be used for other similar concepts defined later on, such as coarse and uniform spaces. For a given metric d, the topology induced by it is denoted by $\mathcal{T}_d$.

2.2 Set theory

This master's thesis uses standard set-theoretical notation for union, intersection, subsets, elements, and the empty set. The notation $X \setminus Y$ is used to denote the difference of two sets X and Y. In addition to this, the *symmetric difference* of two sets X and Y is defined by

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X) .$$

The *power set* of a set X is denoted by $P(X)$. The power set consists of all subsets of the set X.

If X and Y are sets, their product set is denoted by $X \times Y$. In the case of the product of n copies of a set X, the shorthand X^n is used. The *diagonal* Δ_X of a set X is a subset of X^2, defined by

$$\Delta_X = \{(x,x) \mid x \in X\}.$$

Furthermore, let X be a set, and consider the subsets of X^2. Several operations on these subsets will be of use later on. If A is a subset of X^2, the *inverse* of A is defined by

$$A^{-1} = \{(y,x) \in X^2 \mid (x,y) \in A\}.$$

If A and B are subsets of X^2, the *composition* of A and B is defined by

$$A \circ B = \left\{(x,z) \in X^2 \mid \text{ there is a } y \in X \text{ such that } (x,y) \in A \text{ and } (y,z) \in B\right\}.$$

2.3 Functional Analysis

Throughout this thesis, various spaces from functional analysis are required. Recall that a *Banach space* is a complete normed vector space, and a *Hilbert space* is a complete inner product space. Furthermore, recall that an infinite sum of non-negative numbers a_i over an index set I is defined as follows:

$$\sum_{i \in I} a_i = \sup \left\{ \sum_{i \in I'} a_i \;\middle|\; I' \subset I, |I'| < \infty \right\}.$$

Infinite sums for non-positive numbers are defined similarly. For numbers a_i attaining both positive and negative values, if the absolute values $|a_i|$ have a finite sum, one can define the sum by splitting the numbers into positive and negative subsets.

Let X be a set. For a real number $p \geq 1$, one can define the l^p*-space* over X, denoted by $l^p(X)$, as follows:

$$l^p(X) = \left\{ a \in \prod_{x \in X} \mathbb{R} \;\middle|\; \sum_{x \in X} |a_x|^p < \infty \right\}.$$

The space $l^p(X)$ is a Banach space under the usual l^p-norm when summation and scalar multiplication are defined coordinatewise. In the case of $p = 2$, $l^p(X)$ is a Hilbert space with the standard inner product of l^2-spaces. These spaces are a special case of the more general L^p-spaces for spaces X equipped with a measure μ, obtained by selecting μ to be the counting measure. The aforementioned properties in the general case can be found from various functional analysis textbooks, for example [1]. If x is an element of X, the unit vector along the x-coordinate in $l^p(X)$ will be denoted by $\mathbf{e}_x$.

Another construction required later is the *direct sum of Hilbert spaces*. Let I be an index set, and for every i in I, let X_i be a Hilbert space. The direct sum, denoted by $\bigoplus_{i\in I} X_i$, is defined by

$$\bigoplus_{i\in I} X_i = \left\{ x \in \prod_{i\in I} X_i \;\middle|\; \sum_{i\in I} \|x_i\|^2 < \infty \right\}.$$

The elements of $\bigoplus_{i\in I} X_i$ are expressed in the form of $\bigoplus_{i\in I} x_i$, where the coordinates x_i are elements of X_i correspondingly. As in the previous case, the space $\bigoplus_{i\in I} X_i$ is a Hilbert space when using coordinatewise definitions of summation and scalar multiplication, along with the inner product defined by

$$\langle x, y\rangle = \sum_{i\in I} \langle x_i, y_i\rangle .$$

The proofs resemble the ones for regular L^2-spaces and are omitted here.

Chapter 3

Coarse geometry

This chapter is an introduction to the theory of coarse spaces. Coarse spaces are sets equipped with a coarse structure, which describes the behavior of the space at a distance. A coarse space has a well-defined notion of boundedness and bounded subsets. One can obtain some intuition on the concept by considering a extremely zoomed-out view of a space, under which for example the spaces $\mathbb{Z}$ and $\mathbb{R}$ look similar.

After initial definitions, a method of deriving a coarse structure from a metric is obtained. This is similar to how a metric induces a topology or some other topological structure, but the properties described are majorly the opposite of those described by topology. The closest topological counterpart to coarse structures is the concept of uniform structures, which is used for example in generalizing uniform continuity.

Following this, various property-preserved mappings for coarse spaces are defined, along with the notion of coarse equivalence. The end of the chapter presents some additional results, most of which are referred to later on in this thesis.

The concepts presented in this chapter are mostly based on the doctoral thesis [6], although with some influences from other papers. One can also refer to the book [10] for more information.

3.1 Coarse structures

Definition 3.1. Let X be a set, and let $\mathcal{E}$ be a subset of $P(X^2)$. The collection $\mathcal{E}$ is a *coarse structure* on X, if the following conditions apply:

- *Diagonal property:* The diagonal Δ_X is an element of $\mathcal{E}$.
- *Subset property:* If $A \in \mathcal{E}$ and $B \subset A$, then $B \in \mathcal{E}$.
- *Finite union property:* If $A, B \in \mathcal{E}$, then $A \cup B \in \mathcal{E}$.

- *Inverse property:* If $A \in \mathcal{E}$, then $A^{-1} \in \mathcal{E}$.
- *Composition property:* If $A, B \in \mathcal{E}$, then $A \circ B \in \mathcal{E}$.

The elements of the coarse structure $\mathcal{E}$ are commonly referred to as *controlled sets* or *entourages*. The pair $(X, \mathcal{E})$ is called a *coarse space.*

Example 3.2. Let X be a set and let $\mathcal{E}_{\max}$ be the entire power set $P(X^2)$. Then $\mathcal{E}_{\max}$ is a coarse structure.

Furthermore, let $\mathcal{E}_{\min}$ be composed of all the subsets of the diagonal Δ_X. The collection $\mathcal{E}_{\min}$ contains Δ_X, and is closed under subsets and unions. In addition to this, if A is a subset of Δ_X, then $A^{-1} = A$. Furthermore, if B is another subset of Δ_X, then $A \circ B = A \cap B$, which is contained in Δ_X. Therefore, $\mathcal{E}_{\min}$ is a coarse structure.

Due to the diagonal and subset properties of controlled sets, every coarse structure on X contains $\mathcal{E}_{\min}$. Furthermore, since coarse structures on X are subsets of $P(X^2)$, every coarse structure on X is contained within $\mathcal{E}_{\max}$. Hence, $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ are the smallest and largest possible coarse structures on X.

A key property of coarse structures is that they introduce a notion of boundedness on sets.

Definition 3.3. Let X be a set and $\mathcal{E}$ a coarse structure on X. Let B be a nonempty subset of X. We say that X is *bounded* with respect to $\mathcal{E}$, if there is an $x \in X$ such that $B \times \{x\}$ is controlled.

Proposition 3.4. *Let X be a set and $\mathcal{E}$ a coarse structure on X. Let B be a nonempty subset of X. Then B is bounded with respect to $\mathcal{E}$ if and only if the set B^2 is controlled.*

Proof. Assume that B^2 is controlled. Since B is nonempty, there is some element x contained in B. Due to the subset property of controlled sets, $\{x\} \times B$ is controlled.

Next, assume that there is an $x \in X$ such that $B \times \{x\}$ is controlled. The set B^2 can be written in the form

$$B^2 = (B \times \{x\}) \circ (\{x\} \times B) = (B \times \{x\}) \circ (B \times \{x\})^{-1}.$$

Hence, due to the inverse and composition properties of controlled sets, B^2 is controlled. □

Because Δ_X is always controlled, the subset property of controlled sets implies that $\{(x, x)\}$ is controlled for all $x \in X$. Hence, by our definition, singletons are always bounded. In addition to this, subsets of bounded sets are bounded: This follows from the subset property of controlled sets, since if B is bounded and A is a nonempty subset of B, the set A^2 is a subset of B^2.

The finite union of bounded sets is, unlike in the case of metric spaces, not always bounded. A counterexample can be seen in $\mathcal{E}_{\min}$, the bounded sets of which are precisely the singleton sets. Using the subset property, one can see that the entire space X is bounded if and only if the coarse structure is $\mathcal{E}_{\max}$.

3.2 Coarse geometry

This section develops coarse structures on metric spaces. Similarly to the theory of topological spaces, one can find a canonical coarse structure for X using the metric d. This structure describes the bounded sets of X.

Let (X, d) be a metric space. Recall that the *diameter* of a set $A \subset X$ is defined by $d(A) = \sup_{x,y \in A} d(x, y)$, and a set A is *bounded* in (X, d) if it has a finite diameter. This is equivalent to A being contained in a ball of finite radius $B_d(x, r)$, where $x \in X$ and $r \in \mathbb{R}$.

For the sake of convenience, several new notations are adopted. Let r be a nonnegative real number. The *r-diagonal* of (X, d), denoted by $\Delta_d(r)$, is defined by

$$\Delta_d(r) = \{(x, y) \in X^2 \mid d(x, y) \leq r\}.$$

Note that the 0-diagonal $\Delta_d(0)$ is the regular diagonal Δ_X of X.

Furthermore, let E be a subset of X^2. The notation $d[E]$ is adopted for the following value:

$$d[E] = \sup_{(x,y) \in E} d(x, y).$$

The value of $d[E]$ may be infinite. For the previously defined r-diagonals, the value of $d[\Delta_d(r)]$ is at most r. In addition to this, if B is a subset of X, $d[B^2]$ equals the diameter of B.

Proposition 3.5. *Let (X, d) be a metric space. Define $\mathcal{E}_d \subset P(X^2)$ as follows:*

$$\mathcal{E}_d = \left\{ E \subset X^2 \;\middle|\; d[E] < \infty \right\}$$

Then $\mathcal{E}_d$ is a coarse structure.

The structure $\mathcal{E}_d$ defined in Proposition 3.5 is called the *bounded coarse structure* of (X, d).

Proof. The distance $d(x, y)$ is zero for each $(x, y) \in \Delta_X$. Therefore, $d[\Delta_X]$ equals zero, and as such Δ_X is an element of $\mathcal{E}_d$.

Let A be an element of $\mathcal{E}_d$. Because $d(x,y) = d(y,x)$ for every $(x,y) \in A$, $d[A^{-1}]$ equals $d[A]$. Hence, the set A^{-1} is an element of $\mathcal{E}_d$. Let A' be a subset of A. As a supremum over a smaller set, $d[A']$ is at most $d[A]$, and therefore $A' \in \mathcal{E}_d$.

Let A and B be elements of $\mathcal{E}_d$. If (a,b) is an element of $A \cup B$, either $(a,b) \in A$ and $d(a,b) \leq d[A]$, or $(a,b) \in B$ and $d(a,b) \leq d[B]$. Therefore, $d[A \cup B]$ is at most $\max(d[A], d[B])$. This shows that $A \cup B$ is an element of $\mathcal{E}_d$.

Finally, let A and B be as previously, and let (a,c) be an element of $A \circ B$. In this case, there is an element b of X for which $(a,b) \in A$ and $(b,c) \in B$. Using the triangle inequality, one obtains the bound

$$d(a,c) \leq d(a,b) + d(b,c) \leq d[A] + d[B].$$

Hence, $d[A \circ B]$ is at most $d[A] + d[B]$, and therefore $A \circ B$ is an element of $\mathcal{E}_d$. □

Proposition 3.6. *Let (X,d) be a metric space, and let B be a nonempty subset of X. Then B is bounded with respect to d if and only if B is bounded with respect to $\mathcal{E}_d$.*

Proof. Due to Proposition 3.4, B is bounded with respect to $\mathcal{E}_d$ if and only if B^2 is an element of $\mathcal{E}_d$. By the definition of $\mathcal{E}_d$, this is true precisely when $d[B^2]$ is finite, or in other words, when the diameter of B is finite. Therefore, B is bounded with respect to $\mathcal{E}_d$ if and only if B is bounded with respect to d. □

Example 3.7. Let X be the open interval $\{x \in \mathbb{R} | -1 < x < 1\}$. Denote the euclidian metric on X by d_0. Let d_1 be the discrete metric on X, where $d_1(x,y) = 1$ whenever $x \neq y$. Finally, let d_2 be the metric on X defined by $d_2(x,y) = |\tan(\pi x/2) - \tan(\pi y/2)|$.

The metrics d_0 and d_2 induce the same topology on X, while the topology induced by d_1 is different. Note that the entire set X is bounded under d_0 and d_1, but not under d_2. Hence, both d_0 and d_1 induce the maximal coarse structure on X, but d_2 doesn't. As a result, $\mathcal{T}_{d_0} = \mathcal{T}_{d_2}$, but $\mathcal{E}_{d_0} \neq \mathcal{E}_{d_2}$. Conversely, $\mathcal{E}_{d_0} = \mathcal{E}_{d_1}$, but $\mathcal{T}_{d_0} \neq \mathcal{T}_{d_1}$.

This illustrates how the topology and the coarse structure defined by a metric describe different properties of the metric, and similarity of two metrics in one does not guarantee the similarity of them in the other. While openness describes the properties of a set on a small scale, boundedness describes its properties on a large scale.

Example 3.8. The following example shows that even if two metrics have the same bounded sets, they may define different coarse structures. Let $\mathbb{N}$ be the set of natural numbers. If $f : \mathbb{N} \to \mathbb{R}$ is a strictly increasing function, one may define a metric d on $\mathbb{N}$ via $d(x,y) = |f(x) - f(y)|$. Take the functions f_1 and f_2 from $\mathbb{N}$ to $\mathbb{R}$, where $f_1(n) = n$ and $f_2(n) = n^2$, and define d_1 and d_2 as previously.

Let B be a nonempty subset of $\mathbb{N}$, and denote its smallest element by b_0. Assume first that B is finite. Hence, it has a largest element b_1. One sees that the diameter of B with respect to d_1 is $b_1 - b_0$, and the diameter with respect to d_2 is $b_1^2 - b_0^2$.

Next, assume instead that B is infinite. Hence, there is a sequence $(b_i)_{i=1}^{\infty}$ which tends to infinity in B. One sees that both $d_1(b_0, b_i) = b_i - b_0$ and $d_2(b_0, b_i) = b_i^2 - b_0^2$ become arbitrarily large as i tends to infinity. Hence, both d_1 and d_2 have finite sets as their bounded sets.

Observe the set $E = \{(n, n+1) | n \in \mathbb{N}\}$. Since $d_1(n, n+1) = 1$ for all natural numbers n, one sees that E is an element of $\mathcal{E}_{d_1}$. However, E is not an element of $\mathcal{E}_{d_2}$, since $d_2(n, n+1) = 2n + 1$ attains arbitrarily large values as n tends to infinity. Hence, d_1 and d_2 define different coarse structures, despite having the same bounded sets.

The comparisons presented so far have been between coarse and topological spaces. However, the closest topological counterpart to coarse spaces is the concept of a *uniform space*. For the sake of comparison, the basics of uniform spaces will be briefly presented here. For further details on the subject including proofs, see [4].

Definition 3.9. Let X be a set, and let $\mathcal{S}$ be a subset of $P(X^2)$. The collection $\mathcal{S}$ is a *uniform structure* on X, if the following conditions apply:

- Every element of $\mathcal{S}$ contains the diagonal Δ_X.
- If $A \in \mathcal{S}$ and $A \subset B$, then $B \in \mathcal{S}$.
- If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$.
- If $A \in \mathcal{S}$, then $A^{-1} \in \mathcal{S}$.
- If $A \in \mathcal{S}$, then there is an element $B \in \mathcal{S}$ fulfilling $B \circ B \subset A$.

Similarly to coarse structures, the elements of a uniform structure $\mathcal{S}$ are referred to as entourages, and the pair $(X, \mathcal{S})$ is called a uniform space.

One can see how the required properties of coarse structures correspond to the ones of uniform structures, only in most cases being defined in the opposite direction. Readers familiar with the concept may note that uniform structures are filters on $X \times X$. A uniform structure $\mathcal{S}$ on X yields a unique topology on X, where a given point x is in the interior of a set U if $U \times \{x\} = V \cap X \times \{x\}$ for some $V \in \mathcal{S}$.

For a given uniform structure $\mathcal{S}$, a subset $\mathcal{S}'$ of $\mathcal{S}$ will be called a *generating set of* $\mathcal{S}$ if every entourage of $\mathcal{S}$ contains a set of $\mathcal{S}'$. Note that in literature, the term *fundamental system of entourages* of $\mathcal{S}$ may be used for such sets $\mathcal{S}'$. Using the fact that supersets of entourages are entourages, the structure $\mathcal{S}$ can be obtained from the set $\mathcal{S}'$ uniquely.

If (X, d) is a metric space, one can show that the set $\{\Delta_d(r) | r > 0\}$ forms a generating set for a uniform structure $\mathcal{S}_d$. Hence, a metric d on X defines a uniform structure $\mathcal{S}_d$ on X, similarly to how it defines a topology $\mathcal{T}_d$ and a coarse structure $\mathcal{E}_d$.

Uniform spaces are used to generalize the concept of uniform continuity between metric spaces. Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be uniform spaces. A map $f : X \to Y$ is said to be uniformly continuous if for every entourage V' of Y the set $(f \times f)^{-1}V'$ is an entourage of X. This can be shown to coincide with the metric definition of uniform continuity when the uniform structures are induced by metrics.

Example 3.10. Similarly to Example 3.7, this example shows that coarse and uniform structures induced by a metric contain different information on the inducing metric. Let d_1, d_2 and d_3 be three metrics on $\mathbb{N}$ defined as follows for $n, m \in \mathbb{N}$, $n \neq m$:

$$\begin{aligned} d_1(n,m) &= 2^{-n} + 2^{-m}, \\ d_2(n,m) &= 2, \\ d_3(n,m) &= 2^n + 2^m. \end{aligned}$$

One can easily show that the resulting d_i are metrics. Under d_1 and d_2 the space $\mathbb{N}$ is bounded, whereas under d_3 it is not. Hence, one sees that $\mathcal{E}_{d_1} = \mathcal{E}_{d_2} \neq \mathcal{E}_{d_3}$.

Note that the sets $\Delta_{d_2}(1)$ and $\Delta_{d_3}(1)$ are both the diagonal $\Delta_{\mathbb{N}}$. Due to this, both $\mathcal{S}_{d_2}$ and $\mathcal{S}_{d_3}$ contain the diagonal, and the first two properties of uniform structures show that $\mathcal{S}_{d_2} = \mathcal{S}_{d_3} = \{V \in P(X^2) \mid \Delta_{\mathbb{N}} \subset V\}$. Next, it is shown that $\mathcal{S}_{d_1}$ does not contain $\Delta_{\mathbb{N}}$. Assuming the contrary implies that $\Delta_{d_1}(r) \subset \Delta_{\mathbb{N}}$ for some $r > 0$, since $\{\Delta_{d_1}(r) \mid r > 0\}$ is a generating set for S_{d_1}. This is clearly not true, as the distances $d_1(n, n+1)$ become arbitrarily small as n increases. Therefore, one concludes that $\mathcal{S}_{d_1} \neq \mathcal{S}_{d_2} = \mathcal{S}_{d_3}$.

3.3 Maps and coarse equivalence

Property preserving mappings are a key part of most fields of mathematics. For coarse spaces, there are two main classes of mappings which could be considered analogous to the continuous functions of topology.

Definition 3.11. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. A map $f : X \to Y$ is *coarsely uniform* if, whenever E is an X-controlled set, the set $(f \times f)(E)$ is Y-controlled.

A map $f : X \to Y$ is a *coarse map* if, in addition to being coarsely uniform, it is *coarsely proper*: $f^{-1}B$ is bounded in X whenever B is bounded in Y.

The identity mapping id_X is a coarse map. In addition to this, both coarse uniformness and coarse properness are preserved in compositions of mappings.

The next step is defining coarse equivalences, the notion of similarity for two coarse spaces. A desirable property would be that all bounded spaces would be consider coarsely equivalent. This can be motivated by the idea that coarse structures describe large-scale properties, and at a large distance, a bounded set appears similar to a point.

The standard approach of defining equivalence would be to use property-preserving mappings that have a similarly property-preserving inverse. However, this approach will not result in similarity of bounded spaces: If X and Y are bounded spaces with $|Y| < |X|$, there is no bijective map from X to Y. Therefore, no map from X to Y has an inverse.

In order to avoid this problem, an equivalence relation on mappings is introduced.

Definition 3.12. Let X be a set and $(Y, \mathcal{E}_Y)$ a coarse space. Let f and g be maps from X to Y. The maps f and g are *close*, if the set $\{(f(x), g(x)) \mid x \in X\}$ is controlled.

For any two maps f, g from a set X to a set Y, the notation $(f, g)(X)$ is adopted for the set $\{(f(x), g(x)) \mid x \in X\}$.

Proposition 3.13. *Closeness of maps is an equivalence relation.*

Proof. Let f, g and h be maps from set X to coarse space $(Y, \mathcal{E}_Y)$. Because $(f, f)(X)$ is a subset of Δ_Y, closeness is reflexive. Furthermore, $(g, f)(X)$ is the inverse of $(f, g)(X)$, which shows that closeness is symmetric. Finally, transitivity is obtained by seeing that $(f, h)(X)$ is a subset of $(f, g)(X) \circ (g, h)(X)$. $\square$

Remark 3.14. Suppose that Y is a metric space with metric d, and $\mathcal{E}_Y$ is the bounded coarse structure with respect to d. In this case, due to the definition of bounded coarse structures, two maps f and g are close if and only if $\sup_{x \in X} d(f(x), g(x))$ is finite.

Using the notion of closeness, a definition for coarse equivalences is introduced.

Definition 3.15. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. Let f be a coarsely uniform map from X to Y. Then f is a *coarse equivalence* if there is a coarsely uniform map $g : Y \to X$, for which $g \circ f$ is close to id_X, and $f \circ g$ is close to id_Y. The map g is called a *coarse inverse* of f.

Two coarse spaces are said to be *coarsely equivalent*, if there is a coarse equivalence between them.

Proposition 3.16. *Coarse equivalence of spaces is an equivalence relation. Furthermore, the composition of coarse equivalences is a coarse equivalence.*

Proof. Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ and $(Z, \mathcal{E}_Z)$ be coarse spaces. The identity map id_X is a coarse equivalence, with itself as a coarse inverse. If $f : X \to Y$ is a coarse equivalence, then its coarse inverse $g : Y \to X$ is also a coarse equivalence. Hence, coarse equivalence of spaces is reflexive and symmetric.

Next, assume that $f' : Y \to Z$ is another coarse equivalence, with coarse inverse g'. The maps $f' \circ f$ and $g \circ g'$ are both coarsely uniform. Now, let x be a point of X. Because $g' \circ f'$ is close to id_Y, the pair $(f(x), (g' \circ f' \circ f)(x))$ is an element of the controlled set $(\mathrm{id}_Y, g' \circ f')(Y)$. Hence, the pair $((g \circ f)(x), (g \circ g' \circ f' \circ f)(x))$ is an element of the set $(g \times g)\left((\mathrm{id}_Y, g' \circ f')(Y)\right)$, which is controlled due to the coarse uniformness of g. Consequently, we obtain the relation

$$(\mathrm{id}_X, g \circ g' \circ f' \circ f)(X) \subset (\mathrm{id}_X, g \circ f)(X) \circ (g \times g)\left((\mathrm{id}_Y, g' \circ f')(Y)\right),$$

which shows that $(g \circ g') \circ (f' \circ f)$ is close to id_X. The proof for $(f \circ f') \circ (g' \circ g)$ is analoguous. Hence, $f' \circ f$ is a coarse equivalence with coarse inverse $g' \circ g$. As a result, the composition of coarse equivalences is a coarse equivalence, and coarse equivalence of spaces is transitive. □

The definition of coarse equivalences uses coarsely uniform maps. The following Proposition shows that a definition via coarse maps yields the same coarse equivalences.

Proposition 3.17. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. Let $f : X \to Y$ be a coarse equivalence, with g as its coarse inverse. In this case, both f and g are coarse maps.*

Proof. It suffices to show that both f and g are coarsely proper. Since g is also a coarse equivalence, the claim follows merely by showing that f is coarsely proper.

Let B be a bounded subset of Y, and let (x, y) be an element of $(f^{-1}B)^2$. Note that the set $(g(B))^2 = (g \times g)(B^2)$ is controlled. Since $f(x)$ and $f(y)$ are elements of B, the points $(g \circ f)(x)$ and $(g \circ f)(y)$ are elements of $g(B)$. Due to this, the pair (x, y) is an element of the controlled set

$$(\mathrm{id}_X, g \circ f)(X) \circ (g(B))^2 \circ (g \circ f, \mathrm{id}_X)(X).$$

Therefore, $(f^{-1}B)^2$ is a subset of a controlled set, which shows that $f^{-1}B$ is bounded. Hence, f is coarsely proper. □

Let $(X, \mathcal{E}_X)$ be a bounded coarse space. Note that $\mathcal{E}_X$ is the power set $P(X)$, and therefore every subset of X^2 is controlled. Due to this, every map f from a coarse space into X is coarsely uniform, and every pair of maps f, g from a set into X is close. Hence, if $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are bounded coarse spaces, one may select arbitrary maps $f : X \to Y$ and $g : Y \to X$, and conclude that f is a coarse equivalence with coarse inverse g. In other words, all bounded coarse spaces are coarsely equivalent, and all maps between bounded coarse spaces are coarse equivalences.

Furthermore, assume that $(X, \mathcal{E}_X)$ is bounded and $f : Y \to X$ is a coarse equivalence. Proposition 3.17 shows that f is coarsely proper, and therefore $Y = f^{-1}X$ is also a bounded coarse space. Hence, not only are all bounded coarse spaces equivalent with each other, but also no non-bounded coarse space is equivalent with a bounded one. This is the first example of a coarse invariant, and therefore also provides the first example of two spaces not being coarsely equivalent.

3.4 Coarse embeddings

The main mapping class of interest in this thesis is the coarse embedding. As with topological embeddings, a coarse embedding $f : X \to Y$ is a coarse equivalence from X into a coarse subspace of Y. In order to state the definition for general coarse spaces, a notion of coarse subspaces is first required.

Proposition 3.18. *Let $(X, \mathcal{E})$ be a coarse space, and let A be a subset of X. Define the collection $\mathcal{E}|A$ by*

$$\mathcal{E}|A = \{E \in \mathcal{E} \mid E \subset A^2\} = \mathcal{E} \cap P(A^2).$$

The collection $\mathcal{E}|A$ is a coarse structure.

Proof. Since both $\mathcal{E}$ and $P(A^2)$ are closed under composition, inverses, subsets and finite unions, the same properties pass on to $\mathcal{E}|A$. The diagonal Δ_A is a subset of Δ_X, and therefore also an element of $\mathcal{E}$. Since the diagonal Δ_A is also a subset of A^2, it is an element of $\mathcal{E}|A$. □

Note that when A is a subspace of a metric space (X, d), the bounded coarse structure $\mathcal{E}_{d|A}$ is also of the form $\{E \in \mathcal{E}_d \mid E \subset A^2\}$. Hence, the notion of a coarse subspace coincides with the notion of a metric subspace for bounded coarse structures $\mathcal{E}_d$.

Let $(X, \mathcal{E})$ be a coarse space, and let A be a subset of X. Observe the sets of the form $E \cap A^2$, where E is an element of $\mathcal{E}$. Due to the subset property, $E \cap A^2$ is an element of $\mathcal{E}$. Since $E \cap A^2$ is a subset of A^2, it is also an element of $\mathcal{E}|A$. Furthermore, if E is an element of $\mathcal{E}|A$, one obtains that $E = E \cap A^2$. Therefore, every element of $\mathcal{E}|A$ is of this form, and the set $\mathcal{E}|A$ can be written in the form

$$\mathcal{E}|A = \{E \cap A^2 \mid E \in \mathcal{E}\}. \tag{3.19}$$

Next is the first formal definition of coarse embeddings. The definition follows the standard method of defining various types of embeddings.

Definition 3.20. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. A mapping $f : X \to Y$ is a *coarse embedding* if f is a coarse equivalence from $(X, \mathcal{E}_X)$ to $(f(X), \mathcal{E}_Y|f(X))$.

Proposition 3.21. *Let X and Y be coarse spaces, and let $f : X \to Y$ be a coarse equivalence. Then f is a coarse embedding.*

Proof. Since every controlled set of $(f(X), \mathcal{E}_Y|f(X))$ is also a controlled set of $(Y, \mathcal{E}_Y)$, the coarse uniformness of f is preserved when restricting the target space. Denote the coarse inverse of f by g, and let $h : f(X) \to X$ be the restriction $h = g|f(X)$. If E is an element of $\mathcal{E}_X$, $(h \times h)^{-1}E$ can be written in the form $((g \times g)^{-1}E) \cap (f(X))^2$. Hence, it follows from equation 3.19 that h is coarsely uniform. Since $h \circ f$ and $g \circ f$ are equal, $h \circ f$ is close to id_X. Finally, the set $(f \circ h, \mathrm{id}_{f(X)})(f(X))$ is the intersection of $(f \circ h, \mathrm{id}_Y)(Y)$ with $(f(X))^2$, which proves that $f \circ h$ is close to $\mathrm{id}_{f(X)}$. □

The following Proposition yields an alternate definition of coarse embeddings, which is generally easier to check.

Proposition 3.22. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. A mapping $f : X \to Y$ is a coarse embedding if and only if the conditions $\{(f \times f)(E) \mid E \in \mathcal{E}_X\} \subset \mathcal{E}_Y$ and $\{(f \times f)^{-1}E \,|\, E \in \mathcal{E}_Y\} \subset \mathcal{E}_X$ hold. In other words, f is a coarse embedding if and only if f is coarsely uniform, and the pre-image of every controlled set under $(f \times f)$ is controlled.*

Proof. First, note that $f : X \to Y$ is coarsely uniform if and only if $f : X \to f(X)$ is coarsely uniform. This is an immediate consequence of the fact that all sets $(f \times f)(E)$, where E is a subset of X^2, are elements of $P(f(X)^2)$.

Next, assume that there is a coarse inverse $g : f(X) \to X$. Let E be an element of $\mathcal{E}_Y$. Let (x, y) be a pair which $f \times f$ maps into E. Note that the pair $(f(x), f(y))$ is also an element of $E \cap f(X)^2$, which due to equation 3.19 is a controlled set of $\mathcal{E}_Y|f(X)$. Hence, the pair $((g \circ f)(x), (g \circ f)(y))$ is an element of the set $(g \times g)(E \cap f(X)^2)$, which is controlled. From this, one can infer the inclusion

$$(f \times f)^{-1}E \subset (\mathrm{id}_X, g \circ f)(X) \circ (g \times g)(E \cap f(X)^2) \circ (g \circ f, \mathrm{id}_X)(X).$$

Consequently, $(f \times f)^{-1}E$ is controlled for every $E \in \mathcal{E}_Y$.

For the other direction, assume that for every $E \in \mathcal{E}_Y$ the set $(f \times f)^{-1}E$ is controlled. Select $g : f(X) \to X$ as follows: For every element y in $f(X)$, $g(y)$ is an arbitrarily chosen element in $f^{-1}\{y\}$. This definition immediately yields that $f \circ g = \mathrm{id}_{f(X)}$. If E is an element of $\mathcal{E}_Y|f(X)$, the set $(g \times g)(E)$ is a subset of $(f \times f)^{-1}E$. Since E is also an element of $\mathcal{E}_Y$, the set $(g \times g)(E)$ is controlled. Consequently, g is coarsely uniform.

Finally, let x be an element of X. Since the maps $f \circ g$ and $\mathrm{id}_{f(X)}$ are equal, the map $f \circ g \circ f$ maps x to $f(x)$. Hence, the pair $(x, (g \circ f)(x))$ is an element of $(f \times f)^{-1}\Delta_{f(X)}$. Therefore, $(\mathrm{id}_X, g \circ f)(X)$ is controlled, or in other words, $g \circ f$ is close to id_X. In conclusion, g is a coarse inverse of f. □

Coarse embeddings have a useful characterization in metric spaces. In works which only use coarse embeddings in the context of metric spaces, this is commonly used as the definition of a coarse embedding.

Proposition 3.23. *Let (X, d) and (Y, d') be metric spaces. A map f is a coarse embedding from $(X, \mathcal{E}_d)$ to $(Y, \mathcal{E}_{d'})$, if and only if the following condition applies: There exist* *increasing functions ρ_+ and ρ_- from $\mathbb{R}_0$ to itself, for which both $\rho_+(t)$ and $\rho_-(t)$ tend to infinity as t increases, and the inequality*

$$\rho_-(d(x,y)) \leq d'(f(x), f(y)) \leq \rho_+(d(x,y))$$

holds for all pairs $(x,y) \in X^2$.

Proof. Let f be a map from X to Y. Assume first that the functions ρ_+ and ρ_- exist as defined above. Let E be an element of $\mathcal{E}_d$. By the definition of $\mathcal{E}_d$, $d[E]$ is finite. Using the increasingness of ρ_+, one obtains the inequality

$$d'\big[(f \times f)(E)\big] = \sup_{(x,y)\in E} d'(f(x), f(y)) \leq \sup_{(x,y)\in E} \rho_+ (d(x,y)) \leq \rho_+\big(d[E]\big) < \infty.$$

Hence, $(f \times f)(E)$ is controlled.

Next, let F be an element of $\mathcal{E}_{d'}$. Fix an element (x,y) of $(f \times f)^{-1}F$. Since $(f(x), f(y))$ is in F, one obtains the inequality

$$\rho_-(d(x,y)) \leq d'(f(x), f(y)) \leq d'[F].$$

Since ρ_- tends to infinity, there is some real number C for which $\rho_-(C)$ is greater than $d'[F]$. Due to ρ_- being increasing, $d(x,y)$ is less than C. Because this bound holds for all elements (x,y) of $(f \times f)^{-1}F$, the set $(f \times f)^{-1}F$ is controlled. By Proposition 3.22, f is a coarse embedding.

Next, assume that f is a coarse embedding. Define functions $g_- : \mathbb{R}_0 \to \mathbb{R}_0 \cup \{\infty\}$ and $g_+ : \mathbb{R}_0 \to \mathbb{R}_0$ by

$$g_-(r) = \inf \{d'(f(x), f(y)) \mid d(x,y) \geq r\}, \tag{3.24}$$

$$g_+(r) = \sup \{d'(f(x), f(y)) \mid d(x,y) \leq r\} = d'\big[(f \times f)(\Delta_d(r))\big]. \tag{3.25}$$

Note that if X is bounded, $g_-(r)$ may be an infimum over an empty set. In this case, the value of $g_-(r)$ is defined to be infinite. For values of r less than $d(X)$, $g_-(r)$ is guaranteed to be finite-valued. Since f is a coarse embedding, g_+ is finite-valued for all nonnegative r. Furthermore, the sets $\Delta_d(r)$ and $\{(x,y) \mid d(x,y) \geq r\}$ are respectively increasing and decreasing with respect to r. Due to this, both g_- and g_+ are increasing functions.

Using the previously defined g_+ and g_-, define functions ρ_- and ρ_+ as follows:

$$\rho_-(r) = \min(g_-(r), r), \tag{3.26}$$

$$\rho_+(r) = \max(g_+(r), r). \tag{3.27}$$

Both ρ_- and ρ_+ are finite-valued, since $\rho_-(r)$ is at most r and ρ_+ is a maximum of two finite-valued functions. As a minimum and a maximum of two increasing functions, ρ_- and ρ_+ are increasing. It is clear that ρ_+ tends to infinity.

To show that ρ_- tends to infinity, one has to show that g_- obtains arbitrarily large values. Assume to the contrary that g_- has an upper bound C. Hence, for every natural number n, there exists a pair (x_n, y_n) of X^2 which fulfills $d(x_n, y_n) \geq n$ and $d'(f(x_n), f(y_n)) \leq C$. It follows that $d\left[(f \times f)^{-1}\Delta_{d'}(C)\right]$ is infinite, which contradicts the fact that f is a coarse embedding. Hence, the function ρ_- tends to infinity.

Finally, let x and y be two elements of X. The definitions 3.24-3.27 directly yield the inequalities

$$\rho_-(d(x,y)) \leq g_-(d(x,y)) \leq d'(f(x), f(y)) \leq g_+(d(x,y)) \leq \rho_+(d(x,y)).$$

This completes the proof. □

Example 3.28. Equip $\mathbb{R}$ with the standard euclidean metric d_E. Define a map $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x+1 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}.$$

The map f fulfills the following inequality:

$$d_E(x,y) - 1 \leq d_E(f(x), f(y)) \leq d_E(x,y) + 1.$$

Hence, selecting $\rho_-(x) = \max(x-1, 0)$ and $\rho_+(x) = x+1$ shows that f is a coarse embedding. Furthermore, since $f(\mathbb{R}) = \mathbb{R}$, it is a coarse equivalence. Note, however, that f is discontinuous at every point of $\mathbb{R}$. This example shows that, despite the similarity between coarse embeddability and the bilipschitz condition, coarse embeddings between metric spaces are not necessarily continuous anywhere.

Example 3.29. Equip the spaces $\mathbb{R}$ and $\mathbb{Z}$ with the euclidean metric d_E. Let $f : \mathbb{R} \to \mathbb{Z}$ be the *floor function*, which maps x to the largest integer n which is at most x. This function is commonly denoted by $f(x) = \lfloor x \rfloor$. Similarly to Example 3.28, f is a surjection which fulfills the inequality

$$d_E(x,y) - 1 \leq d_E(f(x), f(y)) \leq d_E(x,y) + 1.$$

Hence, the spaces $(\mathbb{R}, \mathcal{E}_{d_E})$ and $(\mathbb{Z}, \mathcal{E}_{d_E})$ are coarsely equivalent.

One can similarly find a coarse equivalence from $\mathbb{R}^n$ to $\mathbb{Z}^n$, by defining $f(x_1, \dots, x_n) = (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$. In this case, the inequality f fulfills is given by

$$d_E(x,y) - n \leq d_E(f(x), f(y)) \leq d_E(x,y) + n.$$

3.5 Discrete metric spaces

A major focus in coarse geometry is the study of discrete metric spaces. Since discreteness is characterized by all subsets being open, topological equivalence doesn't differentiate between discrete spaces beyond their cardinality. Coarse equivalence, on the other hand, provides a way of categorizing various discrete spaces. This section presents several results indicating the significance of discrete metric spaces in coarse geometry.

One question of interest is determining when a coarse space has a compatible metric. It turns out there is a surprisingly simple characterization of this coarse version of metrizability. Assume that $\mathcal{E}'$ is a subset of a coarse structure $\mathcal{E}$ of a coarse space X, and assume that every element $E \in \mathcal{E}$ is a subset of some $E' \in \mathcal{E}'$. In this case, the structure $\mathcal{E}$ can be defined from $\mathcal{E}'$ via $\mathcal{E} = \{E \subset X^2 \mid \exists E' \in \mathcal{E}' : E \subset E'\}$. This behavior is very much counterpart to the generating sets defined for uniform spaces, and hence the same terminology is used for such sets $\mathcal{E}'$.

The following Proposition is from [10], and the proof presented here is essentially similar to the one given there.

Proposition 3.30. *Let $(X, \mathcal{E})$ be a coarse space. The following conditions are equivalent:*

1. *There is a metric d on X for which $\mathcal{E}$ is the bounded coarse structure $\mathcal{E}_d$.*
2. *The structure $\mathcal{E}$ has a countable generating set $\mathcal{E}'$ which fulfills $\cup\mathcal{E}' = X^2$.*

Proof. Condition 2 follows easily from condition 1 by selecting

$$\mathcal{E}' = \big\{\Delta_d(n) \,\big|\, n \in \mathbb{N}\big\}.$$

Assume then that $\mathcal{E}' = \{E_1, E_2, \ldots\}$ is a generating set of $\mathcal{E}$, and that the union $\bigcup_{i \in \mathbb{Z}_+} E_i$ is X^2. For all natural numbers i, define the sets F_i recursively as follows:

$$F_0 = \Delta_X$$

$$F_i = (E_i \cup E_i^{-1}) \cup F_{i-1} \cup \bigcup_{j=1}^{i-1} (F_j \circ F_{i-j}) \quad \text{for } i \in \mathbb{Z}_+$$

Recall that a set $A \subset X^2$ is symmetric if A^{-1} equals A. The diagonal Δ_X is a symmetric element of $\mathcal{E}'$, the unions $(F_j \circ F_{i-j}) \cup (F_{i-j} \circ F_{i-(i-j)})$ are symmetric, and unions of symmetric sets are symmetric. In addition to this, the union used to define F_i is finite. Using these facts, one can see by induction that F_i are symmetric elements of $\mathcal{E}$. Furthermore, since E_i is a subset of F_i for every positive value of i, the sets F_i form a countable generating set of $\mathcal{E}$, and the union $\bigcup_{i=0}^{\infty} F_i$ is X^2.

Now define $d : X^2 \to \mathbb{R}$ as follows: $d(x,y)$ is the smallest integer i for which (x,y) is an element of F_i. Since the union of F_i is X^2, d is well defined and finite-valued. Since F_0 is the diagonal of X, $d(x,y)$ is zero if and only if x and y are equal. Since the sets F_i are symmetric, d is symmetric.

Next, let x, y, and z be elements of X. If any two of these elements are equal, the triangle inequality holds trivially for them. In the remaining case of $x \neq y \neq z \neq x$, note that (x,y) and (y,z) are elements of $F_{d(x,y)}$ and $F_{d(y,z)}$ respectively. Hence, (x,z) is an element of the set $F_{d(x,y)} \circ F_{d(y,z)}$, which is a subset of $F_{d(x,y)+d(y,z)}$. Therefore $d(x,z)$ is at most $d(x,y) + d(y,z)$. This concludes the proof that d is a metric.

Finally, note that by the definition of d, the sets F_i are exactly the sets $\Delta_d(i)$. Hence, the coarse structures $\mathcal{E}$ and $\mathcal{E}_d$ have an equal generating set. Since a coarse structure can be uniquely determined from a generating set, $\mathcal{E}$ and $\mathcal{E}_d$ are equal. □

A similar metrizability result also holds for uniform structures: A uniform space is metrizable precisely when it has a countable generating set and the intersection of the generating set is the diagonal. For further details on this, see [4].

Example 3.31. Define a coarse structure $\mathcal{E}$ on $\mathbb{R}$ as follows: $\mathcal{E}$ consists of all subsets E of $\mathbb{R}^2$ for which $E \setminus \Delta_{\mathbb{R}}$ is finite. It is easily checked that $\mathcal{E}$ is a coarse structure. However, if sets $E_1, E_2, \dots$ form a countable generating set of $\mathcal{E}$, the set $(\bigcup_{i=1}^{\infty} E_i) \setminus \Delta_R$ is countable, and therefore the set $\bigcup_{i=1}^{\infty} E_i$ is not equal to $\mathbb{R}^2$. Hence, $(\mathbb{R}, \mathcal{E})$ is not metrizable.

Note that the metric constructed in Proposition 3.30 is discrete, as its image set is the set of natural numbers. Therefore, the proof of Proposition 3.30 immediately yields the following corollary.

Corollary 3.32. *Let (X, d') be a metric space. Then there is a discrete metric d on X for which the coarse structures $\mathcal{E}_d$ and $\mathcal{E}_{d'}$ are identical.*

Proof. Due to Proposition 3.30, the coarse space $(X, \mathcal{E}_{d'})$ has a countable generating set, and in the proof of the aforementioned Proposition, a discrete metric d is constructed from the generating set fulfilling $\mathcal{E}_d = \mathcal{E}_{d'}$. □

Corollary 3.32 can be thought of as a counterpart to the well known result that every metric has a topologically equivalent bounded metric. Just as discrete metric spaces are similar topologically, bounded metric spaces have similar coarse properties.

Besides Corollary 3.32, Proposition 3.30 has another important corollary.

Corollary 3.33. *Metrizability of coarse spaces is a coarse invariant.*

Proof. Let X be a coarse space, (Y, d) a metric space, and $f : X \to Y$ a coarse equivalence. By Proposition 3.21, f is a coarse embedding, letting one use Proposition 3.22. For every controlled subset E of X^2, $(f \times f)(E)$ is contained in the set $\Delta_d(n)$ for some natural number n. Therefore, the set E is contained in $(f \times f)^{-1}\Delta_d(n)$ for some $n \in \mathbb{N}$. Since the sets $(f \times f)^{-1}\Delta_d(n)$ are controlled and their union is $(f \times f)^{-1}Y^2 = X^2$, Proposition 3.30 implies the metrizability of the coarse space X. This concludes the proof. □

The final result of this section yields another way of reducing an arbitrary metric space to a coarsely equivalent discrete space. This is possible due to the fact that spaces of different cardinality can be coarsely equivalent.

Definition 3.34. Let (X, d) be a metric space, and let c be a positive real number. A subset A of X is called *c-discrete*, if for every pair $(a, b) \in A^2$, either a equals b or $d(a, b)$ is at least c.

Proposition 3.35. *Let (X, d) be a metric space, and fix a positive real number c. Let A be a maximal c-discrete subset of (X, d). Then the spaces X and A are coarsely equivalent.*

Proof. Let x be an element in X. Since A is a maximal c-discrete subset, there is an $a_x \in A$ which fulfills $d(a_x, x) < c$, since otherwise $A \cup \{x\}$ would be a larger c-discrete subset of X. Define a map $p : X \to A$ via $p(x) = a_x$. Note that if a is an element of A, p maps a to itself. Hence, p is a surjection.

Let x and y be elements of X. Using the triangle inequality, one obtains the inequality

$$d(x, y) \leq d(x, p(x)) + d(p(x), p(y)) + d(p(y), y) < 2c + d(p(x), p(y))$$

and the inequality

$$d(p(x), p(y)) \leq d(p(x), x) + d(x, y) + d(y, p(y)) < 2c + d(x, y).$$

Therefore, using the notation of Proposition 3.23, a selection of $\rho_+(t) = t + 2c$ and $\rho_-(t) = \max(0, t - 2c)$ proves that p is a coarse embedding. Since p is also a surjection, p is a coarse equivalence. □

One can use Zorn's lemma to prove that every metric space X contains a maximal c-discrete subset for every positive real value of c.

Chapter 4

Coarse embeddings into Hilbert spaces

This chapter focuses on coarse embeddability of metric spaces into Hilbert spaces. A major reason for the interest into coarse spaces is that coarse embeddability has been of use in trying to solve several complicated conjectures in K-theory. This chapter gives some background information on these applications. Furthermore, a property stronger than coarse embeddability known as Property A is presented, and various results related to it are derived.

4.1 General background and motivation

This section is an informal overview of the connections of coarse embeddability and several K-theoretic conjectures, which stands as the main motivator for research on coarse embeddability into Hilbert spaces. The level of detail will be considerably low, as even most definitions in K-theory are well beyond this thesis. Information on the relevant conjectures can be found for example in [11] and [7].

To start off, several basic concepts of metric spaces have to be introduced. A metric space is *proper* if every closed ball is compact. A metric space (X, d) is said to have *bounded geometry*, if for every positive real number r the supremum $\sup_{x\in X} |B(x, r)|$ is finite. Note that bounded geometry implies discreteness. Since discrete spaces are proper precisely when all closed balls are finite, all spaces of bounded geometry are also proper.

The *Baum-Connes conjecture* concerns a map, referred to as the index map, between two groups derived from a countable group. The value group of the index map is derived purely analytically using tools of K-theory. The domain group, on the other hand, is derived using K-homology, the dual theory of K-theory, and is instead related to geometry and topology. The Baum-Connes conjecture claims that the map in question is an isomorphism, a claim that would provide a link between these two theories. The conjecture is still an open question, having been proven only in specific special cases.

The coarse Baum-Connes conjecture is a variation of the original Baum-Connes conjecture. For proper metric spaces, one can define similar groups and a similar index map to the original Baum-Connes conjecture. The coarse Baum-Connes conjecture states that for spaces with bounded geometry, the defined index map is an isomorphism. The index maps of the two variations are related on discrete groups with a finite generating set, a class for which both maps can be defined.

In the paper [14], Guoliang Yu proved the following major result.

Theorem 4.1 (Yu). *Let X be a metric space with bounded geometry, and assume that X is coarsely embeddable into a Hilbert space. In this case, the coarse Baum-Connes conjecture holds for X.*

One reason for the interest in the coarse Baum-Connes conjecture is that it can be seen as a stronger form of multiple other conjectures. One such conjecture is the *Novikov conjecture*, an important conjecture in topology which is closely linked to the injectivity of the Baum-Connes index map.

Similarly to the Baum-Connes conjecture, one can define a coarse variant of the Novikov conjecture for metric spaces with bounded geometry. This variant has also obtained a partial solution, given by Kasparov and Yu in the paper [8].

Definition 4.2. Let H be a normed vector space. The space H is *uniformly convex* if, for every positive real number ε, there is a positive real number δ fulfilling the following: If x and y are unit vectors of H and $\|x-y\|$ is at least ε, the value of $\|(x+y)/2\|$ is at most $1-\delta$.

Theorem 4.3 (Kasparov, Yu). *Let X be a metric space with bounded geometry, and assume that X is coarsely embeddable into a uniformly convex Banach space. In this case, the coarse Novikov conjecture holds for X.*

Note that every Hilbert space is uniformly convex. To see this, let x and y be unit vectors of a Hilbert space with a real-valued inner product. The norm $\|x-y\|^2$ can be written in the form $2-2\langle x,y\rangle$, and the norm $\|(x+y)/2\|^2$ can be written in the form $1/2+\langle x,y\rangle/2$. Hence, if $\|x-y\|$ is at least ε, $\langle x,y\rangle$ is at most $1-\varepsilon^2/2$, and one may select $\delta = 1-\sqrt{1-\varepsilon^2/4}$. The same method of proof works for complex-valued inner products.

If a coarse space X has a coarse embedding f into a Hilbert space, Corollary 3.33 implies that X has an inducing metric. Due to this, coarse embeddability into Hilbert spaces is of interest only in the context of metric spaces.

4.2 Property A

In the paper [14], Guoliang Yu introduced the *Property A* for metric spaces. The main interest of Property A is that it implies coarse embeddability into a Hilbert space. An in-depth survey of existing results related to Property A can be found in the article [12].

Definition 4.4. Let X be a metric space with a metric d. The space X has *Property A* if, for every $\varepsilon > 0$ and $r > 0$, one can select a family $\{A_x \subset X \times \mathbb{N} \mid x \in X, |A_x| < \infty\}$ which fulfills the following conditions:

1. For every x and y in X, if $d(x,y)$ is at most r, the intersection $A_x \cap A_y$ is nonempty and the sets A_x and A_y fulfill the inequality
$$\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \varepsilon.$$
2. There is an $R > 0$ for which every set A_x is contained within $B_X(x,R) \times \mathbb{N}$.

Yu's original definition of Property A considered only discrete metric spaces. However, nothing in the definition explicitly requires the discreteness of X. The compatibility of the above definition with non-discrete spaces will be analyzed later in detail.

Besides requiring the discreteness of X, Yu's definition also replaces the second condition with the following two conditions:

- There is an $R > 0$ fulfilling $d(\mathrm{pr}_X(A_x)) \leq R$ for every x in X, where pr_X is the usual projection map from $X \times \mathbb{N}$ onto X.
- Every set A_x contains the element $(x, 0)$.

It can be shown that replacing the second condition of Definition 4.4 with the above two conditions yields an equivalent definition.

Theorem 4.5. *Let X be a metric space with metric d. If X has Property A, there is a coarse embedding from X into a Hilbert space H.*

Proof. This proof follows the outline given in the paper [14]. The Hilbert space H is given by
$$H = \bigoplus_{k=1}^{\infty} l^2(X \times \mathbb{N}).$$

In order to define a coarse embedding f, for every positive integer k, select the families $\{A_x^{(k)}\}$ using the definition of Property A with $r = k$ and $\varepsilon = 2^{-k}$. Now, fix an element x_0 of X. The coarse embedding f can be defined via

$$f(x) = \bigoplus_{k=1}^{\infty} \left(\left(\frac{1}{\sqrt{|A_x^{(k)}|}} \sum_{a \in A_x^{(k)}} \mathbf{e}_a \right) - \left(\frac{1}{\sqrt{|A_{x_0}^{(k)}|}} \sum_{a \in A_{x_0}^{(k)}} \mathbf{e}_a \right) \right).$$

What remains is to show that f is a coarse embedding, and that f maps elements of X into H.

Next, let x be an element of X. An estimate is derived for the l^2-norm of $(f(x))_k$, the k:th coordinate of $f(x)$. Assuming that $A_x^{(k)} \cap A_{x_0}^{(k)}$ is nonempty, $\|(f(x))_k\|_{l^2}$ can be written in the form

$$\|(f(x))_k\|_{l^2} = \sqrt{\left(\sum_{a \in A_x^{(k)}} \frac{1}{\left|A_x^{(k)}\right|} \right) + \left(\sum_{a \in A_{x_0}^{(k)}} \frac{1}{\left|A_{x_0}^{(k)}\right|} \right) - \left(\sum_{a \in A_x^{(k)} \cap A_{x_0}^{(k)}} \frac{2}{\sqrt{\left|A_x^{(k)}\right| \left|A_{x_0}^{(k)}\right|}} \right)}$$

$$= \sqrt{1 + 1 - \frac{2\left|A_x^{(k)} \cap A_{x_0}^{(k)}\right|}{\sqrt{\left|A_x^{(k)}\right| \left|A_{x_0}^{(k)}\right|}}} \tag{4.6}$$

$$= \sqrt{\left(\frac{2\sqrt{\left|A_x^{(k)}\right| \left|A_{x_0}^{(k)}\right|} - 2\left|A_x^{(k)} \cap A_{x_0}^{(k)}\right|}{\sqrt{\left|A_x^{(k)}\right| \left|A_{x_0}^{(k)}\right|}} \right)}. \tag{4.7}$$

The form 4.6 immediately yields that $\|(f(x))_k\|_{l^2}$ is at most $\sqrt{2}$. This holds also when $A_x^{(k)} \cap A_{x_0}^{(k)}$ is empty, as in that case $\|(f(x))_k\|_{l^2}$ equals $\sqrt{2}$.

Let A and B be arbitrary sets. Recall the arithmetic-geometric mean inequality, which states that for nonnegative numbers a and b, $\sqrt{ab}$ is at most $(a+b)/2$. By applying this, one obtains the inequality

$$\begin{aligned} 2\sqrt{|A|\,|B|} - 2\,|A \cap B| &\le |A| + |B| - 2\,|A \cap B| \\ &= (|A| - |A \cap B|) + (|B| - |A \cap B|) \\ &= |A \setminus B| - |B \setminus A| \\ &= |A \triangle B|\,. \end{aligned}$$

Further assessment of the form $\sqrt{|A|\,|B|}$ yields the inequality

$$\begin{aligned}\sqrt{|A|\,|B|} &= \sqrt{(|A \cap B| + |A \setminus B|)\,(|A \cap B| + |B \setminus A|)}\\ &\geq \sqrt{|A \cap B|^2} = |A \cap B|\,.\end{aligned}$$

Next, the two previous inequalities are applied to the formula 4.7. With the assumption of $A_x^{(k)} \cap A_{x_0}^{(k)}$ being nonempty, one obtains the inequality

$$\|(f(x))_k\|_{l^2} \leq \sqrt{\frac{\left|A_x^{(k)} \triangle A_{x_0}^{(k)}\right|}{\left|A_x^{(k)} \cap A_{x_0}^{(k)}\right|}}.$$

Hence, due to how the sets $A_x^{(k)}$ were selected, $\|(f(x))_k\|_{l^2}$ is at most $\sqrt{2^{-k}}$ whenever k is at least $d(x, x_0)$. This yields the following upper bound for $\|f(x)\|_{l^2}$:

$$\begin{aligned}\|f(x)\|_{l^2} &= \sqrt{\left(\sum_{k=1}^{\lfloor d(x,x_0)\rfloor} \|(f(x))_k\|_{l^2}^2\right) + \left(\sum_{k=\lfloor d(x,x_0)\rfloor+1}^{\infty} \|(f(x))_k\|_{l^2}^2\right)}\\ &\leq \sqrt{\left(\sum_{k=1}^{\lfloor d(x,x_0)\rfloor} 2\right) + \left(\sum_{k=\lfloor d(x,x_0)\rfloor+1}^{\infty} 2^{-k}\right)}\\ &\leq \sqrt{2d(x, x_0) + 1}.\end{aligned}$$

The resulting bound shows that $f(x)$ is an element of $\bigoplus_{k=1}^{\infty} l^2(X \times \mathbb{N})$, as its l^2-norm is finite.

The bound can also be used to obtain half of the requirements for a coarse embedding. Let x and y be elements of X, and observe the difference $f(x) - f(y)$. The x_0-terms cancel out, resulting in a form otherwise identical to $f(x)$, but with x_0 replaced by y. Therefore, the above calculations yield the bound

$$\|f(x) - f(y)\|_{l^2} \leq \sqrt{2d(x, y) + 1}.$$

Hence, using again the notation of Proposition 3.23, one may select $\rho_+(t) = \sqrt{2t + 1}$, and all that remains is finding a suitable lower bound ρ_-.

For every positive integer k, denote by R_k the bound satisfying the second condition of coarse embeddability for the previously selected values of $r = k$ and $\varepsilon = 2^{-k}$. Fix a positive integer k, and let x and y be elements of X with $d(x, y) \geq 2R_k$. In this case, the open balls $B_X(x, R_k)$ and $B_X(y, R_k)$ don't intersect, and therefore the sets $A_x^{(k)}$ and $A_y^{(k)}$ are separate.

As has been previously stated, if the sets $A_x^{(k)}$ and $A_y^{(k)}$ are separate, the value of $\|(f(x))_k\|_{l^2}$ is $\sqrt{2}$. Hence, one obtains the lower bound

$$\|f(x) - f(y)\|_{l^2} \geq \sqrt{2\sum_{k=1}^{\infty} \mathbf{1}_{2R_k}(d(x, y))}\,,$$

where $\mathbf{1}_{2R_k}$ is the characteristic function of the set $\{t \in \mathbb{R} \mid t \geq 2R_k\}$. This allows one to make the selection

$$\rho_-(t) = \min\left(\sqrt{2\sum_{k=1}^{\infty} \mathbf{1}_{2R_k}(t)}, t\right).$$

The minimum is used to avoid the problem of the first function being potentially infinite, as finiteness is only guaranteed up to the diameter of $f(X)$. The resulting ρ_- is increasing as a minimum of two increasing functions. Lastly, the limit $\lim_{t\to\infty} \rho_-(t)$ is clearly infinite. This concludes the proof. □

Next, it is shown that Property A is a coarse invariant for metric spaces. The proof follows the one presented in the article [12].

Proposition 4.8. *Property A is preserved in coarse equivalences between metric spaces.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and let f be a coarse equivalence from X to Y, with coarse inverse g. Assume that the space X has Property A, and fix nonnegative real numbers r and ε. Since g is coarsely uniform, the set $(g \times g)(\Delta_{d_Y}(r))$ is controlled in X. Denote by r' the finite-valued supremum of the set $d_X\left[(g \times g)(\Delta_{d_Y}(r))\right]$.

Next, use the Property A of X with the values r' and ε. Denote the resulting family by $\{A'_x\}$, and the bound obtained from the second condition of Property A by R'. For every y in Y, define a function $n_y : Y \to \mathbb{N}$ by

$$n_y(y') = \left|A'_{g(y)} \cap (f^{-1}\{y'\} \times \mathbb{N})\right|.$$

Since the set $A'_{g(y)}$ is finite, n_y is finite-valued. Now, one may define the sets A_y as follows:

$$A_y = \bigcup_{y' \in Y} \{(y', 1), (y', 2), \ldots, (y', n_y(y'))\}.$$

It remains to show that the family $\{A_y \mid y \in Y\}$ fulfills the conditions of Property A.

Let y be an element of Y. Note that sets $f^{-1}\{y'\}$ are mutually disjoint and cover X. Thanks to this, the finiteness of $|A_y|$ can be concluded with the equation

$$|A_y| = \sum_{y' \in Y} n_y(y') = \sum_{y' \in Y} \left|A'_{g(y)} \cap (f^{-1}\{y'\} \times \mathbb{N})\right| = \left|A'_{g(y)}\right| < \infty.$$

Next, let (y', k) be an element of A_y. Due to the definition of n_y, the intersection $A'_{g(y)} \cap (f^{-1}\{y'\} \times \mathbb{N})$ is nonempty. Hence, there is an element (x', k') of $A'_{g(y)}$ fulfilling $f(x') = y'$. The distance $d_X(x', g(y))$ is at most R'. Use of the triangle inequality yields the bound

$$\begin{aligned} d_Y(y', y) &\leq d_Y(f(x'), (f \circ g)(y)) + d_Y((f \circ g)(y), y) \\ &\leq d_Y\left[(f \times f)(\Delta_{d_X}(R'))\right] + d_Y\left[(f \circ g, \mathrm{id}_Y)(Y)\right] \end{aligned}$$

The resulting upper bound is finite, since $f \times f$ maps controlled sets of X^2 to controlled sets of Y^2 and $f \circ g$ is close to id_Y. The obtained bound is therefore a suitable value of R which satisfies the second condition of Property A for the family $\{A_y \mid y \in Y\}$.

Finally, let y_1 and y_2 be elements of Y. Assume that $d_Y(y_1, y_2)$ is at most r. Due to how r' was defined, $d_X(g(y_1), g(y_2))$ is at most r'. Derive a lower bound for $|A_{y_1} \cap A_{y_2}|$ as follows:

$$\begin{aligned} |A_{y_1} \cap A_{y_2}| &= \sum_{y' \in Y} \min(n_{y_1}(y'), n_{y_2}(y')) \\ &= \sum_{y' \in Y} \min\left(\left|A'_{g(y_1)} \cap (f^{-1}\{y'\} \times \mathbb{N})\right|, \left|A'_{g(y_2)} \cap (f^{-1}\{y'\} \times \mathbb{N})\right|\right) \\ &\geq \sum_{y' \in Y} \left|A'_{g(y_1)} \cap A'_{g(y_2)} \cap (f^{-1}\{y'\} \times \mathbb{N})\right| \\ &= \left|A'_{g(y_1)} \cap A'_{g(y_2)}\right|. \end{aligned}$$

This also results in the following upper bound for $|A_{y_1} \triangle A_{y_2}|$:

$$\begin{aligned} |A_{y_1} \triangle A_{y_2}| &= |A_{y_1}| + |A_{y_2}| - 2\,|A_{y_1} \cap A_{y_2}| \\ &\leq \left|A'_{g(y_1)}\right| + \left|A'_{g(y_2)}\right| - 2\left|A'_{g(y_1)} \cap A'_{g(y_2)}\right| \\ &= \left|A'_{g(y_1)} \triangle A'_{g(y_2)}\right|. \end{aligned}$$

From this, one may conclude the inequality

$$\frac{|A_{y_1} \triangle A_{y_2}|}{|A_{y_1} \cap A_{y_2}|} \leq \frac{\left|A'_{g(y_1)} \triangle A'_{g(y_2)}\right|}{\left|A'_{g(y_1)} \cap A'_{g(y_2)}\right|} < \varepsilon.$$

Hence, the family $\{A_y \mid y \in Y\}$ fulfills the first condition of Property A. □

Remark 4.9. Assume one has an alternate formulation of Property A given on a more general class of metric spaces than discrete ones,and assume that the formulation coincides with Definition 4.4 on discrete metric spaces. If the formulation is coarsely invariant, one can combine Proposition 4.8 with either Proposition 3.32 or Proposition 3.35 to show that the alternate formulation is equivalent to Definition 4.4 on all metric spaces.

Hence, Proposition 4.8 also supports the compatibility of Definition 4.4 with non-discrete spaces. An example of such an alternate formulation of Property A for non-discrete spaces can be found in chapter 5 of [12].

Beyond the previous remark, Proposition 4.8 also has another useful corollary.

Corollary 4.10. *All bounded metric spaces have Property A.*

Proof. The singleton space $\{0\}$ has Property A, as any nonempty family A_0 fulfills the conditions of Property A for all values of r and ε. Since all bounded spaces are coarsely equivalent and Property A is a coarse invariant, all bounded metric spaces have Property A. □

Finally, an alternate condition for Property A is given for metric spaces with bounded geometry. Let f be a mapping between two metric spaces (X, d) and (Y, d'), and let r, ε be positive real numbers. The mapping f is said to have a *variation* of (r, ε) if $d'(f(x), f(x')) < \varepsilon$ holds for all $x, x' \in X$ satisfying $d(x, x') \leq r$. Furthermore, recall that the *probability space* of X, denoted by $\mathcal{P}(X)$, is the subspace of $l^1(X)$ defined by

$$\mathcal{P}(X) = \{v \in l^1(X) \mid \|v\|_{l^1} = 1, v_x \geq 0 \text{ for every } x \in X\}.$$

Proposition 4.11. *Let (X, d) be a metric space. Consider the following condition:*

- *For every pair r, ε of positive real numbers, there is a map $f : X \to \mathcal{P}(X)$ with a variation of (r, ε), and there is a constant R for which the support of $f(x)$ is contained within $B_X(x, R)$.*

If (X, d) has Property A, then (X, d) has the above condition. Furthermore, if (X, d) has both bounded geometry and the above condition, (X, d) has Property A.

Proof. The proof follows the methods used in article [12]. Assume that (X, d) has Property A. Fix a pair r, ε, and use property A to select the corresponding sets A_x and bound R. Define a map $f_{r,\varepsilon}$ from X to $l^1(X)$ as follows:

$$(f(x))_y = \frac{|A_x \cap \{y\} \times \mathbb{N}|}{|A_x|}.$$

Directly from the definition, one sees that the l^1-norm of $f(x)$ is 1, and the image of f is therefore contained in $\mathcal{P}(X)$. Furthermore, the second condition of Property A implies that $(f(x))_y$ is nonzero only if y is in $B_X(x,R)$.

Next, assume that x and x' are elements of X with $d(x,x')$ at most r. Note that for any two sets A and B, the following inequality holds:

$$|A| - |B| \le |A \setminus B| \le |A \triangle B|.$$

By switching A and B with each other, one concludes the inequality

$$\big||A| - |B|\big| \le |A \triangle B|\,.$$

Applying this inequality, one obtains a bound for the numerator of $|(f(x) - f(x'))_y|$:

$$\begin{aligned}\big||A_x \cap \{y\} \times \mathbb{N}| - |A_{x'} \cap \{y\} \times \mathbb{N}|\big| &\le \big|(A_x \cap \{y\} \times \mathbb{N}) \triangle (A_{x'} \cap \{y\} \times \mathbb{N})\big| \\ &= \big|(A_x \triangle A_{x'}) \cap (\{y\} \times \mathbb{N})\big|.\end{aligned}$$

Using this bound and the first condition of Property A results in the following bound for $\|f(x) - f(x')\|_{l^1}$:

$$\begin{aligned}\|f(x) - f(x')\|_{l^1} &\le \left\|f(x) - \frac{|A_{x'}|}{|A_x|} f(x')\right\|_{l^1} + \left\|\frac{|A_{x'}|}{|A_x|} f(x') - f(x')\right\|_{l^1} \\ &= \frac{\big\||A_x| f(x) - |A_{x'}| f(x')\big\|_{l^1}}{|A_x|} + \frac{\big||A_{x'}| - |A_x|\big|}{|A_x|}\, \|f(x')\|_{l^1} \\ &\le \frac{|A_x \triangle A_{x'}|}{|A_x \cap A_{x'}|} + \frac{|A_{x'} \triangle A_x|}{|A_{x'} \cap A_x|} \\ &< 2\varepsilon.\end{aligned}$$

Hence, f has a variation of $(r, 2\varepsilon)$. Since the chosen r and ε were arbitrary positive reals, (X,d) satisfies the desired condition.

For the other direction, assume that (X,d) has bounded geometry, and assume that (X,d) fulfills the condition of the alternate condition given in the Proposition. Fix a pair (r,ε) fulfilling $\varepsilon \le 1/3$, and denote the resulting map and constant by f and R. Since X has bounded geometry, there is an upper bound to the number of elements in an arbitrary R-sphere $B_X(x,R)$. Denote this bound by N.

Select an integer S for which S^{-1} is less than ε/N. Now, there is a function $g : X \to \mathcal{P}(X)$ for which $(g(x))_y = a_{x,y}/S$ with some integers $a_{x,y} \in \mathbb{N}$, the value of $(g(x))_y$ is zero whenever $(f(x))_y$ equals zero, and $|(g(x) - f(x))_y| \le S^{-1}$ for all elements $x, y \in X$. Note that when x is fixed, $(g(x) - f(x))_y$ is nonzero for at most N values of y, and therefore the inequality $\|f(x) - g(x)\|_{l^1} < NS < \varepsilon$ holds for every $x \in X$.

Using the function g, one can define the sets $A_x \subset X \times \mathbb{N}$ via

$$A_x = \{(y, n) \in X \times \mathbb{N} \mid y < a_{x,y} = (Sg(x))_y\} .$$

Note that every A_x has exactly S elements, since the norm $\|g(x)\|_{l^1}$ equals 1. Since $(g(x))_y$ is nonzero only for elements y in $B_X(x, R)$, the sets A_x are contained within the sets $B_X(x, R) \times \mathbb{N}$.

Next, assume that the distance $d(x, x')$ is at most r. In this case, one obtains the bound

$$\begin{aligned} \|g(x) - g(x')\|_{l^1} &\leq \|g(x) - f(x)\|_{l^1} + \|f(x) - f(x')\|_{l^1} + \|f(x') - g(x')\|_{l^1} \\ &< 3\varepsilon. \end{aligned}$$

Note that for a fixed y, the amount of elements in the set $(A_x \triangle A_{x'}) \cap \{y\} \times \mathbb{N}$ is $|a_{x,y} - a_{x',y}|$. Hence, $|A_x \triangle A_{x'}|$ has the following bound:

$$|A_x \triangle A_{x'}| = \|Sg(x) - Sg(x')\|_{l^1} < 3S\varepsilon.$$

Furthermore, due to the selected upper bound of $1/3$ on ε, the size of the set $A_x \cap A_{x'}$ also has the lower bound

$$|A_x \cap A_{x'}| = \frac{|A_x| + |A_{x'}| - |A_x \triangle A_{x'}|}{2} > \frac{2S - 3S\varepsilon}{2} \geq \frac{S}{2}$$

Due to this, the set $A_x \cap A_{x'}$ is nonempty. Furthermore, the obtained bounds yield the inequality

$$\frac{|A_x \triangle A_{x'}|}{|A_x \cap A_{x'}|} < 6\varepsilon.$$

Since the selected r and ε were arbitrarily large and small respectively, X has Property A. □

As a final remark, it has been shown in [9] that Property A is a stronger condition than coarse embeddability into Hilbert spaces.

Chapter 5

Asymptotic dimension

In this chapter, the concept of asymptotic dimension is introduced. Asymptotic dimension is the coarse counterpart of the Lebesgue covering dimension, also known as the topological dimension, of a space. Afterwards, several properties of asymptotic dimension are derived, and the dimensions of several example spaces are presented. Finally, connections are derived between a finite asymptotic dimension and the concepts of Chapter 4.

5.1 Background

In order to define the topological and asymptotic dimensions of a space, it is convenient to first introduce some covering-related terminology.

Definition 5.1. Let X be a set, and let $\mathcal{V}$ be a cover of X. The *order* of $\mathcal{V}$ is defined by

$$\operatorname{ord}\mathcal{V} = \sup_{x\in X} |\{V \in \mathcal{V} \mid x \in V\}| .$$

Let $\mathcal{U}$ be another cover of X. The cover $\mathcal{U}$ is a *refinement* $\mathcal{V}$ if every U in $\mathcal{U}$ is contained in some element V of $\mathcal{V}$.

Definition 5.2. Let X be a topological space. The *Lebesgue covering dimension* of X, denoted by $\dim X$, is at most n, if the following condition is true: For every open cover $\mathcal{V}$ of X, there is an open refinement $\mathcal{U}$ of $\mathcal{V}$ which has an order of at most $n + 1$.

The covering dimension $\dim X$ is the smallest nonnegative integer for which $\dim X \leq n$ holds. If no nonnegative integer fulfills said inequality, $\dim X$ is infinite.

The Lebesgue covering dimension is a way of assigning a space a dimension based solely on its topological properties. For vector spaces such as $\mathbb{R}^n$, the Lebesgue covering dimension coincides with the standard concept of dimension for vector spaces. As the

Lebesgue dimension is defined with topological properties, homeomorphic spaces end up having the same dimension, making the Lebesgue dimension a topological invariant.

Asymptotic dimension provides a similar concept of dimension for coarse spaces. As has been previously seen, coarse invariants are capable of classifying discrete spaces. The Lebesgue covering dimension of every discrete space is zero. This can be easily seen by selecting $\mathcal{U}$ to be the cover of singletons for every cover $\mathcal{V}$. Asymptotic dimension, on the other hand, yields differing values for spaces such as $\mathbb{Z}^n$ for different values of n.

5.2 Definitions of asymptotic dimension

Definition 5.3. Let (X, d) be a metric space, and let $\mathcal{U}$ be a collection of subsets of X. The collection $\mathcal{U}$ is *uniformly bounded*, if $\sup_{U \in \mathcal{U}} d(U)$ is finite.

This definition can be extended to coarse spaces as follows: Let X be a coarse space, and let $\mathcal{U}$ be a collection of subsets of X. The collection $\mathcal{U}$ is uniformly bounded if the set $\bigcup_{U \in \mathcal{U}} U^2$ is controlled. The aforementioned set will be called the *bounding set* of $\mathcal{U}$.

Asymptotic dimension has many equivalent definitions. As the initial definition, the most clear counterpart to Definition 5.2 is used.

Definition 5.4. Let X be a metric space or a coarse space. The *asymptotic dimension* of X, denoted by $\operatorname{asdim} X$, is at most n, if the following condition is true: For every uniformly bounded cover $\mathcal{U}$ of X, there is a uniformly bounded cover $\mathcal{V}$ with an order of at most $n + 1$, and the initial cover $\mathcal{U}$ is a refinement of $\mathcal{V}$.

The asymptotic dimension $\operatorname{asdim} X$ is the smallest nonnegative integer for which $\operatorname{asdim} X \leq n$ holds. If no nonnegative integer fulfills said inequality, $\operatorname{asdim} X$ is infinite.

The following Proposition yields a convenient reformulation for the asymptotic dimension of metric spaces. Recall that a cover $\mathcal{U}$ of a metric space X has a *Lebesgue number* of λ, if for every element x of X, the open ball $B_X(x, \lambda)$ is contained within some set of $\mathcal{U}$.

Proposition 5.5. *Let X be a metric space, and let n be a nonnegative integer. The following conditions are equivalent:*

1. *The asymptotic dimension of X is at most n.*

2. *For every positive real number λ, there exists a uniformly bounded cover $\mathcal{V}$ of X with Lebesgue number λ and an order of at most $n + 1$.*

Proof. Assume that asdim X is at most n, and fix a positive real number λ. Let $\mathcal{U}$ be the cover of X consisting of all open balls of radius λ. Hence, $\mathcal{U}$ is a refinement of a uniformly bounded cover $\mathcal{V}$ with an order of at most $n+1$. Since every open ball of radius λ is contained in a refinement of $\mathcal{V}$, the cover $\mathcal{V}$ has a Lebesgue number of λ. Hence, the first condition implies the second one.

Next, assume that the second condition holds, and fix a uniformly bounded cover $\mathcal{U}$ of X. One may assume that $\mathcal{U}$ does not contain the empty set. Denote by R the value of the finite bound $\sup_{U \in \mathcal{U}} d(U)$. Using the second condition, select a cover $\mathcal{V}$ with the value $\lambda = R+1$. Note that $\mathcal{V}$ is uniformly bounded and has an order of at most $n+1$. For every set U in the cover $\mathcal{U}$, select an element x_u from U. Now, since $d(U)$ is less than $R+1$, U is contained within the open ball $B_X(x_u, R+1)$, which in turn is contained within some set of $\mathcal{V}$. Due to this, $\mathcal{U}$ is a refinement of $\mathcal{V}$. Therefore, the second condition implies the first one. □

Next, the main alternate definition of asymptotic dimension is formulated. The definition is based on uniformly bounded families where all sets are sufficiently far from each other. This is formalized by the following definition.

Definition 5.6. Let (X, d) be a metric space, and fix a positive real number r. Let $\mathcal{U}$ be a family of subsets of X. The family $\mathcal{U}$ is *r-disjoint* if, for every pair U_1, U_2 of two different elements of $\mathcal{U}$, the inequality $d(U_1, U_2) > r$ holds.

Alternatively, let $(X, \mathcal{E})$ be a coarse space, and fix a controlled set E. Let $\mathcal{U}$ be a family of subsets of X. The family $\mathcal{U}$ is *E-disjoint* if, for every pair U_1, U_2 of two different elements of $\mathcal{U}$, the intersection $E \cap (U_1 \times U_2)$ is empty.

For a metric space X, denote by E_r the controlled set $\{(x, y) \in X^2 \mid d(x, y) \leq r\}$. In this case, r-disjointness is equivalent to E_r-disjointness. If r and r' are two positive reals fulfilling $r' \leq r$, r-disjointness implies r'-disjointness. Similarly, if E and E' are two controlled sets fulfilling $E' \subset E$, E-disjointness implies E'-disjointness.

Proposition 5.7. *Let X be a coarse space, and let n be a nonnegative integer. The following conditions are equivalent:*

1. *The asymptotic dimension of X is at most n.*

2. *For every controlled set E, there exist $n+1$ uniformly bounded families $\mathcal{U}_0, \ldots, \mathcal{U}_n$, where every family $\mathcal{U}_i$ is E-disjoint, and the union $\bigcup_{i=0}^{n} \mathcal{U}_i$ is a cover of X.*

Furthermore, if X is a metric space, the following condition is also equivalent with the above:

3. *For every positive real number r, there exist $n+1$ uniformly bounded families $\mathcal{U}_0, \dots, \mathcal{U}_n$, where every family $\mathcal{U}_i$ is r-disjoint, and the union $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of X.*

Proof. Let X be a metric space, and again denote the controlled set $\{(x,y) \in X^2 | d(x,y) \leq r\}$ by E_r. Since r-disjointness and E_r-disjointness are equivalent, condition 2 implies condition 3. Furthermore, every controlled set E is contained in the set E_r for some r, and in this case, an r-disjoint family is also E-disjoint. Therefore, conditions 2 and 3 are equivalent for metric spaces.

Next, condition 1 is proven from condition 2. Let X be a coarse space, and assume the second condition. Let $\mathcal{U}$ be a uniformly bounded cover of X, and let E be the bounding set of $\mathcal{U}$. Use condition 2 to select uniformly bounded $(E \circ E)$-disjoint families $\mathcal{V}'_0, \dots \mathcal{V}'_n$. Denote by F_i the bounding sets of the covers $\mathcal{V}'_i$, and let F be the union $\bigcup_{i=0}^n F_i$ of bounding sets. The union $\mathcal{V}' = \bigcup_{i=0}^n \mathcal{V}'_i$ is a uniformly bounded cover with bounding set F. Define the sets $\mathcal{V}_i$ and the cover $\mathcal{V}$ as follows:

$$\begin{aligned} \mathcal{V}_{V'} &= \{U \in \mathcal{U} \mid U \cap V' \neq \emptyset\} && \text{for each } V' \in \mathcal{V}', \\ \mathcal{V}_i &= \{\cup \mathcal{V}_{V'} \mid V' \in \mathcal{V}'_i\} && \text{for each } i \in \{0, \dots, n\}, \text{ and} \\ \mathcal{V} &= \bigcup_{i=0}^n \mathcal{V}_i. \end{aligned}$$

If x is an element of X, it is contained in some $U_x \in \mathcal{U}$ and some $V'_x \in \mathcal{V}'$. Hence, the intersection $V'_x \cap U_x$ is nonempty, and due to this, x is an element of $\cup \mathcal{V}_{V'_x}$. Therefore, $\mathcal{V}$ is a cover. Since $\mathcal{V}'$ is a cover, every element U of $\mathcal{U}$ intersects some V'_U of $\mathcal{V}'$. Due to this, $\mathcal{U}$ is a refinement of $\mathcal{V}$.

Let $V = \cup \mathcal{V}_{V'}$ be an element of $\mathcal{V}$, and let (x,y) be an element of V^2. There exist sets U_x and U_y of $\mathcal{U}$ which intersect V' and contain x and y respectively. Let v'_x and v'_y be elements of the respective nonempty intersections $U_x \cap V'$ and $U_y \cap V'$. Since the sets U_x and U_y are elements of $\mathcal{U}$ which has the bounding set E, the pairs (x, v'_x) and (v'_y, y) are elements of E. Similarly, since V' is an element of the cover $\mathcal{V}'$ with the bounding set F, the pair (v'_x, v'_y) is an element of F. Using this, one can conclude that

$$\bigcup_{V \in \mathcal{V}} V^2 \subset E \circ F \circ E,$$

and therefore the cover $\mathcal{V}$ is uniformly bounded.

It remains to verify that $\mathcal{V}$ has an order of at most $n+1$. The claim follows from showing that the families $\mathcal{V}_i$ are disjoint. Let x be an element of X, and let $V_1 = \cup \mathcal{V}_{V'_1}$ and $V_2 = \cup \mathcal{V}_{V'_2}$ be two different elements of a given $\mathcal{V}_i$. Assume to the contrary that x

is contained in both V_1 and V_2. Hence, there exist sets U_1 and U_2 which contain X and intersect V_1' and V_2' respectively. Choose points u_1 and u_2 from these intersections. Now (u_1, x) and (x, u_2) are elements of E. Therefore, (u_1, u_2) is contained in both $E \circ E$ and $V_1' \times V_2'$. This is a contradiction, since the family $\mathcal{V}_i'$ was selected to be $(E \circ E)$-disjoint. The proof that condition 2 implies condition 1 is now finished.

Finally, condition 2 is proven from condition 1. This proof follows the one presented in the article [6]. For a given controlled set E of $\mathcal{E}$, point x of X, and subset A of X, define the *E-ball* at x and the *E-interior* of A as follows:

$$\begin{aligned} B_{\mathcal{E}}(x, E) &= \{x' \in X \mid (x, x') \in E\} \\ \operatorname{int}_E(A) &= \{a \in A \mid B_{\mathcal{E}}(a, E) \subset A\}. \end{aligned}$$

Let E be a controlled set, and denote by F the set $E \cup E^{-1} \cup \Delta_X$. In this case, F is a controlled symmetric set containing the diagonal. Define the sets F_i for positive natural numbers i as follows:

$$F_i = \underbrace{F \circ F \circ \ldots \circ F}_{i \text{ copies}}.$$

The sets F_i are symmetric and contain Δ_X. Next, let $\mathcal{V}'$ be the family of F_{n+1}-balls $\{B_{\mathcal{E}}(x, F_{n+1}) \mid x \in X\}$. Since F_{n+1} contains the diagonal Δ_X, $\mathcal{V}'$ is a cover. Furthermore, if x is an element of X, using the fact that F_{n+1} is symmetric, one concludes that $(B_{\mathcal{E}}(x, F_{n+1}))^2$ is contained within $F_{n+1} \circ F_{n+1}$. Hence, $\mathcal{V}'$ is a uniformly bounded cover. By the definition of asymptotic dimension, one may select a uniformly bounded cover $\mathcal{V}$ which $\mathcal{V}'$ refines and which has an order of at most $n + 1$.

Using the previously defined cover $\mathcal{V}$, one can define for values $i = 0, \ldots, n$ the collections

$$\begin{aligned} \mathcal{V}_i &= \{V_0 \cap \ldots \cap V_i \mid V_j \in \mathcal{V}, V_j \neq V_k \text{ for } j \neq k\}, \\ \mathcal{W}_i &= \left\{\operatorname{int}_{F_{n+1-i}}(V) \mid V \in \mathcal{V}_i\right\}, \text{ and} \\ \mathcal{U}_i &= \{W \setminus \cup \mathcal{W}_{i+1} \mid W \in \mathcal{W}_i\}, \text{ where } \mathcal{W}_{n+1} = \{\emptyset\}. \end{aligned}$$

Note that every set $U \in \mathcal{U}_i$ is a subset of some $V \in \mathcal{V}$. Hence, the collections $\mathcal{U}_i$ are uniformly bounded. Let x be an element of X. There is a set $V \in \mathcal{V}$ for which $B_{\mathcal{E}}(x, F_{n+1})$ is contained in V. This, on the other hand, means that x is an element of $\operatorname{int}_{F_{n+1}}(V)$, which is a set of $\mathcal{W}_0$. Hence, x is an element of $\mathcal{U}_i$, where i is the largest integer for which x is an element of a set of $\mathcal{W}_i$. In conclusion, $\mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i$ is a uniformly bounded cover of X.

It remains to check that the collections $\mathcal{U}_i$ are E-disjoint. For this, it is enough to show that they are F-disjoint. Assume to the contrary that U_1 and U_2 are distinct elements of $\mathcal{U}_i$, and (x, y) is an element of $(U_1 \times U_2) \cap F$. Then there exist distinct intersections of $\mathcal{V}$-sets

$V_{1,0} \cap \ldots \cap V_{1,i}$ and $V_{2,0} \cap \ldots \cap V_{2,i}$ containing $B_{\mathcal{E}}(x, F_{n+1-i})$ and $B_{\mathcal{E}}(y, F_{n+1-i})$ respectively. Note that there are at least $i+2$ different $V_{j,k}$ sets in total, since the intersections are distinct.

Because (x, y) is in F, $B_{\mathcal{E}}(x, F)$ contains y. In the case where i equals n, this means that the sets $V_{1,j}$ all contain y. This is a contradiction, since y would be contained in an intersection of $n+2$ different sets of $\mathcal{V}$. On the other hand, if i is less than n, the ball $B_{\mathcal{E}}(x, F_{n+1-i})$ contains the ball $B_{\mathcal{E}}(y, F_{n-i})$. Hence, the ball $B_{\mathcal{E}}(y, F_{n-i})$ is contained in all sets $V_{j,k}$. This implies that y is an element of $\cup \mathcal{W}_{i+1}$, which is also a contradiction. Hence, the families $\mathcal{U}_i$ are F-disjoint, which concludes the proof. □

5.3 Properties and examples

In this section, a few properties of asymptotic dimension are derived, and afterwards the asymptotic dimensions of several important spaces are computed. Further information beyond the contents of this chapter can be found for example in [2].

Proposition 5.8. *Let X and Y be coarse spaces. If X is coarsely embeddable into Y, the inequality* $\operatorname{asdim} X \leq \operatorname{asdim} Y$ *holds.*

Proof. Let f be a coarse embedding from X to Y. Assume that $\operatorname{asdim} Y$ is at most n. The claim follows by showing that $\operatorname{asdim} X$ is also at most n. The alternate definition from Proposition 5.7 will be used for this.

Let E be a controlled set of X. Due to Proposition 3.22, the set $E' = (f \times f)(E)$ is controlled. Using Proposition 5.7, there are $n+1$ uniformly bounded E'-disjoint families $\mathcal{V}_0, \ldots, \mathcal{V}_n$, the union of which covers Y. Define the families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ via $\mathcal{U}_i = \{f^{-1}V \mid V \in \mathcal{V}_i\}$.

The union of the families $\mathcal{U}_i$ is a cover of X. Fix an index i, and denote by B_i' the bounding set of the cover $\mathcal{V}_i$. If U is an element of a cover $\mathcal{U}_i$, U is of the form $f^{-1}V$, where V^2 is contained in B_i'. Hence, U^2 is contained within the controlled set $(f \times f)^{-1}B_i'$, which shows the uniform boundedness of the families $\mathcal{U}_i$.

Finally, the E-disjointness of the families $\mathcal{U}_i$ is shown. Let U_1 and U_2 be two different sets of a given $\mathcal{U}_i$, and denote by V_1 and V_2 the corresponding sets of $\mathcal{V}_i$ which fulfill $U_j = f^{-1}V_j$ for the values $j = 1, 2$. Assume to the contrary that (x_1, x_2) is an element of $E \cap (U_1 \times U_2)$. The pair $(f(x_1), f(x_2))$ is an element of $(f \times f)(E)$, which was denoted by E'. The same pair is also an element of $f(f^{-1}V_1) \times f(f^{-1}V_2)$, which is a subset of $V_1 \times V_2$. This is a contradiction, since the families $\mathcal{V}_i$ were selected to be E'-disjoint. Hence, the families $\mathcal{U}_i$ are E-disjoint, and consequently they fulfill the requirements of Proposition 5.7. □

The previous Proposition immediately yields two useful results as immediate corollaries.

Corollary 5.9. *Asymptotic dimension is a coarse invariant.*

Proof. Let X and Y be coarse spaces, and let $f : X \to Y$ be a coarse equivalence with coarse inverse g. By Proposition 3.21, both f and g are coarse embeddings, and therefore, Proposition 5.8 yields the inequality $\operatorname{asdim} X \leq \operatorname{asdim} Y \leq \operatorname{asdim} X$. □

Corollary 5.10. *Let X be a coarse space, and let Y be a coarse subspace of X. In this case, the inequality $\operatorname{asdim} Y \leq \operatorname{asdim} X$ holds.*

Proof. Since identity maps are coarse equivalences, Definition 3.20 shows that the inclusion $i : Y \hookrightarrow X$ is a coarse embedding. Therefore, the claim follows from Proposition 5.8. □

Next, the asymptotic dimensions of various example spaces are derived.

Proposition 5.11. *The spaces $\mathbb{R}$ and $\mathbb{Z}$ equipped with the euclidean metric have an asymptotic dimension of 1.*

Proof. It has been shown in Example 3.29 that the spaces $\mathbb{R}$ and $\mathbb{Z}$ are coarsely equivalent. Therefore, they have the same asymptotic dimension. Observe the following covers of $\mathbb{Z}$, where k takes values in positive integers:

$$\mathcal{U}_k = \big\{\{ik, ik+1, \ldots, (i+2)k-1\} \bigm| i \in \mathbb{Z}\big\}.$$

The covers $\mathcal{U}_k$ are totally bounded with bound $2k-1$. For a given value of k, every integer is contained in only 2 sets of $\mathcal{U}_k$, and hence the covers $\mathcal{U}_k$ have an order of 2. Furthermore, for every integer $j \in \mathbb{Z}$ and positive integer k, the set $I_{j,k} = \{j, j+1, \ldots, j+k-1\}$ is contained within a set of $\mathcal{U}_k$. Note that for odd values of k, the set $I_{j,k}$ equals the closed Euclidean ball at $j + (k-1)/2$ with radius $(k-1)/2$. Hence, the covers $\mathcal{U}_k$ attain arbitrarily large Lebesgue numbers as k increases, and therefore by Proposition 5.5, $\mathbb{Z}$ has an asymptotic dimension of at most 1.

Finally, make a counterassumption of $\mathbb{Z}$ having an asymptotic dimension of 0. Hence, there is a uniformly bounded disjoint cover $\mathcal{U}$ of $\mathbb{Z}$ with Lebesgue number greater than 1. Select an element U of $\mathcal{U}$. Since U is bounded, it has a greatest element u. Since the sets of $\mathcal{U}$ are mutually disjoint, no set of it contains the pair $\{u, u+1\}$. This contradicts $\mathcal{U}$ having a Lebesgue number greater than 1. In conclusion, both $\mathbb{Z}$ and $\mathbb{R}$ have an asymptotic dimension of 1. □

Showing that asdim $\mathbb{R}^n$ and asdim $\mathbb{Z}^n$ equal n is considerably more complicated. However, due to its importance, the key points of a proof for it will be sketched here.

A natural way of obtaining the upper bound for asdim $\mathbb{R}^n$ and asdim $\mathbb{Z}^n$ is by proving a product theorem for asymptotic dimension. More specifically, for metric spaces X and Y, one can show that $\operatorname{asdim}(X \times Y)$ is at most the sum $\operatorname{asdim} X + \operatorname{asdim} Y$. A proof for this can be found in the paper [2]. As an alternate approach, one can explicitly construct covers in $\mathbb{R}^n$ or $\mathbb{Z}^n$ which fulfill Proposition 5.5.

The lower bound for asdim $\mathbb{R}^n$ and asdim $\mathbb{Z}^n$, on the other hand, can be reduced to the topological dimension of $[0, 1]^n$ via a trick outlined in [3].

Proposition 5.12. *The asymptotic dimension of* $\mathbb{R}^n$ *is at least the topological dimension of* $[0, 1]^n$.

Proof. Assume that asdim $\mathbb{R}^n$ is k. Let $\mathcal{U}$ be an open cover of the space $[0, 1]^n$. Since $[0, 1]^n$ is compact, the cover $\mathcal{U}$ has a Lebesgue number ε. Let $\mathcal{V}$ be a uniformly bounded cover of $\mathbb{R}^n$ with a Lebesgue number of 1 and an order of at most $k + 1$. Since $\mathcal{V}$ has a positive Lebesgue number, the family

$$\mathcal{V}' = \{\operatorname{int}_{\mathbb{R}^n}(V) \mid V \in \mathcal{V}\}$$

is an open cover of $\mathbb{R}^n$. Furthermore, since $\mathcal{V}'$ is a refinement of $\mathcal{V}$, $\mathcal{V}'$ is uniformly bounded and has an order of at most $k + 1$.

Denote by R the uniform bound of $\mathcal{V}'$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the map defined by $f(x) = \varepsilon x/R$. Note that f is a homeomorphism which shrinks distances by a factor of ε/R. Therefore, the family

$$\mathcal{U}' = \{f(V') \cap [0, 1]^n \mid V' \in \mathcal{V}'\}$$

is an open cover of $[0, 1]^n$ with an uniform bound of ε and an order of at most $k + 1$. Since $\mathcal{U}'$ has an uniform bound of ε and $\mathcal{U}$ has a Lipschitz number of ε, the cover $\mathcal{U}'$ is a refinement of $\mathcal{U}$. Hence, every open cover of $[0, 1]^n$ has an open refinement with an order of at most $k + 1$. This results in the inequality $\dim [0, 1]^n \leq k$, which concludes the proof. □

Proving that the topological dimension of $[0, 1]^n$ is at least n is a non-elementary topological fact. One can see [5, Thms. 1.8.1, 1.7.9, 1.7.7] for a demonstration on how to derive this from the general form of Brouwer's fixed point theorem.

Finally, examples are given to demonstrate some of the more unintuitive aspects of asymptotic dimension.

Example 5.13. Let X be the set $\{2^n \mid n \in \mathbb{N}\}$ equipped with the relative metric from $\mathbb{R}$. For every nonnegative integer n, define a cover $\mathcal{U}_n$ via

$$\mathcal{U}_n = \left\{ \{2^0, 2^1, \ldots, 2^n\}, \{2^{n+1}\}, \{2^{n+2}\}, \ldots \right\}.$$

For any two different sets U_1 and U_2 of $\mathcal{U}_n$, the distance $d(U_1, U_2)$ is at least 2^n. Hence, the cover $\mathcal{U}_n$ is $(2^n - 1)$-disjoint. Furthermore, the cover $\mathcal{U}_n$ is uniformly bounded with bounding diameter 2^n. Due to Proposition 5.7, X has an asymptotic dimension of 0.

Example 5.14. Let X be the vertex set of a possibly infinite tree graph, with root node x_0. For a given vertex x, denote the set of its descendants by $D(x)$. One can obtain a metric d in X by defining $d(x, y)$ as the amount of vertices in the shortest path from x to y. Define a map $f : X \to \mathbb{N}$ by $f(x) = d(x, x_0)$. The value of f at x will be called the root distance of x.

Fix a number r. By Proposition 5.11 and the subset property of asymptotic dimension, there is a uniformly bounded cover $\mathcal{U}$ of $\mathbb{N}$ with Lebesgue number r and an order of 2. For every set $U \in \mathcal{U}$, denote its smallest element by n_U. Now, one may define a collection $\mathcal{V}$ as follows:

$$\mathcal{V} = \left\{ f^{-1}U \cap D(x) \mid U \in \mathcal{U}, f(x) = n_U \right\}.$$

Note that for a given $U \in \mathcal{U}$, the sets $D(x)$ with $f(x) = n_U$ form a mutually disjoint cover of $f^{-1}U$. Hence, $\mathcal{V}$ is a cover of X with an order of at most 2. The cover $\mathcal{V}$ is uniformly bounded by $2R$, where R is the uniform bound of $\mathcal{U}$. Furthermore, it is easy to see that $\mathcal{V}$ also has the Lebesgue number r. Hence, the space X has an asymptotic dimension of at most 1.

Next, observe the case where x_0 has infinitely many descendants, and every other element of x has exactly one descendant. Let Y be a coarse space which is coarsely equivalent to X, and let $g : X \to Y$ be the resulting coarse equivalence. By Proposition 3.21, g is a coarse embedding, and by Corollary 3.33, Y is a metric space. Let ρ_+ and ρ_- be the functions provided by Proposition 3.23 for the map g.

Select an integer k for which $2k$ is greater than $\rho_+(0)$. Hence, if x and x' are two distinct elements with root distance k, g maps them into distinct elements. Now select an $s \in \mathbb{R}$ fulfilling $\rho_-(s) > k$. In this case, there are infinitely many distinct elements y in Y fulfilling $d(y, g(x_0)) < s$. As a result, the coarse equivalence class of the given space X contains no spaces with bounded geometry, despite X having a finite asymptotic dimension.

5.4 Asymptotic dimension and coarse embeddings

This section explores the connections of asymptotic dimension to coarse embeddability into Hilbert spaces. In the paper [13], it was originally proven that a finite asymptotic dimension implies the coarse Baum-Connes conjecture for proper metric spaces. It turns out that this is related to the result from paper [14], as a finite asymptotic dimension implies coarse embeddability into a Hilbert space.

The proofs of the main results use a specific class of functions, which are derived from the covers given by Proposition 5.5. The following lemma proves the main properties of these functions. The proof of the lemma uses ideas from the proof of a common reformulation of asymptotic dimension in terms of simplicial complexes. This proof can be found from [3].

Lemma 5.15. *Let (X, d) be an unbounded metric space, and fix a positive real number λ. Let $\mathcal{V}$ be a uniformly bounded cover of X with Lebesgue number λ and a finite order of at most n. One may define a map $f : X \to \mathbb{R}^{\mathcal{V}}$ as follows:*

$$(f(x))_V = \frac{d(x, X \setminus V)}{\sum_{V' \in \mathcal{V}} d(x, X \setminus V')}.$$

The map f is well defined, and fulfills the following conditions:

- *The image set $f(X)$ is contained within the probability space $\mathcal{P}(\mathcal{V})$.*
- *The image set $f(X)$ is contained within the space $l^2(\mathcal{V})$.*
- *The inequality $\|f(x) - f(y)\|_{l^1} \leq (2n + 1)^2\lambda^{-1}d(x, y)$ holds for all $x, y \in X$.*
- *The inequality $\|f(x) - f(y)\|_{l^2} \leq (2n + 1)^2\lambda^{-1}d(x, y)$ holds for all $x, y \in X$.*
- *There is a constant R_f for which, whenever $d(x, y)$ is greater than R_f, the inequality $\|f(x) - f(y)\|_{l^2} \geq \sqrt{2/n}$ holds.*

Proof. Since X is unbounded, it isn't an element of $\mathcal{V}$. Hence, the distances used to define f are well defined and finite. Note that if V is an element of $\mathcal{V}$, $d(x, X \setminus V)$ is positive only if x is an element of V. Furthermore, since $\mathcal{V}$ has an order of at most n, every x is an element of at most n sets $V \in \mathcal{V}$. Hence, for every $x \in X$, $f(x)$ has at most n nonzero coordinates, and the sum $\sum_{V' \in \mathcal{V}} d(x, X \setminus V')$ is finite.

The final requirement for the well-definedness of f is that the sum $\sum_{V' \in \mathcal{V}} d(x, X \setminus V')$ is nonzero. This follows from the fact that $B_d(x, \lambda)$ is contained in some set $V \in \mathcal{V}$, which implies the inequality

$$\sum_{V' \in \mathcal{V}} d(x, X \setminus V') \geq \lambda.$$

As previously noted, $f(x)$ has at most n nonzero coordinates for every $x \in X$. Hence, the image set $f(X)$ is contained in both $l^2(\mathcal{V})$ and $l^1(\mathcal{V})$. To further see that the image is in $\mathcal{P}(\mathcal{V})$, note that for a fixed $x \in X$, every coordinate of $f(x)$ is nonnegative, and the L^1-norm of $f(x)$ is

$$\|f(x)\|_{l^1} = \frac{\sum_{V\in\mathcal{V}} d(x, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(x, X \setminus V')} = 1.$$

Next, the Lipschitz bounds of f are proven. Using the triangle inequality, one concludes for any set $A \subset X$ the inequality

$$d(x, A) - d(y, A) \leq (d(x, y) + d(y, A)) - d(y, A) = d(x, y).$$

Since the estimate is symmetric with respect to x and y, one obtains the inequality $|d(x, A) - d(y, A)| \leq d(x, y)$. Fix a set $V \in \mathcal{V}$, and compute an upper bound for $|(f(x) - f(y))_V|$ as follows:

$$\begin{aligned}
&|(f(x) - f(y))_V| \\
&= \left| \frac{d(x, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(x, X \setminus V')} - \frac{d(y, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(y, X \setminus V')} \right| \\
&\leq \left| \frac{d(x, X \setminus V) - d(y, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(x, X \setminus V')} \right| + \left| \frac{d(y, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(x, X \setminus V')} - \frac{d(y, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(y, X \setminus V')} \right| \\
&\leq \frac{d(x, y)}{\lambda} + \frac{d(y, X \setminus V)}{\sum_{V'\in\mathcal{V}} d(y, X \setminus V')} \cdot \frac{\sum_{V'\in\mathcal{V}} |d(x, X \setminus V') - d(y, X \setminus V')|}{\sum_{V'\in\mathcal{V}} d(x, X \setminus V')} \\
&\leq \frac{d(x, y)}{\lambda} + 1 \cdot \frac{2n \cdot d(x, y)}{\lambda} \\
&= \frac{2n + 1}{\lambda} d(x, y).
\end{aligned}$$

Since $f(x)$ and $f(y)$ have at most n nonzero coordinates, $|(f_\lambda(x) - f_\lambda(y))_V|$ is nonzero for at most $2n$ different elements V of $\mathcal{V}$. This yields the following Lipschitz bounds for f:

$$\begin{aligned}
\|f_\lambda(x) - f_\lambda(y)\|_{l^1} &\leq 2n \cdot \left(\frac{2n + 1}{\lambda} d(x, y) \right) \leq \frac{(2n + 1)^2}{\lambda} d(x, y), \\
\|f_\lambda(x) - f_\lambda(y)\|_{l^2} &\leq \sqrt{2n} \left(\frac{2n + 1}{\lambda} d(x, y) \right) \leq \frac{(2n + 1)^2}{\lambda} d(x, y).
\end{aligned}$$

Finally, denote by R_f the uniform bound on the family $\mathcal{V}$. For any element x of X, define the vectors $a_x, b_x \in l^2(\mathcal{V})$ as follows:

$$\begin{aligned}
(a_x)_V &= d(x, X \setminus V) \\
(b_x)_V &= \begin{cases} 1, & d(x, X \setminus V) > 0 \\ 0, & d(x, X \setminus V) = 0 \end{cases}
\end{aligned}$$

Since $d(x, X \setminus V)$ is positive only for at most n sets $v \in \mathcal{V}$, both a_x and b_x are clearly in $l^2(\mathcal{V})$. Using these vectors, one can write $f(x)$ in the form

$$f(x) = \frac{a_x}{\langle a_x, b_x \rangle}.$$

Hence, the Cauchy-Schwarz inequality yields a lower bound on the L^2-norm of $f(x)$:

$$\|f(x)\|_{l^2} = \frac{\|a_x\|_{l^2}}{\langle a_x, b_x \rangle} \geq \frac{1}{\|b_x\|_{l^2}} \geq \frac{1}{\sqrt{n}}.$$

Assume that x and y are two elements of X and the distance $d(x, y)$ is greater than R_f. Due to R_f being the uniform bound on $\mathcal{V}$, no set $V \in \mathcal{V}$ contains both x and y. Hence, $f(x)$ and $f(y)$ are orthogonal, which yields the lower bound

$$\|f(x) - f(y)\|_{l^2} = \sqrt{\|f(x)\|_{l^2}^2 + \|f(y)\|_{l^2}^2} \geq \sqrt{\frac{2}{n}}.$$

□

Theorem 5.16. *Let (X, d) be a metric space with a finite asymptotic dimension of n. Then there exists a coarse embedding from X into a Hilbert space.*

Proof. If X is bounded, it has a trivial coarse embedding into a Hilbert space in the form of a constant mapping. One may therefore assume that X is unbounded.

For every $i \in \mathbb{Z}_+$, Proposition 5.5 gives a uniformly bounded cover $\mathcal{V}_i$ with Lebesgue number $2^{-i}/(2n+3)^2$ and an order of at most $n + 1$. Using the covers $\mathcal{V}_i$, Lemma 5.15 yields 2^{-i}-Lipschitz maps $f_i : X \to H_i$, where the spaces $H_i = l^2(\mathcal{V}_i)$ are Hilbert spaces. Denote the corresponding constants R_{f_i} by R_i. Now, fix an element $x_0 \in X$, and define the map $f : X \to \bigoplus_{i=1}^{\infty} H_i$ by

$$(f(x))_i = f_i(x) - f_i(x_0).$$

What remains is checking that f is well defined and finding the functions ρ_+ and ρ_- of Proposition 3.23 for f.

Let x and y be two elements of X. Since the maps f_i are 2^{-i}-Lipschitz, one obtains the bound

$$\|f(x) - f(y)\|_{l^2} \leq \sqrt{\left(\sum_{i=1}^{\infty} 2^{-i}\right) (d(x, y))^2} = d(x, y).$$

By the definition of f, $f(x_0)$ is zero. Hence, a selection of $y = x_0$ proves the well-definedness of f. Furthermore, the above bound permits the selection of $\rho_+(t) = t$.

Next, denote by $\mathbf{1}_{R_i}$ the characteristic function of the set $\{t \in \mathbb{R} \mid t > R_i\}$. Define the function $g : \mathbb{R}_0 \to \mathbb{R}_0 \cup \{\infty\}$ as follows:

$$g = \sum_{i=1}^{\infty} \mathbf{1}_{R_i} .$$

As of now g may assume the value ∞, but it will be shown that g is in fact finite valued. Clearly g is increasing and tends to infinity. By the final condition of Lemma 5.15, one obtains the bound

$$\|f_i(x) - f_i(y)\|_{l^2} \geq \left(\sqrt{\frac{2}{n+1}}\right) \mathbf{1}_{R_i}(d(x,y)).$$

Using this bound, the following lower bound for $\|f(x) - f(y)\|$ is obtained:

$$\|f(x) - f(y)\|_{l^2} \geq \sqrt{\frac{2}{n+1} g(d(x,y))} .$$

Since X is unbounded and f is well-defined, the estimate obtained shows the finite-valuedness of g. Hence, one can make the selection $\rho_-(t) = \sqrt{2g(t)/(n+1)}$. The selected ρ_- and ρ_+ fulfill the conditions of Proposition 3.23. Therefore, f is a coarse embedding from X into a Hilbert space. □

Theorem 5.17. *Let (X, d) be a metric space with bounded geometry. If the space X has a finite asymptotic dimension of n, the space X has Property A.*

Proof. The proof is based on Proposition 4.11 and Lemma 5.15. Let λ be a positive real number. Due to Corollary 4.10, one may assume that X is unbounded.

Proposition 5.5 yields a uniformly bounded family $\mathcal{V}_\lambda$ with Lebesgue number λ and an order of at most $n+1$. For every x in X, select a set $V_x \in \mathcal{V}_\lambda$ for which $B_X(x, \lambda)$ is contained within V_x. Now, define a cover $\mathcal{U}_\lambda$ via

$$\mathcal{U}_\lambda = \{V_x \mid x \in X\} .$$

The cover $\mathcal{U}_\lambda$ retains the Lebesgue number of λ, is uniformly bounded, and has an order of at most $n+1$. However, the crucial difference with $\mathcal{V}_\lambda$ is that the cardinality of $\mathcal{U}_\lambda$ cannot exceed the cardinality of X. In order to see this, select for every $U \in \mathcal{U}_\lambda$ a point x_U for which U is the set V_{x_U}. In this case, the map $h : U \mapsto x_U$ is injective, which shows that $|\mathcal{U}_\lambda|$ is at most $|X|$.

Using the fixed λ and the selected $\mathcal{U}_\lambda$, Lemma 5.15 yields a $((2n+3)^2/\lambda)$-Lipschitz map $f_\lambda : X \to \mathcal{P}(\mathcal{U}_\lambda)$. By identifying every coordinate U with the coordinate $h(U)$, the space $\mathcal{P}(\mathcal{U}_\lambda)$ can be considered to be a subspace of $\mathcal{P}(X)$. Hence, f_λ defines a map from X into $\mathcal{P}(X)$.

Now, if given a pair (r, ε), one may select λ to be $(2n+3)^2\varepsilon/(2r)$. In this case, f_λ is $(\varepsilon/(2r))$-Lipschitz. Hence, if x and x' are elements of X fulfilling $d(x, x') \leq r$, one obtains the bound

$$\|f(x) - f(x')\|_{l^1} \leq \frac{\varepsilon}{2r} d(x, x') \leq \frac{\varepsilon}{2} < \varepsilon.$$

Furthermore, denote by R the uniform bound on $\mathcal{U}_\lambda$. In this case, if x and x' are elements of X fulfilling $d(x, y) \geq R + 1$, x is not contained within V_y. Due to this, $d(x, X \setminus V_y)$ is zero, and therefore, $(f_\lambda(x))_y = (f_\lambda(x))_{V_y}$ is zero. In conclusion, X fulfills the condition of Proposition 4.11, which combined with the bounded geometry of X implies that X has Property A. □

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